

TD n°1

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An isomorphism $f : A \rightarrow B$ such that there exists $g : B \rightarrow A$ such that $g \circ f = Id_A$ and $f \circ g = Id_B$.

1. Show that g is unique
2. Show that in the following situation

$$\begin{array}{ccc}
 & B & \\
 f \nearrow & * & \searrow g \\
 A & \xrightarrow{g \circ f} & C
 \end{array}
 \quad
 \begin{array}{l}
 g \circ f \text{ is a isomorphism when} \\
 g \text{ and } f \text{ are isomorphism}
 \end{array}$$

3. deduce in (*) that g is an isomorphism when f and $g \circ f$ are isomorphism
4. deduce in (*) that f is an isomorphism when g and $g \circ f$ are isomorphism
5. Suppose that in $A \xrightarrow{f} B \xrightarrow{g} A \xrightarrow{h} B$ one has $g \circ f = Id_A$ and $h \circ g = Id_B$ show that $f = h$ in that case
6. Characterize the isomorphisms in the category **Set** of sets and function. **Top** of topological spaces and continuous functions

Correction :

1. Let $h : B \rightarrow A$ a morphism that $h \circ f = Id_A$ and $f \circ h = Id_B$

$$\begin{aligned} h &= h \circ Id_B && \text{By neutrality} \\ &= h \circ (f \circ g) \\ &= (h \circ f) \circ g && \text{By associativity} \\ &= Id_A \circ g \\ &= g && \text{By neutrality} \end{aligned}$$

We can say that g is unique. We will make note f^{-1}

2. We have f^{-1} and g^{-1} the inverse of f and g (they are isomorphism)

We will show that $f^{-1} \circ g^{-1}$ is an inverse of $g \circ f$

$$\begin{aligned} (f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ (g^{-1} \circ g) \circ f && \text{By associativity} \\ &= f^{-1} \circ f \\ &= Id_A \end{aligned}$$

A similar reasoning can be used to show $(g \circ f) \circ (f^{-1} \circ g^{-1})$

3. Let $g' = f \circ (g \circ f)^{-1}$

$$\begin{aligned} g \circ g' &= g \circ (f \circ (g \circ f)^{-1}) && g' \circ g = (f \circ (g \circ f)^{-1}) \circ g \\ &= (g \circ f) \circ (g \circ f)^{-1} && = f \circ (g \circ f)^{-1} \circ g \circ f \circ f^{-1} \\ &= Id_C && = f \circ (g \circ f)^{-1} \circ (g \circ f) \circ f^{-1} \\ &&& = f \circ f^{-1} \\ &&& = Id_B \end{aligned}$$

g is well an isomorphism.

4. Roughly the same proof.
5. We have :

$$\begin{aligned} f &= (h \circ g) \circ f && h \circ g = Id_B \\ &= h \circ (g \circ f) && \text{By associativity} \\ &= h && g \circ f = Id_A \end{aligned}$$

This question implies the first question.

6. Homomorphism (= A map between two structures, that preserves the operations of the structures
 $f(x \bullet y) = f(x) \bullet f(y)$)