Probabilistic Models

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Conditional Probability

Definition 1 (Conditional Probability)**.** The *conditional probability* of an event A, given an event B with $\mathbf{P}(B) > 0$, is defined by

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)},$$

and specifies a new probability law on the same sample space Ω. In particular, all properties of probability laws remain valid for conditional probability laws.

Remark 1. If the possible outcomes are finitely many and equally likely, then

$$\mathbf{P}(A|B) = \frac{\text{number of elements of } A \cap B}{\text{number of elements of } B}$$

Definition 2 (Multiplication Rule)**.** Assuming that all the conditioning events have positive probability, we have

$$\mathbf{P}\left(\cap_{i=1}^{n} A_i\right) = \mathbf{P}(A_1)\,\mathbf{P}(A_2|A_1)\,\mathbf{P}(A_3|A_1 \cap A_2) \cdots \mathbf{P}\left(A_n |\, \cap_{i=1}^{n-1} A_i\right)$$

Total probability theorem and Baye's rule

Definition 3 (Total Probability Theorem)**.** Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space and $\mathbf{P}(A_i) > 0$, for all i. Then, for any event B, we have

$$\begin{aligned}\mathbf{P}(B) &= \mathbf{P}(A_1 \cap B) + \cdots + \mathbf{P}(A_1 \cap B)\\ &= \mathbf{P}(A_1)\,\mathbf{P}(B|A_1) + \cdots + \mathbf{P}(A_n)\,\mathbf{P}(B|A_n)\end{aligned}$$

Definition 4 (Baye's Rule)**.** Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space, and assume that $\mathbf{P}(A_i) > 0$, for all i. Then, for any event B such that $\mathbf{P}(B) > 0$, we have

$$\begin{aligned}\mathbf{P}(A_i|B) &= \frac{\mathbf{P}(A_i)\,\mathbf{P}(B|A_i)}{\mathbf{P}(B)}\\ &= \frac{\mathbf{P}(A_i)\,\mathbf{P}(B|A_i)}{\mathbf{P}(A_1)\,\mathbf{P}(B|A_1) + \cdots + \mathbf{P}(A_n)\,\mathbf{P}(B|A_n)}\end{aligned}$$

Independence

Definition 5 (Independence)**.** Two events A and B are said to be *independent* if

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\,\mathbf{P}(B)$$

Remark 2. If A, B are independent, then A, B^c and A^c, B^c are also independent.

Conditional Independence

Definition 6 (Conditional Independence)**.** Two events A and B are said to be *conditionally independent* given another event C with $\mathbf{P}(C) > 0$, if

$$\mathbf{P}(A \cap B \mid C) = \mathbf{P}(A \mid C)\,\mathbf{P}(B \mid C)$$

Remark 3. If $\mathbf{P}(B \cap C) > 0$, conditional independence is equivalent to the condition

$$\mathbf{P}(A \mid B \cap C) = \mathbf{P}(A \mid C)$$

Remark 4. Independence does not imply conditional independence, and vice versa.

Independence of a Collection of Events

Definition 7 (Independence of Several Events)**.** We say that the events $A_1, A_2, \ldots, A_n$ are *independent* if

$$\mathbf{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathbf{P}(A_i) \quad \text{for every subset } S \text{ of } \{1, 2, \ldots, n\}.$$