Finite Element Method, Unit 3

The Newton-Raphson Method

Motivation: $\sigma_{ji,j} = 0$ $\quad (\vec{t} = 0, \ \vec{f} = 0)$

$$\int_v \sigma_{ji} v_{i,j} \, dv = 0$$

linear elasticity:

- $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ $\Rightarrow$ linear kinematics
- $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$ $\Rightarrow$ linear material

hyperelasticity:

- $E_{IJ} = \frac{1}{2}(F_{kI} F_{kJ} - \delta_{IJ})$

$$= \frac{1}{2}(u_{I,J} + u_{J,I} + u_{k,I} \cdot u_{k,J})$$

$\Rightarrow$ nonlinear kinematics

- $\underline{\underline{S}} = \dfrac{\partial \psi}{\partial \underline{\underline{E}}}(\underline{\underline{E}})$

$\Rightarrow$ nonlinear material

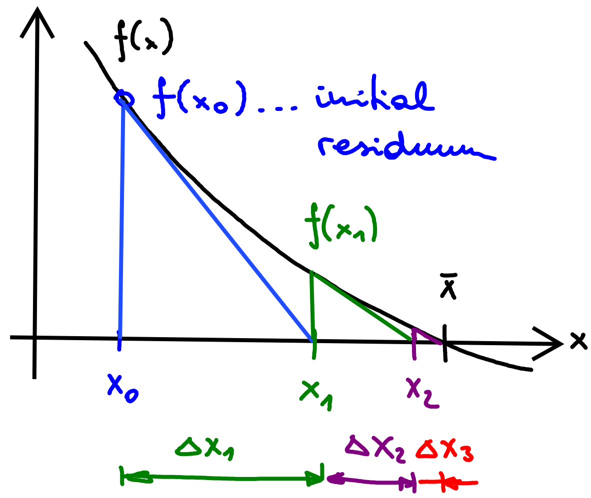
The Newton-Raphson method for scalar equations

given: $f(x)$ ... differentiable, nonlinear

$f'(x)$ ... calculatable

x_0 ... initial guess

wanted: $\bar{x}$ with $f(\bar{x}) = 0$



Taylor series

$$f(x_0 + \Delta x_1) = f(x_0) + \Delta x_1 f'(x_0) + \cancel{\ldots} \overset{!}{=} 0$$

$$\Delta x_1 = -\frac{f(x_0)}{f'(x_0)} = -\left[f'(x_0)\right]^{-1} f(x_0)$$

$$x_1 = x_0 + \Delta x_1 = x_0 - \left[f'(x_0)\right]^{-1} f(x_0)$$

in general: $x_{k+1} = x_k - \left[f'(x_k)\right]^{-1} f(x_k)$

$x_{k+1} = \varphi(x_k)$ ... fixed point iteration

$$\left.\begin{array}{l} f(x_k) \ \ldots \ \text{Residuum} \\ \Delta x_k \ \ldots \ \text{Increment} \end{array}\right\} \text{both should converge to zero}$$

how many steps to go?

$\Rightarrow$ define convergence criteria

e.g. $c_{res} = 10^{-10}$, $c_{\Delta x} = 10^{-10}$

$\Rightarrow$ define max number of iterations

e.g. $k_{max} = 10$

$\Rightarrow$ check convergence in each step

if $|f(x_{k+1})| < c_{res}$ and $|\Delta x_{k+1}| < c_{\Delta x}$

$\bar{x} \approx x_{k+1}$

else if $k+1 = k_{max}$

error : No convergence

end if

Nonlinear Equation Systems

$$\left.\begin{aligned} f^1(x^1, x^2, \dots, x^n) &= 0 \\ f^2(x^1, x^2, \dots, x^n) &= 0 \\ &\ \vdots \\ f^n(x^1, x^2, \dots, x^n) &= 0 \end{aligned}\right\} \vec{f}(\vec{x}) = 0$$

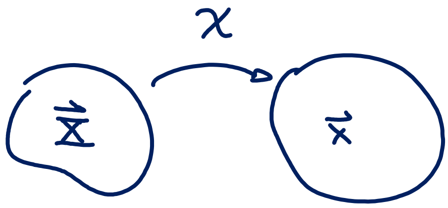
$$\vec{x}_0 = \left[x_0^1, x_0^2, \dots, x_0^n\right]^T$$

$$\vec{x}_{k+1} = \vec{x}_k - \left[\frac{\partial \vec{f}}{\partial \vec{x}}(\vec{x}_k)\right]^{-1} \vec{f}(\vec{x}_k)$$

$$\frac{\partial \vec{f}}{\partial \vec{x}} = \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^n}{\partial x^1} & \cdots & \frac{\partial f^n}{\partial x^n} \end{bmatrix}$$

$\Rightarrow$ everything else stays the same

$$\int \sigma_{ji} \, v_{i,j} \, dv = 0$$



$$\Leftrightarrow$$

deformation χ $\quad \vec{x} = \chi(\vec{X})$