Graph Algorithms
TD1 : Introduction

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1 To begin

Exercice 1.1 – Show that a graph always has an even number of odd degree vertices

Correction 1.1

Let G a graph. Thanks to *Handshaking Lemma* we have:

$$\sum_{v \in V(G)} deg(v) = 2|G(E)|$$

So the number $\sum_{v \in V(G)} deg(v)$ is an even number. To keep this property we must have an even number of odd degree vertices otherwise the sum become odd.

Exercice 1.2 – Show that a graph with at least 2 vertices contains 2 vertices of equal degree

Correction 1.2

Let G a graph with at least 2 vertices.

- If G has no isolated vertex, the degree of a vertex is between 1 and $|V(G)| - 1$ (v a vertex $1 \leq deg(v) < |V(G)|$). So we have $n := |V(G)| - 1$ different values for the degree of a vertex in G. The graph G contains $n + 1$ vertices so by the Pigeonhole principle we have two vertices with the same degree.
- If G has one isolated vertex, it's the same idea that the previous one but the degree of a vertex is between 0 and $|V(G)| - 2$.
- If G has at least two isolated vertices, we have 2 vertices with the same degree.

Exercice 1.3 – Let G be a graph of minimum degree $\delta(G) \leq 2$. Show that G contains a cycle.

Correction 1.3

Let G a graph with $\delta(G) \leq 2$.
Show that G has a cycle. Let $P = v_1 \dots v_l$ be the maximum path in G. Since it cannot be extended into a larger path, we must have $N(v_l) \subseteq V(P)$. So $\exists i \leq l - 1$ $v_i \in N(v_l)$ which yields that $v_i, \dots, v_l$ is a cycle.

Exercice 1.4 – Let G be a graph of minimum degree d, and of girth $2t + 1$. Given any vertex $v \in V(G)$, show that there are at least $d(d - 1)^{i-1}$ vertices at distance exactly i from v in G, for every $1 \leq i \leq t$. Deduce a lower bound on the number of vertices of G.

Correction 1.4

2 Dense subgraphs

Exercice 2.1 – Show that every graph of average degree d contains a subgraph of minimum degree at least $\frac{d}{2}$.

Correction 2.1

Let G a graph with an average degree d. We take the maximum average degree subgraph of G we note them H. We know, that we have $ad(H) = mad(H) \leq d$
If we have a vertex of degree less than $\frac{d}{2}$ in H then we have :

$$\begin{aligned} ad(H\backslash v) = \frac{2|E(H)| - 2deg_H(v)}{|V(H)|-1} &> \frac{2|E(H)| - d}{|V(H)|-1} \leq \frac{2|E(H)| - \frac{2|E(H)|}{|V(H)|}}{|V(H)|-1} \\ &> \frac{2|E(H)| \times |V(H)| - 2|E(H)|}{(|V(H)|-1) \times |V(H)|} = \frac{2[E(H)]}{|V(H)|} = d \end{aligned}$$

So if we have this type of vertex we have a contradiction $ad(H\backslash v) > mad(G)$

Exercice 2.2 – Can you find a similar relation between the maximum degree and the minimum degree ? And between the maximum degree and the average degree ?

Correction 2.2

Exercice 2.3 – Show that every graph of average degree d contains a bipartite subgraph of average degree at least $\frac{d}{2}$.

Correction 2.3

Let $H = (X, Y, E)$ a bipartite of the graph G given by the maximum cut.

3 Cuts and trees

Exercice 3.1 – If G is connected , and $e = uv$ is a bridge in G, how many connected components does $G\backslash e$ contain ? Show that u and v are cut-vertices.

Exercice 3.2 – Show that a graph G is a tree if and only if there exists a unique path from u to v in G, for every pair of vertices $u, v \in G$.

Exercice 3.3 – Let T a BFS tree of a graph G. Show that every edge of G is contained either within a layer of T, or between two consecutive layers of T.

Exercice 3.4 – Let T be a DFS tree of a graph G. Show that, for every edge $e \in E(G)$, ther is a branch of T that contains both extremities of e.