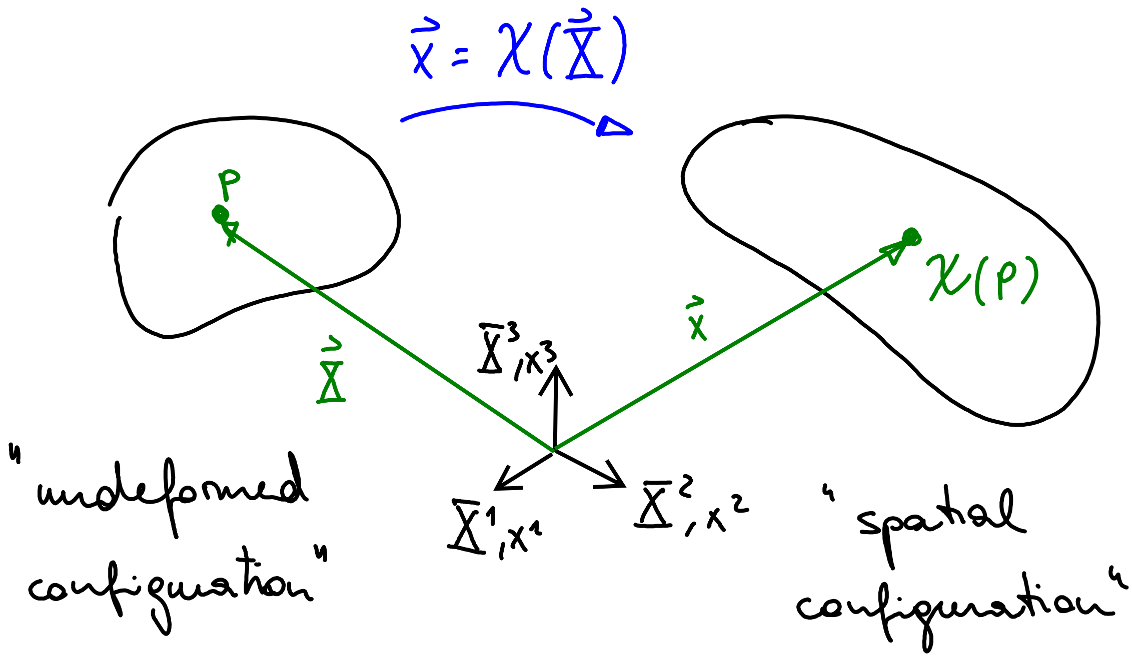
Finite Element Method, Unit 4

Hyperelasticity & Linearization

Kinematics



$\underline{\underline{F}} = D\chi = \dfrac{\partial \vec{x}}{\partial \vec{X}}$ ... deformation gradient

$J = \det \underline{\underline{F}}$

$dv = J\, dV$

$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{1}})$ ... Green-Lagrange strain tensor

$\underline{\underline{A}} = \frac{1}{2}(\underline{\underline{i}} - \underline{\underline{F}}^{-T} \underline{\underline{F}}^{-1})$ ... Euler-Almansi strain tensor

Stress Tensors

Cauchy : $\underline{\underline{\sigma}}$ $\qquad \sigma^{ij}$

Kirchhoff : $\underline{\underline{\tau}} = J \underline{\underline{\sigma}}$ $\qquad \tau^{ij} = J \sigma^{ij}$

1^{st} PK : $\underline{\underline{P}} = J \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-T}$ $\qquad P^{iJ} = \sigma^{ij} J F^{-T}{}_j{}^J$

Piola transformation

2^{nd} PK : $\underline{\underline{S}} = J \underline{\underline{F}}^{-1} \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-T}$ $\qquad S^{IJ} = J F^{-1 I}{}_i \sigma^{ij} F^{-T}{}_j{}^J$

Work conjugate pairs

$$\omega(\chi, \vec{v}) = \int_v \sigma^{ij}(\chi)\, v^i{}_{,j}\, dv = 0 \qquad (\vec{t} = \vec{0},\ \vec{f} = \vec{0})$$

$$= \int_V P^{iJ}\, \delta F^i{}_J\, dV \qquad \delta F^i{}_J = v^i{}_{,J} = v^i{}_{,j} F^j{}_J$$

$$= \int_V \underline{F^i{}_I S^{IJ} \delta F^i{}_J}\, dV$$

we use this formulation

$$= \int_V S^{IJ} \underbrace{F^i{}_I \delta F^i{}_J}_{\delta E_{IJ}}\, dV$$

Material / Constitutive Law

$$\hat{\varphi}(\underline{\underline{F}}) \rightarrow \underline{\underline{P}} = \frac{\partial \hat{\varphi}}{\partial \underline{\underline{F}}} \qquad \hat{\varphi} \text{ ... elastic potential}$$

$$\varphi(\underline{\underline{E}}) \rightarrow \underline{\underline{S}} = \frac{\partial \varphi}{\partial \underline{\underline{E}}}$$

linear el.: $\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$

$$\cancel{S^{IJ} = C^{IJKL} E_{KL}}$$

dir. der.: $\underbrace{D\underline{\underline{S}}(\chi)[\Delta\vec{u}]}_{\Delta\underline{\underline{S}}} = \left[\frac{d}{d\varepsilon} \underline{\underline{S}}(\chi + \varepsilon \Delta\vec{u}) \right]_{\varepsilon = 0}$

$$\Delta \underline{\underline{S}} = \Delta \frac{\partial \psi}{\partial \underline{\underline{E}}} = \underbrace{\frac{\partial^2 \psi}{\partial \underline{\underline{E}}\, \partial \underline{\underline{E}}}}_{\underline{\underline{\underline{\underline{\mathbb{C}}}}}\ \ldots \text{ material elasticity tensor}} : \Delta \underline{\underline{E}} \quad \ldots \text{ chain rule}$$

Linearization - Newton Raphson Method

$$W(\chi, \vec{v}) = 0$$

$$W(\chi_k + \Delta u_{k+1}, \vec{v}) \approx W(\chi_k, \vec{v}) + \underbrace{DW(\chi_k, \vec{v})[\Delta \vec{u}_{k+1}]}_{\Delta W} \overset{!}{=} 0$$

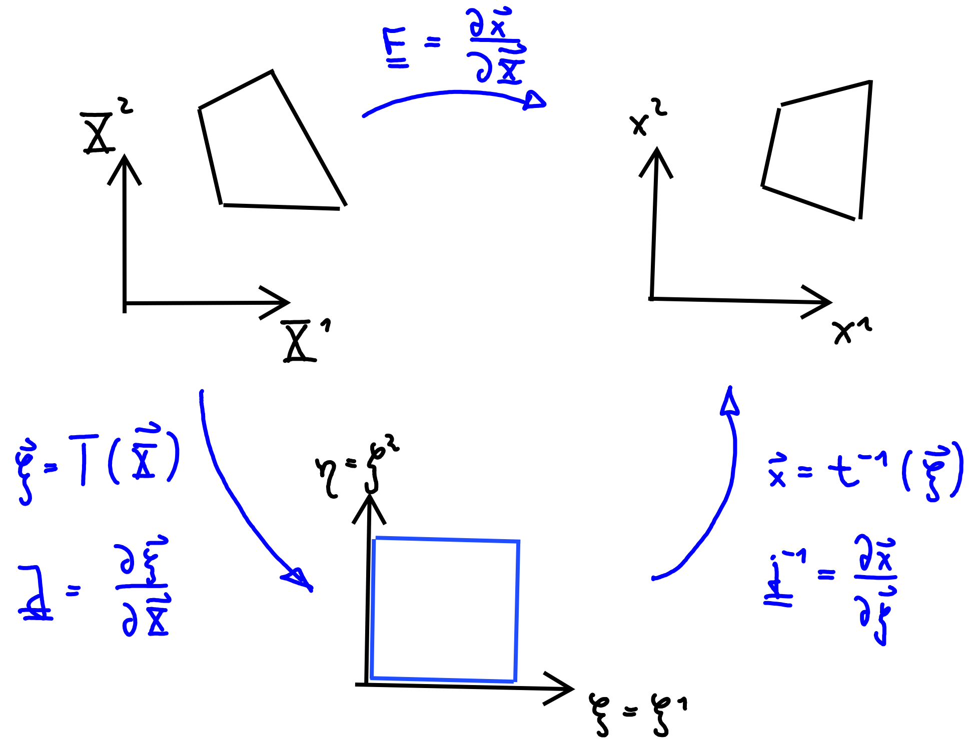
$$W(\chi, \vec{v}) = \int_V \underline{\underline{F}}\, \underline{\underline{S}} : \delta \underline{\underline{F}}\, dV$$

$$\Delta W = \int_V \Delta(\underline{\underline{F}}\, \underline{\underline{S}}) : \delta \underline{\underline{F}}\, dV$$

$$= \underbrace{\int_V \Delta \underline{\underline{F}}\, \underline{\underline{S}} : \delta \underline{\underline{F}}\, dV}_{① \text{ ISC}} + \underbrace{\int_V \underline{\underline{F}} \big(\underbrace{\underline{\underline{\underline{\underline{C}}}}\, \Delta \underline{\underline{E}}}_{\Delta \underline{\underline{S}}}\big) : \delta \underline{\underline{F}}\, dV}_{② \text{ CC}}$$

Finite element discretization

$\vec{x} = \chi(\vec{X})$



$$\underline{\underline{F}} = \frac{\partial \vec{x}}{\partial \vec{X}} = \frac{\partial \vec{x}}{\partial \vec{\xi}} \frac{\partial \vec{\xi}}{\partial \vec{X}} = \underline{\underline{j}}^{-1} \underline{\underline{J}}$$

$\underline{\underline{F}} = \underline{\underline{j}}^{-1} \underline{\underline{J}}$ ... calc. with Jacobian classes

$\Delta \underline{\underline{F}} = \Delta \underline{\underline{j}}^{-1} \underline{\underline{J}}$ $\quad \underline{\underline{J}} = \frac{\partial \vec{\xi}}{\partial \vec{X}} \quad \vec{\xi}, \vec{X}$ ... const.

$$\Delta \underline{\underline{j}}^{-1} = D \frac{\partial \vec{x}}{\partial \vec{\xi}} [\Delta \vec{u}] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\partial(\vec{x} + \varepsilon \Delta \vec{u})}{\partial \vec{\xi}} = \frac{\partial \Delta \vec{u}}{\partial \vec{\xi}}$$

$$\Delta \underline{\underline{F}} = \frac{\partial \Delta \vec{u}}{\partial \vec{\xi}} \underline{\underline{J}}$$

Interpolation

inc. node displacements

Shape. getArray

$$\Delta \vec{u} = \Delta \underline{\underline{\hat{u}}} \cdot \underline{N}$$

$\Delta u_i = \Delta \hat{u}_{ik} N_k \qquad i = 1 \ldots \text{dim.}$

$\qquad\qquad\qquad\qquad k = 1 \ldots \#\text{nodes}$

$$\frac{\partial \Delta u_i}{\partial \xi_j} = \Delta u_{i,j} = \Delta j^{-1}{}_{ij} = \Delta \hat{u}_{ik} N_{k,j}$$ Shape.get Derivative Array

$$\Delta F_{ij} = \Delta j^{-1}{}_{ik} J_{kj} = \Delta \hat{u}_{il} N_{l,k} J_{kj}$$

$$= \Delta \hat{u}_{ml} \, \delta_{mi} N_{l,k} J_{kj}$$

$$\delta F_{ij} = v_{i,j} \longrightarrow \delta_{mi} N_{l,k} J_{kj}$$ discretization of test functions

① Initial Stress Component

$$\int_\Omega (\Delta \underline{F} \, \underline{\underline{S}}) : \delta \underline{F} \, d\bar{V}$$

$$= \int_\Omega (\Delta F S)_{il} \, \delta F_{il} \, d\bar{V}$$

$$= \int_\Omega \Delta F_{ik} S_{kl} \, \delta F_{il}$$ discr. of inc. disp & test functions

$$= \int_{\Omega^{ref}} \underbrace{\Delta \hat{u}_{im} N_{m,j} J_{jk}}_{\Delta F_{ik}} S_{kl} \underbrace{\delta_{ni} N_{o,p} J_{pl}}_{(\delta F_{il})_{no}} \det J^{-1} \, d\bar{V}$$

$$= \underbrace{\int_{\Omega^{ref}} \underset{ik}{\delta_{ni}} \underset{lm}{N_{m,j}} \underset{mn}{J_{jk}} \underset{no}{S_{kl}} \underset{jp}{N_{o,p}} \underset{po}{J_{pl}} \det J^{-1} \, d\bar{V}}_{A^{①}\,{}^{ijkl}_{noim}} \; \underset{kl}{\Delta \hat{u}_{im}}$$

$$A^{(1)}_{ijkl} = \int_{\Omega^{ref}} \delta_{ik} N_{l,m} J_{mn} S_{no} N_{j,p} J_{po} \det J^{-1} \, d\bar{V}$$

$A^{(2)}_{ijkl}$ ... const. comp.

$$\int_{\Omega} (\underline{\underline{F}} \, \Delta \underline{\underline{S}}) : \delta \underline{\underline{F}} \, dV$$

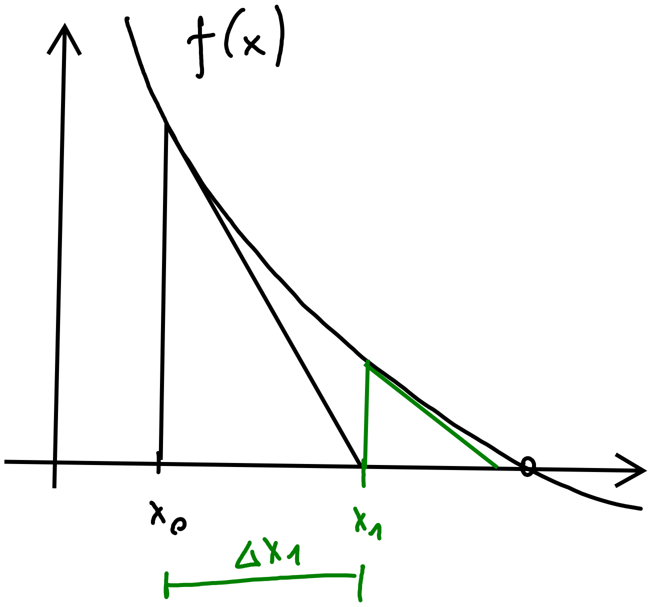
$$= \int_{\Omega} \underline{\underline{F}} \left[\underline{\underline{\underline{\underline{C}}}} : (\Delta \underline{\underline{F}}^T \underline{\underline{F}}) \right] : \delta \underline{\underline{F}} \, dV$$

$$= \int_{\Omega} F_{ik} C_{kjlm} \Delta F_{ol} F_{om} \, \delta F_{ij} \, dV \longrightarrow A^{(2)}_{ijkl} \Delta \hat{u}_{kl}$$

F^{int}_{ij} ... internal forces

$$\int_{\Omega} (\underline{\underline{F}} \, \underline{\underline{S}}) : \delta \underline{\underline{F}} \, dV$$

$$= \int_{\Omega} F_{ik} S_{kj} \, \delta F_{ij} \, dV \longrightarrow F^{int}_{ij}$$



Newton

$$\vec{f}(\vec{x}) = \vec{0}$$

$$\frac{\partial \vec{f}}{\partial \vec{x}}$$

$$\left.\frac{\partial \vec{f}}{\partial \vec{x}}\right|_k \cdot \Delta \vec{x}_{k+1} = - \vec{f}\,\Big|_k$$

Mechanics

$$\omega(\chi, \vec{v}) = 0$$

$$D\omega(\chi, \vec{v})[\Delta \vec{u}] \longrightarrow \text{Newton Tangent}$$

$$D\omega(\chi_k, \vec{v})[\Delta \vec{u}_{k+1}] = -\omega(\chi_k, \vec{v})$$

FEM

$$A_{ijkl}\Big|_k \, \Delta u_{kl}\Big|_{k+1} = - F^{int}_{ij}\Big|_k$$