

Lambda Calculus and category theory

Valeran MAYTIE

Contents

1	Introduction	1
2	Categories	2
3	Functors	3
4	Transformation	4

1 Introduction

Boole :

- If you consider propositions (no quantifiers) of classical logic : $A ::= P | A \wedge B | \neg A | A \vee B | \top | \perp$
- Ordered by logical implication $A \leq B \Leftrightarrow A \Rightarrow B$, A implies B or $A \vdash B$

Observation $A \wedge B \leq A, A \wedge B \leq B$. moreover if $C \leq A$ and $C \leq B$ then $C \leq A \wedge B$ (for all proprieties)
Which means that $A \wedge B$ define a infimum of A and B (greatest lower bound, or glb)

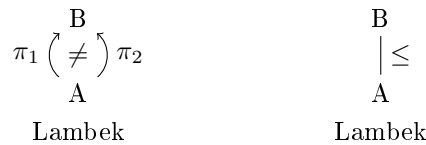
Definition – $A \Rightarrow B = (\neg A) \vee B = \neg(A \wedge \neg B)$.

Observation :

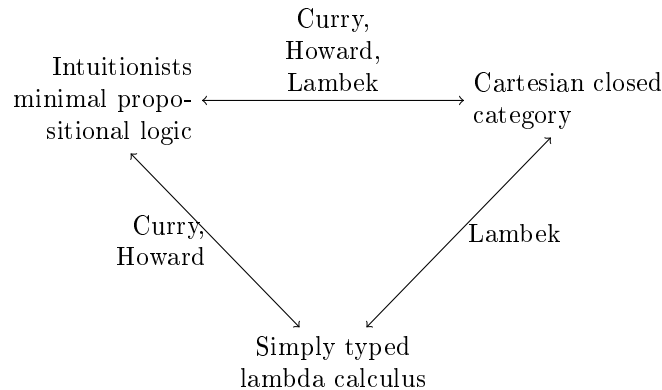
- $A \wedge (A \Rightarrow B) \leq B$
- $A \vee \neg A \leq \text{true}$
- $A \wedge \neg A \geq \text{false}$

Frege Ideography (first proof system)

The idea that a mathematical proof is a mathematical object. In particular there may be different proofs of a proposition A formula.



Lambek understood connection between:



Definition – A monoid (M, \bullet, e) is a set M equipped with a binary operation $\bullet : M \times M \rightarrow M$ with a neutral element $e \in M$: $M^0 \rightarrow M$ satisfying two equations :

- (associativity) $\forall x, y, z \in M, x \bullet (y \bullet z) = (x \bullet y) \bullet z$
- (neutrality) $\forall x \in M, x \bullet e = x = e \bullet x$

Example – $(\mathbb{N}, +, 0), (\mathbb{Z}, +, 0), (\mathbb{N}, \times, 1)$ and any group.

Free monoid on a set (=alphabet) A . A^* contains finite sequences of element A $w = [a_1 \dots a_n]$

- Binary operation is concatenation.
- Neutral element is the empty word.

2 Categories

Definition – A category \mathcal{C} is a graph

- Whose nodes are called objects
- Whose edges are called morphism/maps/arrow.

The objects of \mathcal{C} form a class of objects.

Every pair of object A, B comes with a set $Hom(A, B)$ of morphisms $A \xrightarrow{f} B, f \in Hom(A, B)$

The graph is equipped with:

- A morphism $id_A \in Hom(A, A)$ for all object A of \mathcal{C}
- A composition defined as a function $\circ_{A,B,C} : Hom(B, C) \times Hom(A, B) \rightarrow Hom(A, C)$ for every objects A, B, C of \mathcal{C}

It satisfying the following equation :

– associativity :

$$\begin{array}{ccc}
 & B & \xrightarrow{g} C \\
 f \nearrow & & \searrow h \\
 A & \xrightarrow{g \circ f} & D \\
 & \xrightarrow{h \circ g \circ f} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & B & \xrightarrow{g} C \\
 f \nearrow & & \searrow h \\
 A & \xrightarrow{h \circ g} & D \\
 & \xrightarrow{h \circ g \circ f} &
 \end{array}$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

– neutrality :

$$\begin{array}{ccc}
 Id_A & & Id_B \\
 \downarrow & & \downarrow \\
 A & \xrightarrow{f} & B \\
 Id_B \circ f = f = f \circ Id_A
 \end{array}$$

Definition – A small category is a category whose class of object is a set. What we defined as a category is called “locally small category”.

Example – Ordered Set: Every ordered set A defines a category.

- Objects: elements of A
- Morphisms : $a \rightarrow b \Leftrightarrow a \leq b$

$$Hom(a, b) = \begin{cases} \text{singleton} & a \leq b \\ \emptyset & \end{cases}$$

The composition is defined by transitivity:

$$\begin{array}{ccccc}
a & \xrightarrow{\quad} & b & \xrightarrow{\quad} & c \\
& a \leq b & & b \leq c & \\
a & & & & c \\
& & \leq & & \\
a & \xrightarrow{\quad} & b & &
\end{array}$$

Definition – An ordered category \mathcal{C} is a category where $Hom(A, B)$ is a singleton for all object A, B of \mathcal{C} .

Observation – An ordered category is the same thing as a pre-order (= trans, refl).

Example – Monoid

- A category with one object $*$, $M = Hom(*, *)$ define a monoid.
 - $\circ : Hom(*, *) \times Hom(*, *) \rightarrow Hom(*, *)$
 - $id_* \in M = Hom(*, *)$ define the neutral element
- Conversely every monoid $M = (M, \bullet, e)$ defines a category $\mathcal{B}M$ or ΣM with:
 - One object $*$
 - $Hom(*, *) = M$
 - Composition defined by $y \circ x = y \bullet x$ with e , the neutral element.

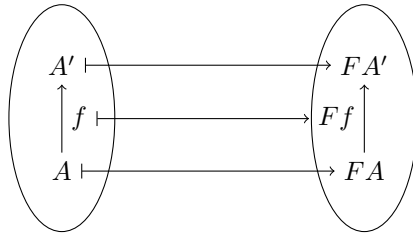
$$\begin{array}{c}
2 = 1 \circ 1 \\
\begin{array}{c} \circlearrowleft \\ 0 \end{array} \\
\downarrow \\
\mathcal{B}(\mathbb{N}, +, 0) \quad * \\
\uparrow \\
\begin{array}{c} \circlearrowright \\ 1 \end{array} \\
3 = 2 \circ 1
\end{array}$$

3 Functors

A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between category \mathcal{A} and \mathcal{B} is a graph homomorphism :

- F associates an object FA in \mathcal{B} to every object A in \mathcal{A}
- F associates a morphism $FA \xrightarrow{Ff} FA'$ in \mathcal{B} to every morphism $A \xrightarrow{f} A'$ in \mathcal{A} .

It preserves composition and identity



$$\begin{array}{ccc}
A & & FA \\
\uparrow Id_A & & \uparrow F(Id_A) = Id_{FA} \\
A & & FA
\end{array}$$

$$\begin{array}{ccc}
A'' & & FA'' \\
g \uparrow & & \uparrow Fg \\
A' & \xrightarrow{g \circ_A f} & FA' \\
f \uparrow & & \uparrow Ff \\
A & & FA
\end{array}
\quad Fg \circ_b Ff = F(g \circ_A f)$$

Example – List of different application :

- A Functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between order category is a same thing as a monotone (=order preserving) function. The reason is that the preservation of composition and identity holds in \mathcal{B} .

- A functor $F : \mathcal{BM} \rightarrow \mathcal{BN}$ between categories with one object is a function $H \rightarrow M$ which defined monoid homomorphism.

$$F(m \circ_M n) = Fm \circ_N Fn$$

- functor $F : \mathcal{BM} \rightarrow \mathcal{Set}$ is a same thing as a \mathcal{Set} $X = F(*)$ equipped with a family of functions $F_m : X \rightarrow X$ satisfying the equation :

- $Fm \circ Fn = F(m \bullet n)$
- $F(e) = Id_X$

\hookrightarrow Equivalently a function

$$\begin{aligned} M \times X &\mapsto X \\ (m, x) &\mapsto m \bullet x \end{aligned}$$

Where we write $F_m(x) = m \bullet x$ and action of Mon X satisfying the equations of $(m \bullet_M n) \bullet x = m \bullet (m \bullet x)$ and $e \bullet x$.

Example – Given a finite set A (alphabet) of letters. We construct the monoid A^* of finite words on A equipped with concatenation as composition and empty element as neutral element. We write $[a_1 \dots a_n] \in A^*$ and ϵ : the empty word noted $[]$. A functor $\mathcal{BA}^* \xrightarrow{F} \mathcal{Set}$ is a set $Q = F(*)$ equipped with an action that it a function :

$$\begin{aligned} A^* \times Q &\rightarrow Q \\ ([w], q) &\mapsto [w] \bullet q \end{aligned}$$

Remark – this action $A \times Q \rightarrow Q$ is equivalent to the data of a family $\delta_a : Q \rightarrow Q$ of function called transition function of a deterministic total automata with Set of Q of states.

Example – Given a functor $\mathcal{BM} \rightarrow \mathcal{Set}$ construct a category $\int F$ (Grothendieck construction) whose object are the elements of $F(*)$, whose morphisms $x \xrightarrow{m} y$ are of the form $x \mapsto m \bullet x$. Show that $\int F$ comes with a functors $\pi : \int F \rightarrow \mathcal{BM}$.

4 Transformation