λ-calculus

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Halting problem

There is no λ-term H such has $H[t] = T$ if T has a normal form and $H[T] = F$ if T has no normal form.

Let N the set of λ-term that have a normal form.
N is not empty (it contains all the variables) and N is not equal to Λ because it not contains the λ-term Ω.
So we have $\Lambda \backslash N$ non-empty and non-equal to Λ
By Scott's theorem, the set N is not recursively separable. So the λ-term H does not exist.

List in pure λ calculus

We define this useful lambda term :

- $[I] = \lambda x.x$
- $[T] = \lambda x\ y.x$
- $[F] = \lambda x\ y.y$
- $\langle t, u \rangle = \lambda x.x\ t\ u$
- $\pi_1 \langle t, u \rangle = \langle t, u \rangle [T]$
- $\pi_2 \langle t, u \rangle = \langle t, u \rangle [F]$

We define our integers as follows :

- $[0] = [I]$
- $[S] = \lambda n. \langle [F], n \rangle$
- $[isZ] = \lambda n. \pi_1\ t$
- $[P] = \lambda n. \pi_2\ t$

We take the following fixpoint :

$$A = (\lambda xy.y(xxy))$$
$$\Theta = A\ A$$

Finally, we define our lists as follows :

- $[\text{Nil}] = \lambda n\ f.\ n$
- $[x :: l] = \lambda n\ f.\ f\ x\ [l]$

The list $0 :: 1 :: [\text{Nil}]$ is represented as follows:

$$\lambda n_0\ f_0.\ f_0\ [0]\ (\lambda n_1\ f_1.\ f_1\ [1]\ (\lambda n\ f.\ n))$$

The function *nth* k l which return the k^{nth} element of the list l. We return an option if k is too large. The options are defined as follows :

- $[None] = \lambda ns.\ n$
- $[Some(x)] = \lambda ns.\ sx$

Now we can define *nth* function :

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nth = Θ (λ f k l. l [None]
                     (λ x l'. [isZ] k [Some(x)])
                                   (f ([P] k) l'))
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Listing 1: Nth function on list

We want to proof this property $\forall k\ l\ x, nth\ [k]\ [l] =_\beta nth\ [k+1]\ [x :: l]$.

Proof – The calculations have been simplified to make the proof easier to read.
We proceed by induction on $[k]$:

- if $k = 0$
  - if $l = [\text{Nil}]$, then

$$\begin{aligned} nth\ [0]\ [\text{Nil}] &\to_\beta^* [\text{Nil}]\ [None]\ (\dots) \\ &\to_\beta [None] \end{aligned}$$

$$\begin{aligned} nth\ [1]\ [x :: [\text{Nil}]] &\to_\beta^* nth\ [0]\ [\text{Nil}] \\ &\to_\beta^* [None] \end{aligned}$$

  - if $l = [x' :: [l]]$, then

$$\begin{aligned} nth\ [0]\ [x' :: [l]] &\to_\beta^* [isZ]\ [0]\ [Some(x')]\ (\dots) \\ &\to_\beta [Some(x')] \end{aligned}$$

$$\begin{aligned} nth\ [1]\ [x :: x' :: [l]] &\to_\beta^* nth\ [0]\ [x' :: [l]] \\ &\to_\beta^* [Some(x')] \end{aligned}$$

- if $k = n + 1$ by induction hypothesis we know that $\forall l\ x, nth\ [n]\ [l] =_\beta nth\ [n+1]\ [x :: l]$

$$\begin{aligned} nth\ [k+2]\ [x' :: x :: l] &\to_\beta^* nth\ [n+1]\ [x :: l] && \text{One step on } nth \\ &=_\beta nth\ [n]\ [l] && \text{Induction Hypothesis} \end{aligned}$$

□