Homework

1 Basic operation and their notation

Exercice 1.1: Inner/outer products in Dirac notation

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} &(\langle 0| + 2\langle 1|) \times (3|0\rangle + 4|1\rangle) \\ =&3\langle 0|0\rangle + 4\langle 0|1\rangle + 6\langle 1|0\rangle + 8\langle 1|1\rangle \\ =&3 + 8 \\ =&11 \end{aligned}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}$$

The last two in Dirac notation :

$$\begin{aligned} &(3|0\rangle + 4|1\rangle) \times (\langle 0| + 2\langle 1|) \\ =&3|0\rangle\langle 0| + 6|0\rangle\langle 1| + 4|1\rangle\langle 0| + 8|1\rangle\langle 1| \end{aligned}$$

Exercice 1.2: Matrix products in Dirac notation

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 23 \\ 34 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} &(|0\rangle\langle 0| + 3|0\rangle\langle 1| + 2|1\rangle\langle 0| + 4|1\rangle\langle 1|) \times (5|0\rangle + 6|1\rangle) \\ =&5|0\rangle\langle 0|0\rangle + 6|0\rangle\langle 0|1\rangle + 15|0\rangle\langle 1|0\rangle + 18|0\rangle\langle 1|1\rangle + \\ &10|1\rangle\langle 0|0\rangle + 12|1\rangle\langle 0|1\rangle + 20|1\rangle\langle 1|0\rangle + 24|1\rangle\langle 1|1\rangle \\ =&5\langle 0|0\rangle|0\rangle + 18\langle 1|1\rangle|0\rangle + 10\langle 0|0\rangle|1\rangle + 24\langle 1|1\rangle|1\rangle \\ =&5|0\rangle + 18|0\rangle + 10|1\rangle + 24|1\rangle \\ =&23|0\rangle + 24|1\rangle \end{aligned}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} &(1/\sqrt{2}|0\rangle\langle 0| + 1/\sqrt{2}|0\rangle\langle 1| + 1/\sqrt{2}|1\rangle\langle 0| - 1/\sqrt{2}|1\rangle\langle 1|)^2 \\ =&1/2|0\rangle\langle 0|0\rangle\langle 0| + 1/2|0\rangle\langle 0|0\rangle\langle 1| + 1/2|0\rangle\langle 1|1\rangle\langle 0| - 1/2|0\rangle\langle 1|1\rangle\langle 1| \\ &1/2|1\rangle\langle 0|0\rangle\langle 1| + 1/2|1\rangle\langle 0|0\rangle\langle 0| - 1/2|1\rangle\langle 1|1\rangle\langle 0| + 1/2|1\rangle\langle 1|1\rangle\langle 1| \\ =&1/2|0\rangle\langle 0| + 1/2|0\rangle\langle 0| + 1/2|1\rangle\langle 1| + 1/2|1\rangle\langle 1| \\ =&|0\rangle\langle 0| + |1\rangle\langle 1| \end{aligned}$$

Exercice 1.3: Tensor products in Dirac/Coecke notation

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The last two in Dirac notation :

$$(|0\rangle + 2|1\rangle) \otimes (3|0\rangle + 4|1\rangle) = 3|00\rangle + 4|01\rangle + 6|10\rangle + 8|11\rangle$$

$$|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle = |00\rangle + |11\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$$

In Dirac notation :

$$\begin{aligned} &(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ =&|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\ =&|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| \end{aligned}$$

$$\begin{aligned} &(|0\rangle\langle 0|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ =&|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \\ =&|0\rangle\langle 0| + |1\rangle\langle 1| \end{aligned}$$

$$\begin{aligned} &1/\sqrt{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \otimes 1/\sqrt{2}(|0\rangle\langle 0| + |1\rangle\langle 0| + |1\rangle\langle 0| - |1\rangle\langle 1|) \\ =&\ 1/2(|0\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 2| + |0\rangle\langle 3| + \\ &|1\rangle\langle 0| - |1\rangle\langle 1| + |1\rangle\langle 2| - |1\rangle\langle 3| + \\ &|2\rangle\langle 0| + |2\rangle\langle 1| - |2\rangle\langle 2| - |2\rangle\langle 3| + \\ &|3\rangle\langle 0| - |3\rangle\langle 1| - |3\rangle\langle 2| + |3\rangle\langle 3|) \end{aligned}$$

We want to prove $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

- Dirac's notation :
- Coecke's notation :

Exercice 1.4: Dagger in Dirac/Coecke notation

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}^\dagger = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \qquad \begin{pmatrix} 1 & 3i \\ 2 & 4i \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 2 \\ -3i & -4i \end{pmatrix}$$

In Dirac notation:

$$\begin{aligned} &1/\sqrt{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)^\dagger \\ =&1/\sqrt{2}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|) \end{aligned}$$

$$\begin{aligned} &(|0\rangle\langle 0| + 3i|0\rangle\langle 1| + 2|1\rangle\langle 0| + 4i|1\rangle\langle 1|)^\dagger \\ =&(|0\rangle\langle 0| - 3i|1\rangle\langle 0| + 2|0\rangle\langle 1| - 4i|1\rangle\langle 1|) \end{aligned}$$

Exercice 1.5: Gates in Dirac notations

$$H = 1/\sqrt{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

$$CNot = |0\rangle\langle 0| + |1\rangle\langle 1| + |3\rangle\langle 2| + |2\rangle\langle 3|$$

$$T = |0\rangle\langle 0| + e^{\frac{i\pi}{4}}|1\rangle\langle 1|$$

Proof that are unitary matrix :

- H : $H^\dagger H = Id_1$ already do in Exercie-1.2
- $CNot$:

$$\begin{aligned} CNot^\dagger CNot &= (|0\rangle\langle 0| + |1\rangle\langle 1| + |3\rangle\langle 2| + |2\rangle\langle 3|)(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|) \\ &= (|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3|) \\ &= Id_4 \end{aligned}$$

- T:

$$\begin{aligned} (|0\rangle\langle 0| + e^{\frac{i\pi}{4}}|1\rangle\langle 1|)^2 &= |0\rangle\langle 0|0\rangle\langle 0| + e^{\frac{i\pi}{4}}|0\rangle\langle 0|1\rangle\langle 1| + e^{\frac{i\pi}{4}}(|1\rangle\langle 1|0\rangle\langle 0|) + e^{\frac{i\pi}{2}}(|1\rangle\langle 1|1\rangle\langle 1|) \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| \end{aligned}$$

Exercice 1.6: Pauli matrices in Dirac/Coecke notation

- For all $i, k \in [0, 3]$ we want to show $\sigma_i\sigma_j = \delta_{ij}I + i\sum_k \epsilon_{ijk}\sigma_k$
  - If $i = j$ then $\sigma_i\sigma_j = I$ and for all k we have $\epsilon_{ijk} = 0$.
    So we have $\delta_{ij}I = I = \sigma_i\sigma_j$

- For all $i, k \in [0, 3]$ we want to show $[\sigma_i, \sigma_j] = 2i \sum_k \epsilon_{ijk}$

$$
\begin{aligned}
[\sigma_i, \sigma_j] &= \sigma_i\sigma_j - \sigma_i\sigma_j \\
&= (\delta_{ij}I + i\sum_k \epsilon_{ijk}\sigma_k) - (\delta_{ji}I + i\sum_k \epsilon_{jik}\sigma_k) \\
&= i\sum_k \epsilon_{ijk}\sigma_k - i\sum_k \epsilon_{jik}\sigma_k \\
&= i\sum_k \epsilon_{ijk}\sigma_k + i\sum_k \epsilon_{ijk}\sigma_k \\
&= 2i\sum_k \epsilon_{ijk}\sigma_k
\end{aligned}
$$

- For all $i, k \in [0, 3]$ with $i \neq j$ we want to show $\{\sigma_i, \sigma_j\} = 0$

$$
\begin{aligned}
\{\sigma_i, \sigma_j\} &= \sigma_i\sigma_j + \sigma_i\sigma_j \\
&= \delta_{ij}I + i\sum_k \epsilon_{ijk}\sigma_k + \delta_{ji}I - i\sum_k \epsilon_{ijk}\sigma_k && \epsilon_{jik} = -\epsilon_{ijk} \\
&= 2\delta_{ij}I \\
&= 0 && i \neq j
\end{aligned}
$$

2 Postulates on pure states

Exercice 2.1: Evolutions
Let $|\psi\rangle = |0\rangle \otimes |0\rangle$ the initial state of two qubits. We want to compute $CNot(H \otimes I)|\psi\rangle$.

$$
\begin{aligned}
CNot(H \otimes I)(|0\rangle \otimes |0\rangle) &= CNot((|0\rangle + |1\rangle)/\sqrt{2} \otimes (|0\rangle + |1\rangle)/\sqrt{2}) \\
&= \frac{1}{2}CNot(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
&= \frac{1}{2}(|00\rangle + |01\rangle + |11\rangle + |10\rangle) := |\psi'\rangle
\end{aligned}
$$

$$
\begin{aligned}
(H \otimes I)CNot|\psi'\rangle &= (H \otimes I)\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
&= |00\rangle && H^2|\psi\rangle = |\psi\rangle
\end{aligned}
$$

We have $I := H^2$

1. $T^4H\ |0\rangle$
2. $HT^4H\ |0\rangle$
3. $CNot(H \otimes HT^4H)(|0\rangle \otimes |0\rangle)$
4. $(I \otimes CNot)(CNot \otimes I)(H \otimes I \otimes I)(|0\rangle \otimes |0\rangle \otimes |0\rangle)$
5. $(H \otimes I)(|0\rangle \otimes |0\rangle))$

Exercice 2.2: Measuring in another basis

- orthogonal :

$$\begin{aligned}\langle+|-\rangle &= (\frac{1}{\sqrt{2}})^2 + \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} \\ &= (\frac{1}{\sqrt{2}})^2 - (\frac{1}{\sqrt{2}})^2 \\ &= 0\end{aligned}$$

- norm one :

$$\begin{aligned}\langle+|+\rangle &= (\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1\end{aligned}$$

$$\begin{aligned}\langle-|-\rangle &= (\frac{1}{\sqrt{2}})^2 + (\frac{-1}{\sqrt{2}})^2 \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1\end{aligned}$$

- generate $\mathbb{C}^2$
  Let $c = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$ we want to find x and y such that $x|+\rangle + y|-\rangle = c$.

$$\begin{aligned}x|+\rangle + y|-\rangle &= \frac{x}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{y}{\sqrt{2}}(|0\rangle - |1\rangle) \\ &= (\frac{x+y}{\sqrt{2}})|0\rangle + (\frac{x-y}{\sqrt{2}})|1\rangle\end{aligned}$$

  So we have this system :

$$\begin{aligned}\begin{cases}(x+y)/\sqrt{2} &= \alpha \\ (x-y)/\sqrt{2} &= \beta\end{cases} &\Rightarrow \begin{cases}x+y &= \sqrt{2}\alpha \\ x-y &= \sqrt{2}\beta\end{cases} \\ &\Rightarrow \begin{cases}2x &= \sqrt{2}(\alpha+\beta) \\ x-y &= \sqrt{2}\beta\end{cases} \\ &\Rightarrow \begin{cases}x &= \frac{\sqrt{2}}{2}(\alpha+\beta) \\ -x+y &= -\sqrt{2}\beta\end{cases} \\ &\Rightarrow \begin{cases}x &= \frac{\sqrt{2}}{2}(\alpha+\beta) \\ y &= \frac{\sqrt{2}}{2}(\alpha-\beta)\end{cases}\end{aligned}$$

$B = \{|0\rangle, |1\rangle\}$ is another o.n.b of $\mathbb{C}^2$
We need to show that $\sum_{M\in\mathcal{M}_\pm} M^\dagger M = 1$

$$\begin{aligned}\sum_{M\in\mathcal{M}_\pm} &= (|+\rangle\langle+|)^\dagger(|+\rangle\langle+|) + (|-\rangle\langle-|)^\dagger(|-\rangle\langle-|) \\ &= (|+\rangle\langle+|+\rangle\langle+|) + (|-\rangle\langle-|-\rangle\langle-|) \\ &= |+\rangle\langle+| + |-\rangle\langle-| \\ &= \frac{1}{2}(|0\rangle\langle0| + |0\rangle\langle1| + |1\rangle\langle0| + |1\rangle\langle1|) + \frac{1}{2}(|0\rangle\langle0| - |0\rangle\langle1| - |1\rangle\langle0| + |1\rangle\langle1|) \\ &= |0\rangle\langle0| + |1\rangle\langle1| \\ &= 1\end{aligned}$$

So $\mathcal{M}_{\pm}$ is a valid measurement.
We have $|\psi\rangle = \frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle$

- For $|+\rangle\langle+|$

$$\begin{aligned}
p(|+\rangle) &= \langle\psi|(|+\rangle\langle+|)^{\dagger}|+\rangle\langle+||\psi\rangle \\
&= \langle\psi||+\rangle\langle+||\psi\rangle \\
&= \langle\psi|+\rangle\langle+|\psi\rangle \\
&= (\frac{1}{3}\times\frac{1}{\sqrt{2}}+\frac{\sqrt{8}}{3}\times\frac{1}{\sqrt{2}})^2 \\
&= \frac{1}{2}\times(\frac{1+\sqrt{8}}{3})^2 = \frac{1}{2}\times\frac{(1+\sqrt{8})^2}{9} = \frac{(1+\sqrt{8})^2}{18}
\end{aligned}$$

- For $|-\rangle\langle-|$

Exercice 2.3: Measuring to distinguish
We defined:

$$M_0 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \quad M_+ = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \quad M_f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3 Some mathematics

Exercice 3.1: Spectral theorems complements
Let D and D' two diagonal matrices and U a unitary matrix.

- Let $A = UDU^{\dagger}$ and $B = UD'U^{\dagger}$

$$\begin{aligned}
AB &= UDU^{\dagger}UD'U^{\dagger} \\
&= UDD'U^{\dagger} && U \text{ is a unitary matrix} \\
&= UD'DU^{\dagger} && D \text{ and } D' \text{ are diagonal} \\
&= UD'U^{\dagger}UDU^{\dagger} \\
&= BA
\end{aligned}$$

- Let $M = UDU^{\dagger}$

$$\begin{aligned}
MM^{\dagger} &= UDU^{\dagger}(UDU^{\dagger})^{\dagger} \\
&= UDU^{\dagger}(U^{\dagger})^{\dagger}D^{\dagger}U^{\dagger} \\
&= UDU^{\dagger}UD^{\dagger}U^{\dagger} \\
&= UDD^{\dagger}U^{\dagger} \\
&= UD^{\dagger}DU^{\dagger} \\
&= UD^{\dagger}U^{\dagger}UDU^{\dagger} \\
&= (UDU^{\dagger})^{\dagger}UDU^{\dagger} \\
&= M^{\dagger}M
\end{aligned}$$

- Let $E = UDU^\dagger$ with having only non-negative value.

  Let $|\psi\rangle \in \mathcal{M}_{n,1}(\mathbb{C})$, d_i such that $E_{i,i} = d_i$

$$\begin{aligned}\langle\psi|E|\psi\rangle &= \langle\psi|UDU^\dagger|\psi\rangle \\ &= (U^\dagger|\psi\rangle)^\dagger D(U^\dagger|\psi\rangle) \\ &= \sum_{i=1}^{n} d_i(U_i|\psi\rangle)^2 \\ &\geq 0\end{aligned}$$

- Let $V = UDU^\dagger$ with D having only modulus one values.

$$\begin{aligned}VV^\dagger &= UDU^\dagger(UDU^\dagger)^\dagger \\ &= UDU^\dagger UD^\dagger U^\dagger \\ &= UDD^\dagger U^\dagger \\ &= UU^\dagger && D \text{ has only modulus one values} \\ &= I\end{aligned}$$

  So V is a unitary matrix.

- Let E a non-negative matrix. E is spectrally decomposable with non-negative eigenvalues. We can take $M := \sqrt{E}$ which is defined by its spectral decomposition being with the square roots of the eigenvalues of E. M is hermitian and $E = MM$, so $E^\dagger = (MM)^\dagger = M^\dagger M^\dagger = MM = E$.

- The follow matrix is not normal :

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$MM^\dagger = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix} \neq \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix} = M^\dagger M$$

Exercice 3.2: Isometry versus unitary versus involution

- Let M a unitary and hermitian matrix.

$$\begin{aligned}MM &= MM^\dagger && M \text{is hermitian} \\ &= I && M \text{ is unitary}\end{aligned}$$

- Matrix 2×2 unitary that is not an involution :

$$M = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

  The inverse of M is :

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

  We have $M \neq M^{-1}$ so M is not an involution.

- Matrix $m \times n$ isometry that is not a unitary:

$$M = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$M^\dagger M = \begin{pmatrix} 1 \end{pmatrix} = I_1 \quad MM^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq I_2$$

- Let M an $n \times n$ isometry matrix ($M^\dagger M = I_n$)

4 On the nature of quantum information

Exercice 4.1: Hadamard

- $a = 0$

$$\begin{aligned} H|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^0|1\rangle) \end{aligned}$$

- $a = 1$

$$\begin{aligned} H|1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^1|1\rangle) \end{aligned}$$

Exercice 4.2: Who controls whom?
We define :

$$NotC = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$NotC|00\rangle = |00\rangle \qquad NotC|01\rangle = |11\rangle$$
$$NotC|10\rangle = |10\rangle \qquad NotC|11\rangle = |01\rangle$$

We want to proof $(H \otimes H)CNot(H \otimes H) = NotC$:

$$\begin{aligned} (H \otimes H)(|x\rangle \otimes |y\rangle) &= (H|x\rangle \otimes H|y\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + (-1)^y|1\rangle) \\ &= \frac{1}{2}(|00\rangle + (-1)^x|10\rangle + (-1)^y|01\rangle + (-1)^{x+y}|11\rangle) \end{aligned}$$

We apply the operator $CNot$:

$$\begin{aligned}
&CNot(\frac{1}{2}(|00\rangle + (-1)^x|10\rangle + (-1)^y|01\rangle + (-1)^{x+y}|11\rangle)) \\
&= \frac{1}{2}(|00\rangle + (-1)^x|11\rangle + (-1)^y|01\rangle + (-1)^{x+y}|10\rangle) \\
&= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x+y}|1\rangle) \otimes (\frac{1}{\sqrt{2}}(|0\rangle + (-1)^y|1\rangle)) \\
&= H|x \oplus y\rangle \otimes H|y\rangle && (-1)^{x+y} = (-1)^{x\oplus y}\ (x \oplus y \in \{0,1\})
\end{aligned}$$

Finlay we apply $(H \otimes H)$:

$$\begin{aligned}
HH|x \oplus y\rangle \otimes HH|y\rangle &= |x \oplus y\rangle \otimes |y\rangle \\
&= NotC(|x\rangle \otimes |y\rangle)
\end{aligned}$$

5 Protocols

6 Quantum error correction

7 Bell