TD n°1

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An isomorphism $f : A \to B$ such that there exists $g : B \to A$ such that $g \circ f = Id_A$ and $f \circ g = Id_B$.

1. Show that g is unique
2. Show that in the following situation

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] \arrow[rr, phantom, shift left=4ex, "*"] & & C \end{tikzcd}$$

$g \circ f$ is a isomorphism when g and f are isomorphism

3. deduce in $(*)$ that g is an isomorphism when f and $g \circ f$ are isomorphism
4. deduce in $(*)$ that f is an isomorphism when g and $g \circ f$ are isomorphism
5. Suppose that in $A \xrightarrow{f} B \xrightarrow{g} A \xrightarrow{h} B$ one has $g \circ f = Id_A$ and $h \circ g = Id_B$ show that $f = h$ in that case
6. Characterize the isomorphisms in the category **Set** of sets and function. **Top** of topological spaces and continuous functions

Correction :

1. Let $h : B \to A$ a morphism that $h \circ f = Id_A$ and $f \circ h = Id_B$

$$\begin{aligned} h &= h \circ Id_B && \text{By neutrality} \\ &= h \circ (f \circ g) \\ &= (h \circ f) \circ g && \text{By associativity} \\ &= Id_A \circ g \\ &= g && \text{By neutrality} \end{aligned}$$

We can say that g is unique. We will make note f^{-1}

2. We have f^{-1} and g^{-1} the inverse of f and g (they are isomorphism)
We will show that $f^{-1} \circ g^{-1}$ is an inverse of $g \circ f$

$$\begin{aligned} (f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ (g^{-1} \circ g) \circ f && \text{By associativity} \\ &= f^{-1} \circ f \\ &= Id_A \end{aligned}$$

A similar reasoning can be used to show :$(g \circ f) \circ (f^{-1} \circ g^{-1}$

3. Let $g' = f \circ (g \circ f)^{-1}$

$$\begin{aligned} g \circ g' &= g \circ (f \circ (g \circ f)^{-1}) \\ &= (g \circ f) \circ (g \circ f)^{-1} \\ &= Id_C \end{aligned}$$

$$\begin{aligned} g' \circ g &= (f \circ (g \circ f)^{-1}) \circ g \\ &= f \circ (g \circ f)^{-1} \circ g \circ f \circ f^{-1} \\ &= f \circ (g \circ f)^{-1} \circ (g \circ f) \circ f^{-1} \\ &= f \circ f^{-1} \\ &= Id_B \end{aligned}$$

g is well an isomorphism.

4. Roughly the same proof.
5. We have :

$$\begin{aligned} f &= (h \circ g) \circ f & h \circ g = Id_B \\ &= h \circ (g \circ f) & \text{By associativity} \\ &= h & g \circ f = Id_A \end{aligned}$$

   This question implies the first question.
6. Homomorphism (= A map between two structures, that preserves the operations of the structures $f(x \bullet y) = f(x) \bullet f(y)$)