Lambda Calculus and category theory

Valeran MAYTIE

Contents

1 Introduction

Boole :

- If you consider propositions (no quantifiers) of classical logic : $A ::= P|A \wedge B|\neg A|A \wedge B|\top|\bot$
- Ordered by logical implication $A \leq B \Leftrightarrow A \Rightarrow B$, A implies B or $A \vdash B$

Observation $A \wedge B \leq A, A \wedge B \leq B$. moreover if $C \leq A$ and $C \leq B$ then $C \leq A \wedge B$ (for all proprieties) Which means that $A \wedge B$ define a infimum of A and B (greatest lower bound, or glb)

Definition – $A \Rightarrow B = (\neg A) \vee B = \neg(A \wedge \neg B)$.

Observation :

- $A \wedge (A \Rightarrow B) \leq B$
- $A \vee \neg A \leq$ true
- $A \wedge \neg A \geq$ false

Frege Ideography (first proof system)

The idea that a mathematical proof is a mathematical object. In particular there may be different proofs of a proposition A formula.

$$\begin{tikzcd} \mathrm{B} \\ \mathrm{A} \arrow[u, bend left, "\pi_1"] \arrow[u, bend right, "\pi_2"'] \arrow[u, phantom, "\neq"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathrm{B} \\ \mathrm{A} \arrow[u, dash, "\leq"'] \end{tikzcd}$$

Lambek Lambek

Lambek understood connection between:

$$\begin{tikzcd} \text{Intuitionists minimal propositional logic} \arrow[rr, leftrightarrow, "\text{Curry, Howard, Lambek}"] \arrow[dr, leftrightarrow, "\text{Curry, Howard}"'] & & \text{Cartesian closed category} \\ & \text{Simply typed lambda calculus} \arrow[ur, leftrightarrow, "\text{Lambek}"'] & \end{tikzcd}$$

Definition – A monoid $(M, \bullet, e)$ is a set M equipped with a binary operation $\bullet : M \times M \to M$ with a neutral element $e \in M_e : M^0 \to M$ satisfying two equations :

- (associativity) $\forall x, y, z \in M, x \bullet (y \bullet z) = (x \bullet y) \bullet z$
- (neutrality) $\forall x, \in M, x \bullet e = x = e \bullet x$

Example – $(\mathbb{N}, +, 0), (\mathbb{Z}, +, 0), (\mathbb{N}, \times, 1)$ and any group.
Free monoid on a set (=alphabet) A. A^* contains finite sequences of element A $w = [a_1 \dots a_n]$

- Binary operation is concatenation.
- Neutral element is the empty word.

2 Categories, functors, natural transformations

Definition – A category $\mathcal{C}$ is a graph

- Whose nodes are called objects
- Whose edges are called morphism/maps/arrow.

The objects of $\mathcal{C}$ form a class of objects.
Every pair of object A, B comes with a set $Hom(A, B)$ of morphisms $A \xrightarrow{f} B$, $f \in Hom(A, B)$
The graph is equipped with:

- A morphism $id_A \in Hom(A, A)$ for all object A of $\mathcal{C}$
- A composition defined as a function $\circ_{A,B,C} : Hom(B, C) \times Hom(A, B) \to Hom(A, C)$ for every objects A, B, C of $\mathcal{C}$

  It satisfying the following equation :

  – associativity :

$$\begin{tikzcd} & B \arrow[r, "g"] & C \arrow[dr, "h"] & \\ A \arrow[ur, "f"] \arrow[urr, "g \circ f"'] \arrow[rrr, "h \circ g \circ f"'] & & & D \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[r, "g"] \arrow[drr, "h \circ g"'] & C \arrow[dr, "h"] & \\ A \arrow[ur, "f"] \arrow[rrr, "h \circ g \circ f"'] & & & D \end{tikzcd}$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

  – neutrality :

$$\begin{tikzcd} A \arrow[loop above, "Id_A"] \arrow[r, "f"'] & B \arrow[loop above, "Id_B"] \end{tikzcd}$$

$$Id_B \circ f = f = f \circ Id_A$$

Definition – A small category is a category whose class of object is a set. What we defined as a category is called "locally small category".

Example – Ordered Set: Every ordered set A defines a category.

- Objects: elements of A
- Morphisms : $a \to b \Leftrightarrow a \leq b$

$$Hom(a, b) = \begin{cases} \text{singleton} & a \leq b \\ \emptyset \end{cases}$$

  The composition is defined by transitivity:

$$\begin{tikzcd} a \arrow[r, "a \leq b"'] & b \arrow[r, "b \leq c"'] & c \end{tikzcd}$$

$$a \qquad \leq \qquad c$$

$$\begin{tikzcd} a \arrow[rr] & & b \end{tikzcd}$$

Definition – An ordered category $\mathcal{C}$ is a category where $Hom(A, B)$ is a singleton for all object A, B of $\mathcal{C}$.

Observation – An ordered category is the same thing as a pre-order (= trans, refl).

Example – <u>Monoid</u>

- A category with one object $*$, $M = Hom(*, *)$ define a monoid.
  - $\circ : Hom(*, *) \times Hom(*, *) \to Hom(*, *)$
  - $id_* \in M = Hom(*, *)$ define the neutral element
- Conversely every monoid $M = (M, \bullet, e)$ defines a category $\mathcal{B}M$ or ΣM with:
  - One object $*$
  - $Hom(*, *) = M$
  - Composition defined by $y \circ x = y \bullet x$ with e, the neutral element.

$$
\begin{array}{cc}
 & 2 = 1 \circ 1 \\
\mathcal{B}(\mathbb{N}, +, 0) & \begin{tikzcd} * \arrow[loop above, "0"] \arrow[loop below, "1"] \end{tikzcd} \\
 & 3 = 2 \circ 1
\end{array}
$$