Probabilistic Models

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Sets

Definition 1 (De Morgan's laws)**.**

$$\left(\bigcup_n S_n\right)^c = \bigcap_n {S_n}^c \tag{1}$$

$$\left(\bigcap_n S_n\right)^c = \bigcup_n {S_n}^c \tag{2}$$

Theorem 1.

$$A \cup \left(\cap_{n=1}^{\infty} B_n\right) = \cap_{n=1}^{\infty} \left(A \cup B_n\right) \tag{3}$$

$$A \cap \left(\cup_{n=1}^{\infty} B_n\right) = \cup_{n=1}^{\infty} \left(A \cap B_n\right) \tag{4}$$

Probability Laws

Definition 2 (Probability Axioms)**.**

1. **Nonnegativity** $\mathbf{P}(A) \geq 0$, for every event A.
2. **Additivity** If A and B are two disjoint events then

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$$

3. **Normalization** $\mathbf{P}(\Omega) = 1$, where Ω is the *sample space*.

Definition 3 (Discrete Probability Law)**.** If the sample space consists of a finite number of possible outcomes, then the probability law is specified by the probabilities of the events that consists of a single element. In particular, the probability of any event $\{s_1, s_2, \ldots s_n\}$ is the sum of the probabilities of its elements:

$$\mathbf{P}(\{s_1, s_2, \ldots s_n\}) = \mathbf{P}(s_1) + \mathbf{P}(s_2) + \cdots + \mathbf{P}(s_n)$$

Definition 4 (Discrete Uniform Probability Law)**.** If the sample space consists of n possible outcomes which are equally likely, then the probability of any event A is given by

$$\mathbf{P}(A) = \frac{\text{number of elements of } A}{n}$$

Properties of Probability Laws

Consider a probability law and let A, B and C be events:

(a) If $A \subset B$, then $\mathbf{P}(A) \leq \mathbf{P}(B)$

(b) $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$

(c) $\mathbf{P}(A \cup B) \leq \mathbf{P}(A) + \mathbf{P}(B)$

(d) $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B) + \mathbf{P}(A^c \cap B^c \cap C)$

Definition 5. A *partition* of the sample space Ω is a collection of disjoint events $S_1, S_2, \dots S_n$ such that $\Omega = \cup_{i=1}^{n} S_i$. Then

$$\mathbf{P}(A) = \sum_{i=1}^{n} \mathbf{P}(A \cap S_i)$$

Definition 6 (Bonferroni's inequality)**.** We have

(a) for any two events A and B

$$\mathbf{P}(A \cap B) \geq \mathbf{P}(A) + \mathbf{P}(B) - 1$$

(b) for n events $A_1, A_2, \dots, A_n$

$$\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_n) \geq \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots + \mathbf{P}(A_n) - (n-1)$$

Definition 7 (The inclusion-exclusion formula)**.** We have

(a) for any two events A and B

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cup B)$$

(b) for n events $A_1, A_2, \dots, A_n$. Let $S_1 = \{i | 1 \leq i \leq n\}$, $S_2 = \{(i_1, i_2) | 1 \leq i_1 < i_2 \leq n\}$, and more generally, let S_m be the set of all m-tuples $(i_1, i_2, \dots, i_m)$ of indices that satisfy $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. Then,

$$\begin{aligned}
\mathbf{P}(\cup_{k=1}^{n} A_k) = & \sum_{i \in S_1} \mathbf{P}(A_i) \\
& - \sum_{(i_1, i_2) \in S_2} \mathbf{P}\left(A_{i_1} \cap A_{i_2}\right) \\
& + \sum_{(i_1, i_2, i_3) \in S_3} \mathbf{P}\left(A_{i_1} \cap A_{i_2} \cap A_{i_3}\right) - \cdots \\
& + (-1)^{n-1} \mathbf{P}(\cap_{k=1}^{n} A_k)
\end{aligned}$$

Continuity property of probabilities

Theorem 2. Let $A_1, A_2, \ldots$ be an infinite sequence of events, which is *monotonically increasing*, meaning $A_n \subset A_{n+1}$ for every n. Then

$$\mathbf{P}\left(\lim_{n\to\infty} \cup_{k=1}^{n} A_k\right) = \lim_{n\to\infty} \mathbf{P}\left(A_n\right)$$

Proof. Let $B_1 = A_1$ and $B_n = A_n \cap A_{n-1}^c$ for $n \geq 2$. The events B_n are disjoint, and we have $An = \cup_{k=1}^{n} B_k$. Let $A = \cup_{n=1}^{\infty} A_n$. Then $A = \cup_{k=1}^{\infty} B_k$. Now by the *additivity axiom* we have

$$\begin{aligned}
\mathbf{P}\left(A\right) &= \sum_{k=1}^{\infty} \mathbf{P}\left(B_k\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{P}\left(B_k\right) \\
&= \lim_{n\to\infty} \mathbf{P}\left(\cup_{k=1}^{n} B_k\right) \\
&= \lim_{n\to\infty} \mathbf{P}\left(A_n\right)
\end{aligned}$$

□

Theorem 3. Let $A_1, A_2, \ldots$ be an infinite sequence of events, which is *monotonically decreasing*, meaning $A_{n+1} \subset A_n$ for every n. Then

$$\mathbf{P}\left(\lim_{n\to\infty} \cap_{k=1}^{n} A_k\right) = \lim_{n\to\infty} \mathbf{P}\left(A_n\right)$$