

# Homework

## 1 Basic operation and their notation

### Exercise 1.1: Inner/outer products in Dirac notation

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} & (\langle 0| + 2\langle 1|) \times (3|0\rangle + 4|1\rangle) \\ &= 3\langle 0|0\rangle + 4\langle 0|1\rangle + 6\langle 1|0\rangle + 8\langle 1|1\rangle \\ &= 3 + 8 \\ &= 11 \end{aligned}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}$$

The last two in Dirac notation :

$$\begin{aligned} & (3|0\rangle + 4|1\rangle) \times (\langle 0| + 2\langle 1|) \\ &= 3|0\rangle\langle 0| + 6|0\rangle\langle 1| + 4|1\rangle\langle 0| + 8|1\rangle\langle 1| \end{aligned}$$

### Exercise 1.2: Matrix products in Dirac notation

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 23 \\ 34 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} & (|0\rangle\langle 0| + 3|0\rangle\langle 1| + 2|1\rangle\langle 0| + 4|1\rangle\langle 1|) \times (5|0\rangle + 6|1\rangle) \\ &= 5|0\rangle\langle 0|0\rangle + 6|0\rangle\langle 0|1\rangle + 15|0\rangle\langle 1|0\rangle + 18|0\rangle\langle 1|1\rangle + \\ & \quad 10|1\rangle\langle 0|0\rangle + 12|1\rangle\langle 0|1\rangle + 20|1\rangle\langle 1|0\rangle + 24|1\rangle\langle 1|1\rangle \\ &= 5\langle 0|0\rangle|0\rangle + 18\langle 1|1\rangle|0\rangle + 10\langle 0|0\rangle|1\rangle + 24\langle 1|1\rangle|1\rangle \\ &= 5|0\rangle + 18|0\rangle + 10|1\rangle + 24|1\rangle \\ &= 23|0\rangle + 24|1\rangle \end{aligned}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} & (1/\sqrt{2}|0\rangle\langle 0| + 1/\sqrt{2}|0\rangle\langle 1| + 1/\sqrt{2}|1\rangle\langle 0| - 1/\sqrt{2}|1\rangle\langle 1|)^2 \\ &= 1/2|0\rangle\langle 0|0\rangle\langle 0| + 1/2|0\rangle\langle 0|0\rangle\langle 1| + 1/2|0\rangle\langle 1|1\rangle\langle 0| - 1/2|0\rangle\langle 1|1\rangle\langle 1| \\ & \quad 1/2|1\rangle\langle 0|0\rangle\langle 1| + 1/2|1\rangle\langle 0|0\rangle\langle 0| - 1/2|1\rangle\langle 1|1\rangle\langle 0| + 1/2|1\rangle\langle 1|1\rangle\langle 1| \\ &= 1/2|0\rangle\langle 0| + 1/2|0\rangle\langle 0| + 1/2|1\rangle\langle 1| + 1/2|1\rangle\langle 1| \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| \end{aligned}$$

**Exercice 1.3: Tensor products in Dirac/Coecke notation**

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The last two in Dirac notation :

$$(|0\rangle + 2|1\rangle) \otimes (3|0\rangle + 4|1\rangle) = 3|00\rangle + 4|01\rangle + 6|10\rangle + 8|11\rangle$$

$$|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle = |00\rangle + |11\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$$

In Dirac notation :

$$\begin{aligned} & (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ &= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| \end{aligned}$$

$$\begin{aligned} & (|0\rangle\langle 0|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ &= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| \end{aligned}$$

$$\begin{aligned} & 1/\sqrt{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \otimes 1/\sqrt{2}(|0\rangle\langle 0| + |1\rangle\langle 0| + |1\rangle\langle 0| - |1\rangle\langle 1|) \\ &= 1/2(|0\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 2| + |0\rangle\langle 3| + \\ & \quad |1\rangle\langle 0| - |1\rangle\langle 1| + |1\rangle\langle 2| - |1\rangle\langle 3| + \\ & \quad |2\rangle\langle 0| + |2\rangle\langle 1| - |2\rangle\langle 2| - |2\rangle\langle 3| + \\ & \quad |3\rangle\langle 0| - |3\rangle\langle 1| - |3\rangle\langle 2| + |3\rangle\langle 3|) \end{aligned}$$

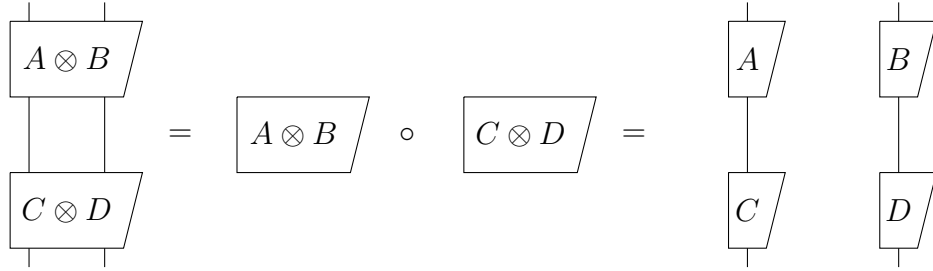
We want to prove  $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

- Dirac's notation :

We have  $A = \sum_{i,j} a_{i,j} |i\rangle \langle j|$  and  $D = \sum_{k,l} d_{k,l} |k\rangle \langle l|$

$$\begin{aligned}
 (A \otimes B)(C \otimes D) &= ((\sum_{i,j} a_{i,j} |i\rangle \langle j|) \otimes B)(C \otimes (\sum_{k,l} d_{k,l} |k\rangle \langle l|)) \\
 &= (\sum_{i,j} a_{i,j} |i\rangle \langle j|)C \otimes B(\sum_{k,l} d_{k,l} |k\rangle \langle l|) && \text{bilinearity of } \otimes \\
 &= (AC) \otimes (BD)
 \end{aligned}$$

- Coecke's notation :



#### Exercice 1.4: Dagger in Dirac/Coecke notation

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}^\dagger = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \quad \begin{pmatrix} 1 & 3i \\ 2 & 4i \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 2 \\ -3i & -4i \end{pmatrix}$$

In Dirac notation:

$$\begin{aligned}
 &1/\sqrt{2}(|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| - |1\rangle \langle 1|)^\dagger \\
 &= 1/\sqrt{2}(|0\rangle \langle 0| + |1\rangle \langle 0| + |0\rangle \langle 1| - |1\rangle \langle 1|)
 \end{aligned}$$

$$\begin{aligned}
 &(|0\rangle \langle 0| + 3i |0\rangle \langle 1| + 2 |1\rangle \langle 0| + 4i |1\rangle \langle 1|)^\dagger \\
 &= (|0\rangle \langle 0| - 3i |1\rangle \langle 0| + 2 |0\rangle \langle 1| - 4i |1\rangle \langle 1|)
 \end{aligned}$$

#### Exercice 1.5: Gates in Dirac notations

$$H = 1/\sqrt{2}(|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| - |1\rangle \langle 1|)$$

$$CNot = |0\rangle \langle 0| + |1\rangle \langle 1| + |3\rangle \langle 2| + |2\rangle \langle 3|$$

$$T = |0\rangle \langle 0| + e^{\frac{i\pi}{4}} |1\rangle \langle 1|$$

Proof that are unitary matrix :

- $H : H^\dagger H = Id_1$  already do in Exercie-1.2
- $CNot$ :

$$\begin{aligned}
 CNot^\dagger CNot &= (|0\rangle \langle 0| + |1\rangle \langle 1| + |3\rangle \langle 2| + |2\rangle \langle 3|)(|0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 3| + |3\rangle \langle 2|) \\
 &= (|0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2| + |3\rangle \langle 3|) \\
 &= Id_4
 \end{aligned}$$

- $T$ :

$$\begin{aligned} (|0\rangle\langle 0| + e^{\frac{i\pi}{4}}|1\rangle\langle 1|)^2 &= |0\rangle\langle 0|0\rangle\langle 0| + e^{\frac{i\pi}{4}}|0\rangle\langle 0|1\rangle\langle 1| + e^{\frac{i\pi}{4}}(|1\rangle\langle 1|0\rangle\langle 0|) + e^{\frac{i\pi}{2}}(|1\rangle\langle 1|1\rangle\langle 1|) \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| \end{aligned}$$

### Exercice 1.6: Pauli matrices in Dirac/Coecke notation

- For all  $i, k \in [0, 3]$  we want to show  $\sigma_i \sigma_j = \delta_{ij} I + i \sum_k \epsilon_{ijk} \sigma_k$ 
  - If  $i = j$  then  $\sigma_i \sigma_j = I$  and for all  $k$  we have  $\epsilon_{ijk} = 0$ .  
So we have  $\delta_{ij} I = I = \sigma_i \sigma_j$
- For all  $i, k \in [0, 3]$  we want to show  $[\sigma_i, \sigma_j] = 2i \sum_k \epsilon_{ijk}$

$$\begin{aligned} [\sigma_i, \sigma_j] &= \sigma_i \sigma_j - \sigma_j \sigma_i \\ &= (\delta_{ij} I + i \sum_k \epsilon_{ijk} \sigma_k) - (\delta_{ji} I + i \sum_k \epsilon_{jik} \sigma_k) \\ &= i \sum_k \epsilon_{ijk} \sigma_k - i \sum_k \epsilon_{jik} \sigma_k \\ &= i \sum_k \epsilon_{ijk} \sigma_k + i \sum_k \epsilon_{ijk} \sigma_k \\ &= 2i \sum_k \epsilon_{ijk} \sigma_k \end{aligned}$$

- For all  $i, k \in [0, 3]$  with  $i \neq j$  we want to show  $\{\sigma_i, \sigma_j\} = 0$

$$\begin{aligned} \{\sigma_i, \sigma_j\} &= \sigma_i \sigma_j + \sigma_j \sigma_i \\ &= \delta_{ij} I + i \sum_k \epsilon_{ijk} \sigma_k + \delta_{ji} I - i \sum_k \epsilon_{jik} \sigma_k & \epsilon_{jik} = -\epsilon_{ijk} \\ &= 2\delta_{ij} I \\ &= 0 & i \neq j \end{aligned}$$

## 2 Postulates on pure states

### Exercice 2.1: Evolutions

Let  $|\psi\rangle = |0\rangle \otimes |0\rangle$  the initial state of two qubits. We want to compute  $CNot(H \otimes I) |\psi\rangle$ .

$$\begin{aligned} CNot(H \otimes I)(|0\rangle \otimes |0\rangle) &= CNot((|0\rangle + |1\rangle)/\sqrt{2} \otimes (|0\rangle + |1\rangle)/\sqrt{2}) \\ &= \frac{1}{2} CNot(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |11\rangle + |10\rangle) := |\psi'\rangle \end{aligned}$$

$$\begin{aligned} (H \otimes I) CNot |\psi'\rangle &= (H \otimes I) \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ &= |00\rangle & H^2 |\psi\rangle = |\psi\rangle \end{aligned}$$

We have  $I := H^2$

1.  $T^4H |0\rangle$
2.  $HT^4H |0\rangle$
3.  $CNot(H \otimes HT^4H)(|0\rangle \otimes |0\rangle)$
4.  $(I \otimes CNot)(CNot \otimes I)(H \otimes I \otimes I)(|0\rangle \otimes |0\rangle \otimes |0\rangle)$
5.  $(H \otimes H)(|0\rangle \otimes |0\rangle)$

### Exercice 2.2: Measuring in another basis

- orthogonal :

$$\begin{aligned}
 \langle +|- \rangle &= \left(\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} \\
 &= \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 \\
 &= 0
 \end{aligned}$$

- norm one :

$$\begin{aligned}
 \langle ++ \rangle &= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 & \langle -- \rangle &= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 \\
 &= \frac{1}{2} + \frac{1}{2} & &= \frac{1}{2} + \frac{1}{2} \\
 &= 1 & &= 1
 \end{aligned}$$

- generate  $\mathbb{C}^2$

Let  $c = \alpha |0\rangle + \beta |1\rangle \in \mathbb{C}^2$  we want to find  $x$  and  $y$  such that  $x|+\rangle + y|-\rangle = c$ .

$$\begin{aligned}
 x|+\rangle + y|-\rangle &= \frac{x}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{y}{\sqrt{2}}(|0\rangle - |1\rangle) \\
 &= \left(\frac{x+y}{\sqrt{2}}\right)|0\rangle + \left(\frac{x-y}{\sqrt{2}}\right)|1\rangle
 \end{aligned}$$

So we have this system :

$$\begin{aligned}
 \begin{cases} (x+y)/\sqrt{2} &= \alpha \\ (x-y)/\sqrt{2} &= \beta \end{cases} &\Rightarrow \begin{cases} x+y &= \sqrt{2}\alpha \\ x-y &= \sqrt{2}\beta \end{cases} \\
 &\Rightarrow \begin{cases} 2x &= \sqrt{2}(\alpha + \beta) \\ x-y &= \sqrt{2}\beta \end{cases} \\
 &\Rightarrow \begin{cases} x &= \frac{\sqrt{2}}{2}(\alpha + \beta) \\ -x+y &= -\sqrt{2}\beta \end{cases} \\
 &\Rightarrow \begin{cases} x &= \frac{\sqrt{2}}{2}(\alpha + \beta) \\ y &= \frac{\sqrt{2}}{2}(\alpha - \beta) \end{cases}
 \end{aligned}$$

$B = \{|0\rangle, |1\rangle\}$  is another o.n.b of  $\mathbb{C}^2$

We need to show that  $\sum_{M \in \mathcal{M}_\pm} M^\dagger M = 1$

$$\begin{aligned}
\sum_{M \in \mathcal{M}_\pm} &= (|+\rangle \langle +|)^\dagger (|+\rangle \langle +|) + (|-\rangle \langle -|)^\dagger (|-\rangle \langle -|) \\
&= (|+\rangle \langle +| + |-\rangle \langle -|) (|+\rangle \langle +| + |-\rangle \langle -|) \\
&= (|+\rangle \langle +| + |-\rangle \langle -|) \\
&= \frac{1}{2}(|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|) + \frac{1}{2}(|0\rangle \langle 0| - |0\rangle \langle 1| - |1\rangle \langle 0| + |1\rangle \langle 1|) \\
&= |0\rangle \langle 0| + |1\rangle \langle 1| \\
&= 1
\end{aligned}$$

So  $\mathcal{M}_\pm$  is a valid measurement.

We have  $|\psi\rangle = \frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle$

- For  $|+\rangle \langle +|$

$$\begin{aligned}
p(|+\rangle \langle +|) &= \langle \psi | (|+\rangle \langle +|)^\dagger |+\rangle \langle +| \psi \rangle \\
&= \langle \psi | |+\rangle \langle +| \psi \rangle \\
&= \langle \psi | \frac{1}{2}(|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|) \psi \rangle \\
&= \frac{1}{2}(\frac{1}{3}\langle 0| + \frac{\sqrt{8}}{3}\langle 1|)(|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| + |1\rangle \langle 1|) \psi \rangle \\
&= \frac{1 + \sqrt{8}}{6}(\langle 0| + \langle 1|)\frac{1}{2}(\frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle) \\
&= \frac{9 + 2\sqrt{8}}{18} = \frac{1}{2} + \frac{\sqrt{8}}{9}
\end{aligned}$$

- For  $|-\rangle \langle -|$

$$\begin{aligned}
p(|-\rangle \langle -|) &= \langle \psi | (|-\rangle \langle -|)^\dagger |-\rangle \langle -| \psi \rangle \\
&= \langle \psi | |-\rangle \langle -| \psi \rangle \\
&= \langle \psi | \frac{1}{2}(|0\rangle \langle 0| - |0\rangle \langle 1| - |1\rangle \langle 0| + |1\rangle \langle 1|) \psi \rangle \\
&= \frac{1}{2}(\frac{1}{3}\langle 0| + \frac{\sqrt{8}}{3}\langle 1|)(|0\rangle \langle 0| - |0\rangle \langle 1| - |1\rangle \langle 0| + |1\rangle \langle 1|) \psi \rangle \\
&= \frac{1}{2}(\frac{1 - \sqrt{8}}{3}\langle 0| + \frac{\sqrt{8} - 1}{3}\langle 1|)(\frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle) \\
&= \frac{1}{2}(\frac{1 - \sqrt{8}}{9} + \frac{8 - \sqrt{8}}{9}) \\
&= \frac{1}{2}(\frac{9 - 2\sqrt{8}}{9}) = \frac{1}{2} - \frac{\sqrt{8}}{9}
\end{aligned}$$

The post measure states are :

$$|\psi_+\rangle = \frac{1}{\sqrt{\frac{1}{2} + \frac{\sqrt{8}}{9}}} \begin{pmatrix} \frac{1+\sqrt{8}}{6} \\ \frac{1+\sqrt{8}}{6} \end{pmatrix}$$

$$|\psi_{-}\rangle = \frac{1}{\sqrt{\frac{1}{2} - \frac{\sqrt{8}}{9}}} \begin{pmatrix} \frac{1-\sqrt{8}}{6} \\ \frac{\sqrt{8}-1}{6} \end{pmatrix}$$

### Exercice 2.3: Measuring to distinguish

We defined:

$$M_0 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad M_+ = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad M_f = \begin{pmatrix} \frac{-1-i}{2} & i \\ \frac{1-i}{2} & i \end{pmatrix}$$

$$\begin{aligned} \langle 0 | M_0^\dagger M_0 | 0 \rangle &= \langle 0 | M_0 | 0 \rangle \\ &= \langle 0 | 0 \rangle = 1 \end{aligned}$$

$$\begin{aligned} \langle + | M_0^\dagger M_0 | + \rangle &= (0 \ 0) M_0 | + \rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} \langle 0 | M_+^\dagger M_+ | 0 \rangle &= (0 \ 0) M_+ | 0 \rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} \langle + | M_+^\dagger M_+ | + \rangle &= \langle + | M_+ | + \rangle \\ &= \frac{2}{\sqrt{2}} \langle 1 | + \rangle \\ &= \frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 1 \end{aligned}$$

The measure  $\mathcal{M}$  is valid :

$$M_0^\dagger M_0 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad M_+^\dagger M_+ = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad M_f^\dagger M_f = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

$$M_0^\dagger M_0 + M_+^\dagger M_+ + M_f^\dagger M_f = I_2$$

### Exercice 2.4: Measuring the phase

a

### Exercice 2.5: Measuring a subsystem

a

## 3 Some mathematics

### Exercice 3.1: Spectral theorems complements

Let  $D$  and  $D'$  two diagonal matrices and  $U$  a unitary matrix.

- Let  $A = UDU^\dagger$  and  $B = UD'U^\dagger$

$$\begin{aligned} AB &= UDU^\dagger UD'U^\dagger \\ &= UDD'U^\dagger \\ &= UD'DU^\dagger \\ &= UD'U^\dagger UDU^\dagger \\ &= BA \end{aligned}$$

$U$  is a unitary matrix  
 $D$  and  $D'$  are diagonal

- Let  $M = UDU^\dagger$

$$\begin{aligned}
 MM^\dagger &= UDU^\dagger(UDU^\dagger)^\dagger \\
 &= UDU^\dagger(U^\dagger)^\dagger D^\dagger U^\dagger \\
 &= UDU^\dagger U D^\dagger U^\dagger \\
 &= U D D^\dagger U^\dagger \\
 &= U D^\dagger D U^\dagger \\
 &= U D^\dagger U^\dagger U D U^\dagger \\
 &= (UDU^\dagger)^\dagger U D U^\dagger \\
 &= M^\dagger M
 \end{aligned}$$

- Let  $E = UDU^\dagger$  with having only non-negative value.

Let  $|\psi\rangle \in \mathcal{M}_{n,1}(\mathbb{C})$ ,  $d_i$  such that  $E_{i,i} = d_i$

$$\begin{aligned}
 \langle \psi | E | \psi \rangle &= \langle \psi | UDU^\dagger | \psi \rangle \\
 &= (U^\dagger | \psi \rangle)^\dagger D (U^\dagger | \psi \rangle) \\
 &= \sum_{i=1}^n d_i (U_i | \psi \rangle)^2 \\
 &\geq 0
 \end{aligned}$$

- Let  $V = UDU^\dagger$  with  $D$  having only modulus one values.

$$\begin{aligned}
 VV^\dagger &= UDU^\dagger(UDU^\dagger)^\dagger \\
 &= UDU^\dagger U D^\dagger U^\dagger \\
 &= U D D^\dagger U^\dagger \\
 &= U U^\dagger && D \text{ has only modulus one values} \\
 &= I
 \end{aligned}$$

So  $V$  is a unitary matrix.

- Let  $E$  a non-negative matrix.  $E$  is spectrally decomposable with non-negative eigenvalues. We can take  $M := \sqrt{E}$  which is defined by its spectral decomposition being with the square roots of the eigenvalues of  $E$ .  $M$  is hermitian and  $E = MM$ , so  $E^\dagger = (MM)^\dagger = M^\dagger M^\dagger = MM = E$ .
- The follow matrix is not normal :

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$MM^\dagger = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix} \neq \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix} = M^\dagger M$$

### Exercise 3.2: Isometry versus unitary versus involution

- Let  $M$  a unitary and hermitian matrix.

$$\begin{aligned}
 MM &= MM^\dagger && M \text{ is hermitian} \\
 &= I && M \text{ is unitary}
 \end{aligned}$$



- Matrix  $2 \times 2$  unitary that is not an involution :

$$M = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

The inverse of  $M$  is :

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

We have  $M \neq M^{-1}$  so  $M$  is not an involution.

- Matrix  $m \times n$  isometry that is not a unitary:

$$M = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$M^\dagger M = \begin{pmatrix} 1 \end{pmatrix} = I_1 \quad MM^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq I_2$$

- Let  $M$  an  $n \times n$  isometry matrix ( $M^\dagger M = I_n$ )

$$\begin{aligned} MM^\dagger &= \sum_{i,j} M_{i,j} |i\rangle \langle j| \sum_{i,j} M_{j,i} |i\rangle \langle j| \\ &= \sum_{i,j} M_{i,j} M_{j,i} |i\rangle \langle i| \\ &= \sum_{i,j} M_{j,i} M_{i,j} |i\rangle \langle j| \\ &= \sum_{i,j} M_{j,i} |i\rangle \langle j| \sum_{i,j} M_{i,j} |i\rangle \langle j| \\ &= M^\dagger M \\ &= I_n \end{aligned}$$

## 4 On the nature of quantum information

### Exercice 4.1: Hadamard

- $a = 0$

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^0|1\rangle) \end{aligned}$$

- $a = 1$

$$\begin{aligned} H|1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^1|1\rangle) \end{aligned}$$

**Exercice 4.2: Who controls whom?**

We define :

$$NotC = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$NotC|00\rangle = |00\rangle$$

$$NotC|01\rangle = |11\rangle$$

$$NotC|10\rangle = |10\rangle$$

$$NotC|11\rangle = |01\rangle$$

We want to proof  $(H \otimes H)CNot(H \otimes H) = NotC$  :

$$\begin{aligned} (H \otimes H)(|x\rangle \otimes |y\rangle) &= (H|x\rangle \otimes H|y\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + (-1)^y|1\rangle) \\ &= \frac{1}{2}(|00\rangle + (-1)^x|10\rangle + (-1)^y|01\rangle + (-1)^{x+y}|11\rangle) \end{aligned}$$

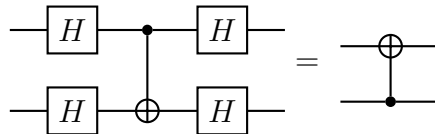
We apply the operator  $CNot$ :

$$\begin{aligned} &CNot\left(\frac{1}{2}(|00\rangle + (-1)^x|10\rangle + (-1)^y|01\rangle + (-1)^{x+y}|11\rangle)\right) \\ &= \frac{1}{2}(|00\rangle + (-1)^x|11\rangle + (-1)^y|01\rangle + (-1)^{x+y}|10\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x+y}|1\rangle) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + (-1)^y|1\rangle)\right) \\ &= H|x \oplus y\rangle \otimes H|y\rangle \end{aligned} \quad (-1)^{x+y} = (-1)^{x \oplus y} \quad (x \oplus y \in \{0, 1\})$$

Finlay we apply  $(H \otimes H)$ :

$$\begin{aligned} (H \otimes H)(H|x \oplus y\rangle \otimes H|y\rangle) &= HH|x \oplus y\rangle \otimes HH|y\rangle \\ &= |x \oplus y\rangle \otimes |y\rangle \\ &= NotC(|x\rangle \otimes |y\rangle) \end{aligned}$$

In quantum circuit :



## 5 Protocols

### Exercice 5.1: Canonical basis versus diagonal basis

- If Bob measures the result in the same basis then he can retrieve the information sent
- If Bob measures in an other basis then he learns nothing about the message.

- If Eve intercepts and measures it in the same basis then Bob can have some information on the message if he read the message in the same basis.
- But if Eve intercepts and measures it in an other basis and Bob read the message in the original basis then he learns nothing about the message.

### Exercise 5.2: BB84

1. Alice will start by producing a random string of bits, encode each of them either into the canonical or the diagonal basis, and send that to Bob.
2. Bob will measure them either using the canonical basis or the diagonal basis, at random.
3. Bob will broadcast which bases he used
4. Alice will know when Bob used the same base. When Bob has used the right base, Bob's information is correct, otherwise it is wrong (previous exercise).
5. Eve does not know the bases like Bob And she has very little chance of having the right basic sequence ( $\frac{1}{2^n}$ ). But she's going to disrupt Bob's measurements.
- 6.

### Exercise 5.3: Quantum random access code

TODO

### Exercise 5.4: The Bell basis

$$|\beta_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\beta_2\rangle = (Y \otimes I) |\beta_0\rangle = \frac{i}{\sqrt{2}}(|10\rangle - |01\rangle)$$

$$|\beta_1\rangle = (X \otimes I) |\beta_0\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$$

$$|\beta_3\rangle = (Z \otimes I) |\beta_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

These four states are orthogonal and orthonormal, orthogonal:

$$\langle\beta_0|\beta_1\rangle = \frac{1}{2} \times 0 = 0$$

$$\langle\beta_0|\beta_3\rangle = \frac{1}{2} \times 0 = 0$$

$$\langle\beta_1|\beta_3\rangle = \frac{1}{2} \times 0 = 0$$

$$\langle\beta_0|\beta_2\rangle = \frac{i}{2} \times 0 = 0$$

$$\langle\beta_1|\beta_2\rangle = \frac{i}{2} \times 0 = 0$$

$$\langle\beta_2|\beta_3\rangle = \frac{i}{2} \times 0 = 0$$

orthonormal:

$$\langle\beta_0|\beta_0\rangle = \frac{1}{2}(0 + 2) = 1$$

$$\langle\beta_2|\beta_2\rangle = \frac{-1}{2}(1 + 1) = -1$$

$$\langle\beta_1|\beta_1\rangle = \frac{1}{2}(1 + 1) = 1$$

$$\langle\beta_3|\beta_3\rangle = \frac{1}{2}(0 + 2) = 1$$

So, this states are an orthonormal basis.

### Exercise 5.5: Superdense coding

TODO

**Exercice 5.6: Discussion: classical description of a single qubit**

A qubit is coded with this formula :  $\alpha|0\rangle + \beta|1\rangle$ . We just need to send 2 complexes numbers. So if a number is encoded with  $n$  bits we send  $4n$  bits.

**Exercice 5.7: Teleportation**

We verify :

**Exercice 5.8: The swap test**

Before the measurement we have this state :

$$\begin{aligned}
 |\kappa\rangle &= (H \otimes I \otimes I) CSwap (H \otimes I \otimes I) |0\rangle \otimes |\phi\rangle \otimes |\psi\rangle \\
 &= (H \otimes I \otimes I) CSwap \left( \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right) \otimes |\phi\rangle \otimes |\psi\rangle \\
 &= (H \otimes I \otimes I) CSwap \frac{1}{\sqrt{2}} (|0\phi\psi\rangle + |1\phi\psi\rangle) \\
 &= (H \otimes I \otimes I) \frac{1}{\sqrt{2}} (|0\phi\psi\rangle + |1\psi\phi\rangle) \\
 &= \frac{1}{2} (|0\rangle \otimes (|\phi\psi\rangle + |\psi\phi\rangle) - |1\rangle \otimes (|\phi\psi\rangle + |\psi\phi\rangle)) \\
 &= \frac{1}{2} (|0\phi\psi\rangle + |0\psi\phi\rangle + |1\phi\psi\rangle - |1\psi\phi\rangle)
 \end{aligned}$$

We apply the measures :

$$\begin{aligned}
 p(0) &= \langle \kappa | 0 \rangle \langle 0 | \kappa \rangle \\
 &= \frac{1}{2} (\langle \phi\psi | + |\psi\phi\rangle) \times \frac{1}{2} (\langle \phi\psi | + |\psi\phi\rangle) \\
 &= \frac{1}{4} (2 + \langle \phi\psi | \psi\phi \rangle + \langle \psi\phi | \phi\psi \rangle) \\
 &= \frac{1}{4} (2 + 2 \langle \phi\psi | \psi\phi \rangle) \\
 &= \frac{1}{2} + \frac{\langle \psi\phi | \psi\phi \rangle}{2} \\
 &= \frac{1}{2} + \frac{1}{2} |\langle \psi | \phi \rangle|^2
 \end{aligned}$$

we have  $p(1) = 1 - p(0)$  so  $p(1) = \frac{1}{2} - \frac{1}{2} |\langle \psi | \phi \rangle|^2$

**Exercice 5.9: Quantum fingerprinting**

TODO

## 6 Quantum error correction

## 7 Bell

**Exercice 7.1: Probability of winning the CHSH game**

We have 4 case :

- $s = r = 0$

$$\begin{aligned}
\mathcal{P}(\text{win}) &= \mathcal{P}(a = b = 0) + \mathcal{P}(a = b = 1) \\
&= |((\cos(0) \langle 0| + \sin(0) \langle 1|) \otimes ((\cos(\frac{\pi}{8}) \langle 0| + \sin(\frac{\pi}{8}) \langle 1|)) |\beta_0\rangle|^2 \\
&\quad + |((\sin(0) \langle 0| - \cos(0) \langle 1|) \otimes ((\sin(\frac{\pi}{8}) \langle 0| - \cos(\frac{\pi}{8}) \langle 1|)) |\beta_0\rangle|^2 \\
&= |(\cos(\frac{\pi}{8}) \langle 00| + \sin(\frac{\pi}{8}) \langle 01|) |\beta_0\rangle|^2 + |(-\sin(\frac{\pi}{8}) \langle 10| + \cos(\frac{\pi}{8}) \langle 11|) |\beta_0\rangle|^2 \\
&= |\frac{1}{\sqrt{2}} \cos(\frac{\pi}{8})|^2 + |\frac{1}{\sqrt{2}} \cos(\frac{\pi}{8})|^2 \\
&= \frac{1}{2} \cos^2(\frac{\pi}{8}) + \frac{1}{2} \cos^2(\frac{\pi}{8}) \\
&= \cos^2(\frac{\pi}{8})
\end{aligned}$$

- the following calculations are similar and we obtain  $\cos^2(\frac{\pi}{8})$

There are 4 different ways to draw  $s$  and  $r$  :

$$\mathcal{P}(\text{win}) = 4 \times \frac{1}{4} \times \cos^2(\frac{\pi}{8}) = \cos^2(\frac{\pi}{8})$$