Counting

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Basics

The Counting Principle

Consider a process that consists of r stages. Suppose that:

(a) There are n_1 possible results at the first stage.

(b) For every possible result at the first stage, there are n_2 possible results at the second stage.

(c) More generally, for an sequence of possible results at the first $i-1$ stages, there are n_i possible results at the i^{th} stage. Then, the total number of possible results of the r-stage process is

$$n_1 n_2 \cdots n_r$$

k-permutations

We start with n distinct objects, and let k be some positive integer, with $0 \leq k \leq n$. We wish to count the number of different ways that we can pick k out of these n objects and arrange them in a sequence.

$$k\text{-permutations} = \frac{n!}{(n-k)!}$$

Combinations

We start with n distinct objects, and let k be some positive integer, with $0 \leq k \leq n$. We wish to count the number of k-element subsets of a given n-element set. The number of possible combinations is

$$\frac{n!}{k!(n-k)!} \text{ which is same as } \binom{n}{k}$$

Partitions

We are given an n-element set and non-negative integers $n_1, n_2, \ldots, n_r$, whose sum is equal to n. We consider partitions of the set into r disjoint subsets, with the i^{th} subset containing exactly n_i elements. The number of partitions is

$$\frac{n!}{n_1! n_2! \cdots n_r!} \text{ which is denoted by } \binom{n}{n_1, n_2, \ldots, n_r}$$

Integer Partitions

Given a non-negative integer n and a positive integer k, consider the equation

$$x_1 + x_2 + \cdots + x_k = n$$

to be solved with respect to non-negative integer variables $x_1, x_2, \ldots, x_k$. Find the total number of solutions?

There is a one-to-one correspondence between non-negative integer solutions of equation $x_1 + x_2 + \cdots + x_k = n$ and sequences of $n + k - 1$ symbols (n "o" and $k - 1$ "$|$"). Hence the total number of solutions equals the number of ways of selecting $k - 1$ places for the symbol "$|$" in a sequence of length $n + k - 1$. Hence the total number of solutions

$$\binom{n+k-1}{k-1}$$

Examples

Example 1. Ninety students, including Joe and Jane, are to be split into three classes of equal size, and this is to be done at random. What is the probability that Joe and Jane end up in the same class?

Solution We place Joe in one class. Regarding Jane, there are 89 possible slots, and only 29 place her in the same class as Joe. Thus the answer is $\frac{29}{89}$. ■

Example 2 (Hypergeometric probabilities)**.** An urn contains n balls, out of which m are red. We select k of the balls at random, without replacement. What is the probability that i of the selected balls are red?

Solution The sample space consists of $\binom{n}{k}$ different ways of selecting k out of n balls. For the event of interest to occur, we have to select i out of the m red balls and also select $k - i$ balls out of the $n - m$ balls, which are not red. Thus the desired probability is

$$\frac{\binom{m}{i}\binom{n-m}{k-i}}{\binom{n}{k}}$$

For $i \geq 0$ satisfying $i \leq m, i \leq k$ and $k - i \leq n - m$. For all other i, the probability is zero. ■

Example 3. Prove the algebraic identity

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

Solution We have a group of n persons. Consider clubs that consists of a special person from the group (the club leader) and a number (possibly zero) of additional club members. Let us count the number of possible clubs of this type in two different ways.

There are n choices for club leader. Once the leader is chosen, we are left with a set of $n-1$ available persons, and we are free to choose any of the 2^{n-1} subsets. Thus the number of possible clubs is $n2^{n-1}$.

Alternatively, for fixed k, we can form a k-person club by first selecting k out of the n available persons, which can be done in $\binom{n}{k}$ ways. We can then select one of the members to be the leader, for which we have k choices. Hence the total mumber of clubs is $\sum_{k=1}^{n} k\binom{n}{k}$

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

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