Expectation, Mean and Variance of a Random Variable

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Expectation

Definition 1. The *Expectation* of a random variable X, with PMF p_X, is given by

$$\mathbf{E}\left[X\right] = \sum_x x p_X\left(x\right)$$

Remark 1. The mean can be interpreted as the *center of gravity* of the PMF. Given a bar with a weight $p_X\left(x\right)$ placed at each point x with $p_X\left(x\right) > 0$, the center of gravity c is the point at which the sum of the torques from the weights to its left is equal to the sum of the torques from the weights to its right:

$$\sum_x (x - c) p_X\left(x\right) = 0$$

Thus $c = \sum_x x p_X\left(x\right)$, i.e the center of gravity is equal to the mean $\mathbf{E}\left[X\right]$.

Remark 2. Let X be a random variable with PMF p_X, and let $g(X)$ be a function of X. Then the expected value of the random variable $g(X)$ is given by

$$\mathbf{E}\left[g(X)\right] = \sum_x g(x) p_X\left(x\right)$$

Variance and Moments

Definition 2 (Variance)**.** The variance $var(X)$ of a random variable X is defined by

$$var(X) = \mathbf{E}\left[(X - \mathbf{E}\left[X\right])^2\right]$$

which can be expressed in terms of moments as

$$var(X) = \mathbf{E}\left[X^2\right] - \left(\mathbf{E}\left[X\right]\right)^2$$

Mean and Variance of a Linear Function of a Random Variable

Definition 3. Let X be a random variable and let

$$Y = aX + b$$

where a and b are given scalars. Then

$$\mathbf{E}\left[Y\right] = a\mathbf{E}\left[X\right] + b \quad \text{and } var(Y) = a^2\, var(X)$$

Mean and Variance of Some Common Random Variables

Bernoulli Random Variable

If X is a *Bernoulli* random variable with PMF

$$p_X(k) = \begin{cases} p & \text{if } k = 1, \\ 1 - p & \text{if } k = 0. \end{cases}$$

The *mean* and *variance* of X are given by

$$\begin{aligned} \mathbf{E}[X] &= p \\ var(X) &= p(1-p) \end{aligned}$$

The Binomial Random Variable

If X is a *Binomial* random variable with PMF

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n,$$

The *mean* and *variance* of X are given by

$$\begin{aligned} \mathbf{E}[X] &= np \\ var(X) &= np(1-p) \end{aligned}$$

The Geometric Random Variable

If X is a *Geometric* random variable with PMF

$$p_X(k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots$$

The *mean* and *variance* of X are given by

$$\begin{aligned} \mathbf{E}[X] &= \frac{1}{p} \\ var(X) &= \frac{1-p}{p^2} \end{aligned}$$

The Poisson Random Variable

If X is a *Poisson* random variable with PMF

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

where λ is a positive parameter characterizing the PMF.

The *mean* and *variance* of X are given by

$$\begin{aligned} \mathbf{E}[X] &= \lambda \\ var(X) &= \lambda \end{aligned}$$

The Discrete Uniform Random Variable

If X is a *Uniform* random variable with PMF

$$p_X(k) = \begin{cases} \frac{1}{b-a+1} & \text{if } k = a, a+1, \ldots, b, \\ 0 & \text{otherwise.} \end{cases}$$

where a and b are two integers with $a < b$.

The *mean* and *variance* of X are given by

$$\mathbf{E}[X] = \frac{a+b}{2}$$
$$var(X) = \frac{(b-a)(b-a+2)}{12}$$

Example 1. Two buses carrying 101 students arrive at a job convention. The buses carry 100 and 1 students respectively. One of the students is randomly selected. Let X be the number of students on the bus carrying this selected student. One of the drivers is also randomly selected. Let Y denote the students on his bus. Which of $\mathbf{E}[X]$ or $\mathbf{E}[Y]$ is bigger?

We expect $\mathbf{E}[X]$ is bigger than $\mathbf{E}[Y]$ since if we choose a student, we are more likely to pick a bus with more students. If we compute the values $\mathbf{E}[X] \approx 100$ while $\mathbf{E}[Y] \approx 50$, which supports our intuition.