λ-calculus

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Halting problem

There is no λ-term H such has $H[t] = T$ if T has a normal form and $H[T] = F$ if T has no normal form.

Let N the set of λ-term that have a normal form.
N is not empty (it contains all the variables) and N is not equal to Λ because it not contains the λ-term Ω.
So we have $\Lambda\backslash N$ non-empty and non-equal to Λ
N and $\Lambda\backslash N$ are closed by the β-reduction.
If $n, n' \in N$ such that $n =_\beta n'$ then $n' \in N$. Because n' has the same normal form as n.
If $n, n' \in \Lambda\backslash N$ such that $n =_\beta n'$ then $n' \in \Lambda\backslash N$. Because n has no normal form so n' has no normal form.
By Scott's theorem, the set N and $\Lambda\backslash N$ are not recursively separable. So the λ-term H does not exist.

List in pure λ calculus

We define this useful lambda term :

- $[I] = \lambda x.x$
- $[T] = \lambda x\ y.x$
- $[F] = \lambda x\ y.y$
- $\langle t, u\rangle = \lambda x.x\ t\ u$
- $\pi_1\langle t, u\rangle = \langle t, u\rangle[T]$
- $\pi_2\langle t, u\rangle = \langle t, u\rangle[F]$

We define our integers as follows :

- $[0] = [I]$
- $[S] = \lambda n.\langle [F], n\rangle$
- $[isZ] = \lambda n.\pi_1\ t$
- $[P] = \lambda n.\pi_2\ t$

We take the following fixpoint :

$$A = (\lambda xy.y(xxy))$$
$$\Theta = A\ A$$

Finally, we define our lists as follows :

- $[\text{Nil}] = \lambda n\ f.\ n$
- $[x :: l] = \lambda n\ f.\ f\ x\ l$

The list $0 :: 1 :: [\text{Nil}]$ is represented as follows:

$$\lambda n_0\ f_0.\ f_0\ [0]\ (\lambda n_1\ f_1.\ f_1\ [1]\ (\lambda n\ f.\ n))$$

The function $nth\ k\ l$ which return the k^{nth} element of the list l in an option type.
The options are defined as follows :

- $[None] = \lambda ns.\ n$
- $[Some(x)] = \lambda ns.\ sx$

Now we can define *nth* function :

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nth = Θ (λ f k l. l [None]
                    (λ x l'. [isZ] k [Some(x)])
                                     (f ([P] k) l'))
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Listing 1: Nth function on list

We want to proof this property $\forall k\ l\ x, nth\ [k]\ [l] =_\beta nth\ [k+1]\ [x :: l]$.

Proof − We just need to compute one step of $nth\ [k+1]\ [x :: l]$

$$\begin{aligned}
nth\ [k+1]\ [x :: [l]] &\rightarrow_\beta^* [x :: [l]]\ [None]\ (\lambda x\ l.\ [isZ]\ [k+1]\ [Some(x)]\ (nth\ [k]\ l)) \\
&\rightarrow_\beta^* [x :: [l]]\ [None]\ (\lambda x\ l.\ [F]\ [Some(x)]\ (nth\ [k]\ l)) \\
&\rightarrow_\beta^* [x :: [l]]\ [None]\ (\lambda x\ l.\ (nth\ [k]\ l)) \\
&\rightarrow_\beta^* (\lambda x\ l.\ (nth\ [k]\ l))\ x\ [l] \\
&\rightarrow_\beta^* nth\ [k]\ [l]
\end{aligned}$$

So by calculation, we have $\forall k\ l\ x, nth\ [k]\ [l] =_\beta nth\ [k+1]\ [x :: l]$.
□