

Independence Basis and Dimension

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Linear independence

Definition 1 (Independence). The set of vectors v_1, v_2, \dots, v_n are *linearly independent* if $c_1v_1 + c_2v_2 + \dots + c_nv_n = 0$ only when c_1, c_2, \dots, c_n are all 0.

Definition 2 (Span of a vector space). Vectors v_1, v_2, \dots, v_k span a vector space, if their linear combination fill the space.

Definition 3 (Basis of a vector space). A *basis* of a vector space is a sequence of vectors v_1, v_2, \dots, v_d with two properties:

- (i) v_1, v_2, \dots, v_d are independent
- (ii) v_1, v_2, \dots, v_d span the vector space

Given a vector space every basis for that space has same number of vectors; that number is the *dimension* of the space.

Definition 4 (Sum of two subspaces). If S and T are two subspaces of a vector space V . Then the sum $S + T$ contains sums $s + t$ for all $s \in S, t \in T$.

Remark 1. If S and T are two subspaces of a vector space V . Then $S \cap T$ is a subspace, while $S \cup T$ is not a subspace. The span of $S \cup T$ is the subspace $S + T$.

Definition 5. If S and T are two subspaces of a vector space V . Then

$$\dim(S) + \dim(T) = \dim(S + T) + \dim(S \cap T)$$

Four fundamental spaces associated with a matrix

For any $m \times n$ matrix A , with $\text{rank}(A) = r$, four fundamental subspaces are:

1. *Column space*, $C(A)$, consists of all combinations of the columns of A and is a subspace of \mathbb{R}^m . $\dim(C(A)) = r$
2. *Nullspace*, $N(A)$, consists of all solutions x of the equation $Ax = 0$ and is a subspace of \mathbb{R}^n . $\dim(N(A)) = n - r$
3. *Row space* of A , $C(A^T)$, consists of combinations of the column vectors of A^T . $\dim(C(A^T)) = r$.
4. *Left Nullspace* of A , $N(A^T)$, is the nullspace of A^T . $\dim(N(A^T)) = m - r$.