

Lambda Calculus and category theory

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1 Introduction

Boole :

- If you consider propositions (no quantifiers) of classical logic : $A ::= P | A \wedge B | \neg A | A \vee B | \top | \perp$
- Ordered by logical implication $A \leq B \Leftrightarrow A \Rightarrow B$, A implies B or $A \vdash B$

Observation $A \wedge B \leq A, A \wedge B \leq B$. moreover if $C \leq A$ and $C \leq B$ then $C \leq A \wedge B$ (for all proprieties)
Which means that $A \wedge B$ define a infimum of A and B greatest lower bound glb

Definition – $A \Rightarrow B = (\neg A) \vee B = \neg(A \wedge \neg B)$.

Observation :

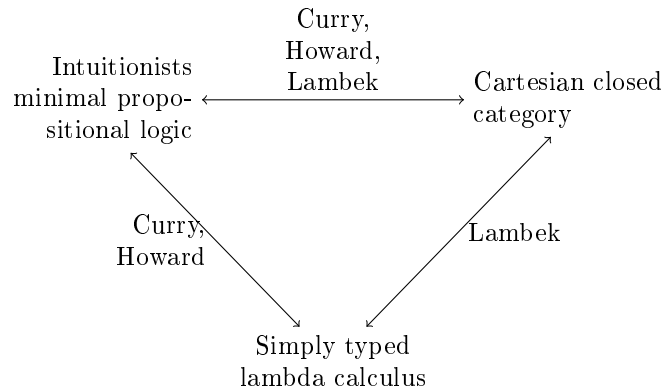
- $A \wedge (A \Rightarrow B) \leq B$
- $A \vee \neg A \leq \text{true}$
- $A \wedge \neg A \geq \text{false}$

Frege : ideography (first proof system)

The idea that a mathematical proof is a mathematical object. In particular there may be different proofs of a proposition A formula.

$$\begin{array}{ccc}
 & B & \\
 \pi_1 \left(\begin{array}{c} \nearrow \\ \neq \\ \searrow \end{array} \right) & & \pi_2 \\
 & A & \\
 \text{Lambek} & & \text{Lambek}
 \end{array}$$

Lambek understood connection between.



Definition – A monoid (M, \bullet, e) is a set M equipped with a binary operation $\bullet : M \times M \rightarrow M$ with a neutral element $e \in M_e : M^0 \rightarrow M$ satisfying two equations :

- (associativity) $\forall x, y, z \in M, x \bullet (y \bullet z) = (x \bullet y) \bullet z$
- (neutrality) $\forall x \in M, x \bullet e = x = e \bullet x$

Example – $(\mathbb{N}, +, 0), (\mathbb{Z}, +, 0), (\mathbb{N}, \times, 1)$ and any group.

Free monoide on a set (=alphabet) A A^* contains finite sequences of element A $w = [a_1 \dots a_n]$

- multiplication is concatenation
- neutral element is empty word

2 Categories, functors, natural transformations

Definition – A category \mathcal{C} is a graph

- whose node are called objects
- whose edges are called morphism/maps/arrow.

The objects of \mathcal{C} form a class of objects. Every pair of objects A, B

Every pair of object A, B comes with a set $Hom(A, B)$ of morphisms $A \xrightarrow{f} B, f \in Home(A, B)$

The graph is equipped with :

- an morphisms $id_A \in Home(A, A)$ for all object A of \mathcal{C}
- a composition defined as a function $\circ_{A,B,C} : Hom(B, C) \times Hom(A, B) \rightarrow Hom(A, C)$ for every objects A, B, C of category \mathcal{C}

It satisfying the following equation :

– associativity :

$$\begin{array}{ccc}
 & B & \xrightarrow{g} C \\
 f \nearrow & & \searrow h \\
 A & \xrightarrow{g \circ f} & D \\
 & \xrightarrow{h \circ g \circ f} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & B & \xrightarrow{g} C \\
 f \nearrow & & \searrow h \\
 A & \xrightarrow{h \circ g} & D \\
 & \xrightarrow{h \circ g \circ f} &
 \end{array}$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

– neutrality :

$$\begin{array}{ccc}
 Id_A & & Id_B \\
 \downarrow & & \downarrow \\
 A & \xrightarrow{f} & B \\
 Id_B \circ f = f = f \circ Id_A
 \end{array}$$

Definition – A small category is a category whose class of object is a set. What we defined as a category is called “locally small category”.

Example – Ordered Set : Claim every ordered set A defines a category

- object : elements of A
- morphisms : $a \rightarrow b \Leftrightarrow a \leq b$

$$Hom(a, b) = \begin{cases} \text{singleton} & a \leq b \\ \emptyset & \end{cases}$$

The composition is defined by transitivity :

$$\begin{array}{ccccc}
 a & \xrightarrow{a \leq b} & b & \xrightarrow{b \leq c} & c \\
 a & & \leq & & c \\
 a & \xrightarrow{\quad \quad \quad} & b & &
 \end{array}$$

Definition – An order category \mathcal{C} is a category when $Hom(A, B)$ is a singleton for all object A, B of \mathcal{C} .

Observation – an order category is the same thing as a pre-order (= trans, refl).

Example – monoid

- A category with one object $*$, $M = Hom(*, *)$ defined a monoid
 - $\circ : Hom(*, *) \times Hom(*, *) \rightarrow Hom(*, *)$
 - $id_* \in M = Hom(*, *)$ defined the neutral element
- Conversely every monoid $M = (M, \bullet, e)$ defined a category $\mathcal{B}M$ or ΣM with one object $*$ and $Hom(*, *) = M$ composition defined $y \circ x = y \bullet x$ id_* defined the neutral element.

