Home work 1

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1 Bell Basis

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Exercise 1 – Show that these 4 vectors form an orthonormal basis

1. We need to show that the norm of the 4 vectors is equal to 1

   This part is simple, because they all have the same constant 2 times, except for a sign which is ignored because it is squared.

$$\frac{1}{\sqrt{2}}^2 + (\mp\frac{1}{\sqrt{2}})^2 = \frac{1}{2} + \frac{1}{2} = 1$$

2. We need to show that they are pairwise orthogonal

   It simple to show that $|\Phi^\mp\rangle$ and $|\Psi^\mp\rangle$ are orthogonal, because the constants are not on the same "ket", so we have $\langle\Phi^\mp|\Psi^\mp\rangle = \frac{1}{\sqrt{2}} \times 0 + 0 \times \frac{1}{\sqrt{2}} \mp 0 \times \frac{1}{\sqrt{2}} \mp \frac{1}{\sqrt{2}} \times 0 = 0$

   Finally, the same calculation is used to show $|\Phi^+\rangle$ is orthogonal to $|\Phi^-\rangle$ and $|\Psi^+\rangle$ is orthogonal to $|\Psi^-\rangle$.

$$\langle\Phi^+|\Phi^-\rangle = \langle\Psi^+|\Psi^-\rangle = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0$$

All norm of the vectors is equal to 1 and they are all orthogonal.
These 4 vectors form an orthonormal basis.

Exercise 2 – Consider the ket vector

$$|\varphi\rangle = \frac{2}{\sqrt{5}}|01\rangle + \frac{i}{\sqrt{5}}|10\rangle$$

(a) Show that this is a vector of norm 1

$$\langle\varphi|\varphi\rangle = \left(\frac{2}{\sqrt{5}}\right)^2 + \frac{i}{\sqrt{5}} \times \overline{\frac{i}{\sqrt{5}}} = \frac{4}{5} + \frac{i \times -i}{5} = \frac{4}{5} + \frac{1}{5} = 1$$

(b) Write it as a linear combination of $|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle$ and $|\Psi^-\rangle$.

   We want to fin α and β such that $\alpha|\Psi^+\rangle + \beta|\Psi^-\rangle = |\varphi\rangle$. We don't need $|\Phi^\mp\rangle$, because $|00\rangle$ and $|11\rangle$ don't appear in $|\varphi\rangle$.

$$\left\{\begin{array}{ll} \alpha\frac{1}{\sqrt{2}}+\beta\frac{1}{\sqrt{2}} & =\frac{2}{\sqrt{5}} \\ \alpha\frac{1}{\sqrt{2}}-\beta\frac{1}{\sqrt{2}} & =\frac{i}{\sqrt{5}} \end{array}\right. \Rightarrow \left\{\begin{array}{llll} \alpha\frac{1}{\sqrt{2}} & + & \beta\frac{1}{\sqrt{2}} & =\frac{2}{\sqrt{5}} \\ \alpha\frac{2}{\sqrt{2}} & & & =\frac{2}{\sqrt{5}}+\frac{i}{\sqrt{5}} \Rightarrow \alpha=\frac{2\sqrt{2}}{2\sqrt{5}}+\frac{i\sqrt{2}}{2\sqrt{5}}=\frac{\sqrt{10}}{5}+\frac{i\sqrt{10}}{10} \end{array}\right.$$

So we have the final equation

$$\beta\frac{1}{\sqrt{2}}=\frac{\sqrt{5}}{5}-\frac{i\sqrt{5}}{10}$$
$$\beta=\frac{\sqrt{10}}{5}-\frac{i\sqrt{10}}{10}$$

Finally we have $|\varphi\rangle=(\frac{\sqrt{10}}{5}+\frac{i\sqrt{10}}{10})\,|\Psi^+\rangle+(\frac{\sqrt{10}}{5}-\frac{i\sqrt{10}}{10})\,|\Psi^-\rangle$

(c) Compute $\langle\varphi|\Psi^+\rangle$

$$\langle\varphi|\Psi^+\rangle=\frac{2}{\sqrt{5}}\times\frac{1}{\sqrt{2}}+\frac{-i}{\sqrt{5}}\times\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{10}}-\frac{i}{\sqrt{10}}$$

2 Reversible Computation

We calculate the circuit C_1 for the values $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$.

$$C_1(|00\rangle)=|00\rangle$$
$$C_1(|01\rangle)=|10\rangle$$
$$C_1(|10\rangle)=|01\rangle$$
$$C_1(|11\rangle)=|11\rangle$$

We can formulate C_1 as follows :

$$C_1:\mathcal{H}\otimes\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$$
$$|x\rangle\otimes|y\rangle\mapsto|y\rangle\otimes|x\rangle$$

Informally, we can say that C_1 swap $|x\rangle$ and $|y\rangle$

3 Play with Controls

We calculate the circuit C_2 for the values $|00\rangle, |01\rangle, |10\rangle$ and $|11\rangle$.

$$C_2(|00\rangle)=|00\rangle$$
$$C_2(|01\rangle)=e^{i\theta}\,|01\rangle$$
$$C_2(|10\rangle)=e^{i\theta}\,|10\rangle$$
$$C_2(|11\rangle)=|11\rangle$$

We can formulate C_2 as follows :

$$C_2:\mathcal{H}\otimes\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$$
$$|x\rangle\otimes|y\rangle\mapsto|x\rangle\otimes(e^{i\theta}((x\oplus y)\oplus(1\oplus x\oplus y)))\,|y\rangle$$

4 Another one

We calculate the circuit C_3 for $|xy\rangle$.

$$\begin{aligned}
\mathcal{H} \otimes \mathcal{H}(|xy\rangle) =& \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + (-1)^y |1\rangle) \\
=& \frac{1}{2}(|00\rangle + (-1)^x |01\rangle + (-1)^y |10\rangle + (-1)^{x+y} |11\rangle) \\
\text{CNOT} \rightarrow& \frac{1}{2}(|00\rangle + (-1)^x |11\rangle + (-1)^y |10\rangle + (-1)^{x+y} |01\rangle) \\
\mathcal{H} \otimes \mathcal{H} \rightarrow& \frac{1}{4}((|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
& (-1)^x(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \\
& (-1)^y(|00\rangle + |01\rangle - |10\rangle - |11\rangle) \\
& (-1)^{x+y}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)) \\
=& \frac{1}{4}((1 + (-1)^x + (-1)^y + (-1)^{x+y}) |00\rangle \\
& (1 + (-1)^{x+1} + (-1)^y + (-1)^{x+y+1}) |01\rangle \\
& (1 + (-1)^{x+1} + (-1)^{y+1} + (-1)^{x+y}) |10\rangle) \\
& (1 + (-1)^x + (-1)^{y+1} + (-1)^{x+y+1}) |11\rangle
\end{aligned}$$

So we have the following results :

$$\begin{aligned}
C_3 |00\rangle &= |00\rangle \\
C_3 |01\rangle &= |11\rangle \\
C_3 |10\rangle &= |10\rangle \\
C_3 |11\rangle &= |01\rangle
\end{aligned}$$

We can formulate C_3 as follows :

$$\begin{aligned}
C_3 : \mathcal{H} \otimes \mathcal{H} &\rightarrow \mathcal{H} \otimes \mathcal{H} \\
|x\rangle \otimes |y\rangle &\mapsto |x \oplus y\rangle \otimes |y\rangle
\end{aligned}$$

We can also define like that :