Homework

1 Basic operation and their notation

Exercice 1.1: Inner/outer products in Dirac notation

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} &(\langle 0| + 2\langle 1|) \times (3|0\rangle + 4|1\rangle) \\ =&3\langle 0|0\rangle + 4\langle 0|1\rangle + 6\langle 1|0\rangle + 8\langle 1|1\rangle \\ =&3 + 8 \\ =&11 \end{aligned}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}$$

The last two in Dirac notation :

$$\begin{aligned} &(3|0\rangle + 4|1\rangle) \times (\langle 0| + 2\langle 1|) \\ =&3|0\rangle\langle 0| + 6|0\rangle\langle 1| + 4|1\rangle\langle 0| + 8|1\rangle\langle 1| \end{aligned}$$

Exercice 1.2: Matrix products in Dirac notation

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 23 \\ 34 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} &(|0\rangle\langle 0| + 3|0\rangle\langle 1| + 2|1\rangle\langle 0| + 4|1\rangle\langle 1|) \times (5|0\rangle + 6|1\rangle) \\ =&5|0\rangle\langle 0|0\rangle + 6|0\rangle\langle 0|1\rangle + 15|0\rangle\langle 1|0\rangle + 18|0\rangle\langle 1|1\rangle + \\ &10|1\rangle\langle 0|0\rangle + 12|1\rangle\langle 0|1\rangle + 20|1\rangle\langle 1|0\rangle + 24|1\rangle\langle 1|1\rangle \\ =&5\langle 0|0\rangle|0\rangle + 18\langle 1|1\rangle|0\rangle + 10\langle 0|0\rangle|1\rangle + 24\langle 1|1\rangle|1\rangle \\ =&5|0\rangle + 18|0\rangle + 10|1\rangle + 24|1\rangle \\ =&23|0\rangle + 24|1\rangle \end{aligned}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} &(1/\sqrt{2}|0\rangle\langle 0| + 1/\sqrt{2}|0\rangle\langle 1| + 1/\sqrt{2}|1\rangle\langle 0| - 1/\sqrt{2}|1\rangle\langle 1|)^2 \\ =&1/2|0\rangle\langle 0|0\rangle\langle 0| + 1/2|0\rangle\langle 0|0\rangle\langle 1| + 1/2|0\rangle\langle 1|1\rangle\langle 0| - 1/2|0\rangle\langle 1|1\rangle\langle 1| \\ &1/2|1\rangle\langle 0|0\rangle\langle 1| + 1/2|1\rangle\langle 0|0\rangle\langle 0| - 1/2|1\rangle\langle 1|1\rangle\langle 0| + 1/2|1\rangle\langle 1|1\rangle\langle 1| \\ =&1/2|0\rangle\langle 0| + 1/2|0\rangle\langle 0| + 1/2|1\rangle\langle 1| + 1/2|1\rangle\langle 1| \\ =&|0\rangle\langle 0| + |1\rangle\langle 1| \end{aligned}$$

Exercice 1.3: Tensor products in Dirac/Coecke notation

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The last two in Dirac notation :

$$(|0\rangle + 2\,|1\rangle) \otimes (3\,|0\rangle + 4\,|1\rangle) = 3\,|00\rangle + 4\,|01\rangle + 6\,|10\rangle + 8\,|11\rangle$$

$$|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle = |00\rangle + |11\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$$

In Dirac notation :

$$\begin{aligned} &(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ =& |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\ =& |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| \end{aligned}$$

$$\begin{aligned} &(|0\rangle\langle 0|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ =& |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \\ =& |0\rangle\langle 0| + |1\rangle\langle 1| \end{aligned}$$

$$\begin{aligned} &1/\sqrt{2}(\,|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \otimes 1/\sqrt{2}(|0\rangle\langle 0| + |1\rangle\langle 0| + |1\rangle\langle 0| - |1\rangle\langle 1|) \\ =&\ 1/2(\,|0\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 2| + |0\rangle\langle 3| + \\ &\quad |1\rangle\langle 0| - |1\rangle\langle 1| + |1\rangle\langle 2| - |1\rangle\langle 3| + \\ &\quad |2\rangle\langle 0| + |2\rangle\langle 1| - |2\rangle\langle 2| - |2\rangle\langle 3| + \\ &\quad |3\rangle\langle 0| - |3\rangle\langle 1| - |3\rangle\langle 2| + |3\rangle\langle 3|) \end{aligned}$$

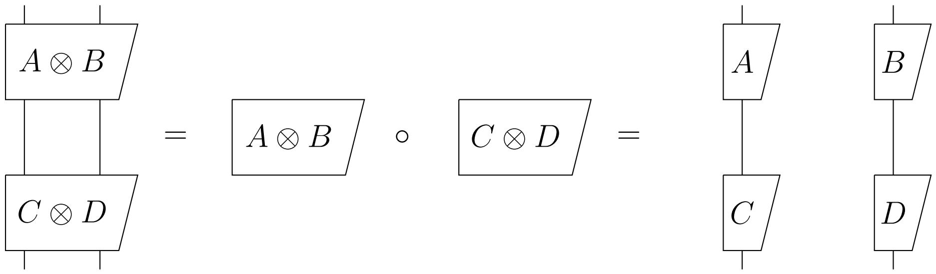
We want to prove $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

- Dirac's notation :

  We have $A = \sum_{i,j} a_{i,j} |i\rangle \langle j|$ and $D = \sum_{k,l} d_{k,l} |k\rangle \langle l|$

$$
\begin{aligned}
(A \otimes B)(C \otimes D) &= ((\sum_{i,j} a_{i,j} |i\rangle \langle j|) \otimes B)(C \otimes (\sum_{k,l} d_{k,l} |k\rangle \langle j|)) \\
&= (\sum_{i,j} a_{i,j} |i\rangle \langle j|)C \otimes B(\sum_{k,l} d_{k,l} |k\rangle \langle j|) && \text{bilinearity of } \otimes \\
&= (AC) \otimes (BD)
\end{aligned}
$$

- Coecke's notation :



Exercice 1.4: Dagger in Dirac/Coecke notation

$$
\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}^\dagger = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \qquad \begin{pmatrix} 1 & 3i \\ 2 & 4i \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 2 \\ -3i & -4i \end{pmatrix}
$$

In Dirac notation:

$$
\begin{aligned}
&1/\sqrt{2}(|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| - |1\rangle \langle 1|)^\dagger \\
=&1/\sqrt{2}(|0\rangle \langle 0| + |1\rangle \langle 0| + |0\rangle \langle 1| - |1\rangle \langle 1|)
\end{aligned}
$$

$$
\begin{aligned}
&(|0\rangle \langle 0| + 3i\,|0\rangle \langle 1| + 2\,|1\rangle \langle 0| + 4i\,|1\rangle \langle 1|)^\dagger \\
=&(|0\rangle \langle 0| - 3i\,|1\rangle \langle 0| + 2\,|0\rangle \langle 1| - 4i\,|1\rangle \langle 1|)
\end{aligned}
$$

Exercice 1.5: Gates in Dirac notations

$$H = 1/\sqrt{2}(|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| - |1\rangle \langle 1|)$$

$$CNot = |0\rangle \langle 0| + |1\rangle \langle 1| + |3\rangle \langle 2| + |2\rangle \langle 3|$$

$$T = |0\rangle \langle 0| + e^{\frac{i\pi}{4}} |1\rangle \langle 1|$$

Proof that are unitary matrix :

- H : $H^\dagger H = Id_1$ already do in Exercie-1.2
- $CNot$:

$$
\begin{aligned}
CNot^\dagger CNot &= (|0\rangle \langle 0| + |1\rangle \langle 1| + |3\rangle \langle 2| + |2\rangle \langle 3|)(|0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 3| + |3\rangle \langle 2|) \\
&= (|0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2| + |3\rangle \langle 3|) \\
&= Id_4
\end{aligned}
$$

- T:

$$\begin{aligned}
&(|0\rangle\langle 0| + e^{\frac{i\pi}{4}}|1\rangle\langle 1|)(|0\rangle\langle 0| + e^{\frac{-i\pi}{4}}|1\rangle\langle 1|)\\
&= |0\rangle\langle 0|0\rangle\langle 0| + e^{\frac{i\pi}{4}}|0\rangle\langle 0|1\rangle\langle 1| + e^{\frac{i\pi}{4}}(|1\rangle\langle 1|0\rangle\langle 0|) + e^{\frac{i\pi}{2}}(|1\rangle\langle 1|1\rangle\langle 1|)\\
&= |0\rangle\langle 0| + |1\rangle\langle 1|
\end{aligned}$$

Exercice 1.6: Pauli matrices in Dirac/Coecke notation

- For all $i, k \in [0,3]$ we want to show $\sigma_i\sigma_j = \delta_{ij}I + i\sum_k \epsilon_{ijk}\sigma_k$
  - If $i = j$ then $\sigma_i\sigma_j = I$ and for all k we have $\epsilon_{ijk} = 0$.
    So we have $\delta_{ij}I = I = \sigma_i\sigma_j$
  - If $j = i + 1$
- For all $i, k \in [0,3]$ we want to show $[\sigma_i, \sigma_j] = 2i\sum_k \epsilon_{ijk}$

$$\begin{aligned}
[\sigma_i, \sigma_j] &= \sigma_i\sigma_j - \sigma_i\sigma_j\\
&= (\delta_{ij}I + i\sum_k \epsilon_{ijk}\sigma_k) - (\delta_{ji}I + i\sum_k \epsilon_{jik}\sigma_k)\\
&= i\sum_k \epsilon_{ijk}\sigma_k - i\sum_k \epsilon_{jik}\sigma_k\\
&= i\sum_k \epsilon_{ijk}\sigma_k + i\sum_k \epsilon_{ijk}\sigma_k\\
&= 2i\sum_k \epsilon_{ijk}\sigma_k
\end{aligned}$$

- For all $i, k \in [0,3]$ with $i \neq j$ we want to show $\{\sigma_i, \sigma_j\} = 0$

$$\begin{aligned}
\{\sigma_i, \sigma_j\} &= \sigma_i\sigma_j + \sigma_i\sigma_j\\
&= \delta_{ij}I + i\sum_k \epsilon_{ijk}\sigma_k + \delta_{ji}I - i\sum_k \epsilon_{ijk}\sigma_k && \epsilon_{jik} = -\epsilon_{ijk}\\
&= 2\delta_{ij}I\\
&= 0 && i \neq j
\end{aligned}$$

2 Postulates on pure states

Exercice 2.1: Evolutions
Let $|\psi\rangle = |0\rangle \otimes |0\rangle$ the initial state of two qubits. We want to compute $CNot(H \otimes I)|\psi\rangle$.

$$\begin{aligned}
CNot(H \otimes I)(|0\rangle \otimes |0\rangle) &= CNot((|0\rangle + |1\rangle)/\sqrt{2} \otimes |0\rangle)\\
&= \frac{1}{\sqrt{2}}CNot(|00\rangle + |10\rangle)\\
&= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) := |\psi'\rangle
\end{aligned}$$

$$
\begin{aligned}
(H \otimes I)CNot\,|\psi'\rangle &= (H \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \\
&= |00\rangle && H^2\,|\psi\rangle = |\psi\rangle
\end{aligned}
$$

We have $I := H^2$

1. $T^4H\ \ |0\rangle$
2. $HT^4H\ \ |0\rangle$
3. $CNot(H \otimes HT^4H)(|0\rangle \otimes |0\rangle)$
4. $(I \otimes CNot)(CNot \otimes I)(H \otimes I \otimes I)(|0\rangle \otimes |0\rangle \otimes |0\rangle)$
5. $(H \otimes H)(|0\rangle \otimes |0\rangle))$

Exercice 2.2: Measuring in another basis

- orthogonal :

$$
\begin{aligned}
\langle +|-\rangle &= (\frac{1}{\sqrt{2}})^2 + \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} \\
&= (\frac{1}{\sqrt{2}})^2 - (\frac{1}{\sqrt{2}})^2 \\
&= 0
\end{aligned}
$$

- norm one :

$$
\begin{aligned}
\langle +|+\rangle &= (\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}
\qquad
\begin{aligned}
\langle -|-\rangle &= (\frac{1}{\sqrt{2}})^2 + (\frac{-1}{\sqrt{2}})^2 \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}
$$

- generate $\mathbb{C}^2$

  Let $c = \alpha\,|0\rangle + \beta\,|1\rangle \in \mathbb{C}^2$ we want to find x and y such that $x\,|+\rangle + y\,|-\rangle = c$.

$$
\begin{aligned}
x\,|+\rangle + y\,|-\rangle &= \frac{x}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{y}{\sqrt{2}}(|0\rangle - |1\rangle) \\
&= (\frac{x+y}{\sqrt{2}})\,|0\rangle + (\frac{x-y}{\sqrt{2}})\,|1\rangle
\end{aligned}
$$

So we have this system :

$$\begin{cases} (x+y)/\sqrt{2} &= \alpha \\ (x-y)/\sqrt{2} &= \beta \end{cases} \Rightarrow \begin{cases} x+y &= \sqrt{2}\alpha \\ x-y &= \sqrt{2}\beta \end{cases}$$
$$\Rightarrow \begin{cases} 2x &= \sqrt{2}(\alpha+\beta) \\ x-y &= \sqrt{2}\beta \end{cases}$$
$$\Rightarrow \begin{cases} x &= \frac{\sqrt{2}}{2}(\alpha+\beta) \\ -x+y &= -\sqrt{2}\beta \end{cases}$$
$$\Rightarrow \begin{cases} x &= \frac{\sqrt{2}}{2}(\alpha+\beta) \\ y &= \frac{\sqrt{2}}{2}(\alpha-\beta) \end{cases}$$

$B = \{|0\rangle, |1\rangle\}$ is another o.n.b of $\mathbb{C}^2$
We need to show that $\sum_{M\in\mathcal{M}_\pm} M^\dagger M = 1$

$$\begin{aligned}
\sum_{M\in\mathcal{M}_\pm} &= (|+\rangle\langle+|)^\dagger(|+\rangle\langle+|) + (|-\rangle\langle-|)^\dagger(|-\rangle\langle-|) \\
&= (|+\rangle\langle+|+\rangle\langle+|) + (|-\rangle\langle-|-\rangle\langle-|) \\
&= |+\rangle\langle+| + |-\rangle\langle-| \\
&= \frac{1}{2}(|0\rangle\langle0| + |0\rangle\langle1| + |1\rangle\langle0| + |1\rangle\langle1|) + \frac{1}{2}(|0\rangle\langle0| - |0\rangle\langle1| - |1\rangle\langle0| + |1\rangle\langle1|) \\
&= |0\rangle\langle0| + |1\rangle\langle1| \\
&= 1
\end{aligned}$$

So $\mathcal{M}_\pm$ is a valid measurement.
We have $|\psi\rangle = \frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle$

- For $|+\rangle\langle+|$

$$\begin{aligned}
p(|+\rangle\langle+|) &= \langle\psi|(|+\rangle\langle+|)^\dagger|+\rangle\langle+|\psi\rangle \\
&= \langle\psi||+\rangle\langle+||\psi\rangle \\
&= \langle\psi|\frac{1}{2}(|0\rangle\langle0| + |0\rangle\langle1| + |1\rangle\langle0| + |1\rangle\langle1|)|\psi\rangle \\
&= \frac{1}{2}(\frac{1}{3}\langle0| + \frac{\sqrt{8}}{3}\langle1|)(|0\rangle\langle0| + |0\rangle\langle1| + |1\rangle\langle0| + |1\rangle\langle1|)|\psi\rangle \\
&= \frac{1+\sqrt{8}}{6}(\langle0| + \langle1|)\frac{1}{2}(\frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle) \\
&= \frac{9+2\sqrt{8}}{18} = \frac{1}{2} + \frac{\sqrt{8}}{9}
\end{aligned}$$

- For $|-\rangle\langle-|$

$$
\begin{aligned}
p(|-\rangle\langle-|) &= \langle\psi|\,(|-\rangle\langle-|)^\dagger\,|-\rangle\langle-|\,|\psi\rangle \\
&= \langle\psi|\,|-\rangle\langle-|\,|\psi\rangle \\
&= \langle\psi|\,\frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)\,|\psi\rangle \\
&= \frac{1}{2}(\frac{1}{3}\langle 0| + \frac{\sqrt{8}}{3}\langle 1|)(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)\,|\psi\rangle \\
&= \frac{1}{2}(\frac{1-\sqrt{8}}{3}\langle 0| + \frac{\sqrt{8}-1}{3}\langle 1|)(\frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle) \\
&= \frac{1}{2}(\frac{1-\sqrt{8}}{9} + \frac{8-\sqrt{8}}{9}) \\
&= \frac{1}{2}(\frac{9-2\sqrt{8}}{9}) = \frac{1}{2} - \frac{\sqrt{8}}{9}
\end{aligned}
$$

The post measure states are :

$$
|\psi_+\rangle = \frac{1}{\sqrt{\frac{1}{2}+\frac{\sqrt{8}}{9}}}\begin{pmatrix} \frac{1+\sqrt{8}}{6} \\ \frac{1+\sqrt{8}}{6} \end{pmatrix}
$$

$$
|\psi_-\rangle = \frac{1}{\sqrt{\frac{1}{2}-\frac{\sqrt{8}}{9}}}\begin{pmatrix} \frac{1-\sqrt{8}}{6} \\ \frac{\sqrt{8}-1}{6} \end{pmatrix}
$$

Exercice 2.3: Measuring to distinguish

We defined:

$$
M_0 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad M_+ = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad M_f = \begin{pmatrix} \frac{-1-i}{2} & i \\ \frac{1-i}{2} & i \end{pmatrix}
$$

$$
\begin{aligned}
\langle 0|\, M_0^\dagger M_0\,|0\rangle &= \langle 0|\, M_0\,|0\rangle \\
&= \langle 0|0\rangle = 1
\end{aligned}
$$

$$
\begin{aligned}
\langle +|\, M_0^\dagger M_0\,|+\rangle &= (0\ 0)M_0\,|+\rangle \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle 0|\, M_+^\dagger M_+\,|0\rangle &= (0\ 0)M_+\,|0\rangle \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle +|\, M_+^\dagger M_+\,|+\rangle &= \langle +|\, M_+\,|+\rangle \\
&= \frac{2}{\sqrt{2}}\langle 1|\,|+\rangle \\
&= \frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 1
\end{aligned}
$$

The measure $\mathcal{M}$ is valid :

$$
M_0^\dagger M_0 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad M_+^\dagger M_+ = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \qquad M_f^\dagger M_f = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}
$$

$$
M_0^\dagger M_0 + M_+^\dagger M_+ + M_f^\dagger M_f = I_2
$$

Exercice 2.4: Measuring the phase
The probability to get a result x when we measure $e^{i\theta}|0\rangle$ is :

$$\begin{aligned} p(x) &= e^{-i\theta} M_x^\dagger M_x e^{i\theta}|0\rangle \\ &= e^{-i\theta} \times e^{i\theta}(\overline{\alpha}\langle 0| + \overline{\beta}\langle 1|)(\alpha|0\rangle + \beta|1\rangle) \\ &= \langle\alpha|\alpha\rangle + \langle\beta|\beta\rangle \end{aligned}$$

The result does not depend on θ. So, we don't have a measure who can distinguish $|0\rangle$ from $e^{i\theta}$
We have the measurement M_+ who can sometimes tell the difference between $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle)$

$$\begin{aligned} p(|+\rangle) &= \langle\psi| M_+^\dagger M_+ |\psi\rangle \\ &= \frac{1}{2} + \frac{1}{4}e^{i\theta} + \frac{1}{4}e^{i\theta} \end{aligned}$$

The result depends on θ :)

Exercice 2.5: Measuring a subsystem
a

3 Some mathematics

Exercice 3.1: Spectral theorems complements
Let D and D' two diagonal matrices and U a unitary matrix.

- Let $A = UDU^\dagger$ and $B = UD'U^\dagger$

$$\begin{aligned} AB &= UDU^\dagger UD'U^\dagger \\ &= UDD'U^\dagger && U \text{ is a unitary matrix} \\ &= UD'DU^\dagger && D \text{ and } D' \text{ are diagonal} \\ &= UD'U^\dagger UDU^\dagger \\ &= BA \end{aligned}$$

- Let $M = UDU^\dagger$

$$\begin{aligned} MM^\dagger &= UDU^\dagger(UDU^\dagger)^\dagger \\ &= UDU^\dagger(U^\dagger)^\dagger D^\dagger U^\dagger \\ &= UDU^\dagger UD^\dagger U^\dagger \\ &= UDD^\dagger U^\dagger \\ &= UD^\dagger DU^\dagger \\ &= UD^\dagger U^\dagger UDU^\dagger \\ &= (UDU^\dagger)^\dagger UDU^\dagger \\ &= M^\dagger M \end{aligned}$$

- Let $E = UDU^\dagger$ with having only non-negative value.

  Let $|\psi\rangle \in \mathcal{M}_{n,1}(\mathbb{C})$, d_i such that $E_{i,i} = d_i$

$$\begin{aligned}
\langle\psi| E |\psi\rangle &= \langle\psi| UDU^\dagger |\psi\rangle \\
&= (U^\dagger |\psi\rangle)^\dagger D(U^\dagger |\psi\rangle) \\
&= \sum_{i=1}^{n} d_i(U_i |\psi\rangle)^2 \\
&\geq 0
\end{aligned}$$

- Let $V = UDU^\dagger$ with D having only modulus one values.

$$\begin{aligned}
VV^\dagger &= UDU^\dagger(UDU^\dagger)^\dagger \\
&= UDU^\dagger UD^\dagger U^\dagger \\
&= UDD^\dagger U^\dagger \\
&= UU^\dagger && D \text{ has only modulus one values} \\
&= I
\end{aligned}$$

  So V is a unitary matrix.

- Let E a non-negative matrix. E is spectrally decomposable with non-negative eigenvalues. We can take $M := \sqrt{E}$ which is defined by its spectral decomposition being with the square roots of the eigenvalues of E. M is hermitian and $E = MM$, so $E^\dagger = (MM)^\dagger = M^\dagger M^\dagger = MM = E$.

- The follow matrix is not normal :

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$MM^\dagger = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix} \neq \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix} = M^\dagger M$$

Exercice 3.2: Isometry versus unitary versus involution

- Let M a unitary and hermitian matrix.

$$\begin{aligned}
MM &= MM^\dagger && M\text{is hermitian} \\
&= I && M \text{ is unitary}
\end{aligned}$$

- Matrix 2×2 unitary that is not an involution :

$$M = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

  The inverse of M is :

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

  We have $M \neq M^{-1}$ so M is not an involution.

- Matrix $m \times n$ isometry that is not a unitary:

$$M = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$M^\dagger M = \begin{pmatrix} 1 \end{pmatrix} = I_1 \quad MM^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq I_2$$

- Let M an $n \times n$ isometry matrix ($M^\dagger M = I_n$)

$$\begin{aligned} MM^\dagger &= \sum_{i,j} M_{i,j} \left|i\right\rangle \left\langle j\right| \sum_{i,j} M_{j,i} \left|i\right\rangle \left\langle j\right| \\ &= \sum_{i,j} M_{i,j} M_{j,i} \left|i\right\rangle \left\langle i\right| \\ &= \sum_{i,j} M_{j,i} M_{i,j} \left|i\right\rangle \left\langle j\right| \\ &= \sum_{i,j} M_{j,i} \left|i\right\rangle \left\langle j\right| \sum_{i,j} M_{i,j} \left|i\right\rangle \left\langle j\right| \\ &= M^\dagger M \\ &= I_n \end{aligned}$$

4 On the nature of quantum information

Exercice 4.1: Hadamard

- $a = 0$

$$\begin{aligned} H\left|0\right\rangle &= \frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle) \\ &= \frac{1}{\sqrt{2}}(\left|0\right\rangle + (-1)^0 \left|1\right\rangle) \end{aligned}$$

- $a = 1$

$$\begin{aligned} H\left|1\right\rangle &= \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle) \\ &= \frac{1}{\sqrt{2}}(\left|0\right\rangle + (-1)^1 \left|1\right\rangle) \end{aligned}$$

Exercice 4.2: Who controls whom?
We define :

$$NotC = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$NotC\left|00\right\rangle = \left|00\right\rangle \qquad NotC\left|01\right\rangle = \left|11\right\rangle$$
$$NotC\left|10\right\rangle = \left|10\right\rangle \qquad NotC\left|11\right\rangle = \left|01\right\rangle$$

We want to proof $(H \otimes H)CNot(H \otimes H) = NotC$:

$$\begin{aligned}(H \otimes H)(|x\rangle \otimes |y\rangle) &= (H|x\rangle \otimes H|y\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + (-1)^y |1\rangle) \\ &= \frac{1}{2}(|00\rangle + (-1)^x |10\rangle + (-1)^y |01\rangle + (-1)^{x+y} |11\rangle)\end{aligned}$$

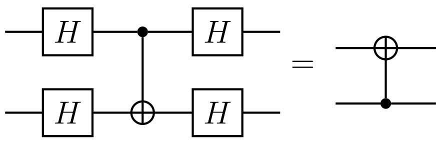
We apply the operator $CNot$:

$$\begin{aligned}&CNot(\frac{1}{2}(|00\rangle + (-1)^x |10\rangle + (-1)^y |01\rangle + (-1)^{x+y} |11\rangle)) \\ &= \frac{1}{2}(|00\rangle + (-1)^x |11\rangle + (-1)^y |01\rangle + (-1)^{x+y} |10\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x+y} |1\rangle) \otimes (\frac{1}{\sqrt{2}}(|0\rangle + (-1)^y |1\rangle)) \\ &= H|x \oplus y\rangle \otimes H|y\rangle \qquad (-1)^{x+y} = (-1)^{x \oplus y} \; (x \oplus y \in \{0, 1\})\end{aligned}$$

Finlay we apply $(H \otimes H)$:

$$\begin{aligned}(H \otimes H)(H|x \oplus y\rangle \otimes H|y\rangle) &= HH|x \oplus y\rangle \otimes HH|y\rangle \\ &= |x \oplus y\rangle \otimes |y\rangle \\ &= NotC(|x\rangle \otimes |y\rangle)\end{aligned}$$

In quantum circuit :



5 Protocols

Exercice 5.1: Canonical basis versus diagonal basis

- If Bob measures the result in the same basis then he can retrieve the information sent
- If Bob measures in an other basis then he learns nothing about the message.
- If Eve intercepts and measures it in the same basis then Bob can have some information on the message if he read the message in the same basis.
- But if Eve intercepts and measures it in an other basis and Bob read the message in the original basis then he learns nothing about the message.

Exercice 5.2: BB84

1. Alice will start by producing a random string of bits, encode each of them either into the canonical or the diagonal basis, and send that to Bob.
2. Bob will measure them either using the canonical basis or the diagonal basis, at random.
3. Bob will broadcast which bases he used

4. Alice will know when Bob used the same base. When Bob has used the right base, Bob's information is correct, otherwise it is wrong (previous exercise).

5. Eve does not know the bases like Bob And she has very little chance of having the right basic sequence ($\frac{1}{2^n}$). But she's going to disrupt Bob's measurements.

6.

They can use common measurements bases to create an encryption key. For example, a basic measurements base sequence can become a binary code with 0 when we have the base $\mathcal{M}$ and 1 if we have the base $\mathcal{M}'$. With this generate key we can communicate with an existing encryption protocol.

Exercice 5.3: Quantum random access code
TODO

Exercice 5.4: The Bell basis

$$|\beta_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\beta_1\rangle = (X \otimes I)\,|\beta_0\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$$
$$|\beta_2\rangle = (Y \otimes I)\,|\beta_2\rangle = \frac{i}{\sqrt{2}}(|10\rangle - |01\rangle) \qquad |\beta_3\rangle = (Z \otimes I)\,|\beta_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

These four states are orthogonal and orthonormal,
orthogonal:

$$\langle\beta_0|\beta_1\rangle = \frac{1}{2} \times 0 = 0 \qquad \langle\beta_0|\beta_2\rangle = \frac{i}{2} \times 0 = 0$$
$$\langle\beta_0|\beta_3\rangle = \frac{1}{2} - \frac{1}{2} = 0 \qquad \langle\beta_1|\beta_2\rangle = \frac{i}{2} - \frac{i}{2} = 0$$
$$\langle\beta_1|\beta_3\rangle = \frac{1}{2} \times 0 = 0 \qquad \langle\beta_2|\beta_3\rangle = \frac{i}{2} \times 0 = 0$$

orthonormal:

$$\langle\beta_0|\beta_0\rangle = \frac{1}{2} + \frac{1}{2} = 1 \qquad \langle\beta_1|\beta_1\rangle = \frac{1}{2} + \frac{1}{2} = 1$$
$$\langle\beta_2|\beta_2\rangle = \frac{1}{2} + \frac{1}{2} = 1 \qquad \langle\beta_3|\beta_3\rangle = \frac{1}{2} + \frac{1}{2} = 1$$

So, this states are an orthonormal basis.
It is also a valid measurement :

$$\begin{aligned}\sum_i \mathcal{M}_i &= |\beta_0\rangle\langle\beta_0| + |\beta_1\rangle\langle\beta_1| + |\beta_2\rangle\langle\beta_2| + |\beta_3\rangle\langle\beta_3| \\ &= I_4\end{aligned}$$

Exercice 5.5: Superdense coding
At the beginning, Alice and Bob share an entangled state $|\beta_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle$ Alice can change $|\beta_0\rangle$ in $|\beta_k\rangle$ with this operation :

- $|\beta_0\rangle$: do nothing

- $|\beta_1\rangle$: apply the matrix X

$$\begin{aligned} X|\beta_0\rangle &= (X\otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \\ &= |\beta_1\rangle \end{aligned}$$

- $|\beta_2\rangle$: apply the matrix Y

$$\begin{aligned} Y|\beta_0\rangle &= (Y\otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ &= \frac{i}{\sqrt{2}}(|10\rangle - |01\rangle) \\ &= |\beta_2\rangle \end{aligned}$$

- $|\beta_3\rangle$: apply the matrix Z

$$\begin{aligned} Y|\beta_0\rangle &= (Z\otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ &= |\beta_3\rangle \end{aligned}$$

So Alice can encode the four possible pairs of bits (00, 01, 10 and 11) with the 4 Bell states. We have shown that Alice can modify $|B_0\rangle$ on her own, so with qubit she can change the communication state to one of the 4 states. Bob can measure the result and retrieve the information from Alice.

Exercice 5.6: Discussion: classical description of a single qubit
A qubit is coded with this formula : $\alpha|0\rangle + \beta|1\rangle$. We just need to send 2 complexes numbers. So if a number is encoded with n bits we send $4n$ bits.

Exercice 5.7: Teleportation

$$\begin{aligned} \frac{1}{2}\sum_i |\beta_i\rangle \otimes \sigma_i|\psi\rangle =& \frac{1}{2\sqrt{2}}((|00\rangle + |11\rangle)\otimes(\alpha|0\rangle + \beta|1\rangle)+ \\ & (|10\rangle + |01\rangle)\otimes(\alpha|1\rangle + \beta|0\rangle)+ \\ & (i|10\rangle - i|01\rangle)\otimes(i\alpha|1\rangle - i\beta|0\rangle)+ \\ & (|00\rangle - |11\rangle)\otimes(\alpha|0\rangle - \beta|1\rangle)) \\ =& \frac{1}{\sqrt{2}}(\alpha|000\rangle + \beta|100\rangle + \alpha|011\rangle + \beta|111\rangle \\ =& \frac{1}{2\sqrt{2}}(2\alpha(|000\rangle + |011\rangle) + 2\beta(|100\rangle + |111\rangle)) \\ =& \frac{1}{\sqrt{2}}(\alpha(|000\rangle + |011\rangle) + \beta(|100\rangle + |111\rangle)) \\ =& \frac{1}{\sqrt{2}}((\alpha|0\rangle + \beta|1\rangle)\otimes|0\rangle\otimes|0\rangle + (\alpha|0\rangle + \beta|1\rangle)\otimes|1\rangle\otimes|1\rangle) \\ =& \frac{1}{\sqrt{2}}(|\psi\rangle\otimes|0\rangle\otimes|0\rangle) + (|\psi\rangle\otimes|1\rangle\otimes|1\rangle) \end{aligned}$$

Exercice 5.8: The swap test
Before the measurement we have this state :

$$
\begin{aligned}
|\kappa\rangle &= (H \otimes I \otimes I)CSwap(H \otimes I \otimes I)\,|0\rangle \otimes |\phi\rangle \otimes |\psi\rangle \\
&= (H \otimes I \otimes I)CSwap((\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) \otimes |\phi\rangle \otimes |\psi\rangle) \\
&= (H \otimes I \otimes I)CSwap\frac{1}{\sqrt{2}}(|0\phi\psi\rangle + |1\phi\psi\rangle) \\
&= (H \otimes I \otimes I)\frac{1}{\sqrt{2}}(|0\phi\psi\rangle + |1\psi\phi\rangle) \\
&= \frac{1}{2}(|0\rangle \otimes (|\phi\psi\rangle + |\psi\phi\rangle) - |1\rangle \otimes (|\phi\psi\rangle + |\psi\phi\rangle)) \\
&= \frac{1}{2}(|0\phi\psi\rangle + |0\psi\phi\rangle + |1\phi\psi\rangle - |1\psi\phi\rangle)
\end{aligned}
$$

We apply the measures :

$$
\begin{aligned}
p(0) &= \langle\kappa|\,|0\rangle\,\langle 0|\,|\kappa\rangle \\
&= \frac{1}{2}(\langle\phi\psi| + |\psi\phi\rangle) \times \frac{1}{2}(\langle\phi\psi| + |\psi\phi\rangle) \\
&= \frac{1}{4}(2 + \langle\phi\psi|\psi\phi\rangle + \langle\psi\phi|\phi\psi\rangle) \\
&= \frac{1}{4}(2 + 2\,\langle\phi\psi|\psi\phi\rangle) \\
&= \frac{1}{2} + \frac{\langle\psi\phi|\psi\phi\rangle}{2} \\
&= \frac{1}{2} + \frac{1}{2}|\,\langle\psi|\phi\rangle\,|^2
\end{aligned}
$$

we have $p(1) = 1 - p(0)$ so $p(1) = \frac{1}{2} - \frac{1}{2}|\,\langle\psi|\phi\rangle\,|^2$

Exercice 5.9: Quantum fingerprinting
TODO

6 Quantum error correction

Exercice 6.1: Proof of the 3 qubit code against bit flip errors
Let $P = |000\rangle\,\langle 000| + |111\rangle\,\langle 111|$
We define the corresponding CPTP map $\mathcal{E}$ (course not definition):

$$\mathcal{E}(\rho) = p^3\rho + p^2(1-p)[X_1\rho X_1 + X_2\rho X_2 + X_3\rho X_3]$$

p is the probability that a qubit will flip and X_i represent i^{th}qubit flip.
We decompose $\mathcal{E}$:

$$E = \{\sqrt{(1-p)^2}X_1, \sqrt{(1-p)^2}X_2, \sqrt{(1-p)^2}X_3, \sqrt{(1-p)^2}I\}$$

Now we will check the quantum error-correction conditions:

$$
\begin{aligned}
&PE_i^\dagger E_j P = 0 && i \neq j \\
&PE_i^\dagger E_i P = (1-p)^2 p && i \in \{0,1,2\} \\
&PE_3^\dagger E_3 P = (1-p)^3
\end{aligned}
$$

This equation define the matrix

$$
\begin{pmatrix}
(1-p)^2 p & 0 & 0 & 0 \\
0 & (1-p)^2 p & 0 & 0 \\
0 & 0 & (1-p)^2 p & 0 \\
0 & 0 & 0 & (1-p)^3
\end{pmatrix}
$$

This matrix is clearly Hermitian, so the code corrector P can correctsd the noisy channel $\mathcal{E}$

7 Bell

Exercice 7.1: Probability of winning the CHSH game
We have 4 case :

- s = r = 0

$$
\begin{aligned}
\mathcal{P}(\text{win}) =& \mathcal{P}(a=b=0) + \mathcal{P}(a=b=1) \\
=& |((\cos(0)\langle 0| + \sin(0)\langle 1|) \otimes ((\cos(\frac{\pi}{8})\langle 0|) + \sin(\frac{\pi}{8})\langle 1|))\,|\beta_0\rangle\,|^2 + \\
& |((\sin(0)\langle 0| - \cos(0)\langle 1|) \otimes ((\sin(\frac{\pi}{8})\langle 0|) - \cos(\frac{\pi}{8})\langle 1|))\,|\beta_0\rangle\,|^2 \\
=& |(\cos(\frac{\pi}{8})\langle 00| + \sin(\frac{\pi}{8})\langle 01|)\,|\beta_0\rangle\,|^2 + |(-\sin(\frac{\pi}{8})\langle 10| + \cos(\frac{\pi}{8})\langle 11|)\,|\beta_0\rangle\,|^2 \\
=& |\frac{1}{\sqrt{2}}\cos(\frac{\pi}{8})|^2 + |\frac{1}{\sqrt{2}}\cos(\frac{\pi}{8})|^2 \\
=& \frac{1}{2}\cos^2(\frac{\pi}{8}) + \frac{1}{2}\cos^2(\frac{\pi}{8}) \\
=& \cos^2(\frac{\pi}{8})
\end{aligned}
$$

- the following calculations are similar and we obtain $\cos^2(\frac{\pi}{8})$

There are 4 different ways to draw s and r :

$$
\mathcal{P}(\text{win}) = 4 \times \frac{1}{4} \times \cos^2(\frac{\pi}{8}) = \cos^2(\frac{\pi}{8})
$$