Homework

1 Basic operation and their notation

Exercice 1.1: Inner/outer products in Dirac notation

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} &(\langle 0| + 2\langle 1|) \times (3|0\rangle + 4|1\rangle) \\ =&3\langle 0|0\rangle + 4\langle 0|1\rangle + 6\langle 1|0\rangle + 8\langle 1|1\rangle \\ =&3 + 8 \\ =&11 \end{aligned}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}$$

The last two in Dirac notation :

$$\begin{aligned} &(3|0\rangle + 4|1\rangle) \times (\langle 0| + 2\langle 1|) \\ =&3|0\rangle\langle 0| + 6|0\rangle\langle 1| + 4|1\rangle\langle 0| + 8|1\rangle\langle 1| \end{aligned}$$

Exercice 1.2: Matrix products in Dirac notation

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 23 \\ 34 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} &(|0\rangle\langle 0| + 3|0\rangle\langle 1| + 2|1\rangle\langle 0| + 4|1\rangle\langle 1|) \times (5|0\rangle + 6|1\rangle) \\ =&5|0\rangle\langle 0|0\rangle + 6|0\rangle\langle 0|1\rangle + 15|0\rangle\langle 1|0\rangle + 18|0\rangle\langle 1|1\rangle + \\ &10|1\rangle\langle 0|0\rangle + 12|1\rangle\langle 0|1\rangle + 20|1\rangle\langle 1|0\rangle + 24|1\rangle\langle 1|1\rangle \\ =&5\langle 0|0\rangle|0\rangle + 18\langle 1|1\rangle|0\rangle + 10\langle 0|0\rangle|1\rangle + 24\langle 1|1\rangle|1\rangle \\ =&5|0\rangle + 18|0\rangle + 10|1\rangle + 24|1\rangle \\ =&23|0\rangle + 24|1\rangle \end{aligned}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned} &(1/\sqrt{2}|0\rangle\langle 0| + 1/\sqrt{2}|0\rangle\langle 1| + 1/\sqrt{2}|1\rangle\langle 0| - 1/\sqrt{2}|1\rangle\langle 1|)^2 \\ =&1/2|0\rangle\langle 0|0\rangle\langle 0| + 1/2|0\rangle\langle 0|0\rangle\langle 1| + 1/2|0\rangle\langle 1|1\rangle\langle 0| - 1/2|0\rangle\langle 1|1\rangle\langle 1| \\ &1/2|1\rangle\langle 0|0\rangle\langle 1| + 1/2|1\rangle\langle 0|0\rangle\langle 0| - 1/2|1\rangle\langle 1|1\rangle\langle 0| + 1/2|1\rangle\langle 1|1\rangle\langle 1| \\ =&1/2|0\rangle\langle 0| + 1/2|0\rangle\langle 0| + 1/2|1\rangle\langle 1| + 1/2|1\rangle\langle 1| \\ =&|0\rangle\langle 0| + |1\rangle\langle 1| \end{aligned}$$

Exercice 1.3: Tensor products in Dirac/Coecke notation

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The last two in Dirac notation :

$$(|0\rangle + 2|1\rangle) \otimes (3|0\rangle + 4|1\rangle) = 3|00\rangle + 4|01\rangle + 6|10\rangle + 8|11\rangle$$

$$|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle = |00\rangle + |11\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$$

In Dirac notation :

$$\begin{aligned} &(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ =&|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\ =&|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| \end{aligned}$$

$$\begin{aligned} &(|0\rangle\langle 0|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ =&|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \\ =&|0\rangle\langle 0| + |1\rangle\langle 1| \end{aligned}$$

$$\begin{aligned} &1/\sqrt{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \otimes 1/\sqrt{2}(|0\rangle\langle 0| + |1\rangle\langle 0| + |1\rangle\langle 0| - |1\rangle\langle 1|) \\ =&\ 1/2(|0\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 2| + |0\rangle\langle 3| + \\ &\quad |1\rangle\langle 0| - |1\rangle\langle 1| + |1\rangle\langle 2| - |1\rangle\langle 3| + \\ &\quad |2\rangle\langle 0| + |2\rangle\langle 1| - |2\rangle\langle 2| - |2\rangle\langle 3| + \\ &\quad |3\rangle\langle 0| - |3\rangle\langle 1| - |3\rangle\langle 2| + |3\rangle\langle 3|) \end{aligned}$$

We want to prove $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

- Dirac's notation :
- Coecke's notation :

Exercice 1.4: Dagger in Dirac/Coecke notation

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}^\dagger = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \qquad \begin{pmatrix} 1 & 3i \\ 2 & 4i \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 2 \\ -3i & -4i \end{pmatrix}$$

In Dirac notation:

$$\begin{aligned} &1/\sqrt{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)^\dagger \\ =&1/\sqrt{2}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|) \end{aligned}$$

$$\begin{aligned} &(|0\rangle\langle 0| + 3i|0\rangle\langle 1| + 2|1\rangle\langle 0| + 4i|1\rangle\langle 1|)^\dagger \\ =&(|0\rangle\langle 0| - 3i|1\rangle\langle 0| + 2|0\rangle\langle 1| - 4i|1\rangle\langle 1|) \end{aligned}$$

Exercice 1.5: Gates in Dirac notations

$$H = 1/\sqrt{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

$$CNot = |0\rangle\langle 0| + |1\rangle\langle 1| + |3\rangle\langle 2| + |2\rangle\langle 3|$$

$$T = |0\rangle\langle 0| + e^{\frac{i\pi}{4}}|1\rangle\langle 1|$$

Exercice 1.6: Pauli matrices in Dirac/Coecke notation .

2 Postulates on pure states

Let $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ the initial state of two qubits. We want to compute $CNot(H \otimes I)|psi\rangle$.

1. $(|0\rangle - |1\rangle)/\sqrt{2}$

3 Some mathematics

Exercice 3.1: Spectral theorems complements Let D and D' two diagonal matrices and U a unitary matrix.

- Let $A = UDU^\dagger$ and $B = UD'U^\dagger$

$$\begin{aligned} AB &= UDU^\dagger UD'U^\dagger \\ &= UDD'U^\dagger && U \text{ is a unitary matrix} \\ &= UD'DU^\dagger && D \text{ and } D' \text{ are diagonal} \\ &= UD'U^\dagger UDU^\dagger \\ &= BA \end{aligned}$$

- Let $M = UDU^\dagger$

4 On the nature of quantum information

5 Protocols

6 Quantum error correction

7 Bell