Lambda Calculus and category theory

Valeran MAYTIE

Contents

1 Introduction 1

2 Categories, functors, natural transformations

 $\mathbf{2}$

1 Introduction

Boole:

- If you consider propositions (no quantifiers) of classical logic : $A ::= P|A \wedge B| \neg A|A \wedge B| \top |\bot|$
- Ordered by logical implication $A \leq B \Leftrightarrow A \Rightarrow B$, A implies B or $A \vdash B$

Observation $A \wedge B \leq A, A \wedge B \leq B$. moreover if $C \leq A$ and $C \leq B$ then $C \leq A \wedge B$ (for all proprieties) Which means that $A \wedge B$ define a infimum of A and B greatest lower bound glb

Definition $-A \Rightarrow B = (\neg A) \lor B = \neg (A \land \neg B).$

Observation:

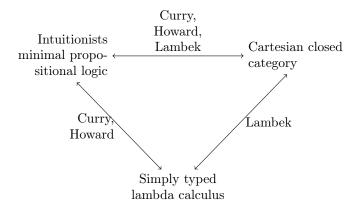
- $A \wedge (A \Rightarrow B) \leq B$
- $A \lor \neg A \le \text{true}$
- $A \wedge \neg A \ge \text{false}$

Frege: ideography (first proof system)

The idea that a mathematical proof is a mathematical object. In particular there may be different proofs of a proposition A formula.

$$\begin{array}{ccc}
B & & B \\
\pi_1 & \neq & \pi_2 & & \leq \\
A & & A
\end{array}$$
Lambek Lambek

Lambek understood connection between.



Definition – A monoid (M, \bullet, e) is a set M equipped with a binary operation $\bullet : M \times M \to M$ with a neutral element $e \in M_e : M^0 \to M$ satisfying two equations :

- (associativity) $\forall x, y, z \in M, x \bullet (y \bullet z) = (x \bullet y) \bullet z$
- (neutrality) $\forall x, \in M, x \bullet e = x = e \bullet x$

Example $-(\mathbb{N},+,0),(\mathbb{Z},+,0),(\mathbb{N},\times,1)$ and any group.

Free monoide on a set (=alphabet) A A^* contains finite sequences of element A $w = [a_1 \dots a_n]$

- multiplication is concatenation
- neutral element is empty word

2 Categories, functors, natural transformations

Definition – A category C is a graph

- whose node are called objects
- whose edges are called morphism/maps/arrow.

The objects of C form a class of objects. Every pair of objects A, B

Every pair of object A, B comes with a set Hom(A, B) of morphisms $A \xrightarrow{f} B, f \in Home(A, B)$ The graph is equipped with :

- an morphisms $id_A \in Home(A, A)$ for all object A of C
- a composition defined as a function $\circ_{A,B,C}: Hom(B,C) \times Hom(A,B) \to Hom(A,C)$ for every objects A,B,C of category \mathcal{C}

It satisfying the following equation :

- associativity:

- neutrality:

$$\begin{array}{ccc} Id_A & & Id_B \\ \bigcirc & & \bigcirc \\ \mathbf{A} & & f \end{array}$$

$$Id_B \circ f = f = f \circ Id_A$$

Definition – A small category is a category whose class of object is a set. What we defined as a category is called "locally small category".

Example - Ordered Set: Claim every ordered set A defines a category

- \bullet object : elements of A
- morphisms : $a \to b \Leftrightarrow a \le b$

$$Hom(a,b) = \begin{cases} singleton & a \le b \\ \emptyset & \end{cases}$$

The composition is defined by transitivity:

$$\begin{array}{cccc}
a & \xrightarrow{a \leq b} & b & \xrightarrow{b \leq c} & c \\
a & & \leq & c \\
a & \xrightarrow{b} & b
\end{array}$$

Definition – An order category \mathcal{C} is a category when Hom(A, B) is a singleton for all object A, B of \mathcal{C} .

Observation – an order category is the some thing as a pre-order (= trans, refl).

Example - monoid

- A category with one object *, M = Hom(*, *) defined a monoid
 - $-\circ: Hom(*,*) \times Hom(*,*) \rightarrow Hom(*,*)$
 - $-id_* \in M = Hom(*,*)$ defined the neutral element
- Conversely every monoid $M = (M, \bullet, e)$ defined a category $\mathcal{B}M$ or ΣM with on object * and Hom(*, *) = M composition defined $y \circ x = y \bullet x \ id_*$ defined the neutral element.

