TD n°2

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1 Tree automata

Exercise 1.1 – Give a bottom-up tree automaton recognizing the language of all trees on $\mathcal{T}(\Sigma)$, with $\Sigma = \{f^2, g^2, \#\}$ for which the following holds:

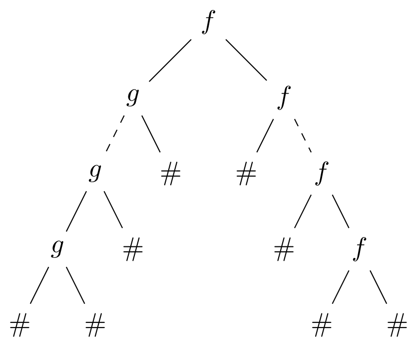
- a path with symbol g has two children f
- the root is a g

Correction 1.1

Let the automaton $\mathcal{A} = \{Q = \{q_0, q_1, q_2\}, \Sigma, \delta, \{q_2\}\}$ with :

$$\delta = \begin{cases} \# & \rightarrow & q_0 \\ f(q, q') & \rightarrow & q_1 & \forall q, q' \in Q \\ g(q_1, q_1) & \rightarrow & q_2 \end{cases}$$

Exercise 1.2 – Give a bootom-up tree automaton which recognizes all trees over (Σ) with $\Sigma = \{f^2, g^2, \#\}$ with the following shape :

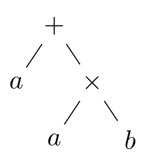


Correction 1.2

Let the automaton $\mathcal{A} = \{Q = \{q_0, q_1, q_2, q_3\}, \Sigma, \delta, \{q_3\}\}$ with :

$$\delta = \begin{cases} \# & \rightarrow & q_0 \\ f(q_0, q_0) & \rightarrow & q_1 & g(q_0, q_0) & \rightarrow & q_2 \\ f(q_0, q_1) & \rightarrow & q_1 & g(q_2, q_0) & \rightarrow & q_2 \\ f(q_2, q_2) & \rightarrow & q_3 \end{cases}$$

Exercise 1.3 – Let $\Sigma)\{+^2, \times^2, a, b\}$, consider the language on Σ of non ambiguous arithmetic expressions (that is, expressions which do not require parenthesis). For instance the tree :



is non ambiguous. Is this language regular ? Is it top-down recognizable ?

Correction 1.3

Let the top-down automaton $\mathcal{A} = \{\{q_0, q_1\}, \Sigma, \delta, \{q_0\}\}$ with :

$$\delta = \begin{cases} q_0, a & \rightarrow & \{\} & q_0, b & \rightarrow & \{\} \\ q_1, a & \rightarrow & \{\} & q_1, b & \rightarrow & \{\} \\ \\ q_0, \times & \rightarrow & \{(q_1, q_1)\} & q_0, + & \rightarrow & \{(q_0, q_0)\} \\ q_1, \times & \rightarrow & \{(q_1, q_1)\} \end{cases}$$

An expression is non ambiguous, if below the $\times$ there are only $\times$ or constants (a, b). We have a top-down regular automaton so this language is regular.

Exercise 1.4 – Let T be a tree language. Define *leaves*(t) as the word formed by taking all the symbols with arity 0 encountered during a depth first, left to right traversal of the tree. Formally, such a word can be written as the longest sequence of symbols : $t(l_1), t(l_2), \ldots, t(l_k)$ for which the following hold :

- $\forall i, |t(l_i)| = 0$
- $\forall i, j, i < j \Rightarrow l_i <_{\text{lex}} l_j$ ($<_{\text{lex}}$: the lexicographic order on paths).

We define *leaves*$(T) = \{$*leaves*$(t) | t \in T\}$. Show that even in the case where T is a recognizable tree language, *leaves*(T) is not necessarily a regular word language.

Correction 1.4

We can consider the language $\mathcal{T}$:

$$\begin{aligned} l((,), t) &\in \mathcal{T} \\ r((, t,)) &\in \mathcal{T} \\ n((,)) &\in \mathcal{T} \end{aligned}$$

$\mathcal{T}$ is a tree language recognized by $\mathcal{A} = \{\{q_0, q_1, q_2\}, \Sigma = \{l^3, r^3, n^2, (,)\}, \delta, \{q_2\}\}$ with :

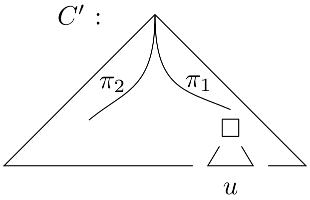
$$\delta = \begin{cases} (& \rightarrow & q_0 &) & \rightarrow & q_1 \\ n(q_0, q_1) & \rightarrow & q_2 & r(q_0, q_2, q_1) & \rightarrow & q_2 \\ l(q_0, q_1, q_2) & \rightarrow & q_2 \end{cases}$$

The language *leaves*(T) represent the Dyck's word. But this language is not regular.

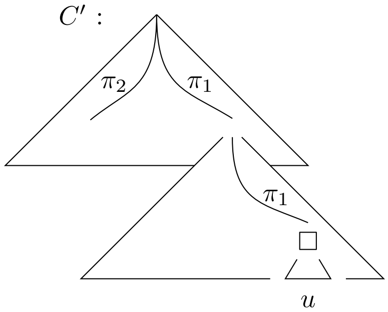
Exercise 1.5 – Show that the set of perfect binary trees over $\Sigma = \{f^2, a\}$ is not regular. A prefect binary tree is a tree for which all leaves have the same depth (that is, all paths to leafs have the same length).

Correction 1.5

We suppose that this language $\mathcal{T}$ is regular. So by the pumping lemma we have a $p \geq 1$ such that for all $t \in \mathcal{T}$ of $\mathtt{height}(t) \geq p$, there exists $C, C' \in \mathcal{T}(\Sigma, \{\square\})$ with C' non trivial and $u \in \mathcal{T}$ such that $t = C[C'[u]]$ and for all $n \geq 0$, $C[C'^n[u]] \in \mathcal{T}$.
C' is non trivial (it can not be $\square$) so it contains at least one f :



$C'[u]$ is perfect so the height of the tree on π_1 and π_2 are equals. But if we consider the tree $C'[C'[u]]$ the height of the tree on π_1 and π_2 are different :



$C[C'^2[u]] \notin \mathcal{T}$ which contradicts the pumping lemma. So the language $\mathcal{T}$ is not regular.

Exercise 1.6 – Let $\mathcal{T}$ be a language over $\Sigma = \{f^2, a, b\}$. Consider the congruence $f(x, y) \equiv f(y, x)$ for $x, y \in \mathcal{T}(\Sigma)$. Show that if $\mathcal{T}$ is regular, the set $\mathcal{T}' = \{t' | \exists t \in T, t \equiv t'\}$ is regular.

Correction 1.6

Let $\mathcal{A} = \{Q, \Sigma, \delta, \mathcal{I}\}$ a top-down tree automaton who recognizes the language $\mathcal{T}$.
We can construct a top-down tree automaton which recognizes the language $\mathcal{T}'$:

$\mathcal{A}' = \{Q, \Sigma, \delta', \mathcal{I}\}$ with :

$$\delta' = \begin{cases} a & \rightarrow & \delta(a) \\ b & \rightarrow & \delta(b) \\ f(q, q') & \rightarrow & \delta(f(q', q)) \end{cases}$$