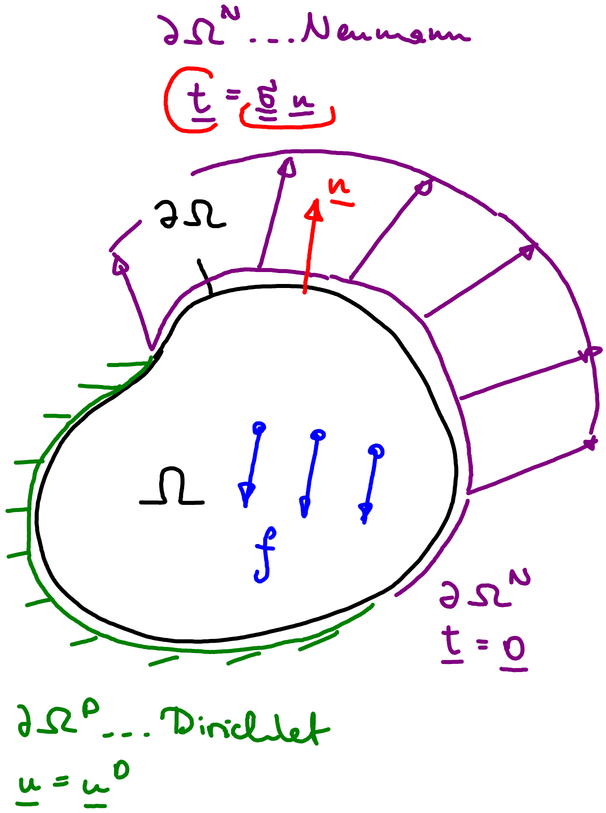
# Finite Element Method, Unit 2

## The Linear Finite Element Method

$$\nabla \underline{\underline{\sigma}} + \cancel{\underline{f}} = \rho \cancel{\underline{a}}$$

$$\sigma_{ji,j} + \cancel{f_i} = \rho \cancel{a_i}$$

$$\sigma_{ji,j} = \frac{\partial \sigma_{ji}}{\partial x_j}$$




strong form

$$\boxed{\begin{array}{ll} \sigma_{ji,j} = 0 & \text{in } \Omega \\ u_i = u_i^D & \text{on } \partial\Omega^D \\ \sigma_{ji} n_j = t_i & \text{on } \partial\Omega^N \end{array}}$$

6 unknowns $\sigma_{ji} = \sigma_{ij}$

3 equations $i = 1,2,3$

### Material law

elasticity tensor

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$$

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} \cdot \underline{\underline{\varepsilon}}$$

6 unknowns $\varepsilon_{kl} = \varepsilon_{lk}$

6 new equations

### Kinematic law

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + [\nabla \underline{u}]^T\right)$$

3 unknowns $u_i$

6 equations

Unknowns: $6\ \sigma_{ij} = \sigma_{ji}$

$6\ \varepsilon_{ij} = \varepsilon_{ji}$

$3\ u_i$

$15$

Equations: 3 Balance

6 Material

6 Kinematic

15 ✓

## Weak form

$$\int_\Omega \sigma_{ji,j} \cdot v_i \, d\Omega = \cancel{\int_\Omega 0 \, v_i \, d\Omega}$$

$$[u v]_{,i} = u_{,i} v + u v_{,i}$$

$$\int_\Omega [\sigma_{ji} v_i]_{,j} \, dV - \int_\Omega \sigma_{ji} v_{i,j} \, dV = 0$$

Gauß: $\int_\Omega (\ldots)_{,i} \, dV = \int_{\partial\Omega} (\ldots) n_i \, dA$

$$\underbrace{\int_{\partial\Omega} \sigma_{ji} v_i n_j \, dA}_{= \int_{\partial\Omega^N} \underline{\sigma_{ji} n_j}_{t_i} v_i \, dA + \int_{\partial\Omega^D} \sigma_{ji} n_j \underline{v_i}_{\emptyset \text{ on } \partial\Omega^D} \, dA} - \int_\Omega \sigma_{ji} v_{i,j} \, dV = 0$$

$$\int_\Omega \sigma_{ji} v_{i,j} \, dV = \int_{\partial\Omega^N} t_i v_i \, dA$$

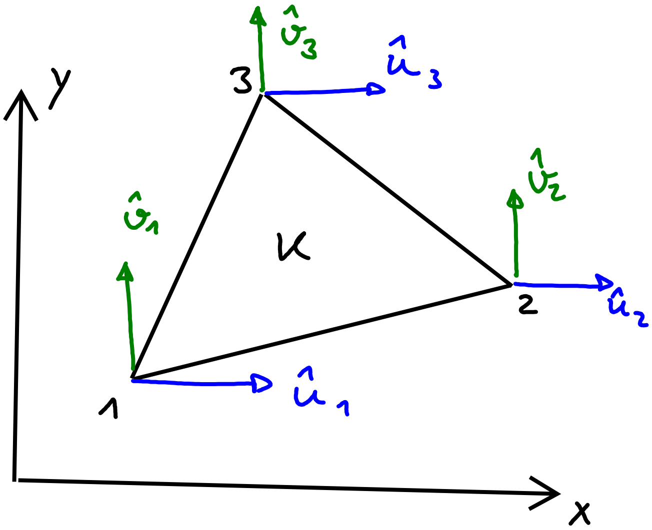
$$\sigma_{ji} = C_{jikl} \varepsilon_{kl} = C_{jikl} \tfrac{1}{2}(u_{k,l} + u_{l,k})$$

$$= C_{jikl} u_{k,l}$$

$$\boxed{\int_\Omega v_{i,j} C_{ijkl} u_{k,l} \, dV = \int_{\partial\Omega^N} t_i v_i \, dA \qquad u_i = u_i^D \text{ on } \partial\Omega^D}$$

weak form

## Interpolation




$$u(x,y) = N^{K,i}(x,y)\ \hat{u}_i$$

$$v(x,y) = N^{K,i}(x,y)\ \hat{v}_i$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \\ \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \end{bmatrix}}_{\text{node displacements}} \underbrace{\begin{bmatrix} N^{K,1} \\ N^{K,2} \\ N^{K,3} \end{bmatrix}}_{\text{shape functions}}$$

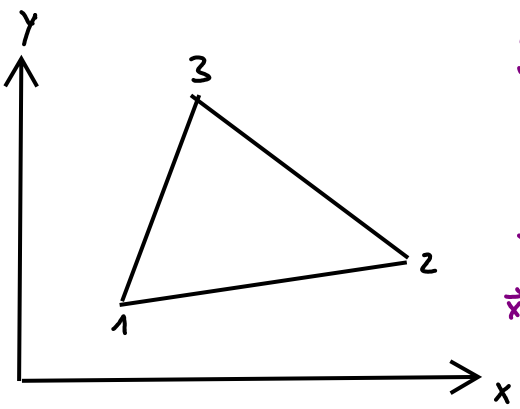
$$u_i = \hat{u}_{ij}\, N^{K,j}$$

$u_1 = u$
$u_2 = v$

## Derivatives & Transformations

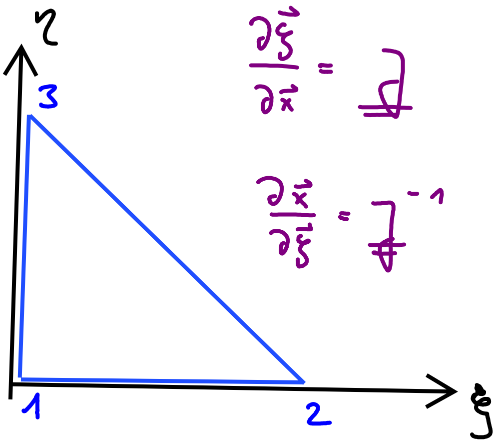
$$\begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \\ \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \end{bmatrix} \begin{bmatrix} N^{K,1}{}_{,x} & N^{K,1}{}_{,y} \\ N^{K,2}{}_{,x} & N^{K,2}{}_{,y} \\ N^{K,3}{}_{,x} & N^{K,3}{}_{,y} \end{bmatrix}$$

$$u_{i,j} = \hat{u}_{ik}\, N^{K,k}{}_{,j}$$




$$\vec{\xi} = T^K(\vec{x})$$

$$\vec{x} = T^{K-1}(\vec{\xi})$$




$$\frac{\partial \vec{\xi}}{\partial \vec{x}} = \underline{\underline{J}}$$

$$\frac{\partial \vec{x}}{\partial \vec{\xi}} = \underline{\underline{J}}^{-1}$$

$$f(x,y) \rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} \quad \text{... chain rule}$$

$$\begin{bmatrix} N^{k,1}{}_{,x} & N^{k,1}{}_{,y} \\ N^{k,2}{}_{,x} & N^{k,2}{}_{,y} \\ N^{k,3}{}_{,x} & N^{k,3}{}_{,y} \end{bmatrix} = \begin{bmatrix} N^1{}_{,\xi} & N^1{}_{,\eta} \\ N^2{}_{,\xi} & N^2{}_{,\eta} \\ N^3{}_{,\xi} & N^3{}_{,\eta} \end{bmatrix} \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix}}_{\text{Jacobian } J = \frac{\partial \vec{\xi}}{\partial \vec{x}}}$$

$$u_i = \hat{u}_{ij} N^{k,j}$$

$$u_{i,k} = \hat{u}_{ij} N^{k,j}{}_{,k}$$

rename indices

$$u_{k,e} = \hat{u}_{km} N^m{}_{,n} J_{ne}$$

$$= \hat{u}_{ij} N^j{}_{,e} J_{ek}$$

## Discretization of test functions

$$u_i = \hat{u}_{io} N^o = \hat{u}_{po} \delta_{pi} N^o$$

$$v_i \longrightarrow \delta_{pi} N^o$$

$$v_{i,j} \longrightarrow \delta_{pi} N^o{}_{,q} J_{qj}$$

# Linear Equation System

$$\int_\Omega v_{i,j} C_{ijkl} u_{k,l}\, d\bar{V} = \int_{\partial\Omega^N} t_i v_i \, dA$$

$$\int_\Omega v_{i,j} C_{ijkl} u_{k,l}\, d\bar{V}$$

$$= \int_{\underbrace{\Omega}_{\Omega^{ref}}} \underbrace{\delta_{pi} N^o_{,q} J_{qj}}_{v_{i,j}} C_{ijkl} \underbrace{\hat{u}_{km} N^m_{,n} J_{nl}}_{u_{k,l}} \underbrace{d\bar{V}}_{\det J^{-1} dV}$$

$$\underbrace{\int_{\Omega^{ref}} C_{pjkl} N^o_{,q} J_{qj} N^m_{,n} J_{nl} \det J^{-1}\, dV}_{A_{pokm} \text{ ... stiffness matrix}} \hat{u}_{km}$$

imkn j p pm l o on kl

$A_{ijkl}$

$$A_{ijkl} = \int_{\Omega^{ref}} C_{imkn} N^j_{,p} J_{pm} N^l_{,o} J_{on} \det J^{-1}\, d\bar{V}$$

$$\int_{\partial\Omega^N} t_i v_i \, dA = \int_{\partial\Omega^N} t_i \delta_{io} N^p \, dA = \underbrace{\int_{\partial\Omega^N} t_o N^p \, dA}_{F_{op} \text{ ... force vector}}$$

$$F_{ij} = \int_{\partial\Omega^N} t_i N^j \, dA$$

$$A_{ijkl} \hat{u}_{kl} = F_{ij}$$