Problem 1

Show that the following distributions belong to the exponential family. Find the natural parameter θ, scale parameter ϕ and convex function $b(\theta)$. Also find the $E(Y)$ and $Var(Y)$ as functions of the natural parameter. Specify the canonical link functions.

1. Exponential distribution $Exp(\lambda)$, $f(y;\lambda) = \lambda e^{-\lambda y}$;

$f(y;\lambda) = \exp\{-\lambda y + \log\lambda\}$ forms an EF distribution

with $\begin{cases} \theta = -\lambda \\ b(\theta) = -\log\lambda = -\log -\theta \\ \phi = 1 \end{cases}$

$EY = b'(\theta) = -\frac{1}{\theta}$

$VarY = b''(\theta) = \frac{1}{\theta^2}$

$g(u) = b'^{-1}(\mu) = -\frac{1}{u} \quad (u = \frac{1}{\lambda})$

2. Binomial distribution $Bin(n,\pi)$, $f(y;\pi) = \binom{n}{y}\pi^y(1-\pi)^{n-y}$, where n is known;

$f(y;\pi) = \exp\{y\log\pi + (n-y)\log(1-\pi) + \log\binom{n}{y}\}$

$= \exp\{y\log\frac{\pi}{1-\pi} + n\log(1-\pi) + \log\binom{n}{y}\}$ forms an EF distribution

with $\begin{cases} \theta = \log\frac{\pi}{1-\pi} \\ b(\theta) = -n\log(1-\pi) = n\log(1+e^\theta) \\ \phi = 1 \end{cases}$

$EY = b'(\theta) = \frac{ne^\theta}{1+e^\theta}$

$VarY = b''(\theta) = \frac{ne^\theta}{(1+e^\theta)^2}$

$g(u) = b'^{-1}(u) = \log\frac{u}{n-u} \quad (u = n\pi)$

3. Poisson distribution $Pois(\lambda)$, $f(y;\lambda) = \frac{1}{y!}\lambda^y e^{-\lambda}$;

$f(y;\lambda) = \exp\{y\log\lambda - \lambda - \log y!\}$ forms an EF distribution

with $\begin{cases} \theta = \log\lambda \\ b(\theta) = \lambda = e^\theta \\ \phi = 1 \end{cases}$

$EY = b'(\theta) = e^\theta$

$VarY = b''(\theta) = e^\theta$

$g(\lambda) = b'^{-1}(\lambda) = \log\lambda$

4. Chi-squared distribution $\chi^2_{(k)}$, $f(y;k) = \frac{1}{\Gamma(\frac{k}{2})2^{\frac{k}{2}}} y^{\frac{k}{2}-1} e^{-\frac{y}{2}}$;

$f(y;k) = \exp\{\frac{k}{2}\log y - \frac{y}{2} - \log(\Gamma(\frac{k}{2})) - \frac{k}{2}\log 2 - \log y\}$ form an EF distribution

with $\begin{cases} \theta = \frac{k}{2} \\ b(\theta) = \log(\Gamma(\frac{k}{2})) + \frac{k}{2}\log 2 = \log(\Gamma(\theta)) + \theta\log 2 \\ \phi = 1 \end{cases}$

$E(Y) = 2\theta$, $Var(Y) = 4\theta$

$g(k) = b'^{-1}(k) = \frac{k}{2}$

5. Negative binomial distribution $NB(m, \beta)$, $f(y; \beta) = \binom{y+m-1}{m-1}\beta^m(1-\beta)^y$, where m is known;

$f(y;\beta) = \exp\{m\log\beta + y\log(1-\beta) + \log\binom{y+m-1}{m-1}\}$ forms EF distribution

with $\begin{cases} \theta = \log(1-\beta) \\ b(\theta) = -m\log\beta = -m\log(1-e^\theta) \\ \phi = 1 \end{cases}$

$EY = b'(\theta) = \frac{me^\theta}{1-e^\theta}$

$VarY = b''(\theta) = \frac{me^\theta}{(1-e^\theta)^2}$

$g(u) = b'^{-1}(u) = \log\frac{u}{m+u} \quad (u = \frac{m}{\beta}(1-\beta))$

6. The Gamma distribution $Gamma(\alpha, \beta)$, $f(y; \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\beta y}$, where the shape parameter α is known.

$f(y;\beta) = \exp\{(\alpha-1)\log y - \beta y + \alpha\log\beta - \log(\Gamma(\alpha))\}$ forms EF distribution

with $\begin{cases} \theta = -\beta \\ b(\theta) = -\alpha\log\beta = -\alpha\log -\theta \\ \phi = 1 \end{cases}$

$EY = b'(\theta) = -\frac{\alpha}{\theta}$

$VarY = b''(\theta) = \frac{\alpha}{\theta^2}$

$g(u) = b'^{-1}(u) = -\frac{\alpha}{u} \quad (u = \frac{\alpha}{\beta})$

Problem 2

Assume $Y_1, Y_2, ..., Y_n$ are independent and follow a binomial distribution where $Y_i \sim Bin(m, \pi_i)$ and m is known. Furthermore, assume $log\frac{\pi_i}{1-\pi_i} = X_i\beta$. What are the expressions of deviance residuals and Pearson residuals respectively (use $\hat{\beta}$ to represent the MLE)? What are the expressions of the deviance and Pearson's χ^2 statistic?

$f(y_i; \pi_i) = \exp\{y_i\log\frac{\pi_i}{1-\pi_i} + m\log(1-\pi_i) + \log\binom{m}{y_i}\}$

$l(y; \pi) = \sum_{i=1}^n [y_i\log\frac{\pi_i}{1-\pi_i} + m\log(1-\pi_i) + \log\binom{m}{y_i}]$

The maximum likelihood is when $E(y_i) = y_i = m\pi_i$, therefore $\pi_{max} = \frac{y_i}{m}$

So the maximum value is $l(y; \pi_{max}) = \sum_{i=1}^n [y_i\log\frac{y_i}{m-y_i} + m\log(1-\frac{y_i}{m}) + \log\binom{m}{y_i}]$

For other models with $\hat{\beta}$, $X_i\hat{\beta} = \log\frac{\hat{\pi}_i}{1-\hat{\pi}_i}$, $E(y_i) = m\hat{\pi}_i = m\cdot\frac{e^{x_i\hat{\beta}}}{1+e^{x_i\hat{\beta}}}$

So the log-likelihood value is $l(y; \hat{\pi}) = \sum_{i=1}^n [y_ix_i\hat{\beta} - m\log(1+e^{x_i\hat{\beta}}) + \log\binom{m}{y_i}]$

$D(\pi_{max}, \hat{\pi}) = 2[l(y; \pi_{max}) - l(y; \hat{\pi})]$

$= 2\sum_{i=1}^n [y_i\log\frac{y_i}{m-y_i} - y_ix_i\hat{\beta} + m\log((1-\frac{y_i}{m})(1+e^{x_i\hat{\beta}}))]$

$r_{Di} = sign(y_i - m\frac{e^{x_i\hat{\beta}}}{1+e^{x_i\hat{\beta}}})\cdot\sqrt{2[y_i\log\frac{y_i}{m-y_i} - y_ix_i\hat{\beta} + m\log((1-\frac{y_i}{m})(1+e^{x_i\hat{\beta}}))]}$

$$E(y_i) = m\hat{\pi}_i = m\frac{e^{x_i\hat{\beta}}}{1+e^{x_i\hat{\beta}}}, \quad Var(y_i) = m\hat{\pi}_i(1-\hat{\pi}_i) = m\cdot\frac{e^{x_i\hat{\beta}}}{(1+e^{x_i\hat{\beta}})^2}$$

$$G = \sum_{i=1}^{n}\frac{\left(y_i - m\frac{e^{x_i\hat{\beta}}}{1+e^{x_i\hat{\beta}}}\right)^2}{m\frac{e^{x_i\hat{\beta}}}{(1+e^{x_i\hat{\beta}})^2}} = \sum_{i=1}^{n}\frac{[y_i(1+e^{x_i\hat{\beta}}) - me^{x_i\hat{\beta}}]^2}{m\cdot e^{x_i\hat{\beta}}}$$

$$r_{pi} = \frac{y_i(1+e^{x_i\hat{\beta}}) - me^{x_i\hat{\beta}}}{\sqrt{m\cdot e^{x_i\hat{\beta}}}}$$

Problem 3

Consider the binary response variable $Y \sim Bernoulli$ with $P(Y = 1) = \pi$ and $P(Y = 0) = 1 - \pi$. Observations Y_i, $i = 1, \ldots, n$, are independent and identically distributed as Y.

1. Find the Wald test statistic, the score test statistic, and the likelihood ratio test statistic to test hypotheses on $\pi = \pi_0$.

$$f(y_i;\pi) = \pi^{y_i}(1-\pi)^{1-y_i}$$
$$= exp\{y_i\log\pi + (1-y_i)\log(1-\pi)\}$$

$$l(y;\pi) = \sum_{i=1}^{n}[y_i\log\pi + (1-y_i)\log(1-\pi)]$$

$$S(\pi) = \frac{\partial l(y;\pi)}{\partial\pi} = \sum_{i=1}^{n}\left[\frac{y_i}{\pi} - \frac{1-y_i}{1-\pi}\right] = \sum_{i=1}^{n}\frac{y_i-\pi}{\pi(1-\pi)}$$

$$I(\pi) = -E(S'(\pi)) = -E\left(\sum_{i=1}^{n}\left[-\frac{y_i}{\pi^2} - \frac{1-y_i}{(1-\pi)^2}\right]\right)$$
$$= \frac{\sum_{i=1}^{n}Ey_i}{\pi^2} + \frac{\sum_{i=1}^{n}E(1-y_i)}{(1-\pi)^2}$$
$$= \frac{n}{\pi} + \frac{n}{1-\pi} = \frac{n}{\pi(1-\pi)}$$

Therefore:

$$TS_W = (\hat{\pi} - \pi_0)\frac{n}{\hat{\pi}(1-\hat{\pi})}(\hat{\pi} - \pi_0) = \frac{n(\hat{\pi}-\pi_0)^2}{\hat{\pi}(1-\hat{\pi})}$$

$$TS_S = \frac{\sum_{i=1}^{n}y_i - \pi_0}{\pi_0(1-\pi_0)}\cdot\frac{\pi_0(1-\pi_0)}{n}\cdot\frac{\sum_{i=1}^{n}y_i-\pi_0}{\pi_0(1-\pi_0)}$$
$$= \frac{[\sum_{i=1}^{n}(y_i-\pi_0)]^2}{n\pi_0(1-\pi_0)}$$

$$TS_{LR} = 2[l(y;\hat{\pi}) - l(y;\pi_0)]$$
$$= 2\sum_{i=1}^{n}\left[y_i\log\frac{\hat{\pi}}{\pi_0} + (1-y_i)\log\frac{1-\hat{\pi}}{1-\pi_0}\right]$$

2. With large samples, the Wald test statistic, score test statistic and the likelihood ratio test statistic approximately have the $\chi^2(1)$ distribution. For $n = 10$ and data (0, 1, 0, 0, 1, 0, 0, 0, 1, 0), use these statistics to test null hypotheses on for (i) $\pi = 0.1$, (ii) $\pi = 0.3$, (iii) $\pi = 0.5$.

For large samples, $\hat{\pi} = Ey_i = 0.3$
$\chi^2(1, 0.05) = 3.841$

i) $H_0: \pi = 0.1$, $H_1: \pi \neq 0.1$
under H_0, calculate $TS_W = 1.9 < 3.841$
therefore for 0.05 significant level, we failed to reject H_0,
and conclude that $\pi = 0.1$

$H_0: \pi = 0.1$, $H_1: \pi \neq 0.1$
under H_0, calculate $TS_S = 4.44 > 3.841$
therefore for 0.05 significant level, we reject H_0
and conclude that $\pi \neq 0.1$

$H_0: \pi = 0.1$, $H_1: \pi \neq 0.1$
under H_0, calculate $TS_{LR} = 3.1 < 3.841$
therefore for 0.05 significant level, we failed to reject H_0,
and conclude that $\pi = 0.1$

ii) $H_0: \pi = 0.3$, $H_1: \pi \neq 0.3$
under H_0, calculate $TS_W = 0 < 3.841$
therefore for 0.05 significant level, we failed to reject H_0,
and conclude that $\pi = 0.3$

$H_0: \pi = 0.3$, $H_1: \pi \neq 0.3$
under H_0, calculate $TS_S = 0 < 3.841$
therefore for 0.05 significant level, we failed to reject H_0,
and conclude that $\pi = 0.3$

$H_0: \pi = 0.3$, $H_1: \pi \neq 0.3$
under H_0, calculate $TS_{LR} = 0 < 3.841$
therefore for 0.05 significant level, we failed to reject H_0,
and conclude that $\pi = 0.3$

iii) $H_0: \pi = 0.5$, $H_1: \pi \neq 0.5$
under H_0, calculate $TS_W = 1.9 < 3.841$
therefore for 0.05 significant level, we failed to reject H_0,
and conclude that $\pi = 0.5$

$H_0: \pi = 0.5$, $H_1: \pi \neq 0.5$
under H_0, calculate $TS_S = 1.6 < 3.841$
therefore for 0.05 significant level, we failed to reject H_0,
and conclude that $\pi = 0.5$

$H_0: \pi = 0.5$, $H_1: \pi \neq 0.5$
under H_0, calculate $TS_{LR} = 1.65 < 3.841$
therefore for 0.05 significant level, we failed to reject H_0,
and conclude that $\pi = 0.5$

3. Do the test statistics lead to the same conclusions?

These statistics lead to different conclusions at $\pi = 0.1$,
but other than that, it all drew the same conclusions.