Lambda Calculus and category theory

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1 Introduction

Boole:

- If you consider propositions (no quantifiers) of classical logic : $A ::= P|A \wedge B| \neg A|A \wedge B| \top |\bot|$
- Ordered by logical implication $A \leq B \Leftrightarrow A \Rightarrow B$, A implies B or $A \vdash B$

Observation $A \wedge B \leq A, A \wedge B \leq B$. moreover if $C \leq A$ and $C \leq B$ then $C \leq A \wedge B$ (for all proprieties) Which means that $A \wedge B$ define a infimum of A and B (greatest lower bound, or glb)

Definition $-A \Rightarrow B = (\neg A) \lor B = \neg (A \land \neg B).$

Observation:

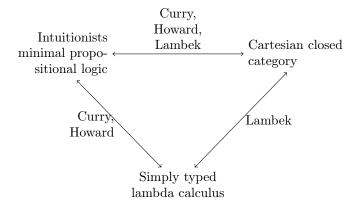
- $A \wedge (A \Rightarrow B) \leq B$
- $A \vee \neg A \leq \text{true}$
- $A \land \neg A \ge \text{false}$

Frege Ideography (first proof system)

The idea that a mathematical proof is a mathematical object. In particular there may be different proofs of a proposition A formula.

$$\begin{array}{ccc}
B & & B \\
\pi_1 & \neq & \pi_2 & & \leq \\
A & & A
\end{array}$$
Lambek Lambek

Lambek understood connection between:



Definition – A monoid (M, \bullet, e) is a set M equipped with a binary operation $\bullet : M \times M \to M$ with a neutral element $e \in M_e : M^0 \to M$ satisfying two equations :

- (associativity) $\forall x, y, z \in M, x \bullet (y \bullet z) = (x \bullet y) \bullet z$
- (neutrality) $\forall x, \in M, x \bullet e = x = e \bullet x$

Example $-(\mathbb{N},+,0),(\mathbb{Z},+,0),(\mathbb{N},\times,1)$ and any group.

Free monoid on a set (=alphabet) A. A^* contains finite sequences of element $A w = [a_1 \dots a_n]$

- Binary operation is concatenation.
- Neutral element is the empty word.

2 Categories, functors, natural transformations

Definition – A category C is a graph

- Whose nodes are called objects
- Whose edges are called morphism/maps/arrow.

The objects of \mathcal{C} form a class of objects.

Every pair of object A, B comes with a set Hom(A, B) of morphisms $A \xrightarrow{f} B, f \in Hom(A, B)$ The graph is equipped with:

- A morphism $id_A \in Hom(A, A)$ for all object A of C
- A composition defined as a function $\circ_{A,B,C}: Hom(B,C) \times Hom(A,B) \to Hom(A,C)$ for every objects A,B,C of \mathcal{C}

It satisfying the following equation :

- associativity:

- neutrality:

$$\begin{array}{ccc} Id_A & & Id_B \\ \bigcap & & \bigcap \\ \mathbf{A} & & f \end{array}$$

$$Id_B \circ f = f = f \circ Id_A$$

Definition - A small category is a category whose class of object is a set. What we defined as a category is called "locally small category".

Example – Ordered Set: Every ordered set A defines a category.

- ullet Objects: elements of A
- Morphisms : $a \to b \Leftrightarrow a \le b$

$$Hom(a,b) = \begin{cases} singleton & a \le b \\ \emptyset & \end{cases}$$

The composition is defined by transitivity:

$$\begin{array}{cccc}
a & \xrightarrow{a \leq b} & b & \xrightarrow{b \leq c} & c \\
a & & \leq & c \\
a & \xrightarrow{b} & b
\end{array}$$

Definition – An ordered category \mathcal{C} is a category where Hom(A, B) is a singleton for all object A, B of \mathcal{C} .

Observation – An ordered category is the same thing as a pre-order (= trans, refl).

Example - Monoid

- A category with one object *, M = Hom(*, *) define a monoid.
 - $-\circ: Hom(*,*) \times Hom(*,*) \rightarrow Hom(*,*)$
 - $-id_* \in M = Hom(*,*)$ define the neutral element
- Conversely every monoid $M=(M,\bullet,e)$ defines a category $\mathcal{B}M$ or ΣM with:
 - One object \ast
 - Hom(*,*) = M
 - Composition defined by $y \circ x = y \bullet x$ with e, the neutral element.

