λ-calculus

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Lemma – $\forall t, |t| \to_\beta^* \varphi(t)$

$$\begin{tikzcd}
 & t \arrow[dl, "|\ |"'] \arrow[dr, "\varphi"] & \\
|t| \arrow[rr, dotted, two heads, "\beta"'] & & \varphi(t)
\end{tikzcd}$$

Proof: We will show our property by induction on the term t.

- Case $t = x$, $|x| = x = \varphi(x)$, so we have $|t| \to_\beta^0 \varphi(t)$.
- Case $t = \lambda x.t_0$, by the induction hypothesis we know that $|t_0| \to_\beta^* \varphi(t_0)$.

$$\begin{aligned}
|(\lambda x.t_0)| &= \lambda x.|t_0| \\
&\to_\beta^* \lambda x.\varphi(t_0) && \text{By induction hypothesis} \\
&= \varphi(\lambda x.t_0)
\end{aligned}$$

- Case $t = \underline{\lambda} x.t_0$, by the induction hypothesis we know that $|t_0| \to_\beta^* \varphi(t_0)$.

$$\begin{aligned}
|(\underline{\lambda} x.t_0)| &= \lambda x.|t_0| \\
&\to_\beta^* \lambda x.\varphi(t_0) && \text{By induction hypothesis} \\
&= \varphi(\underline{\lambda} x.t_0)
\end{aligned}$$

- Case $t = t_1\ t_2$ where $t_1 \neq \underline{\lambda} x.t_0$, by the induction hypothesis we know that $|t_1| \to_\beta^* \varphi(t_1)$ and $|t_2| \to_\beta^* \varphi(t_2)$.

$$\begin{aligned}
|t_1\ t_2| &= |t_1|\ |t_2| \\
&\to_\beta^* \varphi(t_1)\ \varphi(t_2) && \text{By induction hypothesis} \\
&= \varphi(t_1\ t_2)
\end{aligned}$$

- Case $t = (\underline{\lambda} x.t_0)\ t_1$, by the induction hypothesis we know that $|t_0| \to_\beta^* \varphi(t_0)$ and $|t_1| \to_\beta^* \varphi(t_1)$.

$$\begin{aligned}
|(\underline{\lambda} x.t_0)\ t_1| &= (\lambda x.|t_0|)\ |t_1| \\
&\to_\beta^* (\lambda x.\varphi(t_0))\ \varphi(t_1) && \text{By induction hypothesis} \\
&\to \varphi(t_0)\{x \leftarrow \varphi(t_1)\} \\
&= \varphi((\underline{\lambda} x.t_0)\ t_1)
\end{aligned}$$

$\square$