Projections onto subspaces

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Projections

The equation $Ax = b$ is solvable only when b is in the column space of A. If b is not in the column space of A, then $A\hat{x}$ is closest to b if

$$A^T(b - A\hat{x}) = 0 \quad \text{or} \quad A^T A\hat{x} = A^T b$$

The *projection* of b onto the subspace is

$$p = A\hat{x} = A(A^T A)^{-1} A^T b$$

The *projection matrix* P that produces $p = Pb$ is

$$P = A(A^T A)^{-1} A^T$$

Remark 1. When A has *independent* columns, $A^T A$ is square, symmetric and invertible.

Remark 2. If P is a *projection matrix* then

$$P^T = P \quad \text{and} \quad P^2 = P$$

Remark 3. When P projects onto one subspace $I - P$ projects onto the *perpendicular subspace*.

Remark 4. For a 2×2 matrix

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$