

# Lambda Calculus and category theory

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## 1 Introduction

**Boole** :

- If you consider propositions (no quantifiers) of classical logic :  $A ::= P | A \wedge B | \neg A | A \vee B | \top | \perp$
- Ordered by logical implication  $A \leq B \Leftrightarrow A \Rightarrow B$ ,  $A$  implies  $B$  or  $A \vdash B$

Observation  $A \wedge B \leq A, A \wedge B \leq B$ . moreover if  $C \leq A$  and  $C \leq B$  then  $C \leq A \wedge B$  (for all proprieties)  
Which means that  $A \wedge B$  define a infimum of  $A$  and  $B$  (greatest lower bound, or glb)

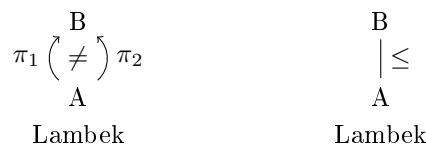
**Definition** –  $A \Rightarrow B = (\neg A) \vee B = \neg(A \wedge \neg B)$ .

Observation :

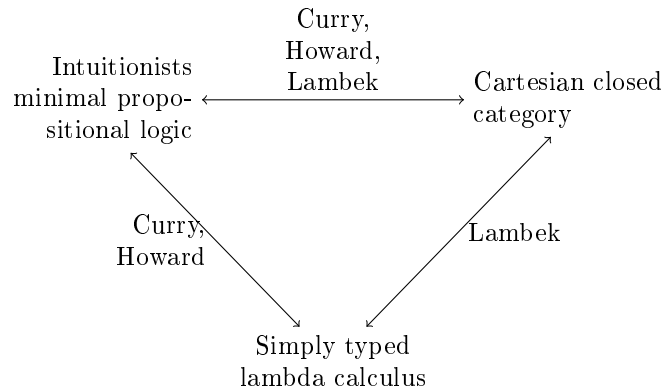
- $A \wedge (A \Rightarrow B) \leq B$
- $A \vee \neg A \leq \text{true}$
- $A \wedge \neg A \geq \text{false}$

**Frege** Ideography (first proof system)

The idea that a mathematical proof is a mathematical object. In particular there may be different proofs of a proposition  $A$  formula.



Lambek understood connection between:



**Definition** – A monoid  $(M, \bullet, e)$  is a set  $M$  equipped with a binary operation  $\bullet : M \times M \rightarrow M$  with a neutral element  $e \in M_e : M^0 \rightarrow M$  satisfying two equations :

- (associativity)  $\forall x, y, z \in M, x \bullet (y \bullet z) = (x \bullet y) \bullet z$
- (neutrality)  $\forall x, \in M, x \bullet e = x = e \bullet x$

**Example** –  $(\mathbb{N}, +, 0), (\mathbb{Z}, +, 0), (\mathbb{N}, \times, 1)$  and any group.

Free monoid on a set (=alphabet)  $A$ .  $A^*$  contains finite sequences of element  $A$   $w = [a_1 \dots a_n]$

- Binary operation is concatenation.
- Neutral element is the empty word.

## 2 Categories, functors, natural transformations

**Definition** – A category  $\mathcal{C}$  is a graph

- Whose nodes are called objects
- Whose edges are called morphism/maps/arrow.

The objects of  $\mathcal{C}$  form a class of objects.

Every pair of object  $A, B$  comes with a set  $Hom(A, B)$  of morphisms  $A \xrightarrow{f} B, f \in Hom(A, B)$

The graph is equipped with:

- A morphism  $id_A \in Hom(A, A)$  for all object  $A$  of  $\mathcal{C}$
- A composition defined as a function  $\circ_{A,B,C} : Hom(B, C) \times Hom(A, B) \rightarrow Hom(A, C)$  for every objects  $A, B, C$  of  $\mathcal{C}$

It satisfying the following equation :

– associativity :

$$\begin{array}{ccc}
 & B & \xrightarrow{g} C \\
 f \nearrow & & \searrow h \\
 A & \xrightarrow{g \circ f} & D \\
 & \xrightarrow{h \circ g \circ f} & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 & B & \xrightarrow{g} C \\
 f \nearrow & & \searrow h \\
 A & \xrightarrow{h \circ g} & D \\
 & \xrightarrow{h \circ g \circ f} & 
 \end{array}$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

– neutrality :

$$\begin{array}{ccc}
 Id_A & & Id_B \\
 \downarrow & & \downarrow \\
 A & \xrightarrow{f} & B \\
 Id_B \circ f = f = f \circ Id_A
 \end{array}$$

**Definition** – A small category is a category whose class of object is a set. What we defined as a category is called “locally small category”.

**Example** – Ordered Set: Every ordered set  $A$  defines a category.

- Objects: elements of  $A$
- Morphisms :  $a \rightarrow b \Leftrightarrow a \leq b$

$$Hom(a, b) = \begin{cases} \text{singleton} & a \leq b \\ \emptyset & \end{cases}$$

The composition is defined by transitivity:

$$\begin{array}{ccccc}
 a & \xrightarrow{a \leq b} & b & \xrightarrow{b \leq c} & c \\
 a & & \leq & & c \\
 a & \xrightarrow{\quad\quad\quad} & b & & 
 \end{array}$$

**Definition** – An ordered category  $\mathcal{C}$  is a category where  $Hom(A, B)$  is a singleton for all object  $A, B$  of  $\mathcal{C}$ .

**Observation** – An ordered category is the same thing as a pre-order (= trans, refl).

**Example** – Monoid

- A category with one object  $*$ ,  $M = Hom(*, *)$  define a monoid.
  - $\circ : Hom(*, *) \times Hom(*, *) \rightarrow Hom(*, *)$
  - $id_* \in M = Hom(*, *)$  define the neutral element
- Conversely every monoid  $M = (M, \bullet, e)$  defines a category  $\mathcal{B}M$  or  $\Sigma M$  with:
  - One object  $*$
  - $Hom(*, *) = M$
  - Composition defined by  $y \circ x = y \bullet x$  with  $e$ , the neutral element.

