Independence Basis and Dimension

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Linear independence

Definition 1 (Independence)**.** The set of vectors $v_1, v_2, \ldots v_n$ are *linearly independent* if $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$ only when $c_1, c_2, \ldots c_n$ are all 0.

Definition 2 (Span of a vector space)**.** Vectors $v_1, v_2, \ldots v_k$ span a vector space, if their linear combination fill the space.

Definition 3 (Basis of a vector space)**.** A *basis* of a vector space is a sequence of vectors $v_1, v_2, \ldots v_d$ with two properties:

(i) $v_1, v_2, \ldots v_d$ are independent

(ii) $v_1, v_2, \ldots v_d$ span the vector space

Given a vector space every basis for that space has same number of vectors; that number is the *dimension* of the space.

Definition 4 (Sum of two subspaces)**.** If S and T are two subspaces of a vector space V. Then the sum $S + T$ contains sums $s + t$ for all $s \in S$, $t \in T$.

Remark 1. If S and T are two subspaces of a vector space V. Then $S \cap T$ is a subspace, while $S \cup T$ is not a subspace. The span of $S \cup T$ is the subspace $S + T$.

Definition 5. If S and T are two subspaces of a vector space V. Then

$$dim(S) + dim(T) = dim(S + T) + dim(S \cap T)$$

Four fundamental spaces associated with a matrix

For any $m \times n$ matrix A, with $rank(A) = r$, four fundamental subspaces are:

1. *Column space*, $\mathrm{C}(A)$, consists of all combinations of the columns of A and is a subspace of R^m. $dim(\mathrm{C}(A)) = r$
2. *Nullspace*, $\mathrm{N}(A)$, consists of all solutions x of the equation $Ax = 0$ and is a subspace of R^n. $dim(\mathrm{N}(A)) = n - r$
3. *Row space* of A, $\mathrm{C}(A^T)$, consists of combinations of the column vectors of A^T. $dim(\mathrm{C}(A^T)) = r$.
4. *Left Nullspace* of A, $\mathrm{N}(A^T)$, is the nullspace of A^T. $dim(\mathrm{N}(A^T)) = m - r$.