Homework

1 Basic operation and their notation

Exercice 1.1: Inner/outer products in Dirac notation

$$\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned}
&(\langle 0| + 2\langle 1|) \times (3|0\rangle + 4|1\rangle) \\
=&3\langle 0|0\rangle + 4\langle 0|1\rangle + 6\langle 1|0\rangle + 8\langle 1|1\rangle \\
=&3 + 8 \\
=&11
\end{aligned}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 4 & 8 \end{pmatrix}$$

The last two in Dirac notation :

$$\begin{aligned}
&(3|0\rangle + 4|1\rangle) \times (\langle 0| + 2\langle 1|) \\
=&3|0\rangle\langle 0| + 6|0\rangle\langle 1| + 4|1\rangle\langle 0| + 8|1\rangle\langle 1|
\end{aligned}$$

Exercice 1.2: Matrix products in Dirac notation

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 23 \\ 34 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned}
&(|0\rangle\langle 0| + 3|0\rangle\langle 1| + 2|1\rangle\langle 0| + 4|1\rangle\langle 1|) \times (5|0\rangle + 6|1\rangle) \\
=&5|0\rangle\langle 0|0\rangle + 6|0\rangle\langle 0|1\rangle + 15|0\rangle\langle 1|0\rangle + 18|0\rangle\langle 1|1\rangle + \\
&10|1\rangle\langle 0|0\rangle + 12|1\rangle\langle 0|1\rangle + 20|1\rangle\langle 1|0\rangle + 24|1\rangle\langle 1|1\rangle \\
=&5\langle 0|0\rangle|0\rangle + 18\langle 1|1\rangle|0\rangle + 10\langle 0|0\rangle|1\rangle + 24\langle 1|1\rangle|1\rangle \\
=&5|0\rangle + 18|0\rangle + 10|1\rangle + 24|1\rangle \\
=&23|0\rangle + 24|1\rangle
\end{aligned}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The last one in Dirac notation :

$$\begin{aligned}
&(1/\sqrt{2}|0\rangle\langle 0| + 1/\sqrt{2}|0\rangle\langle 1| + 1/\sqrt{2}|1\rangle\langle 0| - 1/\sqrt{2}|1\rangle\langle 1|)^2 \\
=&1/2|0\rangle\langle 0|0\rangle\langle 0| + 1/2|0\rangle\langle 0|0\rangle\langle 1| + 1/2|0\rangle\langle 1|1\rangle\langle 0| - 1/2|0\rangle\langle 1|1\rangle\langle 1| \\
&1/2|1\rangle\langle 0|0\rangle\langle 1| + 1/2|1\rangle\langle 0|0\rangle\langle 0| - 1/2|1\rangle\langle 1|1\rangle\langle 0| + 1/2|1\rangle\langle 1|1\rangle\langle 1| \\
=&1/2|0\rangle\langle 0| + 1/2|0\rangle\langle 0| + 1/2|1\rangle\langle 1| + 1/2|1\rangle\langle 1| \\
=&|0\rangle\langle 0| + |1\rangle\langle 1|
\end{aligned}$$

Exercice 1.3: Tensor products in Dirac/Coecke notation

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The last two in Dirac notation :

$$(|0\rangle + 2|1\rangle) \otimes (3|0\rangle + 4|1\rangle) = 3|00\rangle + 4|01\rangle + 6|10\rangle + 8|11\rangle$$

$$|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle = |00\rangle + |11\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$$

In Dirac notation :

$$\begin{aligned} &(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ =&|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\ =&|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| \end{aligned}$$

$$\begin{aligned} &(|0\rangle\langle 0|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ =&|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \\ =&|0\rangle\langle 0| + |1\rangle\langle 1| \end{aligned}$$

$$\begin{aligned} &1/\sqrt{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \otimes 1/\sqrt{2}(|0\rangle\langle 0| + |1\rangle\langle 0| + |1\rangle\langle 0| - |1\rangle\langle 1|) \\ = &\ 1/2(|0\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 2| + |0\rangle\langle 3| + \\ &\quad |1\rangle\langle 0| - |1\rangle\langle 1| + |1\rangle\langle 2| - |1\rangle\langle 3| + \\ &\quad |2\rangle\langle 0| + |2\rangle\langle 1| - |2\rangle\langle 2| - |2\rangle\langle 3| + \\ &\quad |3\rangle\langle 0| - |3\rangle\langle 1| - |3\rangle\langle 2| + |3\rangle\langle 3|) \end{aligned}$$

We want to prove $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

- Dirac's notation :
- Coecke's notation :

Exercice 1.4: Dagger in Dirac/Coecke notation

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}^\dagger = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \qquad \begin{pmatrix} 1 & 3i \\ 2 & 4i \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 2 \\ -3i & -4i \end{pmatrix}$$

In Dirac notation:

$$\begin{aligned} &1/\sqrt{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)^\dagger \\ =&1/\sqrt{2}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|) \end{aligned}$$

$$\begin{aligned} &(|0\rangle\langle 0| + 3i|0\rangle\langle 1| + 2|1\rangle\langle 0| + 4i|1\rangle\langle 1|)^\dagger \\ =&(|0\rangle\langle 0| - 3i|1\rangle\langle 0| + 2|0\rangle\langle 1| - 4i|1\rangle\langle 1|) \end{aligned}$$

Exercice 1.5: Gates in Dirac notations

$$H = 1/\sqrt{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$$

$$CNot = |0\rangle\langle 0| + |1\rangle\langle 1| + |3\rangle\langle 2| + |2\rangle\langle 3|$$

$$T = |0\rangle\langle 0| + e^{\frac{i\pi}{4}}|1\rangle\langle 1|$$

Exercice 1.6: Pauli matrices in Dirac/Coecke notation .

2 Postulates on pure states

Exercice 2.1: Evolutions
Let $|\psi\rangle = |0\rangle \otimes |0\rangle$ the initial state of two qubits. We want to compute $CNot(H \otimes I)|\psi\rangle$.

$$\begin{aligned} CNot(H \otimes I)(|0\rangle \otimes |0\rangle) &= CNot((|0\rangle + |1\rangle)/\sqrt{2} \otimes (|0\rangle + |1\rangle)/\sqrt{2}) \\ &= \frac{1}{\sqrt{2}} CNot(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |11\rangle + |10\rangle) := |\psi'\rangle \end{aligned}$$

$$\begin{aligned} (H \otimes I)CNot|\psi'\rangle &= (H \otimes I)\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\ &= |00\rangle \qquad\qquad H^2|\psi\rangle = |\psi\rangle \end{aligned}$$

1. $T^4 H\ |0\rangle$
2. $HT^4 H\ |0\rangle$
3. $CNot(H|0\rangle \otimes HT^4 H|0\rangle)$

4.

5. $(H \otimes (H^2))(|0\rangle \otimes |0\rangle))$

Exercice 2.2: Measuring in another basis

- orthogonal :

$$\begin{aligned}\langle +|-\rangle &= (\frac{1}{\sqrt{2}})^2 + \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} \\ &= (\frac{1}{\sqrt{2}})^2 - (\frac{1}{\sqrt{2}})^2 \\ &= 0\end{aligned}$$

- norm one :

$$\begin{aligned}\langle +|+\rangle &= (\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1\end{aligned}$$

$$\begin{aligned}\langle -|-\rangle &= (\frac{1}{\sqrt{2}})^2 + (\frac{-1}{\sqrt{2}})^2 \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1\end{aligned}$$

- generate $\mathbb{C}^2$

  Let $c = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$ we want to find x and y such that $x|+\rangle + y|-\rangle = c$.

$$\begin{aligned}x|+\rangle + y|-\rangle &= \frac{x}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{y}{\sqrt{2}}(|0\rangle - |1\rangle) \\ &= (\frac{x+y}{\sqrt{2}})|0\rangle + (\frac{x-y}{\sqrt{2}})|1\rangle\end{aligned}$$

  So we have this system :

$$\begin{cases} (x+y)/\sqrt{2} & = \alpha \\ (x-y)/\sqrt{2} & = \beta \end{cases} \Rightarrow \begin{cases} x+y & = \sqrt{2}\alpha \\ x-y & = \sqrt{2}\beta \end{cases}$$
$$\Rightarrow \begin{cases} 2x & = \sqrt{2}(\alpha+\beta) \\ x-y & = \sqrt{2}\beta \end{cases}$$
$$\Rightarrow \begin{cases} x & = \frac{\sqrt{2}}{2}(\alpha+\beta) \\ -x+y & = -\sqrt{2}\beta \end{cases}$$
$$\Rightarrow \begin{cases} x & = \frac{\sqrt{2}}{2}(\alpha+\beta) \\ y & = \frac{\sqrt{2}}{2}(\alpha-\beta) \end{cases}$$

$B = \{|0\rangle, |1\rangle\}$ is another o.n.b of $\mathbb{C}^2$

We need to show that $\sum_{M\in\mathcal{M}_\pm} M^\dagger M = 1$

$$
\begin{aligned}
\sum_{M\in\mathcal{M}_\pm} &= (|+\rangle\langle+|)^\dagger(|+\rangle\langle+|) + (|-\rangle\langle-|)^\dagger(|-\rangle\langle-|) \\
&= (|+\rangle\langle+|+\rangle\langle+|) + (|-\rangle\langle-|-\rangle\langle-|) \\
&= |+\rangle\langle+| + |-\rangle\langle-| \\
&= \frac{1}{2}(|0\rangle\langle0| + |0\rangle\langle1| + |1\rangle\langle0| + |1\rangle\langle1|) + \frac{1}{2}(|0\rangle\langle0| - |0\rangle\langle1| - |1\rangle\langle0| + |1\rangle\langle1|) \\
&= |0\rangle\langle0| + |1\rangle\langle1| \\
&= 1
\end{aligned}
$$

So $\mathcal{M}_\pm$ is a valid measurement.
We have $|\psi\rangle = \frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle$

- For $|+\rangle\langle+|$

$$
\begin{aligned}
p(|+\rangle) &= \langle\psi|(|+\rangle\langle+|)^\dagger|+\rangle\langle+||\psi\rangle \\
&= \langle\psi||+\rangle\langle+||\psi\rangle \\
&= \langle\psi|+\rangle\langle+|\psi\rangle \\
&= (\frac{1}{3}\times\frac{1}{\sqrt{2}} + \frac{\sqrt{8}}{3}\times\frac{1}{\sqrt{2}})^2 \\
&= \frac{1}{2}\times(\frac{1+\sqrt{8}}{3})^2 = \frac{1}{2}\times\frac{(1+\sqrt{8})^2}{9} = \frac{(1+\sqrt{8})^2}{18}
\end{aligned}
$$

  For $|-\rangle\langle-|$

Exercice 2.3: Measuring to distinguish
a

3 Some mathematics

Exercice 3.1: Spectral theorems complements
Let D and D' two diagonal matrices and U a unitary matrix.

- Let $A = UDU^\dagger$ and $B = UD'U^\dagger$

$$
\begin{aligned}
AB &= UDU^\dagger UD'U^\dagger \\
&= UDD'U^\dagger && U \text{ is a unitary matrix} \\
&= UD'DU^\dagger && D \text{ and } D' \text{ are diagonal} \\
&= UD'U^\dagger UDU^\dagger \\
&= BA
\end{aligned}
$$

- Let $M = UDU^\dagger$

4 On the nature of quantum information

5 Protocols

6 Quantum error correction

7 Bell