CALCULUS

1. Limits and Continuity
2. Differentiation
3. Definite Integrals
   Improper Integrals
4. Partial Differentiation
5. Multiple Integrals
6. Vector Differentiation
7. Vector Integration
8. Fourier Series

Calculus is defined as science of acceleration, retardation and variation.

__Function__ :→ A relation between two sets 'A' & 'B' if $\forall \in A$ $\exists$ a unique $y \in B$ s.t. $f(x) = y$.

(i) __Explicit Function__:→ $z = f(x_1, x_2, \dots, x_n)$

$$\underset{\text{Dependent variable}}{z} = f(\underset{\text{Independent variable}}{x_1, x_2, \dots, x_n})$$

(ii) __Implicit function__:→ $\phi(z, x_1, x_2, \dots, x_n) = C$

(iii) __Composite function__:→ If $z = f(x, y)$ where $x = \phi(t)$ & $y = \psi(t)$

i.e. z is function of some function.

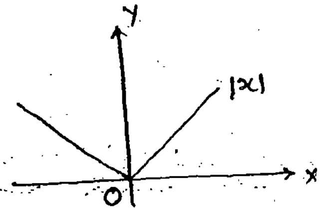
Some Special Functions →

(i) Even Function :→ $f(-x) = f(x)$ Eg:- $\cos x$, $|x|$, ---

(ii) Odd Function :→ $f(-x) = -f(x)$ Eg:- $\sin x$, x, ----

(iii) Modulus Function :→

$$f(x) = |x| = \begin{cases} x & ; \quad x > 0 \\ -x & ; \quad x < 0 \\ 0 & ; \quad x = 0 \end{cases}$$



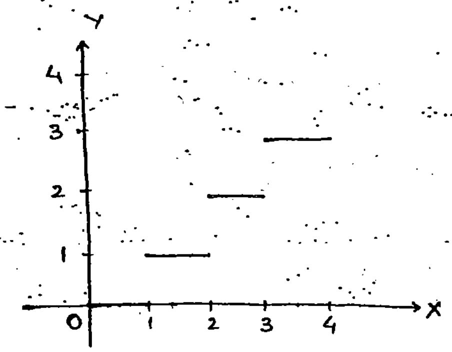
$$\frac{d}{dx}|x| = \frac{|x|}{x} \quad \text{for } x \neq 0$$

(iv) Step-Function / Greatest Integer Function /

$$f(x) = [x] = n \in \mathbb{Z}$$

where, $n \leq x < n+1$

Eg:- $[7.2] = 7$; $[7.999] = 7$; $[-1.2] = -2$



Symmetric Properties of the curve :→

Let $f(x,y) = c$ be the eqn of the curve

(i) If $f(x,y)$ contains only even powers of x i.e. $f(-x,y) = f(x,y)$ then it is symmetric about y-axis.

(ii) If $f(x,y)$ contains only even powers of y i.e. $f(x,-y) = f(x,y)$ then it is symmetric about x-axis.

(iii) If $f(x,y) = f(y,x)$, then, the curve is symmetric about $y = x$.

1. Limit of a function:$\longrightarrow$ Let $f(x)$ be defined in deleted neighbour-hoo of $a \in R$, then, $l \in R$ is said to be limit of $f(x)$ as x approaches a if for given $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$$|f(x) - l| < \epsilon \text{ whenever } |x-a| < \delta.$$

$$\boxed{\lim_{x \to a} f(x) = l}$$

Left limit:$\rightarrow$ when $x < a$, $x \to a$

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

Right Limit:$\rightarrow$ when $x > a$, $x \to a$

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

A limit exists iff LHL = RHL

Indeterminate form:$\rightarrow$ $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0

whenever we have $\lim_{x \to a} \frac{f(x)}{g(x)} \left[\text{as } \frac{0}{0} \text{ or } \frac{\infty}{\infty}\right] = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

This rule is applied until we are free from indeterminate form. (This rule is called L'Hospital Rule)

Standard Limits:→

(i) $$\lim_{x\to a} \frac{x^n - a^n}{x-a} = n a^{n-1}$$

(ii) $$\lim_{x\to 0} \frac{e^{mx}-1}{x} = e$$

(iii) $$\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$$

(iv) $$\lim_{x\to 0} [1+ax]^{1/x} = e^a$$

(v) $$\lim_{x\to \infty} \left[1+\frac{a}{x}\right]^x = e^a$$

(vi) $$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \text{or} \quad \lim_{x\to 0} \frac{\tan x}{x} = 1$$

(vii) $$\lim_{x\to \infty} \frac{\sin x}{x} = 0$$

(viii) $$\lim_{x\to 0} \frac{\sin mx}{x} = m$$

(ix) $$\lim_{x\to 0} \left[\frac{a^x + b^x}{2}\right]^{1/x} = \sqrt{ab}$$

(x) $$\lim_{x\to 0} [\cos x + a \sin bx]^{1/x} = e^{ab}$$

(xi) $$\lim_{x\to 0} \left[\frac{1-\cos ax}{x^2}\right] = \frac{a^2}{2}$$

Questions:→

1. $\lim_{x\to 0} \frac{1-\cos 3x}{x \sin 2x}$.

Solⁿ:→ $\lim_{x\to 0} \frac{1-\cos 3x}{x\sin 2x} = \lim_{x\to 0} \frac{3\sin 3x}{\sin 2x + 2x\cos 2x}$

$= \lim_{x\to 0} \frac{9\cos 3x}{2\cos 2x + 2\cos 2x - 4x\sin 2x}$

$= \frac{9}{4}$.

OR

$\lim_{x\to 0} \frac{(1-\cos 3x)/x^2}{(\sin 2x)/x} = \frac{3^2/2}{2} = \frac{9}{4}$.

2. $\lim_{x\to \frac{\pi}{2}} \frac{\sin^2 2x}{(x-\frac{\pi}{2})^2}$ is ______.

(a) 1 (b) 2 (c) 4 (d) −4

Solⁿ:→ $\lim_{x\to\frac{\pi}{2}} \frac{\left[\sin 2\left(\frac{\pi}{2}-x\right)\right]^2}{\left(\frac{\pi}{2}-x\right)^2}$

Take, $\frac{\pi}{2} - x = t$, then

$\lim_{t\to 0}\left[\frac{\sin 2t}{t}\right]^2 = (2)^2 = 4$.

3. $\lim_{x\to 0} \frac{\log x}{\cot x}$.

Solⁿ:→ $\lim_{x\to 0}\frac{\log x}{\cot x} = \lim_{x\to 0}\frac{1/x}{-\operatorname{cosec}^2 x} = -\left[\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot \sin x\right)\right]$

$= -1\times 0$

$= 0$.

4. $\lim_{x\to\frac{\pi}{2}} \dfrac{\tan x}{\log(\cos x)}$

Sol^n:→ $\lim_{x\to\frac{\pi}{2}} \dfrac{\tan x}{\log(\cos x)} = \lim_{x\to\frac{\pi}{2}} \dfrac{\sec^2 x}{-\sec x \sin x} = -\lim_{x\to\frac{\pi}{2}} \dfrac{1}{\sin x \cos x}$

$= -\infty.$

5. $\lim_{x\to\frac{\pi}{2}} \tan x$

Sol^n:→ $\cancel{\lim_{x\to\frac{\pi}{2}} \tan x = \lim_{x\to\frac{\pi}{2}} \frac{\sin x}{\cos x}}$ limit doesn't exists ($\because$ LHL $\neq$ RHL)

6. $\lim_{x\to 0} \sin x \log x^2$

Sol^n:→ $\lim_{x\to 0} \sin x \log x^2 = \lim_{x\to 0} \dfrac{\log x^2}{\operatorname{cosec} x} = \lim_{x\to 0} \dfrac{2/x}{-\operatorname{cosec} x \cot x}$

$= \lim_{x\to 0} \left(-2 \times \dfrac{\sin x}{x} \times \tan x\right)$

$= -2 \times 1 \times 0 = 0.$

Note:→ $\log_a 0 = \begin{cases} \infty & ; \ a<1 \\ -\infty & ; \ a>1 \end{cases}$

7. $\lim_{x\to 1} (x-1) \tan \dfrac{\pi x}{2}$

Sol^n:→ $\lim_{x\to 1} (x-1)\tan\dfrac{\pi x}{2} = \lim_{x\to 1} \dfrac{(x-1)}{\cot \frac{\pi x}{2}} = \lim_{x\to 1}\left[\dfrac{1}{-\frac{\pi}{2}\operatorname{cosec}^2\left(\frac{\pi x}{2}\right)}\right]$

$= -\dfrac{2}{\pi}.$

Note:→ (i) $\lim_{x\to 0} x \sin(1/x) = 0$

(ii) $\lim_{x\to\infty} x \sin(1/x) = 1$

(iii) $\lim_{x\to 0} \sin(1/x)$ doesnot exists.

8. $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2} \log(1+x) \right]$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $-\frac{1}{2}$ (d) $-\frac{1}{3}$

Solⁿ:→ $\lim_{x \to 0} \left[\frac{x - \log(1+x)}{x^2} \right] = \lim_{x \to 0} \left[\frac{1 - \frac{1}{1+x}}{2x} \right]$

$= \lim_{x \to 0} \frac{1/(1+x)^2}{2} = \frac{1}{2}$

9. $\lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x} \right]$ is

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Solⁿ:→ $\lim_{x \to 0} \left[\frac{\sin^2 x - x^2}{x^2 \sin^2 x} \right] = \lim_{x \to 0} \left[\frac{\sin^2 x - x^2}{x^4 \cdot \cancel{\frac{\sin^2 x}{x^2}}} \right]^1$

$= \lim_{x \to 0} \left[\frac{\sin^2 x - x^2}{x^4} \right]$

$= \lim_{x \to 0} \left[\frac{\sin 2x - 2x}{4x^3} \right]$

$= \lim_{x \to 0} \frac{2\cos 2x - 2}{12x^2}$

$= -\frac{2}{12} \lim_{x \to 0} \left(\frac{1 - \cos 2x}{x^2} \right)$

$= -\frac{2}{12} \times \frac{4}{2} = -\frac{1}{3}$

10. $\lim_{x \to 0} x^x$

Solⁿ:→ Let $y = x^x$

$\Rightarrow \log y = x \log x$

$\Rightarrow \lim_{x \to 0} [\log y] = \lim_{x \to 0} (x \log x) = \log \left(\lim_{x \to 0} x^x \right) = 0$

11. $\lim_{x \to \frac{\pi}{2}} (\cos x)^{\cos x}$

12. $\lim_{x \to 1} [\log x]^{\log x}$

<u>Note</u>:→ If we have 1^{∞} form, $\lim_{x \to a} [f(x)]^{g(x)} = e^{\lim_{x \to a} g(x)[f(x)-1]}$

13. $\lim_{x \to 0} (1-\sin x)^{1/\sin x}$

<u>Solⁿ</u>:→ $e^{\lim_{x \to 0} \frac{1}{\sin x}[1-\sin x-1]} = e^{-1} = \frac{1}{e}$

14. $\lim_{x \to 0} [\cos x]^{1/x^2}$

<u>Solⁿ</u>:→ $e^{\lim_{x \to 0} \frac{1}{x^2}[\cos x-1]} = e^{-\lim_{x \to 0}\left[\frac{1-\cos x}{x^2}\right]} = e^{-1/2} = \frac{1}{\sqrt{e}}$

15. $\lim_{x \to 0} \left(\frac{2^x + 8^x}{2}\right)^{1/x} = \sqrt{ab} = \sqrt{2 \times 8} = \sqrt{16} = 4.$

16. $\lim_{x \to 0} \left(\frac{2^x + 4^x + 8^x}{3}\right)^{1/x} = \sqrt[3]{abc} = \sqrt[3]{2 \times 4 \times 8} = 4.$

17. $\lim_{x \to 0} [\cos x + 2\sin 3x]^{1/x} = e^{ab} = e^{2 \times 3} = e^{6}.$

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18. $\lim_{x \to 0} \left[2\cos x + 3 \sin 4x \right]^{1/x}$

$= \lim_{x\to 0} 2^{1/x} \left[\cos x + \frac{3}{2} \sin 4x\right]^{1/x} = \lim_{x\to 0} 2^{1/x} \cdot \lim_{x\to 0} \left[\cos x + \frac{3}{2} \sin 4x\right]^{1/x}$

$=$ doesnot exists. ($\because$ LHL $\neq$ RHL for $\lim_{x\to 0} 2^{1/x}$)

19. $\lim_{x\to 0} \frac{|x|}{x} = ?$

Solⁿ :→ LHL $= \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{h\to 0} \frac{|-h|}{-h} = -1$

RHL $= \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{h\to 0} \frac{|h|}{h} = 1$

$\because$ LHL $\neq$ RHL

$\therefore$ limit doesn't exists

20. $\lim_{x\to a} [x]$ doesn't exists when a is ——

(a) Real no. (b) Rational no. (c) ✓ Integer (d) all of these

Solⁿ :→ let $a = 2$

LHL $= \lim_{x\to 2^-} [x] = 1$

RHL $= \lim_{x\to 2^+} [x] = 2$

$\because$ LHL $\neq$ RHL $\Rightarrow$ limit doesn't exists.

Continuity of a function:→

(i) Continuity at a point:→ A funⁿ is said to be continuous at a point $x = a$, if $\lim_{x\to a} f(x) = f(a)$

(ii) Continuity in an interval:→ A funⁿ $f(x)$ is said to be continuous in $[a, b]$ if it satisfies the following three conditions:–

(a) $f(x)$ is continuous $\forall\, x \in (a, b)$

(b) $\lim_{x\to a^+} f(x) = f(a)$

(c) $\lim_{x\to b^-} f(x) = f(b)$

Eg:– (i) If $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 2 & \text{if } x = 2 \end{cases}$ check its continuity at $x = 2$.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right) = \lim_{x\to 2}\left(\frac{2x}{1}\right) = 4$$

But at $x = 2$, $f(x) = 2$

$\therefore$ $f(x)$ is not continuous at $x = 2$.

(ii) If $f(x) = \begin{cases} (1+3x)^{1/x} & ; x \neq 0 \\ e^{3} & ; x = 0 \end{cases}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (1+3x)^{1/x} = e^{\lim_{x\to 0} \frac{1}{x}(1+3x-1)} = e^{3}$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

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Questions: →

1. If $f(x) = \begin{cases} 0 & ; \quad x = 0 \\ \frac{1}{2} - x & ; \quad 0 < x < \frac{1}{2} \\ \frac{1}{2} & ; \quad x = 1/2 \\ \frac{3}{2} - x & ; \quad \frac{1}{2} < x < 1 \\ 1 & ; \quad x \geq 1 \end{cases}$

Then which of the following is true :-

(a) $f(x)$ is right continuous at $x=0$

✓(b) $f(x)$ is discontinuous at $x = \frac{1}{2}$

(c) $f(x)$ is continuous at $x=1$

(d) b & c

Sol^n: → (a) $f(0) = 0$.

2. <u>Differentiation</u>:→ A funⁿ f(x) is said to be differentiable at a pt. $x = c$, if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists & finite. & is represented by $f'(c)$.

<u>Left Hand Derivative</u>:→ $\lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$

<u>Right Hand Derivative</u>:→ $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

Necessary condⁿ for a funⁿ to be differentiable is LHD = RHD.

<u>Note</u>:→(i) $f(x) = |x|$ is not differentiable at $x=0$

$$LHD = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{|-h| - 0}{-h} = -1$$

$$RHD = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h} = 1$$

$\therefore$ LHD $\neq$ RHD

$\Rightarrow$ Not differentiable

(ii) $|x-a|$ is not differentiable at $x = a$.

(iii) $|ax+b|$ is not differentiable at $x = -b/a$.

<u>Questions</u>:→

1. $f(x) = |x| + |x+1| + |x-2|$ is differentiable at $x =$ ______.

(a) 0 (b) ✓ 1 (c) −1 (d) 2

2. Let $f(x) = |x+1|$ be defined in the interval $[0,4]$ then,

✓ (a) f(x) is continuous & differentiable

(b) f(x) is continuous but non-differentiable

(c) f(x) is not continuous but differentiable

(d) f(x) is neither differentiable nor continuous

3. If $f(x) = |x|^3$ where $x \in R$, then, $f(x)$ at $x=0$ is ____ 7

(a) continuous but not differential

(b) Once differentiable but not twice

✓(c) Twice differentiable but not thrice

(d) Thrice differentiable ~~be~~

Solⁿ:→

$$f(x) = |x|^3 = \begin{cases} x^3 & ; \; x>0 \\ -x^3 & ; \; x<0 \\ 0 & ; \; x=0 \end{cases}$$

$f(0)=0$, LHL $=0$, RHL $=0$ $\Rightarrow$ continuous

$$LHD = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|-h|^3-0}{-h} = 0$$

$$RHD = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|^3-0}{h} = 0$$

$\because$ LHD = RHD $\Rightarrow$ differentiable

$$f'(x) = \begin{cases} 3x^2 & , \; x>0 \\ -3x^2 & , \; x<0 \\ 0 & , \; x=0 \end{cases}$$

$$f''(x) = \begin{cases} 6x & , \; x>0 \\ -6x & , \; x<0 \\ 0 & , \; x=0 \end{cases} = 6|x|$$

$$f'''(x) = \begin{cases} 6 & , \; x>0 \\ -6 & , \; x<0 \\ \text{not diff.} & \; x=0 \end{cases}$$

4. If $f(x) = \begin{cases} 2+x & x \geq 0 \\ 2-x & x < 0 \end{cases}$ then $f(x)$ at $x=0$ is ______

(a) Continuous & differentiable

(b) continuous but not differentiable

(c) Differentiable but not continuous

(d) Neither diff. nor continuous

Solⁿ:→ $f(0) = 2+0 = 2$, LHL $= 2-0 = 2$, RHL $= 2+0 = 2$ ⇒ continuous

$$\left.\begin{aligned} \text{LHD} &= \lim_{h\to 0} \frac{[2-(-h)]-2}{-h} = -1 \\ \text{RHD} &= \lim_{h\to 0} \frac{[2+h-2]}{h} = 1 \end{aligned}\right\} \text{Non-Differentiable}$$

Note:→ (i) Every differentiable funⁿ is a continuous funⁿ.

(ii) But every continuous funⁿ is not differentiable.

Mean Value Theorem:→

(i) Rolle's Theorem:→ Let $f(x)$ be defined in $[a,b]$ s.t. it satisfies three condⁿ:-

(a) $f(x)$ is continuous funⁿ in $[a,b]$

(b) $f(x)$ is differentiable funⁿ in (a,b)

(c) $f(a) = f(b)$

then, there exists atleast one point $c \in (a,b)$ ~~where~~ s.t.

$f'(c) = 0$

(ii) Lagrange's Mean Value Theorem:→ Let $f(x)$ be defined in $[a,b]$ s.t. it satisfies two condⁿ:-

(a) $f(x)$ is continuous funⁿ in $[a,b]$

(b) $f(x)$ is differential funⁿ in (a,b)

then, $\exists$ atleast one point c. in (a,b) s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Note: $\rightarrow$ If $f(x)$ is defined in $[a, a+h]$ s.t.

(a) $f(x)$ is continuous in $[a, a+h]$

(b) $f(x)$ is differentiable in $(a, a+h)$

then $\exists$ $\theta \in (0,1)$ s.t. $\boxed{\theta = \dfrac{c-a}{a-b}}$

$f(a+h) = f(a) + h f'(a+\theta h)$

Questions: $\rightarrow$

1. The mean-value 'c' for the fu^n $f(x) = e^x(\sin x - \cos x)$ in $[\frac{\pi}{4}, \frac{5\pi}{4}]$ is ______

(a) 0 (b) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (d) ✓ π

Sol^n: $\rightarrow$ $f(\frac{\pi}{4}) = e^{\pi/4}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = 0$

$f(\frac{5\pi}{4}) = e^{5\pi/4}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) = 0$

$f'(x) = e^x[\cos x + \sin x] + e^x[\sin x - \cos x] = 2e^x \sin x$

$\Rightarrow f'(c) = 2e^c \sin c = 0$

$\Rightarrow c = 0, \pm\pi, \pm 2\pi, \ldots$

2. The mean-value 'c' for the fu^n $f(x) = x^3 - 6x^2 + 11x - 6$ in $[0,4]$ is -

(a) $2 + \frac{2}{\sqrt{3}}$ (b) $2 - \frac{2}{\sqrt{3}}$ (c) ✓ $2 \pm \frac{2}{\sqrt{3}}$ (d) None

Sol^n: $\rightarrow$ $f(0) = -6$

$f(4) = 64 - 96 + 44 - 6 = 6$

$f'(x) = 3x^2 - 12x + 11$

$\therefore f'(c) = 3c^2 - 12c + 11 = \dfrac{f(4) - f(0)}{4 - 0}$

$\Rightarrow 12c^2 - 48c + 44 = 12$

$\Rightarrow c = 2 \pm \frac{2}{\sqrt{3}}$.

3. The value of ζ of $f(b) - f(a) = (b-a) f'(\zeta)$ for the fu^n $f(x) = Ax^2 + Bx + C$, in $[a, b]$ is ____.

(a) ✓ $\frac{b+a}{2}$ (b) $\frac{b-a}{2}$ (c) $b-a$ (d) $b+a$

Sol^n:→ $f'(x) = 2Ax + B$

$$f'(\zeta) = \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow 2A\zeta + B = \frac{f(b) - f(a)}{b-a} = \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b-a}$$

$$\Rightarrow 2A\zeta + B = (b+a)A + B$$

$$\Rightarrow \zeta = \frac{b+a}{2}$$

4. The mean-value 'c' for the fu^n $f(x) = 3x^2 + 5x + 11$ in the $\left[\frac{11}{2}, \frac{16}{2}\right]$ is ____.

Sol^n:→ $$c = \frac{\frac{11}{2} + \frac{16}{2}}{2} = \frac{27}{4}$$

Note:→ If the fu^n $f(x)$ is polynomial of degree 2 i.e. quadratic then, c will be the average value of the extreme value of the fu^n in given interval i.e. $c = \frac{a+b}{2}$ if $f(x)$ is defined in $[a,b]$

5. The value of $\theta \in (0,1)$ for the fu^n $f(x) = \log_e x$ in $[1, e]$ using an appropriate mean-value theorem is ____.

Sol^n:→ $f'(x) = \frac{1}{x} \Rightarrow f'(c) = \frac{1}{c}$

$f(1) = \log_e 1 = 0$

$f(e) = \log_e e = 1$

∴ Using LMVT, $\frac{1}{c} = \frac{1-0}{e-1} \Rightarrow c = e-1$

$$\therefore \theta = \frac{c-a}{b-a} = \frac{e-1-1}{e-1} = \frac{e-2}{e-1} \in (0,1)$$

6. LMVT cannot be applied for $f(x) = x^{1/3}$ in $[-1, 1]$ because 9'

(a) $f(x)$ is not continuous in $[-1, 1]$

(b) ✓ $f(x)$ is not differentiable in $(-1, 1)$

(c) a & b

(d) $f(-1) \neq f(1)$

Solⁿ:→ $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

$\therefore$ $f(x)$ is not differentiable at $x = 0 \in (-1, 1)$

$f(0) = 0$

$LHL = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} (-h)^{1/3} = 0$

$RHL = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} h^{1/3} = 0.$

$LHL = RHL = f(0) = 0 \Rightarrow$ continuous.

7. Rolle's Theorem cannot be applied for $f(x) = |x+2|$ in $[-2, 0]$.

(a) $f(x)$ is not continuous in $[-2, 0]$.

(b) $f(x)$ is not differentiable in $(-2, 0)$

(c) ✓ $f(-2) \neq f(0)$

(d) b & c

8. If $f'(x) = \frac{1}{5-x^2}$ and $f(0) = 1$ then the lower & upper bounds of $f(1)$ are _____.

Solⁿ:→ Let $f(x)$ be defined in $[0, 1]$

By LMVT, $\exists\ c \in (0, 1)$ s.t.

$f'(c) = \frac{f(1) - f(0)}{1 - 0}$

$\Rightarrow f'(c) = f(1) - 1$

$\text{Min}\{f'(x)\} < f'(c) < \text{Max}\{f'(x)\} \qquad 0 \leq x \leq 1$

$$\frac{1}{5} < f(1) - 1 < \frac{1}{4}$$

$$\Rightarrow \quad 1 + \frac{1}{5} < f(1) < \frac{1}{4} + 1$$

(iii) <u>Cauchy's Mean Value Theorem</u> :→ Let $f(x)$ & $g(x)$ be defined in a closed interval $[a, b]$ s.t. they satisfy the condⁿ:-

(a) $f(x)$ & $g(x)$ are continuous in $[a, b]$

(b) $f(x)$ & $g(x)$ are differentiable in (a, b)

(c) $g'(x) \neq 0 \quad \forall\, x \in (a, b)$, then,

$$\exists \; c \in (a, b) \text{ s.t. } \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Questions :→

1. The mean-value 'c' for the funⁿ $f(x) = e^x$ & $g(x) = e^{-x}$ in $[0, 1]$ is ______.

<u>Solⁿ</u>:→ $f'(x) = e^x \Rightarrow f'(c) = e^c$

$g'(x) = -e^{-x} \Rightarrow g'(c) = -e^{-c}$

$\because \; f(1) = e \; , \; f(0) = 1$

$g(1) = \frac{1}{e} \; , \; g(0) = 1$

$$\therefore \quad \frac{e^c}{-e^{-c}} = \frac{e-1}{(1/e)-1} \Rightarrow -e^{2c} = \left[\frac{e-1}{1-e}\right] e$$

$$\Rightarrow +e^{2c} = e \quad \Rightarrow \quad c = \frac{1}{2} \in [0, 1].$$

2. $f(x) = \sin x$ & $g(x) = \cos x$ in $[-\frac{\pi}{2}, 0]$ is ______.

<u>Solⁿ</u>:→ $f'(x) = \cos x$, $g'(x) = -\sin x \neq 0 \;\; \forall\; x \in (-\frac{\pi}{2}, 0)$

$$\frac{f'(c)}{g'(c)} = \frac{f(0) - f(-\pi/2)}{g(0) - g(-\pi/2)}$$

$$\Rightarrow \frac{\cos c}{-\sin c} = \frac{0-(-1)}{1-0} \Rightarrow -\cot c = 1 \Rightarrow \cot c = -1$$

$$\Rightarrow c = -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, 0\right)$$

(iv) Taylor's Theorem :→ OR (Generalised Mean-Value Theorem)

Let $f(x)$ be defined in $[a, a+h]$ s.t.

(a) $f, f', f'', f''', ----, f^{n-1}$ are continuous in $[a, a+h]$

(b) $f, f', f'', ---, f^{n-1}$ are differentiable in $(a, a+h)$

then $\exists\ \theta \in (0,1)$ s.t.

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + --- + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

where, $R_n = \dfrac{h^n}{(n-1)!\, P} (1-\theta)^{n-P} f^n(a+\theta h)$

for $P = n \rightarrow$ ~~Lang~~ Lagrange's form of remainder

$P = 1 \rightarrow$ Cauchy's form of remainder

Case I :- when $p = n$, $R_n = \dfrac{h^n}{n!} f^n(a+\theta h)$

Case II :- when $p = 1$, $R_n = \dfrac{h^n}{(n-1)!} (1-\theta)^{n-1} f^n(a+\theta h)$

Taylor's Series :→ As $n \to \infty$, $R_n \to 0$, then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + ---$$

(i) $f(x) = f(a) + (x-a) f'(a) + \dfrac{(x-a)^2}{2!} f''(a) + ---$ is a Taylor's series expansion of $f(x)$ about $x = a$.

(ii) $f(x) = f(0) + x f'(0) + \dfrac{x^2}{2!} f''(0) + ---$ is a Taylor's series expansion of $f(x)$ about $x = 0$. (Maclaurian's Series)

Questions:→

1. The coeff. of x^2 in the Taylor's Series expansion of $\cos^2 x$ about $x = 0$ is ——

(a) 0 (b) 1 (c) ✓ -1 (d) 2

Sol^n:→ coeff. of $x^2 = \frac{f''(0)}{2!} = \frac{-2}{2} = -1$.

where, $f'(x) = -\sin 2x$

$f''(x) = -2\cos 2x \Rightarrow f''(0) = -2$

2. The coeff. of $(x-2)^4$ in the Taylor's series expansion of e^x about $x = 2$ is ——

(a) $\frac{e^2}{2!}$ (b) ✓ $\frac{e^2}{4!}$ (c) $\frac{e^4}{2!}$ (d) $\frac{e^4}{4!}$

Sol^n:→ coeff. of $(x-2)^4 = \frac{f''''(2)}{4!} = \frac{e^2}{4!}$

3. The coeff. of $(x-\pi)^3$ in the power series expansion of $e^x + \sin x$ in the ascending power of $(x-\pi)$ is ——.

(a) $\frac{e^\pi}{6}$ (b) $\frac{e^\pi+1}{3}$ (c) $\frac{e^\pi-1}{3}$ (d) ✓ None

Sol^n:→ coeff. of $(x-3)^3 = \frac{f^3(\pi)}{3!} = \frac{e^\pi+1}{6}$

where, $f'(x) = e^x + \cos x$

$f''(x) = e^x - \sin x$

$f'''(x) = e^x - \cos x \Rightarrow f^3(\pi) = e^\pi + 1$

4. Which of the following fu^n ~~who~~ would have only odd powers of x in its Taylor's Series expansion about $x = 0$,

(a) $\sin x^2$ (b) $\cos x^2$ (c) $\cos x^3$ (d) ✓ $\sin x^3$

Sol^n:→ $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$

$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$

5. The Taylor's Series expansion of $f(x) = \tan^{-1}x$ about $x=0$ is ______.

Sol^n:→ $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$

$f(x) = \tan^{-1}x$

$f(0) = \tan^{-1} 0 = 0$

$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(0) = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2} \Rightarrow f''(x) = 0$

$$f'''(x) = -2\left[\frac{(1+x^2)^2 + x\cdot 2(1+x^2)2x}{(1+x^2)^4}\right] = -2\left[\frac{1+x^2-4x^2}{(1+x^2)^3}\right] = -2\left[\frac{1-3x^2}{(1+x^2)^3}\right]$$

$\Rightarrow f'''(0) = -2.$

$\therefore \tan^{-1}x = x - 2\frac{x^3}{3!} + \cdots$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

6. The power series expansion of $\frac{\sin x}{x-\pi}$ about $x = \pi$ is ______.

Sol^n:→ Let, $x - \pi = t$, then $f = \frac{\sin(\pi+t)}{t}$ about $t=0$

$= -\frac{\sin t}{t}$ about $t=0$

$= -\frac{1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right]$

$= -1 + \frac{t^2}{2!} - \frac{t^4}{5!} + \frac{t^6}{7!} - \cdots$

$= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{5!} + \frac{(x-\pi)^6}{7!} - \cdots$

7. For the f^n e^{-x}, linear approximation around $x=2$ is ______

(a) $(3-x)\,e^{-2}$ (b) $(1-x)\,e^{-2}$ (c) $[\sqrt{3}+2\sqrt{2}-(1+\sqrt{2})x]\,e^{-2}$

Soln:→ $$f(x) = \underbrace{f(a) + (x-a)\,f'(a)}_{\text{Linear approx.}} + \frac{(x-a)^2}{2!}\,f''(a) + \cdots$$

$$\therefore\ e^{-x} = f(2) + (x-2)\frac{d}{dx}e^{-x}\Big|_{x=2} = e^{-2} + (x-2)(-1)\,e^{-2}$$

$$= e^{-2}\,(3-x)$$

3. Definite Integrals:→

Theorem:→ Let $f(x)$ is a continuous f^n defined in $[a,b]$ & $F(x)$ be the anti-derivative of $f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Note:→ $$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(x)\,dx\right] = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$$

Properties of Definite Integrals:→

(i) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(ii) If $c \in (a,b)$ then, $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(iii) $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(iv) $$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$$

(v) $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ;\ \text{if } f(x) \text{ is even} \\ 0 & ;\ \text{if } f(x) \text{ is odd} \end{cases}$$

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(vi) $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & ; \text{ if } f(2a-x) = f(x) \\ 0 & ; \text{ if } f(2a-x) = -f(x) \end{cases}$$

(vii) $$\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx \quad ; \text{ if } f(a-x) = f(x)$$

(viii) $$\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx \quad ; \text{ if } f(x+a) = f(x)$$

(ix) $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \left[\frac{n-1}{n}\times\frac{n-3}{n-2}\times\cdots\times\frac{2}{3}\text{ (or) }\frac{1}{2}\right]k$$

where, $$k = \begin{cases} 1 & ; \text{ if } n \text{ is odd} \\ \pi/2 & ; \text{ if } n \text{ is even} \end{cases}$$

(x) $$\int_0^{\pi/2} \sin^m x\,\cos^n x\,dx = \left\{\frac{[(m-1)(m-3)\cdots 2\text{ (or) }1][(n-1)(n-3)\cdots 2\text{ (or) }1]}{[(m+n)(m+n-2)\cdots 2\text{ (or) }1]}\right\}$$

where, $$k = \begin{cases} \pi/2 & ; \text{ when } m \,\&\, n \text{ are even} \\ 01 & ; \text{ otherwise} \end{cases}$$

Questions:→

1. $$\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x}\,dx = \underline{\qquad}.$$

Solⁿ:→ Let $f(x) = \tan x \Rightarrow f(0+\frac{\pi}{2}-x) = \cot x$

$$\therefore I = \int_0^{\pi/2} \frac{f(x)}{f(x) + f(0+\frac{\pi}{2}-x)} = \frac{\frac{\pi}{2}-0}{2} = \frac{\pi}{4}.$$

2. $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\cot x}}$ is ______.

Sol^n:→ $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \frac{\frac{\pi}{2} - 0}{2} = \frac{\pi}{4}$

3. $\int_2^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{5-x}} dx$ is ______.

Sol^n:→ $I = \frac{3-2}{2} = \frac{1}{2}$

4. $\int_0^{\pi} |\cos x| \, dx$ is ______.

(a) 1 (b) 0 (c) ✓ 2 (d) None

Sol^n:→ $I = \int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{\pi} -\cos x \, dx$

$= \sin x \Big|_0^{\pi/2} - \sin x \Big|_{\pi/2}^{\pi}$

$= 1 - (-1) = 2$

5. $\int_0^4 (|x| + |3-x|) \, dx$ is ______.

Sol^n:→ $I = \int_0^4 x \, dx + \int_0^3 (3-x) \, dx + \int_3^4 (x-3) \, dx$

$= \frac{x^2}{2}\Big|_0^4 + \left[3x - \frac{x^2}{2}\right]_0^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$

$= 8 + \left[9 - \frac{9}{2}\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$

$= 34 - 21$

$= 13.$

6. $\int_0^n [x]\, dx$ is ____.

(a) $\frac{n(n+1)}{2}$ (b) ✓ $\frac{n(n-1)}{2}$ (c) $\frac{n}{2}$ (d) None.

Solⁿ: $\longrightarrow$ $I = \int_0^1 0.\, dx + \int_1^2 1.\, dx + \int_2^3 2.\, dx + \cdots + \int_{n-1}^n (n-1)\, dx$

$= 0 + [2-1] + 2[3-2] + \cdots + (n-1)[n-n+1]$

$= 1 + 2 + \cdots + (n-1)$.

$= \frac{n(n-1)}{2}$

7. $\int_0^1 x(1-x)^5\, dx$ is ____.

(a) ✓ $\frac{1}{42}$ (b) $\frac{1}{30}$ (c) $\frac{1}{24}$ (d) None

Solⁿ: $\longrightarrow$ $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

$\therefore \int_0^1 x(1-x)^5\, dx = \int_0^1 (1-x).x^5\, dx = \frac{x^6}{6}\Big|_0^1 - \frac{x^7}{7}\Big|_0^1 = \frac{1}{42}$.

8. $\int_0^{\pi/2} \log(\tan x)\, dx$ is ____.

Solⁿ: $\longrightarrow$ Replace x by $(\frac{\pi}{2}-x)$.

$I = \int_0^{\pi/2} \log \cot x$

$\therefore 2I = \int_0^{\pi/2} [\log \tan x + \log \cot x]\, dx = \int_0^{\pi/2} \log[\tan x . \cot x]\, dx = 0$

$\Rightarrow I = 0$.

9. $\int_0^{\pi/4} \log(1+\tan x)\, dx$ is ______.

(a) ✓ $\frac{\pi}{8}\log 2$ (b) $\frac{\pi}{4}\log 2$ (c) $\frac{\pi}{2}\log 2$ (d) None

Sol^n:→ $I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right) dx$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right) dx$$

$$= \int_0^{\pi/4} \log\left(2/1+\tan x\right) dx$$

$$= \int_0^{\pi/4} \left[\log 2 - \log(1+\tan x)\right] dx = \int_0^{\pi/4} \log 2\, dx - \underbrace{\int_0^{\pi/4} \log(1+\tan x)\, dx}_{I}$$

$$\Rightarrow 2I = \int_0^{\pi/4} \log 2\, dx = \frac{\pi}{4}\log 2$$

$$\Rightarrow I = \frac{\pi}{8}\log 2.$$

10. $\int_{-1}^{1} \frac{|x|}{x}\, dx$ is ____.

Sol^n:→ $I = 0$ (odd fu^n)

11. $\int_{-\pi}^{\pi} \log\frac{(1+\sin x)}{(1-\sin x)}\, dx$ is ______.

Sol^n:→ $f(-x) = \log\frac{(1-\sin x)}{(1+\sin x)} = -\log\left(\frac{1+\sin x}{1-\sin x}\right) = -f(x) \Rightarrow$ odd fu^n

$\therefore I = 0.$

12. $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$ is ______.

Sol^n:→ $\int_0^a x f(x)\, dx = \frac{a}{2}\int_0^a f(x)\, dx$; if $f(a-x) = f(x)$

$$f(\pi-x) = \frac{\sin x}{1+\cos^2 x} = f(x)$$

$$\therefore \; I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Put. $\cos x = t \Rightarrow -\sin x \, dx = dt$

$$\Rightarrow I = \frac{\pi}{2}\int_1^{-1} \frac{-dt}{1+t^2} = -\frac{\pi}{2}\left[\tan^{-1} t\right]_1^{-1} = -\frac{\pi}{2}\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$= -\frac{\pi}{2}\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{4}.$$

13. $\displaystyle\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$ is ______.

(a) ✓ $\dfrac{\pi}{ab}$ (b) πab (c) $\dfrac{2\pi}{ab}$ (d) None

Sol^n: → $$I = \int_0^{\pi} \frac{\sec^2 x \, dx}{a^2 + b^2 \tan^2 x}$$

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx & ; \text{ if } f(2a-x) = f(x) \\ 0 & ; \text{ if } f(2a-x) = -f(x) \end{cases}$$

$$f(\pi - x) = f(x)$$

$$\therefore \; I = 2\int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\,dx$$

Let. $\tan x = t$

$\sec^2 x \, dx = dt$

$$\therefore \; I = 2\int_0^{\infty} \frac{dt}{a^2 + (bt)^2} = 2 \times \frac{1}{a}\left[\frac{\tan^{-1}(bt/a)}{b}\right]_0^{\infty} = \frac{2}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{ab}$$

14. If $f(t)$ is a continuous fu^n defined in $[0, 1]$ then $\lim\limits_{t\to 0} \frac{1}{t}\int_0^t f(t)\,dt$ —

(a) 0 (b) ∞ (c) ✓ $f(0)$ (d) $f(1)$

<u>Sol^n</u>:→ $$\lim_{t\to 0} \frac{\int_0^t f(t)\,dt}{t} = \lim_{t\to 0} \frac{\frac{d}{dt}\left\{\int_0^t f(t)\,dt\right\}}{1}$$

$$= \lim_{t\to 0} \left[f(t)\times 1 - f(0)\times 0\right]$$

$$= f(0)$$

15. $\int_0^{\pi/2} \sin^8 x \, dx$ ———.

<u>Sol^n</u>:→ $$I = \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} =$$

16. $\int_0^{\pi/2} \cos^7 x \, dx$ ———.

<u>Sol^n</u>:→ $$I = \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times 1 =$$

17. $\int_0^{\pi/2} \sin^5 x \cos^9 x \, dx$ ———.

<u>Sol^n</u>:→ $$I = \frac{(4\times 2)\times(8\times 6\times 4\times 2)}{14\times 12\times 10\times 8\times 6\times 4\times 2}\times 1 =$$

18. $\int_0^{\pi/2} \sin^6 x \cos^3 x \, dx$ ———.

<u>Sol^n</u>:→ $$I = \frac{(5\times 3\times 1)\times(2)}{9\times 7\times 5\times 3\times 1}\times 1 =$$

19. $\int_0^{\pi/2} \sin^5 x \cos^8 x \, dx$ ———.

<u>Sol^n</u>:→ $$I = \frac{(4\times 2)\times(7\times 5\times 3\times 1)}{13\times 11\times 9\times 7\times 5\times 3\times 1}\times 1 =$$

20. $\int_0^{\pi/2} \sin^6 x \cdot \cos^8 x \, dx$ ______.

Solⁿ: → $I = \dfrac{(5\times3\times1)(7\times5\times3\times1)}{14\times12\times10\times8\times6\times4\times2} \times \dfrac{\pi}{2} =$

21. $\int_{-\pi}^{\pi} \sin^4 x \, dx$ ______.

Solⁿ: → $I = 2\int_0^{\pi} \sin^4 x \, dx$

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx & ; \text{ if } f(2a-x) = f(x) \\ 0 & ; \text{ if } f(2a-x) = -f(x) \end{cases}$$

$\because \sin^4(\pi - x) = \sin^4 x$

$\therefore I = 2\times2\times\int_0^{\pi/2} \sin^4 x \, dx = 4\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} =$

22. $\int_0^{\pi} \sin^4 x \cos^3 x \, dx$ ______.

Solⁿ: → $f(\pi - x) = \sin^4 x \, (-\cos x)^3 = -f(x)$

$\therefore I = 0.$

23. $\int_0^{\pi} \sin^3 x \cos^4 x \, dx$ ______.

Solⁿ: → $f(\pi - x) = \sin^3 x \cos^4 x = f(x)$

$\therefore I = 2\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx = 2\times\dfrac{2\times(3\times1)}{7\times5\times3\times1}\times1 =$

24. $\int_{-2\pi}^{2\pi} \sin^4 x \cos^8 x \, dx$ ______.

Solⁿ:→ $I = 2\int_0^{2\pi} \sin^4 x \cos^8 x \, dx = 2\times 2\times \int_0^{\pi} \sin^4 x \cos^8 x \, dx$

$= 2\times 2\times 2\times \int_0^{\pi/2} \sin^4 x \cos^8 x \, dx$

$= 8\times \frac{(3\times 1)\times(7\times 5\times 3\times 1)}{12\times 10\times 8\times 6\times 4\times 2} \times \frac{\pi}{2}$

Note:→ $\int_a^b \sin^4 x \cos^8 x \, dx = k\int_0^{\pi/2} \sin^4 x \cos^8 x \, dx$

where, $k = \frac{b-a}{\pi/2}$

25. $\int_0^{\pi/2} \sqrt{\sin x} \times \cos^3 x \, dx$ ______.

Solⁿ:→ Let $\sin x = t \Rightarrow \cos x \, dx = dt$

$\therefore I = \int_0^1 \sqrt{t}\,(1-t^2)\, dt = \left.\frac{t^{3/2}}{3/2}\right|_0^1 - \left.\frac{t^3}{3}\right|_0^1 = \frac{2}{3} - \frac{2}{7} =$

Improper Integrals:→

First Kind:→ $\int_a^b f(x)\,dx$ if $a=-\infty$ (or) $b=\infty$ (or) both

i.e $\int_{-\infty}^{b} f(x)\,dx$, $\int_a^{\infty} f(x)\,dx$, $\int_{-\infty}^{\infty} f(x)\,dx$.

Second Kind:$\rightarrow$ $\int_a^b f(x)\,dx$ if a & b are finite but $f(x)$ is infinite for some $x \in [a,b]$.

Eg:- $\int_{-1}^{1} \log(1+x)\,dx$, $\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx$, $\int_0^3 \frac{1}{x^2-5x+4}\,dx$

Convergence of an Improper Integrals:$\rightarrow$

(i) If $\int_a^b f(x)\,dx$ = finite, then, it is a convergent improper integral.

(ii) If $\int_a^b f(x)\,dx$ = infinite, then, it is a divergent improper integral.

Questions:$\rightarrow$

1. Find the convergence of following improper integrals

(i) $\int_0^\infty \frac{1}{a^2+x^2}\,dx$ ——.

Sol^n:$\rightarrow$ $I = \frac{1}{a}\tan^{-1}\frac{x}{a}\Big|_0^\infty = \frac{1}{a}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{2a}$ = finite

which is convergent.

(ii) $\int_0^\infty x\sin x\,dx$ ——.

Sol^n:$\rightarrow$ $I = x\Big[[-\cos x] - (1)[-\sin x]\Big]_0^\infty$ = infinite

which is divergent improper integral.

(iii) $\int_{-\infty}^{0} e^{ax} \cos px \, dx =$ ______.

Sol^n: → $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$

$$\therefore \quad I = \left[\frac{e^{ax}}{a^2+p^2} [a \cos px + p \sin px] \right]_{-\infty}^{0}$$

$$= \frac{a}{a^2+p^2} - 0 = \text{finite}$$

i.e. convergence

(iv) $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} \, dx$

Sol^n: → $I = \int_{-1}^{1} \frac{1+x}{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx + \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} dx$

$$= 2 \int_{0}^{1} \frac{1}{\sqrt{1-x^2}} dx + 0$$

$$= 2 \sin^{-1} x \Big|_0^1 = 2 \cdot \frac{\pi}{2} = \pi = \text{finite}$$

i.e. convergent improper integral.

(v) $\int_{-1}^{1} \frac{1}{x^2} dx$ ______.

Sol^n: → $I = \int_{-1}^{0} \frac{1}{x^2} dx + \int_{0}^{1} \frac{1}{x^2} dx = \left[\frac{-1}{x} \right]_{-1}^{0} + \left[-\frac{1}{x} \right]_{0}^{1} = \text{infinite}$

i.e. divergent improper integral

(vi) $\displaystyle\int_0^3 \frac{1}{x^2-3x+2}\,dx$

Solⁿ: $\rightarrow$ $\displaystyle I = \int_0^3 \frac{1}{(x-1)(x-2)}\,dx = \int_0^1 \frac{dx}{(x-1)(x-2)} + \int_1^2 \frac{dx}{(x-1)(x-2)} + \int_2^3 \frac{dx}{(x-1)(x-2)}$

$$\int \frac{dx}{(x-1)(x-2)} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1} = \log\left(\frac{x-2}{x-1}\right)$$

$$\therefore\ I = \log\left(\frac{x-2}{x-1}\right)\Big|_0^1 + \log\left(\frac{x-2}{x-1}\right)\Big|_1^2 + \log\left(\frac{x-2}{x-1}\right)\Big|_2^3 = \text{infinite}$$

i.e. divergent improper integral.

Comparison Test : $\rightarrow$

For first kind of improper integrals: $\rightarrow$

(a) Let $0 \le f(x) \le g(x)$, then,

(i) $\int_a^b f(x)\,dx$ converges if $\int_a^b g(x)$ is convergent

(ii) $\int_a^b g(x)\,dx$ diverges if $\int_a^b f(x)$ is divergent

(b) Limit form: $\rightarrow$ Let $f(x)$ & $g(x)$ be two positive fuⁿ s.t.

$\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$ (non-zero, finite) then $\int_a^b f(x)\,dx$ &

$\int_a^b g(x)\,dx$ both are convergent or divergent together.

Questions: $\rightarrow$

1. $\displaystyle\int_1^\infty e^{-x^2}\,dx$ ______.

Solⁿ: $\rightarrow$ $e^{x^2} \geq e^{x} \quad \forall\ x \geq 1$

$$\Rightarrow e^{-x^2} \le e^{-x} \quad \forall \quad x \geq 1$$

$$\int_1^\infty e^{-x}\, dx = \left.\frac{e^{-x}}{-1}\right|_1^\infty = 1 \Rightarrow \int_1^\infty e^{-x}\, dx \text{ is convergent}$$

$$\therefore \int_1^\infty e^{-x^2}\, dx \text{ is also convergent.}$$

2. $\displaystyle\int_2^\infty \frac{1}{\log x}\, dx$ ______.

Sol^n:→ $\log x < x \quad \forall \quad x \geq 2$

$$\Rightarrow \frac{1}{\log x} > \frac{1}{x} \quad \forall \quad x \geq 2$$

$$\int_2^\infty \frac{1}{x}\, dx = \log x \Big|_2^\infty = \text{infinite} \Rightarrow \text{divergent}$$

$$\therefore \int_2^\infty \frac{dx}{\log x} \text{ is also divergent.}$$

3. $\displaystyle\int_1^\infty \frac{1}{x^2(e^{-x}+1)}\, dx$ ______.

Sol^n:→ $x^2(e^{-x}+1) > x^2(0+1) \quad \forall \quad x > 1$

$$\Rightarrow \frac{1}{x^2(e^{-x}+1)} < \frac{1}{x^2} \quad \forall \quad x > 1$$

} Method- 1

Method- II :- Let $g(x) = x^2$, $\dfrac{f(x)}{g(x)} = \dfrac{1}{e^{-x}+1}$

so that, $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ thus, $g(x)$ will gives the nature of $f(x)$.

$$\therefore \int_1^\infty \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^\infty = 1 \Rightarrow \text{Convergent}$$

Hence, $f(x)$ is also convergent.

4. $\int_1^\infty \frac{x \tan^{-1}x}{\sqrt{4+x^3}}\, dx$ ______.

Soln:→ $f(x) = \frac{x \tan^{-1}x}{x\sqrt{x}\sqrt{\frac{4}{x^3}+1}} = \frac{\tan^{-1}x}{\sqrt{x}\sqrt{\frac{4}{x^3}+1}}$

Let $g(x) = \frac{1}{\sqrt{x}}$

$\frac{f(x)}{g(x)} = \frac{\tan^{-1}x}{\sqrt{\frac{4}{x^3}+1}} \Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \pi/2$

$\int_1^\infty \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}\Big|_1^\infty = \text{infinite} \Rightarrow \text{divergent}$

$\therefore$ f(x) is also divergent

Comparison Test for the second kind of improper integral :→

(b) Limit Form:→ Let f(x) & g(x) be two +ve fun s.t.

(i) 'a' is a point of discontinuity and $\lim_{x\to a^+} \frac{f(x)}{g(x)} = l_1$

(non-zero & finite)

(ii) 'b' is a point of discontinuity and $\lim_{x\to b^-} \frac{f(x)}{g(x)} = l_2$

(non-zero & finite)

then, $\int_a^b f(x)\, dx$ & $\int_a^b g(x)\, dx$ both converge (or) diverge

together.

Questions:→

1. $\int_0^{\pi/2} \frac{\sin x}{x\sqrt{x}}\,dx$ ______.

Sol^n:→

$$\frac{\sin x}{x} \le 1 \quad \forall \quad x > 0$$

$$\Rightarrow \frac{\sin x}{x\sqrt{x}} \le \frac{1}{\sqrt{x}} \quad \forall \quad x > 0$$

$$\int_0^{\pi/2} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_0^{\pi/2} = 2\times\sqrt{\frac{\pi}{2}} = \text{finite} \Rightarrow \text{convergent}$$

$$\therefore \int_0^{\pi/2} \frac{\sin x}{x\sqrt{x}}\,dx \text{ is also convergent}$$

2. $\int_1^{2} \frac{\sqrt{x}}{\log x}\,dx$ ______.

Sol^n:→

$$\frac{1}{\log x} > \frac{1}{x} \quad \forall \quad x > 1$$

$$\Rightarrow \frac{\sqrt{x}}{\log x} > \frac{1}{\sqrt{x}} \quad \forall \quad x > 1$$

$$\int_1^{2} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_1^{2} = 2\sqrt{2} - 2 = \text{finite} \Rightarrow \text{convergent}$$

$$\therefore \int_1^{2} \frac{\sqrt{x}}{\log x}\,dx \text{ may / may not be convergent.}$$

Thus, first method fails.

$$\lim_{x\to 1^+} \frac{f(x)}{g(x)} = 1$$

Let, $g(x) = \frac{1}{x\log x}$

$$\Rightarrow \frac{f(x)}{g(x)} = x\sqrt{x}$$

$\int_1^{2} \frac{1}{x\log x}\,dx$; Put $\log x = t$

$$\Rightarrow \frac{1}{x}\,dx = dt$$

$$\therefore \int_1^2 \frac{1}{x\log x}\,dx = \int_0^{\log 2} \frac{1}{t}\,dt = \log t \Big|_0^{\log 2} = \text{infinite} \Rightarrow \text{divergent}$$

$\therefore \int_1^2 \frac{\sqrt{x}}{\log x}\,dx$ is also divergent

3. which of the following f^n is strictly bounded :-

(a) x^2 (b) e^x (c) $\frac{1}{x}$ (d)✓ e^{-x^2}

4. which of the following integrals is unbounded :-

(a) $\int_0^{\pi/4} \tan x\,dx$ (b) $\int_0^{\infty} \frac{1}{1+x^2}\,dx$ (c)✓ $\int_0^1 \frac{1}{1-x}\,dx$ (d) $\int_0^{\infty} x e^{-x}\,dx$

Sol^n :- (a) $I = \log \sec x \Big|_0^{\pi/4} = \log\sqrt{2}$

(b) $I = \tan^{-1} x \Big|_0^{\infty} = \pi/2$

(c) $I = -\log(1-x) \Big|_0^1 = \text{infinite}$

(d) $I = x\frac{e^{-x}}{-1}\Big|_0^{\infty} - (1)\frac{e^{-x}}{(-1)^2}\Big|_0^{\infty} = 1$

5. Consider the integrals $I_1 = \int_1^{\infty} \frac{1}{x^2(e^x+1)}\,dx$ & $I_2 = \int_1^{\infty} \frac{x+1}{x\sqrt{x}}\,dx$

then which of the following is true :-

(a) I_1 & I_2 are convergent ~~(b) I_1 is co~~

(b)✓ I_1 is convergent, I_2 is divergent

(c) I_1 is divergent, I_2 is convergent

(d) I_1 & I_2 are divergent

$\underline{Sol^n}$:→ $x^2(e^x+1) > x^2(0+1) \quad \forall\ x \geq 1$

$$\Rightarrow \frac{1}{x^2(e^x+1)} < \frac{1}{x^2} \quad \forall\ x \geq 1$$

$$\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = 1 \Rightarrow \text{convergent}$$

$\therefore$ I_1 is convergent.

$$I_2 = \int_1^\infty \frac{x}{x\sqrt{x}}\,dx + \int_1^\infty \frac{1}{x\sqrt{x}}\,dx = \int_1^\infty \frac{1}{\sqrt{x}}\,dx + \int_1^\infty \frac{1}{x\sqrt{x}}\,dx$$

$$\downarrow$$

$$2\sqrt{x}\Big|_1^\infty \Rightarrow \text{Divergent}$$

$\therefore$ I_2 is divergent.

6. Consider, $I_1 = \int_0^1 \frac{1}{x^{1/3}}\,dx$, $I_2 = \int_{-1}^1 \frac{1}{x}\,dx$, $I_3 = \int_0^1 x\log x\,dx$

then which of the following is convergent:-

(a) I_1 & I_2

(b) I_2 & I_3

✓(c) I_1 & I_3

(d) Only I_1

$\underline{Sol^n}$:→ $I_1 = \frac{x^{2/3}}{2/3}\Big|_0^1 = \text{finite}$

$$I_2 = \int_{-1}^0 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx = \log x\Big|_{-1}^0 + \log x\Big|_0^1 = \text{infinite}$$

$$I_3 = \log x \cdot \frac{x^2}{2}\Big|_0^1 - \int_0^1 \frac{1}{x}\cdot\frac{x^2}{2}\,dx = \left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1$$

$$= -\frac{1}{4} - \left[\lim_{x\to 0} \frac{x^2}{2}\log x\right]$$

$$\lim_{x\to 0} \frac{\log x}{2/x^2} = \lim_{x\to 0} \frac{1/x}{-4/x^3} = \lim_{x\to 0} -\frac{x^2}{4} = 0$$ 20

$\therefore \quad I_3 = \text{finite}$

Gamma Function :→

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx \qquad (n>0)$$

Note :→ (i) $\Gamma 1 = 1$

(ii) $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

(iii) $\Gamma n+1 = n\Gamma n \quad \forall \; n>0$

(iv) $\Gamma n+1 = n! \quad \forall \; n \in z^+$

(v) $\displaystyle\int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\Gamma n}{a^n}$

Questions :→

1. $\displaystyle\int_0^\infty e^{-x^2}\, dx = \_\_\_\_\_\_.$

Sol^n :→ Let $x^2 = t \Rightarrow 2x\, dx = dt \Rightarrow dx = \frac{1}{2} t^{-1/2}\, dt$

$$\therefore I = \int_0^\infty e^{-t} \cdot \frac{1}{2} t^{-1/2}\, dt = \frac{1}{2}\int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\, dt = \frac{1}{2}\Gamma \tfrac{1}{2} = \frac{\sqrt{\pi}}{2}.$$

Note :→ $\displaystyle\int_{-\infty}^\infty e^{-x^2}\, dx = 2\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}$

2. $\displaystyle\int_0^\infty e^{-2x^2} x^7\, dx = \_\_\_\_\_\_.$

Sol^n :→ Let $2x^2 = t \Rightarrow 4x\, dx = dt$

$$\therefore I = \int_0^\infty e^{-t} \left(\frac{t}{2}\right)^3 \frac{dt}{4} = \frac{1}{32}\int_0^\infty e^{-t}\, t^3\, dt = \frac{1}{32}\Gamma 4 = \frac{3!}{32} = \frac{6}{32}.$$

3. $\int_0^1 (x \log x)^4\, dx = \_\_\_\_\_\_$.

Sol^n: → Let, $\log x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}\, dt$

$$\therefore I = \int_{-\infty}^{\infty} [e^{-t}(-t)]^4 (-e^{-t})\, dt = \int_0^{\infty} e^{-5t}\, t^{5-1}\, dt = \frac{\Gamma 5}{5^5} = \frac{4!}{5^5}.$$

4. $\int_0^{\infty} 5^{-4x^2}\, dx = \_\_\_\_\_\_$.

Sol^n: → Let. $5^{-4x^2} = e^{-t} \Rightarrow -4x^2 \log 5 = -t \Rightarrow x = \frac{1}{2\sqrt{\log 5}} \sqrt{t}$

$$\Rightarrow dx = \frac{1}{2\sqrt{\log 5}} \frac{1}{2\sqrt{t}}\, dt$$

$$\therefore I = \int_0^{\infty} e^{-t} \frac{1}{2\sqrt{\log 5}} \frac{t^{-1/2}}{2}\, dt = \frac{1}{4\sqrt{\log 5}} \int_0^{\infty} e^{-t}\, t^{\frac{1}{2}-1}\, dt$$

$$= \frac{\sqrt{\pi}}{4\sqrt{\log 5}}$$

Note: → $\log_a 0 = \begin{cases} \infty & ; \; a < 1 \\ -\infty & ; \; a > 1 \end{cases}$

Beta function: →

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx \qquad (m>0,\; n>0)$$

Note: → (i) $\beta(m,n) = \beta(n,m)$

(ii) $\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m{+}n}$

(iii) $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$

(iv) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$

$$\text{i.e.} \int_0^{\pi/2} \sin^p\theta \, \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\left(p > -\tfrac{1}{2},\ q > -\tfrac{1}{2}\right)$$

Questions:→

1. $\int_0^2 x^7(16-x^4)^{10}\, dx = $ ______.

Sol^n:→ Let, $x^4 = 16t \Rightarrow 4x^3\,dx = 16\,dt \Rightarrow x^3\,dx = 4\,dt$

$$\therefore I = \int_0^1 16t\,(16-16t)^{10}\, 4\,dt = 16^{11}\times 4\int_0^1 t\,(1-t)^{10}\, dt$$

$$= 4\times 16^{11} \times \beta(2,11)$$

$$= 4\times 16^{11} \times \frac{\Gamma 2\ \Gamma 11}{\Gamma 13}$$

$$= 4\times 16^{11} \times \frac{1\times 10!}{12!}$$

2. $\int_0^\infty \frac{x^3(1+x^5)}{(1+x)^{13}}\, dx = $ ______.

Sol^n:→ $\int_0^\infty \frac{x^3}{(1+x)^{13}}\,dx + \int_0^\infty \frac{x^8}{(1+x)^{13}}\,dx = \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+9}}\,dx + \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+4}}\,dx$

$$= \beta(4,9) + \beta(9,4)$$

$$= 2\beta(4,9) = 2\times\frac{\Gamma 4 \times \Gamma 9}{\Gamma 13} = 2\times\frac{3!\times 8!}{12!}$$

3. $\int_0^\infty \left(\frac{x}{1+x^2}\right)^3 dx =$ ______.

Sol^n:→ Let $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$$\therefore I = \int_0^{\pi/2} \left(\frac{\tan\theta}{\sec^2\theta}\right)^3 \sec^2\theta \, d\theta = \int_0^{\pi/2} (\tan^3\theta/\sec^4\theta) \, d\theta$$

$$= \int_0^{\pi/2} \sin^3\theta \cos\theta \, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{3+1}{2}, \frac{1+1}{2}\right) = \frac{1}{2}\beta(2,1)$$

$$= \frac{1}{2} \times \frac{\sqrt{2}\,\sqrt{1}}{\sqrt{3}} = \frac{1}{2} \times \frac{1 \times 1}{2!} = \frac{1}{4}$$

4. Partial Differentiation :→

Let $z = f(x,y)$ then

$$z_x = \frac{\partial z}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$z_y = \frac{\partial z}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

Similarly, $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$ and so on.

Homogeneous function :→

Eg:- (i) $x^2 + xy + y^2$; $n = 2$

(ii) $2x^3 + xy^2 + z^3$; $n = 3$

(iii) $\frac{xy^2 - y^3}{2x+3y}$; $n = 3-1 = 2$

If $f(kx, ky) = k^n f(x,y)$ then $f(x,y)$ is a homogeneous fu^n with degree 'n'.

Note :→ (i) If $f(x, y)$ is a homogeneous funⁿ with degree 'n', then,

$$f(x,y) = \begin{cases} x^n \phi(y/x) \\ y^n \psi(x/y) \end{cases}$$

Euler's Theorem :→ If $f(x,y)$ is a homogeneous funⁿ with degree 'n', then,

(a) $x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y} = nf$

(b) $x^2 \dfrac{\partial^2 f}{\partial x^2} + 2xy \dfrac{\partial^2 f}{\partial x \partial y} + y^2 \dfrac{\partial^2 f}{\partial y^2} = n(n-1) f$

Note :→ If $u(x,y) = f(x,y) + g(x,y) + h(x,y)$, where, f, g & h are homogeneous funⁿ with degree m, n & p respectively, then,

(a) $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = mf + ng + ph$

(b) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = m(m-1)f + n(n-1)g + p(p-1)h$

Note :→ If $f(u)$ is a homogeneous funⁿ in two variables x & y with degree 'n', then,

(a) $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = n \dfrac{f(u)}{f'(u)} = F(u)$

(b) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = F(u)\left[F'(u) - 1\right]$

Total differentiation :→ If $z = f(x,y)$ where $x = \phi(t)$, $y = \psi(t)$, then, the total derivative of 'z' w.r.t. 't' is

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Total differential coefficient,

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Note: → (i) If $f(x,y) = c$ is an implicit funⁿ, then,

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

(ii) If $z = f(x,y)$ where $x = \phi(u,v)$ & $y = \psi(u,v)$, then,

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

& $$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Questions: →

1. If $z = e^x \sin y$, where $x = \log t$ & $y = t^2$, then, $\frac{dz}{dt} = ?$

Solⁿ: → $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= e^x \sin y \times \frac{1}{t} + e^x \cos y \times 2t$$

$$= \frac{e^x \sin y}{t} + 2te^x \cos y$$

$$= \frac{e^x}{t}(\sin y + 2t^2 \cos y)$$

$$= \sin y + 2y\cos y \qquad \{\because t = e^x \ \& \ t^2 = y\}$$

2. The Total derivative of x^2y w.r.t. x, where, x & y are connected by the relation $x^2 + xy + y^2 = 1$ is ______.

Solⁿ: → Let, $u = x^2y$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 2xy + x^2\frac{dy}{dx}$$

Now, $f(x,y) = x^2+xy+y^2 = 1$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x+y}{x+2y}$$

$$\therefore \frac{du}{dx} = 2xy + x^2\left(-\frac{2x+y}{x+2y}\right) = 2xy - x^2\left(\frac{2x+y}{x+2y}\right).$$

3. If $u = f(x+cy) + g(x-cy)$, then, $\frac{u_{xx}}{u_{yy}} =$ ______.

(a) c^{-2} ✓ (b) c^2 (c) $-c^{-2}$ (d) $-c^2$

Soln: → Let, $r = x+cy$, $s = x-cy$

$$u = f(r) + g(s)$$

$$u_x = f'(r)\frac{\partial r}{\partial x} + g'(s)\frac{\partial s}{\partial x} = f'(r) + g'(s)$$

$$u_{xx} = f''(r) + g''(s)$$

$$u_y = f'(r)\frac{\partial r}{\partial y} + g'(s)\cdot\frac{\partial s}{\partial y} = c f'(r) - c g'(s)$$

$$u_{yy} = c^2 f''(r) + c^2 g''(s)$$

$$\therefore \frac{u_{xx}}{u_{yy}} = \frac{1}{c^2} = c^{-2}$$

4. If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$, then, $6u_x + 4u_y =$ ______.

(a) $3u_z$ (b) $4u_z$ (c) $-3u_z$ ✓ (d) $-4u_z$

Soln: → $r = 2x-3y$, $s = 3y-4z$, $t = 4z-2x$

$$u = f(r,s,t)$$

$$u_x = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= f_r(2) + f_s(0) + f_t(-2)$$

$$\therefore 6u_x = 12f_r - 12f_t$$

$$u_y = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= f_r(-3) + f_s(3) + f_t\ (0)$$

$$\therefore\ 4u_y = -12 f_r + 12 f_s$$

$$\therefore\ 6u_x + 4u_y = 12 f_s - 12 f_t$$

$$u_z = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= f_r\ (0) + f_s\ (-4) + f_t\ (4)$$

$$= -4 f_s + 4 f_t$$

5. If $u = \frac{y}{z} + \frac{z}{x}$, then, $x u_x + y u_y + z u_z =$ ______.

(a) $\frac{xy}{z^2}$ (b) $\frac{yz}{x^2}$ (c) $\frac{xz}{y^2}$ (d) ✓ 0

Solⁿ:→ $x\ u_x + y\ u_y + z u_z = 0 \cdot u = 0$ $(\because n = 0)$

6. If $\mu = \frac{x^2 y}{x^{5/2} + y^{5/2}}$, then, $x^2 \mu_{xx} + 2xy\ \mu_{xy} + y^2 \mu_{yy} =$ ______.

(a) $\frac{3}{4}\mu$ (b) ✓ $-\frac{1}{4}\mu$ (c) $-\frac{3}{4}\mu$ (d) $\frac{1}{4}\mu$

Solⁿ:→ $n = 3 - \frac{5}{2} = \frac{1}{2}$

$$\therefore\ x^2 \mu_{xx} + 2xy\ \mu_{xy} + y^2 \mu_{yy} = n(n-1)\mu = -\frac{1}{4}\mu.$$

7. If $u = \operatorname{cosec}^{-1}\left[\frac{x^{1/4} - y^{1/4}}{x^{1/5} + y^{1/5}}\right]$, then, $x u_x + y u_y =$ ______.

(a) $-\frac{1}{20}u$ (b) $-\frac{1}{20}\cot u$ (c) $\frac{1}{20}\tan u$ (d) ✓ $-\frac{1}{20}\tan u$

Solⁿ:→ $\operatorname{cosec} u = \frac{x^{1/4} - y^{1/4}}{x^{1/5} + y^{1/5}}$

$$\Rightarrow\ n = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

$\therefore\ x u_x + y u_y = n \dfrac{f(u)}{f'(u)} = \dfrac{1}{20} \times \dfrac{\operatorname{cosec} u}{-\operatorname{cosec} u \cot u} = -\dfrac{1}{20} \tan u.$

8. If $u = \log\left(\dfrac{x^2}{y}\right)$, then, $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} =$ ______.

(a) u (b) 0 (c)✓ -1 (d) 1

Solⁿ:→ $e^u = \left(\dfrac{x^2}{y}\right) \Rightarrow n = 2-1 = 1$

$$x u_x + y u_y = n \frac{f(u)}{f'(u)} = \frac{e^u}{e^u} = 1 = F(u).$$

$$\therefore\ x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = F(u)\left[F'(u) - 1\right] = -1.$$

9. If $z = x^n f(y/x) + y^{-n} g(x/y)$, then, $x z_x + y z_y + x^2 z_{xx} + 2xy\, z_{xy} + y^2$

is ______.

(a) $n(n-1) z$ (b)✓ $n^2 z$ (c) $n(n+1) z$ (d) nz

Solⁿ:→ The given funⁿ is the sum of two homogeneous funⁿ having degree n & $-n$ respectively.

$$\therefore\ x z_x + y z_y + x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = (nf - ng) + [n(n-1)f] + [(-n)(-n-1)g]$$

$$= n^2 z$$

10. If $u = \sin^{-1}(x/y) + \cos^{-1}(y/x)$, then, $\dfrac{u_x}{u_y} =$ ______.

(a)✓ $-\dfrac{y}{x}$ (b) $-\dfrac{x}{y}$ (c) $\dfrac{y}{x}$ (d) $\dfrac{x}{y}$

Solⁿ:→ $x u_x + y u_y = 0 \cdot u = 0 \quad (\because n = 0)$

$$\Rightarrow \frac{u_x}{u_y} = -\frac{y}{x}.$$

Maxima & Minima:→

For function of Single Variable:→

$f(x) \to \max \to x = c$ if $f(x) \le f(c)$ $\forall x$

$f(x) \to \min \to x = c$ if $f(x) \ge f(c)$ $\forall x$.

Method:→ (i) find $f'(x)$

(ii) Equate $f'(x) = 0$ for obtaining the stationary points

(iii) At each stationary pt. find $f''(x)$

(a) If $f''(x_0) > 0$ then $f(x)$ has minima at $x = x_0$.

(b) If $f''(x_0) < 0$ then $f(x)$ has maxima at $x = x_0$.

(c) If $f''(x_0) = 0$ then $f(x)$ has no extreme at $x = x_0$

and it is called critical point.

Questions:→

1. The funⁿ $f(x) = 2x^3 - 3x^2 - 36x + 10$ has a minimum value at $x =$ ______.

(a) 2 (b) ✓ 3 (c) −2 (d) −3.

Solⁿ:→ $f'(x) = 6x^2 - 6x - 36 = 0$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow x = -2, 3$

$f''(x) = 12x - 6$

$f''(x)\big|_{x=-2} = 12(-2) - 6 = -24 - 6 < 0 \Rightarrow$ maxima at $x = -2$.

$f''(x)\big|_{x=3} = 12(3) - 6 > 0 \Rightarrow$ minima at $x = 3$.

2. The maximum value of fu^n $\frac{e^{\sin x}}{e^{\cos x}}$ is ______. where, $x \in R$

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(a) $e^{1/\sqrt{2}}$ (b) $e^{-\sqrt{2}}$ (c)✓ $e^{\sqrt{2}}$ (d) $e^{-1/\sqrt{2}}$

Sol^n:→ $f(x) = \frac{e^{\sin x}}{e^{\cos x}} = e^{(\sin x - \cos x)}$

f(x) will have max. value when $(\sin x - \cos x)$ will be max.

let, $g(x) = \sin x - \cos x$

$g'(x) = \cos x + \sin x = 0 \Rightarrow x = -\pi/4, \frac{3\pi}{4}$

$g''(x) = -\sin x + \cos x$

$\therefore g''(x)\big|_{x=-\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} > 0 \Rightarrow$ minima at $\frac{-\pi}{4}$

$g''(x)\big|_{x=\frac{3\pi}{4}} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} < 0 \Rightarrow$ maxima at $\frac{3\pi}{4}$

$\therefore f\left(\frac{3\pi}{4}\right) = e^{\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)} = e^{\sqrt{2}}$

3. For the fu^n $f(x) = x^x$, minimum appears at $x =$ ______.

(a) e (b)✓ $\frac{1}{e}$ (c) $e+1$ (d) $e-1$

Sol^n:→ let, $y = x^x \Rightarrow \log y = x \log x$

$\Rightarrow \frac{1}{y} y' = x \cdot \frac{1}{x} + \log x = 1 + \log x$

$\Rightarrow y' = y(1 + \log x) = 0$

$\Rightarrow x = \frac{1}{e}$

4. Consider $f(x) = (x^2 - 4)^2$ where $x \in R$, then, f(x) has

(a) Only one minima (b)✓ Only two minima

(c) three minima (d) three maxima

Solⁿ:→ $f(x) = (x^2-4)^2$

$f'(x) = 2\cdot 2x(x^2-4) = 0 \Rightarrow x = \pm 2, 0$

$f''(x) = 3x^2-4$

$f''(0) = -4 < 0 \Rightarrow$ maxima at $x=0$

$f''(2) = 3\times 4-4 > 0 \Rightarrow$ minima at $x=2$

$f''(-2) = 3\times 4-4 > 0 \Rightarrow$ minima at $x=-2$

5. If $f(x) = a\log x + bx^2 - x$ has an extreme value at $x=-1, 2$ then a & b is ______.

(a) $2, \frac{1}{2}$ (b) $2, -\frac{1}{2}$ (c) ✓ $-2, \frac{1}{2}$ (d) $-2, -\frac{1}{2}$

Solⁿ:→ $f'(x) = \frac{a}{x} + 2bx - 1 = 0$

$\Rightarrow a + 2bx^2 - x = 0$

$\Rightarrow 2bx^2 - x + a = 0$

$\Rightarrow x^2 - \frac{1}{2b}x + \frac{a}{2b} = 0$

$\therefore \frac{1}{2b} = -1+2 = 1 \Rightarrow b = \frac{1}{2}$

$\frac{a}{2b} = -2 \Rightarrow a = -2$

6. The maximum value of $f(x) = x^2 - x - 2$ in $[-4, 4]$ is ______.

(a) ✓ 18 (b) 10 (c) -2.25 (d) indeterminate

Solⁿ:→ $f'(x) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$f''(x) = 2 > 0 \Rightarrow$ minima at $x = \frac{1}{2}$

$f(-4) = 18$

$f(4) = 10$

Maxima & Minima for fu^n of two variables :→

Let $z = f(x, y)$,

consider, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

Method :→ (i) Find p, q, r, s & t

(ii) Equate p & q to zero for obtaining the stationary points

(iii) At each stationary point find r, s & t.

(a) If $rt - s^2 > 0$, $r > 0$ then the fu^n $f(x,y)$ has a minima at that stationary point.

(b) If $rt - s^2 > 0$, $r < 0$ then $f(x,y)$ has a maxima at that stationary point.

(c) If $rt - s^2 < 0$, then $f(x,y)$ has no extreme at that stationary point & it is known as Saddle point

Questions :→

1. The fu^n $f(x,y) = x^2 + y^2 + 6x = 0$ has

(a) ✓ min. at $(-3, 0)$

(b) max. at $(-3, 0)$

(c) $(-3, 0)$ is a saddle point

(d) none

Sol^n :→

2. The fu^n $f(x,y) = x^3 - 3x^2 + 4y^2 + 6$ has a minimum value at $x =$ ______.

(a) $(0,0)$ (b) ✓ $(2,0)$ (c) $(2,1)$ (d) $(-2,0)$

Solⁿ:→ $p = \frac{\partial f}{\partial x} = 3x^2 - 6x = 0 \Rightarrow x = 0, 2.$

$q = \frac{\partial f}{\partial y} = 8y = 0 \Rightarrow y = 0$

$r = \frac{\partial^2 f}{\partial x^2} = 6x - 6$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0$

$t = \frac{\partial^2 f}{\partial y^2} = 8$

At $(0,0)$	& $(2,0)$
$r = -6 < 0$	$r = 6 > 0$
$s = 0$	$s = 0$
$t = 8$	$t = 8$
$rt - s^2 < 0$	$rt - s^2 > 0$
Saddle point	minima at $(2,0)$

3. The fu^n $f(x,y) = x^3 + y^3 - 3axy$ has

(a) max. at (a,a)

(b) ✓ max. at (a,a) if $a < 0$

(c) min. at (a,a)

(d) max. at (a,a) if $a > 0$

Solⁿ:→ $p = 3x^2 - 3ay = 0 \Rightarrow x^2 = ay$

$q = 3y^2 - 3ax = 0 \Rightarrow y^2 = ax$

$\left.\begin{matrix} x^2 = ay \\ y^2 = ax \end{matrix}\right\}$ solving

4. A rectangular box open at the top is to have a volume of $32\ ft^3$, then, the dimensions of the box requiring the least material for its construction is ______.

(a) 8, 2, 2 (b) ✓ 4, 4, 2 (c) 16, 1, 2 (d) 8, 8, $\frac{1}{2}$

Solⁿ:→ Let the dimension of the box is $x, y, z \Rightarrow xyz = 32$

$$\therefore \quad S = xy + 2yz + 2xz$$

$$\text{i.e. } f(x,y) = xy + 2y \cdot \frac{32}{xy} + 2x \cdot \frac{32}{xy}$$

$$= xy + \frac{64}{x} + \frac{64}{y}$$

$$\left.\begin{aligned} p &= y - \frac{64}{x^2} = 0 \\ q &= x - \frac{64}{y^2} = 0 \end{aligned}\right\} \text{solving } x = 4,\ y = 4$$

5. The distance between the origin and a point nearest to it on the surface $z^2 = 1 + xy$ is ______.

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) ✓ 1 (d) None

Solⁿ:→ Let $P(x, y, z)$ be on the surface $z^2 = 1 + xy$

$$\therefore D = \sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + y^2 + xy + 1}$$

Consider, $f = x^2 + y^2 + xy + 1$

$$\left.\begin{aligned} p &= 2x + y = 0 \\ q &= 2y + x = 0 \end{aligned}\right\} \text{solving } x = 0,\ y = 0$$

$$\left.\begin{aligned} r &= 2 \\ s &= 1 \\ t &= 2 \end{aligned}\right\} \begin{aligned} &rt - s^2 = 4 - 1 = 3 > 0 \quad \text{also } r > 0 \\ &\text{i.e. minima at } x = 0, y = 0 \end{aligned}$$

at $x = 0, y = 0$ we have $z^2 = 1 + 0 = 1 \Rightarrow z = \pm 1$

$$\therefore D = \sqrt{1} = 1.$$

Constrained Maxima & Minima :→

Lagrange's method of undetermined multipliers:→

Let $f(x,y,z)$ & $\phi(x,y,z) = c$, then

$$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z) = 0$$

$$\left.\begin{aligned} F_x = 0 \quad \text{i.e.} \quad & \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \\ F_y = 0 \quad \text{i.e.} \quad & \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \\ F_z = 0 \quad \text{i.e.} \quad & \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \end{aligned}\right\} \text{Lagrange's Eq}^n$$

$\lambda \to$ Lagrange's Multiplier

Questions:→

1. The volume of the greatest parallelopiped that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is ——.

(a) ✓ $\frac{8abc}{3\sqrt{3}}$ (b) $\frac{4abc}{3\sqrt{3}}$ (c) $\frac{abc}{\sqrt{3}}$ (d) $\frac{abc}{3\sqrt{3}}$

Sol^n:→ Let $2x, 2y, 2z$ be dimension of parallelopiped.

$$\text{volume} = 8xyz = f(x,y,z)$$

$$\phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

$$\therefore F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = 0$$

$$F_x = 8yz + \frac{2\lambda x}{a^2} = 0 \Rightarrow -\frac{\lambda}{4} = \frac{a^2yz}{x}$$

$$F_y = 8xz + \frac{2\lambda y}{b^2} = 0 \Rightarrow -\frac{\lambda}{4} = \frac{b^2xz}{y}$$

$$F_z = 8xy + \frac{2\lambda z}{c^2} = 0 \Rightarrow -\frac{\lambda}{4} = \frac{c^2xy}{z}$$

$$\therefore \frac{a^2yz}{x} = \frac{b^2xz}{y} \;\&\; \frac{b^2xz}{y} = \frac{c^2xy}{z} \;\&\; \frac{a^2yz}{x} = \frac{c^2xy}{z}$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} \;\&\; \frac{y^2}{b^2} = \frac{z^2}{c^2} \;\&\; \frac{z^2}{c^2} = \frac{x^2}{a^2}$$

$$\Rightarrow \frac{3x^2}{a^2} = 1 \Rightarrow x = \frac{a}{\sqrt{3}} \;\&\; y = \frac{b}{\sqrt{3}},\; z = \frac{c}{\sqrt{3}}$$

$\therefore$ volume, $V = 8xyz = \frac{8abc}{3\sqrt{3}}$.

2. The min value of $x^2+y^2+z^2$ where $x+y+z=1$ is ____.

(a) $\frac{1}{3}$ ✓ (b) $\frac{1}{9}$ (c) $\frac{1}{27}$ (d) 1

Solⁿ:→ $f = (x^2+y^2+z^2)$ & $\phi = x+y+z-1$

$\therefore F = f + \lambda\phi$

$\Rightarrow F_x = 2x+\lambda \Rightarrow -\lambda = 2x$

$F_y = 2y+\lambda \Rightarrow -\lambda = 2y$

$F_z = 2z+\lambda \Rightarrow -\lambda = 2z$

$\Rightarrow x = y = z$

$\therefore x = \frac{1}{3},\ y = \frac{1}{3},\ z = \frac{1}{3}$

$\therefore f_{min} = x^2+x^2+x^2 = 3x^2 = \frac{1}{3}$

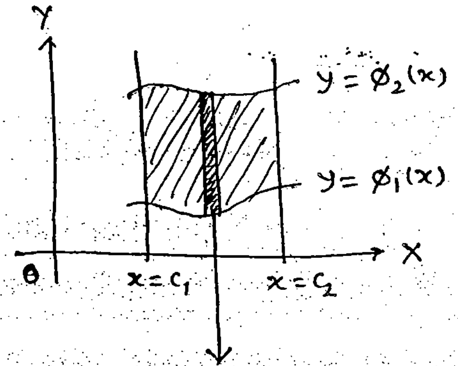
5. Multiple Integrals:→

Double Integral:→ Let $f(x,y)$ be defined at each point in the given region R, then its double integral is, $\iint_R f(x,y)\,dx\,dy$, where, continuity of $f(x,y)$ in the region R ensures the existence of the integral.

Methods of Evaluation:→

Case 1:→ Let the limits of integration be $y=\phi_1(x)$ to $y=\phi_2(x)$ & $x = c_1$ to c_2.

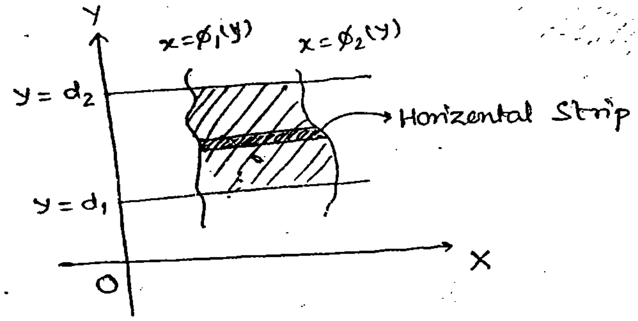
$$\iint_R f(x,y)\,dx\,dy = \int_{x=c_1}^{c_2} \int_{y=\phi_1(x)}^{\phi_2(x)} \{f(x,y)\,dy\}\,dx$$



Vertical strip slided over c_1 to c_2 to get area

Case 2:— when the limits of integration are $x = \phi_1(y)$ & $x = \phi_2(y)$, and $y = d_1$ to $y = d_2$,

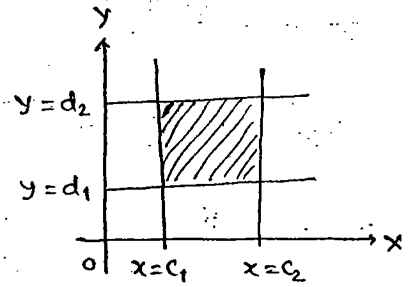
$$\iint_R f(x,y)\,dx\,dy = \int_{y=d_1}^{d_2} \int_{x=\phi_1(y)}^{\phi_2(y)} \{f(x,y)\,dx\}\,dy$$



Case 3:— when the limits are $x = c_1$ to $x = c_2$ & $y = d_1$ to $y = d_2$,

$$\iint_R f(x,y)\,dx\,dy = \int_{x=c_1}^{c_2} \left[\int_{y=d_1}^{d_2} f(x,y)\,dy\right] dx$$

$$= \int_{y=d_1}^{d_2} \left[\int_{x=c_1}^{c_2} f(x,y)\,dx\right] dy$$



Questions:→

1. Evaluate the following

(i) $\displaystyle\int_0^1 \int_0^2 (xy + x^3)\,dx\,dy$ (ii) $\displaystyle\int_0^4 \left[\int_0^{x^2} e^{y/x}\,dy\right] dx$

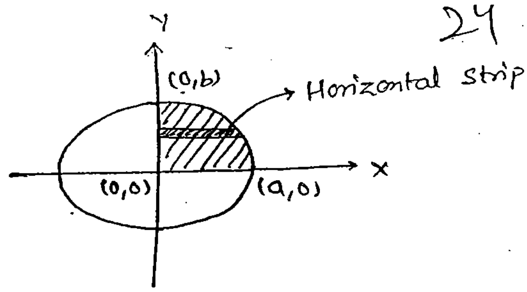
Sol^n:→ (i) $\displaystyle\int_0^1 \left(\frac{x^2}{2}y + \frac{x^4}{4}\right)_0^2 dy = \int_0^1 (2y+4)\,dy = [y^2+4y]_0^1 = 5.$

(ii) $\displaystyle\int_0^4 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx = \int_0^4 (xe^x - x)\,dx = \left[xe^x - e^x - \frac{x^2}{2}\right]_0^4$

$$= [4e^4 - e^4 - 8] - [-1] = 3e^4 - 7.$$

2. The value of $\iint_R xy\,dx\,dy$, where, R is a region in the ~~I~~ +ve quad-rant at the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is ______.

(a)✓ $\frac{a^2b^2}{8}$ (b) $\frac{ab}{8}$ (c) $\frac{a^3b^3}{8}$ (d) $\frac{a^2b^2}{4}$



$\underline{\text{Sol}^n}$:→ Consider the horizontal strip,

$x = 0$ to $x = \frac{a}{b}\sqrt{b^2-y^2}$

$y = 0$ to $y = b$

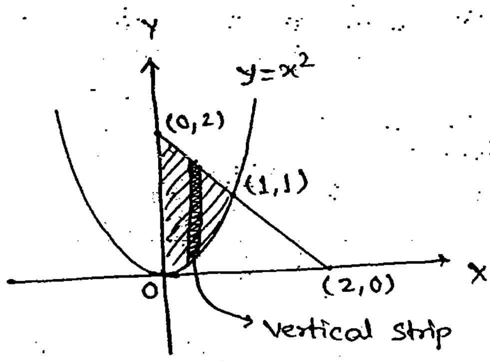
$$\therefore \iint_R xy \, dx\, dy = \int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} xy \, dx\, dy$$

$$= \int_{y=0}^{b} \left[\frac{x^2 y}{2}\right]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= \int_0^b \frac{\frac{a^2}{b^2}(b^2-y^2)\,y}{2}\, dy$$

$$= \left[\frac{a^2}{2}\cdot\frac{y^2}{2} - \frac{a^2}{2b^2}\cdot\frac{y^4}{4}\right]_0^b = \frac{a^2b^2}{4} - \frac{a^2b^2}{8} = \frac{a^2b^2}{8}$$

3. The value of $\iint_R y\, dx\, dy$, where, R is a region $y = x^2$, $x+y=2$ & $x=0$ is ______.



$\underline{\text{Sol}^n}$:→ $x + x^2 = 2$

$\Rightarrow x^2 + x - 2 = 0$

$\Rightarrow (x-1)(x+2) = 0$

$\Rightarrow x = 1, \cancel{-2}$

$\therefore$ pt. of intersection is $(1,1)$

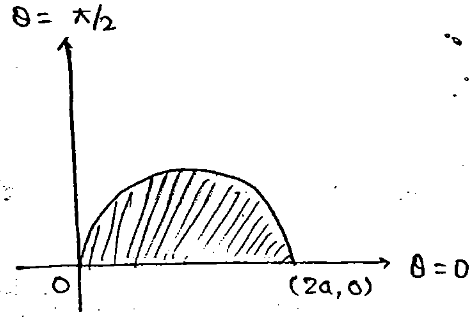
Consider the vertical strip,

$y = x^2$ to $y = 2-x$

$x = 0$ to 1

$$\therefore \iint_R y\, dx\, dy = \int_{x=0}^{1}\int_{y=x^2}^{2-x} y\, dy\, dx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{2-x} dx = \frac{16}{15}.$$

4. The value of $\iint_R r^2 \sin\theta \, dr \, d\theta$, where, R is the region, bounded by the semicircle $r = 2a\cos\theta$ above the initial line is _____.



Solⁿ:→ $r = 0$ to $2a\cos\theta$

$\theta = 0$ to $\pi/2$

$$\therefore \iint_R r^2 \sin\theta \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin\theta \int_{r=0}^{2a\cos\theta} r^2 \, dr \, d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

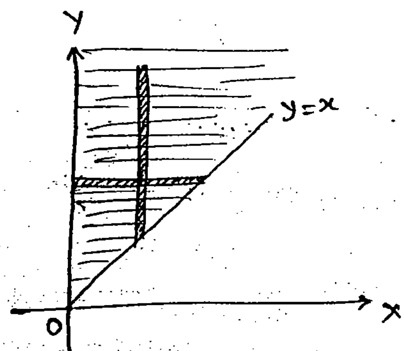
$$= \int_0^{\pi/2} \frac{8a^3\cos^3\theta \sin\theta}{3} \, d\theta$$

$$= -\frac{8a^3}{3}\left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2} = \frac{2a^3}{3}.$$

Change of order of Integration:→

Questions:→

1. The value of $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$ is _____.



Solⁿ:→ Given limits are $y = x$ to $y = \infty$

$x = 0$ to $x = \infty$

Horizontal Strip:–

$x = 0$, $x = y$

$y = 0$, $y = \infty$

$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx = \int_{y=0}^{\infty} \left[\int_{x=0}^{y} \frac{e^{-y}}{y} \, dx\right] dy = \int_0^\infty \left[\frac{e^{-y} x}{y}\right]_0^y dy = \int_0^\infty e^{-y} \, dy$$

$$= 1.$$

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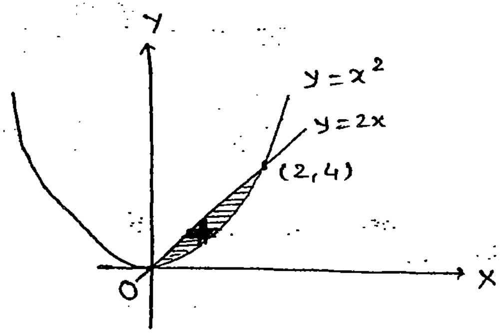
2. By reversing the order of integration $\int_0^2 \int_{x^2}^{2x} f(x,y)\,dy\,dx$ may be represented as,

(a) $\int_0^2 \int_{x^2}^{2x} f(x,y)\,dy\,dx$ (b) $\int_0^2 \int_y^{\sqrt{y}} f(x,y)\,dx\,dy$ ✓(c) $\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$

(d) $\int_{x^2}^{2x} \int_0^2 f(x,y)\,dx\,dy$.



Solⁿ:→ Given limits are

$x=0$ to $x=2$

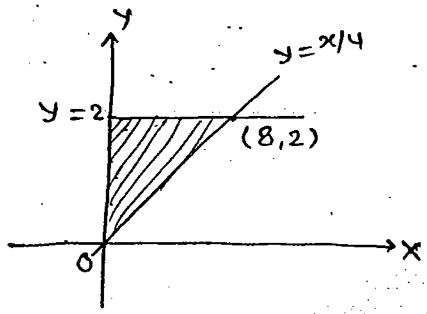
$y=x^2$ to $y=2x$

Horizontal Strip:-

$y=0$ to $y=4$

$x=\frac{y}{2}$ to $x=\sqrt{y}$.

3. By changing the order of integration $\int_0^8 \int_{x/4}^2 f(x,y)\,dy\,dx$ leads to $\int_r^s \int_p^q f(x,y)\,dx\,dy$ then q is ______.



Solⁿ:→ Given limits are

$y=\frac{x}{4}$ to $y=2$

$x=0$ to $x=8$

By changing limits,

$y=0$ to $y=2$

$x=0$ to $x=4y$

$\therefore\ q=4y$.

<u>Triple Integrals</u> :→ Let $f(x,y,z)$ be defined at each point in the region 'R' of space then its triple integral is $\iiint_R f(x,y,z)\,dx\,dy\,dz$.

Let $z = \phi_1(x,y)$ to $z = \phi_2(x,y)$

$y = \psi_1(x)$ to $y = \psi_2(x)$

$x = c_1$ to $x = c_2$, then

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_{x=c_1}^{c_2}\int_{y=\psi_1(x)}^{\psi_2(x)}\int_{z=\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

<u>Questions</u> :→

1. Evaluate $\displaystyle\int_0^2\int_0^x\int_0^{\sqrt{x+y}} z\,dz\,dy\,dx$.

<u>Solⁿ</u> :→ $$I = \int_0^2\int_0^x\left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}} dy\,dx = \int_0^2\left[\frac{(x+y)^2}{4}\right]_0^x dx = 2.$$

2. The value of $\iiint_R y\,dx\,dy\,dz$; where, R is the region bounded by the plane $x=0$, $y=0$, $z=0$ & $x+y+z=1$ is ——

<u>Solⁿ</u> :→ $$\int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dz\,dy\,dx = \int_0^1\int_0^{1-x} y(1-x-y)\,dy\,dx$$

$$\cancel{= \int_0^1\left[y - xy - \frac{y^2}{2}\right]_0^{1-x} dx}$$

$$\cancel{= \int_0^1\left[1-x - x + x^2 - \frac{(1-x)^2}{2}\right] dx}$$

$$= \int_0^1\left[(1-x)\frac{y^2}{2} - \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1\left[\frac{(1-x)^3}{2} - \frac{(1-x)^3}{3}\right] dx = \left.\frac{(1-x)^4}{-24}\right|_0^1 = \frac{1}{24}$$

Change of variables :→ In a double integral, if $x = f(u,v)$ & $y = g(u,v)$

then,
$$\iint_R \phi(x,y) = \iint_R \phi(f,g)\ |J|\ du\ dv = \iint_R \phi(u,v)\ |J|\ du\ dv$$

where, $|J| \to$ Jacobian of transformation used to transform one system to another.

$$|J| = J\left(\frac{x,y}{u,v}\right) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Cartesian form $\longrightarrow$ Polar form :→
(x,y) $\qquad$ (r,θ)

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\Rightarrow \iint_R \phi(x,y)\ dx\ dy = \iint_R \psi(r,\theta)\ r\ dr\ d\theta$$

In a triple integral, if $x = f(u,v,w)$; $y = g(u,v,w)$ & $z = h(u,v,w)$

then,
$$\iiint_R f(x,y,z)\ dx\ dy\ dz = \iiint_R \psi(u,v,w)\ |J|\ du\ dv\ dw$$

where, $$|J| = J\left(\frac{x,y,z}{u,v,w}\right) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Cartesian to Cylindrical Polar Form :$\rightarrow$

(x,y,z) $\qquad$ (r,θ,z)

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$|J| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\therefore \iiint_R \phi(x,y,z)\, dx\, dy\, dz = \iiint_R \psi(r,\theta,z)\, r\, dr\, d\theta\, dz$$

Cartesian to Spherical Polar Form:$\rightarrow$

(x,y,z) $\qquad$ (r,θ,ϕ)

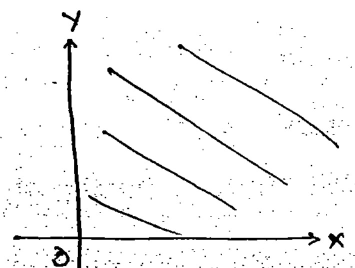
$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$|J| = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

$$\therefore \iiint_R \phi(x,y,z)\, dx\, dy\, dz = \iiint_R \psi(r,\theta,\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

Questions:$\rightarrow$

1. Evaluate $\displaystyle\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$.



$r \rightarrow 0$ to ∞

$\theta \rightarrow 0$ to $\frac{\pi}{2}$

Sol^n:$\rightarrow$ Let $x = r\cos\theta$, $y = r\sin\theta$, then $|J| = r$

& $x^2+y^2 = r^2$

$$\therefore \int_0^\infty \int_0^{\pi/2} e^{-r^2} . r\, dr\, d\theta$$

Let $r^2 = t \Rightarrow r\,dr = \frac{dt}{2}$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\infty} e^{-t}\,\frac{dt}{2}\,d\theta = \int_0^{\pi/2}\left[\frac{e^{-t}}{2}\right]_0^{\infty} d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}.$$

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2. Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$.

Solⁿ:→ $z=0$ to $z=\sqrt{1-x^2-y^2} \Rightarrow z^2 = 1-x^2-y^2 \Rightarrow x^2+y^2+z^2=1$

Using spherical coordinates,

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$x^2+y^2+z^2 = r^2$

$\Rightarrow r \to 0$ to 1

$\theta \to 0$ to $\pi/2$

$\phi \to 0$ to $\pi/2$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{1-r^2}} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \sin\theta\left[\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2}\right] dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left[\sin^{-1}r - \left(\frac{r\sqrt{1-r^2}}{2} + \frac{1}{2}\sin^{-1}(r)\right)\right]_0^1 d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left[\frac{\pi}{4}\right] d\theta\,d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2}\left[-\cos\theta\right]_0^{\pi/2} d\phi = \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

3. By a change of variable $x(u,v) = uv$ & $y(u,v) = \frac{v}{u}$ in double integr the integrand $f(x,y)$ changes to $f(uv, \frac{v}{u})\,\phi(u,v)$ then $\phi(u,v)$ is ___

(a) $2\frac{v}{u}$ ✓ (b) $2uv$ (c) v^2 (d) $=1$

Solⁿ:→ $\phi(u,v) = |J| = \begin{vmatrix} v & u \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{v}{u} + \frac{v}{u} = \frac{2v}{u}.$

Lengths, Areas & Volumes:→

(i) Length of an arc of a curve $y = f(x)$ between the lines $x = x_1$ & $x = x_2$

is, $$l = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

(ii) length of an arc of a curve $x = f(t)$ & $y = g(t)$ between $t = t_1$ to $t = t_2$

is, $$l = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

(iii) Area of the region bounded by the curve $y = f(x)$ & $y = g(x)$ between $x = x_1$ & $x = x_2$ is,

$$A = \int_{x_1}^{x_2} [g(x) - f(x)]\, dx \quad (\text{or}) \quad \int_{x_1}^{x_2} \int_{f(x)}^{g(x)} dy\, dx$$

(iv) The volume of solid generated by revolving $y = f(x)$ between $x = x_1$ & $x = x_2$ about x-axis is,

$$V = \int_{x_1}^{x_2} \pi y^2\, dx$$

about y-axis is, $$V = \int_{y_1}^{y_2} \pi x^2\, dy$$

In polar form:— (i) about initial line (i.e. $\theta = 0^\circ$)

$$V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin\theta\, d\theta$$

(ii) about line $\theta = \frac{\pi}{2}$

$$V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \cos\theta\, d\theta$$

Questions:→

1. The length of the curve $y = \frac{2}{3}x^{3/2}$ between $x=0$ & 1 is ____ .

(a) 0.27 (b) 0.67 (c) 1 (d) 1.22 ✓

Sol^n:→ $\frac{dy}{dx} = \frac{2}{3}\cdot\frac{3}{2}x^{\frac{3}{2}-1} = x^{1/2}$

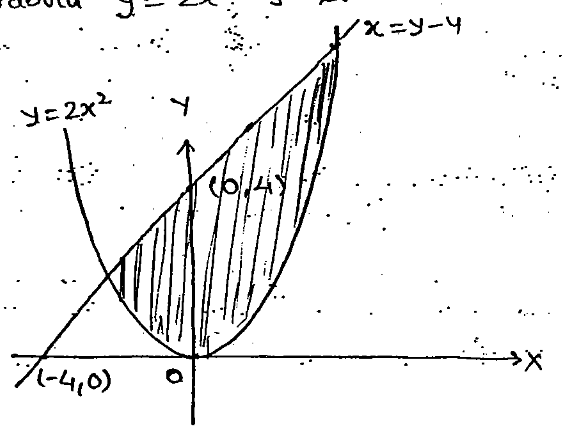
$$\therefore\ l = \int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1\sqrt{1+x}\,dx = \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^1 = 1.22$$

2. The length of the curve, $x = \cos^3\theta$, $y = \sin^3\theta$ between $\theta = 0$ & $\pi/2$ is –

Sol^n:→ $$l = \int_0^{\pi/2}\sqrt{(3\cos^2\theta(-\sin\theta))^2 + (3\sin^2\theta\cos\theta)^2}\,d\theta$$

$$= \int_0^{\pi/2}(3\sin\theta\cos\theta)\sqrt{\sin^2\theta+\cos^2\theta}\,d\theta = \frac{3}{2}.$$

3. The area bounded by the ~~pra~~ parabola $y = 2x^2$ & st. line $x = y-4$ is ____ .



Sol^n:→ Pt. of intersection:–

$y = 2(y-4)^2$

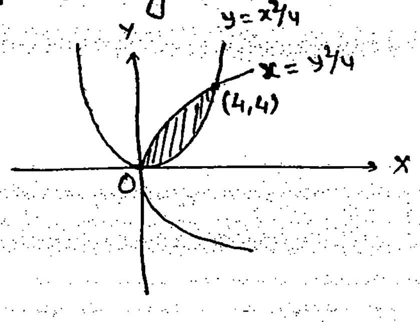
$\Rightarrow y = 2(y^2+16-8y)$

$\Rightarrow 2y^2 - 17y + 32 = 0$

$\Rightarrow y = \frac{17\pm\sqrt{289-256}}{4} = \frac{17\pm\sqrt{33}}{4}$

$\cancel{A = \int_{y=\frac{17-\sqrt{33}}{4}}^{\frac{17+\sqrt{33}}{4}}[(x+4)-2x^2]}$ $\therefore\ A = \int_{\frac{17-\sqrt{33}}{4}}^{\frac{17+\sqrt{33}}{4}}\left[(y-4)-\sqrt{\frac{y}{2}}\right]dy$

4. The area between the curves $y^2 = 4x$ & $x^2 = 4y$ is, ____ .



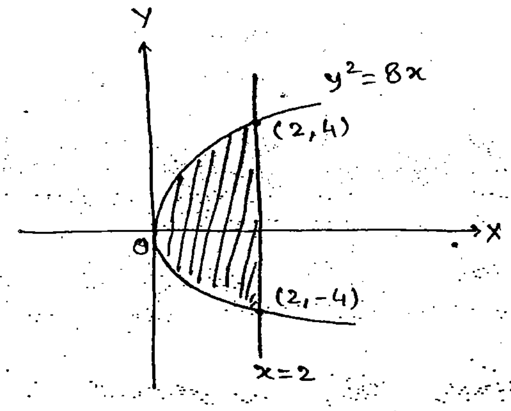
Sol^n:→ $A = \int_0^4\left(2\sqrt{x} - \frac{x^2}{4}\right)dx = \frac{16}{3}.$

5. The volume generated by revolving the area bounded by parabola $y^2 = 8x$ and st. line $x = 2$ about y-axis is ______.

(a) $128\frac{\pi}{5}$ (b) $\frac{5\pi}{128}$ (c) $\frac{127\pi}{5}$ (d) none

Sol^n: →

$$V = \int_{y_1}^{y_2} \pi x^2 \, dy$$



$$= \int_{-4}^{4} \pi \frac{y^4}{64} \, dy$$

$$= \frac{\pi}{64} \left[\frac{y^5}{5}\right]_{-4}^{4} = \frac{32\pi}{5}$$

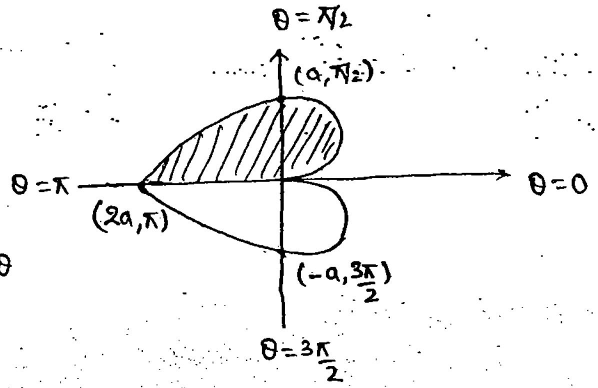
Total volume $= \int_{-4}^{4} \pi x^2 \, dy = \int_{-4}^{4} \pi (2)^2 dy = 4\pi [y]_{-4}^{4} = 32\pi$

∴ Remaining volume $= 32\pi - \frac{32\pi}{5} = \frac{128\pi}{5}$.

6. The volume of solid generated by revolving the cardiac $r = a(1-\cos\theta)$ about the initial line is ______.

Sol^n: →

$$V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin\theta \, d\theta$$



$$= \int_{0}^{\pi} \frac{2\pi}{3} a^3 (1-\cos\theta)^3 \sin\theta \, d\theta$$

Let $1 - \cos\theta = t \Rightarrow \sin\theta \, d\theta = dt$

as $\theta = 0$, $t = 0$

$\theta = \pi$, $t = 2$

$$\therefore V = \int_{0}^{2} \frac{2\pi}{3} a^3 t^3 \, dt = \frac{2\pi a^3}{3} \left[\frac{t^4}{4}\right]_{0}^{2} = \frac{8\pi a^3}{3}.$$

Vector Calculus: →

Position Vector: → The position vector of the point $P(x,y,z)$ in the space is

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

In parametric form,

$$\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

(i) $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos(\vec{a}\cdot\vec{b}) = a_1b_1 + a_2b_2 + a_3b_3$

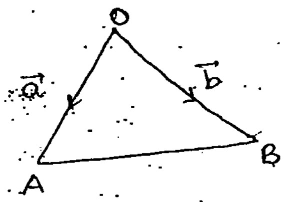
(ii) $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin(\vec{a}\cdot\vec{b})\,\hat{n}$; where $\vec{n}$ is vector of unit length perpendicular to the plane contains $\vec{a}$ & $\vec{b}$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note: → (i) Area of ΔOAB

$= \frac{1}{2}|\vec{OA}\times\vec{OB}|$

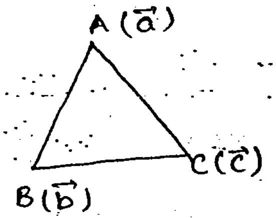
$= \frac{1}{2}|\vec{a}\times\vec{b}|$



(ii) Area of ΔABC

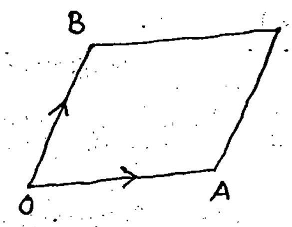
$= \frac{1}{2}|\vec{AB}\times\vec{AC}|$

$= \frac{1}{2}|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})|$



(iii) Area of parallelogram

$= |\vec{a}\times\vec{b}|$



Scalar Triple Products: →

$(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}]$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note:$\rightarrow$ (i) Volume of parallelopiped, $V = |[\vec{a}\ \vec{b}\ \vec{c}]|$

(ii) Volume of tetrahadron, $V = \frac{1}{6}|[\vec{a}\ \vec{b}\ \vec{c}]|$

where, $\vec{a}, \vec{b}$ & $\vec{c}$ are edge vectors of ~~parallelopiett~~ parallelopipe.

Vector Triple Product :$\rightarrow$

$$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{c}\cdot\vec{a})\vec{b} - (\vec{b}\cdot\vec{a})\vec{c}$$

6. Vector Differentiation :$\rightarrow$ Let $\vec{r}(t) = \vec{f}(t)$ be the vector can of the curve

than $\frac{d\vec{r}}{dt} = \lim\limits_{\Delta t\to 0} \frac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t}$.

If t is a time variable then $\frac{d\vec{r}}{dt}$ represents a velocity vector.

Note:$\rightarrow$ (i) $\frac{d\vec{r}}{dt}$ is a vector in the direction of tangent to the curve

at that point.

(ii) $\vec{F}(t)$ is constant in magnitude then, $\vec{F}\cdot\frac{d\vec{F}}{dt} = 0$

(iii) $\vec{F}(t)$ vector with fixed direction then, $\vec{F}\times\frac{d\vec{F}}{dt} = 0$

Properties:$\rightarrow$ Let $\vec{a}(t)$ & $\vec{b}(t)$ be the vector funⁿ of the scalar variable 't'

and ϕ be a scalar funⁿ then,

(i) $\frac{d}{dt}(\vec{a}\pm\vec{b}) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$

(ii) $\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \frac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\frac{d\vec{b}}{dt}$

(iii) $\frac{d}{dt}(\vec{a}\times\vec{b}) = \left(\frac{d\vec{a}}{dt}\times\vec{b}\right) + \left(\vec{a}\times\frac{d\vec{b}}{dt}\right)$

(iv) $\frac{d}{dt}(\phi\vec{a}) = \frac{d\phi}{dt}\vec{a} + \phi\frac{d\vec{a}}{dt}$

Point Function:→ If the value of function depends on position of point, then it is said to be point funⁿ.

Scalar Point Function:→ If to each point $P(x,y,z)$ in region R of space ∃ a unit scalar associated with it, then, $\phi(x,y,z)$ is a scalar point function.

The set of all points in the region R of space together with the scalar values forms a scalar field.

Eg:→ The temp. $T(x,y,z)$ at any point on a body is a scalar point function and the medium it self is a scalar field.

Vector Point Function:→ The velocity at any time 't' of a particle in a fluid flow is a vector point funⁿ.

Vector Differential Operator:→ (del) $\vec{\nabla}$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Level Surface:→ Let $\phi(x,y,z)$ be a scalar field in the region 'R', then the set of points satisfying $\phi(x,y,z) = c$, where, 'c' is an arbitrary const, constitutes a family of surfaces called level surfaces.

Gradient of a Scalar Function:→ Let $\phi(x,y,z)$ be a differentiable scalar pt. funⁿ then gradient of scalar is denoted by grad ϕ

(or) $$\vec{\nabla}\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

↳ vector normal to surface ϕ

$\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$ → unit vector normal to surface ϕ.

~~(i) $\vec{\nabla}$, if $\vec{r} = x\hat{i}$~~

Questions:→

1. $\vec{\nabla} r$ is ______. if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solⁿ:→ ∵ $r = \sqrt{x^2+y^2+z^2} \Rightarrow \vec{\nabla} r = \frac{\vec{r}}{|\vec{r}|}$

Note:→ $\vec{\nabla} f(r) = f'(r) \frac{\vec{r}}{r}$

Eg:- $\vec{\nabla}(\log r) = \frac{1}{r} \cdot \frac{\vec{r}}{r} = \frac{\vec{r}}{r^2}$

$\vec{\nabla}(\sin \log r) = \frac{\cos \log r}{r} \cdot \frac{\vec{r}}{r} = \cos\log r \cdot \frac{\vec{r}}{r^2}$

2. The unit vector normal to the surface $y^2 = 8x$ at $(1, 2)$ is ______.

Solⁿ:→ Let $\phi = y^2 - 8x$

$\vec{\nabla}\phi = -8\hat{i} + 2y\hat{j}$

$\vec{\nabla}\phi\big|_{(1,2)} = -8\hat{i} + 4\hat{j}$

$\therefore \hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{-8\hat{i}+4\hat{j}}{\sqrt{64+16}} = \frac{-8\hat{i}+4\hat{j}}{\sqrt{80}}$.

3. A sphere of unit radius is centred at origin the unit normal at a point $P(x,y,z)$ to the surface of the sphere is the vector

(a) $\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$ (b) $\left(\frac{x}{\sqrt3}, \frac{y}{\sqrt3}, \frac{z}{\sqrt3}\right)$ (c) ✓ (x, y, z) (d) $\left(\frac{x}{\sqrt2}, \frac{y}{\sqrt2}, \frac{z}{\sqrt2}\right)$

Solⁿ:→ $x^2+y^2+z^2=1$

Let $\phi = x^2+y^2+z^2$

$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\therefore \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\hat{i}+y\hat{j}+z\hat{k}$.

Directional Derivative:→ The directional derivative of a differentiable scale fun $\phi(x,y,z)$ in the direction of $\vec{a}$ is given by,

$$D.D. = \vec{\nabla}\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

Let, $\vec{a} = \hat{i}$, then,

$$D.D. = \vec{\nabla}\phi \cdot \frac{\hat{i}}{|\hat{i}|} = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot\hat{i}$$

$$= \frac{\partial\phi}{\partial x}.$$

Let, $\frac{\vec{a}}{|\vec{a}|} = \hat{b}$ then, $D.D. = \vec{\nabla}\phi\cdot\hat{b} = |\vec{\nabla}\phi||\hat{b}|\cos\theta = |\vec{\nabla}\phi|\cos\theta$

The max. value of $\cos\theta = 1$ i.e. when $\theta = 0$ i.e. when $\hat{b}$ is paral to $\vec{\nabla}\phi$. Therefore, the value of the directional derivative is maximum in the direction of normal to the surface ϕ and the maximum value of directional derivative is $|\vec{\nabla}\phi|$.

Angle Between Surfaces:→ Angle between the normal to the surfaces at the pt. of intersection is taken as the angle between the surface. Let θ be the angle b/w the surfaces $\phi_1(x,y,z) = C_1$ & $\phi_2(x,y,z) = C_2$ th

$$\cos\theta = \frac{\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2}{|\vec{\nabla}\phi_1|\,|\vec{\nabla}\phi_2|}$$

Questions:→

1. The directional derivative of $f(x,y) = x^2 - y^2$ at $(1,2)$ in the direction of $4\hat{i}+3\hat{j}$ is

(a) $\frac{4}{5}$ (b) $\frac{3}{5}$ (c) ✓ $-\frac{4}{5}$ (d) $-\frac{3}{5}$

Soln:→ $D.D. = \vec{\nabla}f \cdot \frac{\vec{a}}{|\vec{a}|} = (2x\hat{i} - 2y\hat{j})\frac{(4\hat{i}+3\hat{j})}{5} = \frac{8x - 6y}{5}\Big|_{(1,2)} = \frac{8-12}{5} = -$

2. The directional derivative $\phi = x^2 - y^2 + 2z^2$ at $P(1,2,3)$ in direction $\overrightarrow{PQ}$ where $Q = (5,0,4)$ ______.

Solⁿ:→ $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (5,0,4) - (1,2,3) = (4,-2,1)$

$$\therefore \ D.D. = \vec{\nabla}\phi \cdot \frac{|\overrightarrow{PQ}|}{|\overrightarrow{PQ}|} = (2x\hat{i} - 2y\hat{j} + 4z\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{16+4+1}}$$

$$= \left.\frac{8x + 4y + 4z}{\sqrt{21}}\right|_{(1,2,3)} = \frac{28}{\sqrt{21}}.$$

3. The directional derivative $f = xy^2 + yz^2 + zx^2$ along to the tangent to the curve $x = t$, $y = t^2$ & $z = t^3$ at $(1,1,1)$ is ______.

Solⁿ:→ Vector eqⁿ of the curve is given by

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

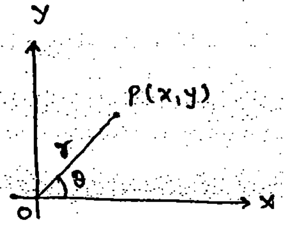
$$= t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{(1,1,1)} = \left.\hat{i} + 2t\hat{j} + 3t^3\hat{k}\right|_{t=1} = \hat{i} + 2\hat{j} + 3\hat{k} = \vec{a}$$

$$\therefore \ D.D. = \vec{\nabla}f \cdot \frac{\vec{a}}{|\vec{a}|} = \left[(y^2 + 2zx)\hat{i} + (2xy + z^2)\hat{j} + (2yz + x^2)\hat{k}\right] \cdot \frac{(\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{1+4+9}}$$

$$= \frac{18}{\sqrt{14}}.$$

4. The directional derivative of $\phi = \frac{y}{x^2+y^2}$ at $(0,1)$ along the line which makes angle 30° with positive x-axis is ______.



Solⁿ:→ $\vec{r} = x\hat{i} + y\hat{j}$

$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$

$\Rightarrow \ \frac{\vec{r}}{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

<u>Note</u>:→ The vector eqⁿ of a st. line which makes an angle θ with +ve x-axis is $\cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\therefore \frac{\vec{r}}{r} = \cos 30^\circ\, \hat{i} + \sin 30^\circ\, \hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

$\therefore$ D.D. $= \vec{\nabla}\phi \cdot \frac{\vec{r}}{r} = -\frac{1}{2}$.

5. The greatest rate of increase of $\phi = x^2yz$ at $(2,-1,2)$ is ______.

Solⁿ:→ $\vec{\nabla}\phi = 2xyz\,\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k}$

$\therefore \vec{\nabla}\phi\big|_{(2,-1,2)} = -8\hat{i} + 8\hat{j} - 4\hat{k}$

Greatest rate of increase (or) max. value of D.D. $= |\vec{\nabla}\phi|$

$= \sqrt{64+64+16}$

$= \sqrt{144} = 12$

6. The angle b/w the surfaces $x^2+y^2+z^2=9$ & $x^2+y^2-z=3$ at $(2,-1,2)$ is ______.

Solⁿ:→ Let, $\phi_1 = x^2+y^2+z^2 \Rightarrow \vec{\nabla}\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\Rightarrow \vec{\nabla}\phi_1\big|_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$

Let, $\phi_2 = x^2+y^2-z \Rightarrow \vec{\nabla}\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$\Rightarrow \vec{\nabla}\phi_2\big|_{(2,-1,2)} = 4\hat{i} - 2\hat{j} - \hat{k}$

$$\cos\theta = \frac{\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2}{|\vec{\nabla}\phi_1|\,|\vec{\nabla}\phi_2|} = \frac{16+4-4}{\sqrt{16+4+16}\;\sqrt{16+4+1}} = \frac{16}{\sqrt{36}\,\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$\Rightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$.

7. If $\vec{a}$ & $\vec{b}$ are two arbitrary vectors with magnitudes a & b respectively then, $|\vec{a}\times\vec{b}|^2 =$ ______.

Solⁿ:→ $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$

$|\vec{a}\times\vec{b}|^2 = a^2b^2\sin^2\theta = a^2b^2(1-\cos^2\theta) = a^2b^2\left(1 - \frac{(\vec{a}\cdot\vec{b})^2}{a^2b^2}\right)$

$= a^2b^2 - (\vec{a}\cdot\vec{b})^2$

Divergence of a Vector Function: $\longrightarrow$ Let $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be the differential vector point function then

$$\text{div } \vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Note: $\rightarrow$ If $\vec{\nabla}\cdot\vec{F} = 0$ then $\vec{F}$ is called solenoidal vector.

Curl of a Vector Function: $\rightarrow$

$$\text{curl } \vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note: $\rightarrow$ (i) If $\vec{\nabla}\times\vec{F} = \vec{0}$ then $\vec{F}$ is called Irrotational vector.

(ii) If $\vec{v} \rightarrow$ velocity vector

$\vec{\omega} \rightarrow$ angular velocity

then, $\vec{v} = \vec{\omega}\times\vec{r}$

$$\text{curl } \vec{v} = \vec{\nabla}\times(\vec{\omega}\times\vec{r}) = 2\vec{\omega}$$

$$\Rightarrow \boxed{\vec{\omega} = \frac{1}{2}\text{curl } \vec{v}}$$

Scalar Potential function: $\longrightarrow$ For every rotational vector, $\exists$ a scalar function ϕ s.t. $\vec{F} = \vec{\nabla}\phi$, then ϕ is said to be scalar potential fun.

Note: $\rightarrow$ (i) $\text{curl}(\text{grad } \phi) = \vec{0}$

(ii) $\text{div}(\text{curl } \vec{F}) = 0$

(iii) $\text{div}(\text{grad } \phi) = \nabla(\nabla\phi) = \nabla^2\phi$

where, $\nabla^2 \rightarrow$ Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Questions:→

1. The value of $\vec{\nabla}\cdot(r^n \vec{r}) =$ ______, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

✓(a) $(n+3)r^n$ (b) $(n-2)r^n$ (c) nr^{n-3} (d) $(n+2)r^{n-1}$

and hence which of the following is solenoidal

(a) $r^3\vec{r}$ (b) $r\vec{r}$ ✓(c) $\frac{\vec{r}}{r^3}$ (d) $\frac{\vec{r}}{r^2}$

Solⁿ:→ $r^n\vec{r} = \underbrace{r^n x}_{f_1}\hat{i} + \underbrace{r^n y}_{f_2}\hat{j} + \underbrace{r^n z}_{f_3}\hat{k}$

$$\vec{\nabla}\cdot(r^n\vec{r}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}(r^n x) = r^n + x\, n\, r^{n-1}\frac{\partial r}{\partial x}$$

$$= r^n + n x r^{n-1}\frac{x}{r}$$

$$= r^n + n r^{n-2}x^2$$

Similarly, $\frac{\partial f_2}{\partial y} = r^n + n r^{n-2}y^2$, $\frac{\partial f_3}{\partial z} = r^n + n r^{n-2}z^2$

$$\therefore \vec{\nabla}\cdot(r^n\vec{r}) = 3r^n + nr^{n-2}(x^2+y^2+z^2) = 3r^n + nr^{n-2}\cdot r^2$$

$$= 3r^n + nr^n = (n+3)r^n$$

Now, for solenoidal $\vec{\nabla}\cdot(r^n\vec{r}) = 0$

$$\Rightarrow (n+3)r^n = 0 \Rightarrow n = -3$$

$$\therefore \vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = 0.$$

2. If $\vec{F} = (3x^2+2y)\hat{i} - 4xz\hat{j} + 3xy^2\hat{k}$ represents a velocity vector then corresponding angular velocity at $(2,2,-1)$ is ______.

Solⁿ:→ $\text{curl}\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2+2y & -4xz & 3xy^2 \end{vmatrix}_{(2,2,-1)} = 32\hat{i} - 12\hat{j} + 2\hat{k}$

$$\therefore \vec{\omega} = \frac{1}{2}\,\text{curl}\ \vec{F} = 16\hat{i} - 6\hat{j} + \hat{k}$$

3. If $\phi(x,y) = ax^2y - y^3$ & $\nabla^2\phi = 0$ then, $a =$ ______.

(a) 2 (b) 3 (c) -2 (d) -3

Solⁿ:→ $\nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2} = 0$

$\Rightarrow \quad 2ay - 6y = 0 \Rightarrow a = 3.$

4. If $\vec{F} = (5x+7z^2)\hat{i} + (4x^2+\lambda y)\hat{j} + (7z-2xy)\hat{k}$ is solenoidal then $\lambda =$ ___.

Solⁿ:→ $\vec{\nabla}\cdot\vec{F} = 0$

$\Rightarrow \quad 5+\lambda+7 = 0 \Rightarrow \lambda = -12.$

5. If $\vec{F} = 5x^2z\,\hat{i} - 7xy^2\,\hat{j} + (12x+7z)\hat{k}$ then $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})$ at $(5,3,-2)$ is $\underline{0}$.

6. If $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational vector funⁿ then $a, b, c =$ ______.

Solⁿ:→ $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$

$= \hat{i}(c+1) + \hat{j}(4-a) + \hat{k}(b-2)$

$= 0$

$\therefore \; c = -1, \; a = 4, \; b = 2.$

7. <u>Vector Integration</u>:→

<u>Line Integral</u>:→ An integral evaluated over a curve is called line integral.

Let, $\vec{F}(x,y,z) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a differentiable point ~~vector~~ function defined at each point on the curve 'c' then its line integral is

$$\int_c \vec{F}\cdot d\vec{r} = \int_c (f_1\,dx + f_2\,dy + f_3\,dz)$$

Note:→ If c is a closed curve then line integral of $\vec{F}$ over 'c' is called circulation of $\vec{F}$ i.e. $\oint_c \vec{F}\cdot d\vec{r}$.

<u>Work done by a force:</u>→ The total work done by a force $\vec{F}$ in moving a particle along a curve 'c' is $\int_c \vec{F}\cdot d\vec{r}$

Note:→ If $\vec{F}$ is irrotational, then, the line integral of $\vec{F}$ is independent of the path.

i.e. when $\vec{F}$ is irrotational we have $\vec{F} = \vec{\nabla}\phi$, where ϕ is a scalar potential fun then,

$$\int_a^b \vec{F}\cdot d\vec{r} = \phi_b - \phi_a .$$

<u>Questions:</u>→

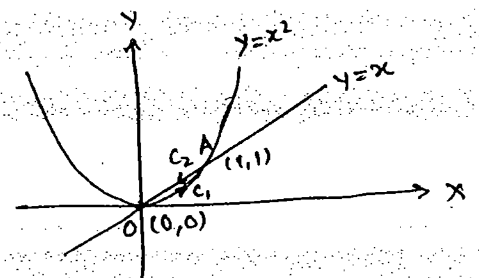
1. The value of $\int_c \vec{F}\cdot d\vec{r}$, where, $\vec{F} = 3xy\,\hat{i} - y^2\hat{j}$ & 'c' is the curve $y = 2$… joining pts $(0,0)$ & $(1,2)$ is ______.

<u>Soln</u>:→ $\int_c \vec{F}\cdot d\vec{r} = \int_c (F_1\,dx + F_2\,dy + F_3\,dz) = \int_c (3xy\,dx - y^2 dy)$

$$= \int_0^1 [3x(2x^2)\,dx - 4x^4 . 4x\,dx] \qquad \left\{ \begin{aligned} &\Rightarrow y = 2x^2 \\ &\Rightarrow dy = 4x\,dx \end{aligned} \right\}$$

$$= \frac{6}{4} - \frac{16}{6} =$$

2. The value of $\int_c \vec{F}\cdot d\vec{r}$, where, $\vec{F} = 3xy\,\hat{i} - y^2\hat{j}$ & 'c' is the curve bounded by $y = x$ & $y = x^2$ is ______.



<u>Soln</u>:→ $\int_c \vec{F}\cdot d\vec{r} = \int_{C_1} (\ ) + \int_{C_2} (\ )$

along C_1, $y = x^2 \Rightarrow dy = 2x\,dx$

$\therefore \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} (3xy\,dx - y^2 dy)$

$$= \int_0^1 [3x(x^2)\,dx - x^4 \cdot 2x\,dx] = \frac{5}{12}$$

Now, along C_2. $y = x \Rightarrow dy = dx$

$$\therefore \int_{C_2} \bar{F}\cdot d\bar{x} = \int_1^0 [3x(x)\,dx - x^2\,dx] = -\frac{2}{3}$$

$$\therefore \int \bar{F}\cdot d\bar{x} = \frac{5}{12} - \frac{2}{3} = -\frac{3}{12} = -\frac{1}{4}.$$

3. The value of $\int_C [(3x+4y)\,dx + (2x-3y)\,dy]$ where 'C' is the circle of radius 2, with centre at origin in X-Y plane is ___.

(a) 4π (b) 8π (c) -4π ✓(d) -8π.

Solⁿ:→ Here, $C \rightarrow x^2+y^2 = 2^2$

whenever the curve is circle we will go for polar form.

let, $x = 2\cos\theta$; $y = 2\sin\theta$

$\Rightarrow dx = -2\sin\theta\,d\theta$, $dy = 2\cos\theta\,d\theta$

$$\therefore \int_0^{2\pi} [(3\times 2\cos\theta + 4\times 2\sin\theta)(-2\sin\theta\,d\theta) + (2\times 2\cos\theta - 3\times 2\sin\theta)(2\cos\theta\,d\theta)]$$

$$= \int_0^{2\pi} [-12\sin\theta\cos\theta\,d\theta - 16\sin^2\theta\,d\theta + 8\cos^2\theta\,d\theta - 12\sin\theta\cos\theta\,d\theta]$$

$$= \int_0^{2\pi} [-24\sin\theta\cos\theta\,d\theta -$$

4. The value of $\int_C \bar{F} \cdot d\bar{r}$, where, $\bar{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$ along the line joining

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(i) $(0,0,1)$ & $(0,1,1)$

(ii) $(0,1,1)$ & $(2,1,1)$

is ________.

Sol^n:→ (i) Along that st. line $x=0$, $z=1$ ⇒ $dx=0$, $dz=0$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C F_2\, dy = \int_0^1 xz\, dy = 0.$$

(ii) $y=1, z=1$ ⇒ $dy=0$, $dz=0$

$$\int_C \vec{F}\cdot d\bar{r} = \int_C F_1\, dx = \int_0^2 (2y+3)\, dx = \int_0^2 5\, dx = 5x\Big|_0^2 = 10.$$

5. The total work done by a force $\bar{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$ in moving a particle along a st. line joining the pts. $(0,0,0)$ & $(1,2,$

is ________.

Sol^n:→ $$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \Rightarrow \frac{x-0}{1-0} = \frac{y-0}{2-0} = \frac{z-0}{3-0} = t$$

$$\Rightarrow x = t,\quad y = 2t,\quad z = 3t$$

$$dx = dt,\quad dy = 2dt,\quad dz = 3dt.$$

$$\int_C \vec{F}\cdot d\bar{r} = \int_C (F_1\, dx + F_2\, dy + F_3\, dz)$$

$$= \int_0^1 \left[(3t^2+12t)\, dt + (-14\times 2t\times 3t)(2\,dt) + (20\times t\times 9t^2)(3dt)\right]$$

$$= \frac{540}{4}.$$

6. The line integral $\int \vec{F} \cdot d\vec{r}$ of vector f^n $\vec{F} = 2xyz\,\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k}$ from the origin $(0,0,0)$ to the pt. $(1,1,1)$ is ——.

(a) 0 (b) 1 (c) -1

(d) cannot be determined without specifying the path.

Sol^n:→
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{vmatrix} = \hat{i}(x^2 - x^2) - \hat{j}(2xy - 2xy) + \hat{k}(2xz - 2xz)$$
$$= \vec{0}.$$

$\Rightarrow F$ is irrotational.

$$\Rightarrow \vec{F} = \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 2xyz, \quad \frac{\partial \phi}{\partial y} = x^2z, \quad \frac{\partial \phi}{\partial z} = x^2y$$

The total differentiation of ϕ,
$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$
$$= 2xyz\,dx + x^2z\,dy + x^2y\,dz$$
$$= d(x^2yz)$$
$$\Rightarrow \phi = x^2yz$$

$$\therefore \int_a^b \vec{F} \cdot d\vec{r} = \phi_b - \phi_a = \phi_{(1,1,1)} - \phi_{(0,0,0)} = 1.$$

<u>Green's Theorem in a Plane</u>:→ Let $M(x,y)$ & $N(x,y)$ be continuous f^n having continuous 1st order partial derivative defined in the closed region R bounded by the closed curve 'c' then,
$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Questions:→

1. Evaluate $\oint_c (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where, c is a curve bounde

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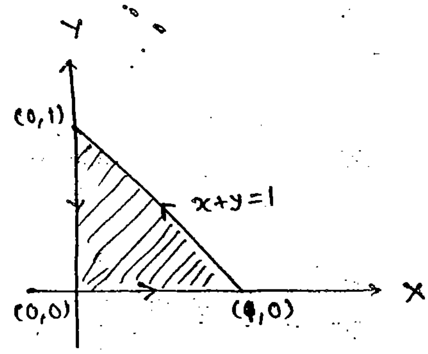
by $x=0,\ y=0$ & $x+y=1$.



Solⁿ:→ $M = 3x^2 - 8y^2$

$$\Rightarrow \frac{\partial M}{\partial y} = -16y$$

$$N = 4y - 6xy$$

$$\Rightarrow \frac{\partial N}{\partial x} = -6y$$

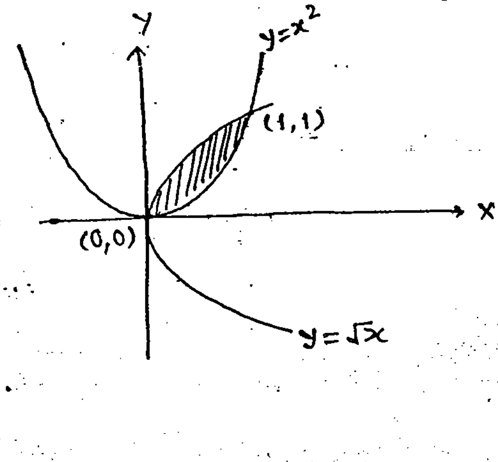
$$\therefore \frac{\partial M}{\partial x} - \frac{\partial M}{\partial y} = 10y$$

$$\therefore \oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1\int_0^{1-x} 10y\ dx\,dy = 5\int_0^1 (1-x)^2\,dx = \frac{5}{3}.$$

2. Evaluate $\oint_c (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where, c is a curve bounded by

$y = \sqrt{x}$ & $y = x^2$.



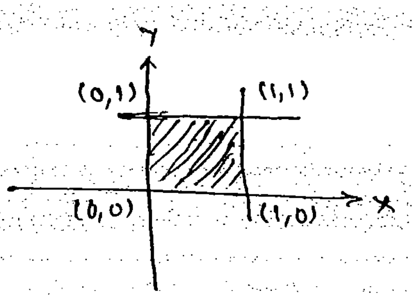
Solⁿ:→ $\cancel{M = (3x^2 - 8y^2)}$

$$\oint_c M\,dx + N\,dy = \int_0^1\int_{x^2}^{\sqrt{x}} 10y\ dx\,dy$$

$$= 5\int_0^1 (x - x^4)\,dx = \frac{3}{2}.$$

3. Evaluate $\oint_c xy\,dy - y^2\,dx$, where, c is a square cut from the first quadra

~~from~~ by the lines $x=1$ & $y=1$.



Solⁿ:→ $M = -y^2 \Rightarrow \frac{\partial M}{\partial y} = -2y$

$N = xy \Rightarrow \frac{\partial N}{\partial x} = y$

$$\therefore \quad \cancel{\frac{\partial M}{\partial x}} \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3y$$

$$\therefore \oint_C M\,dx + N\,dy = \int_0^1\int_0^1 3y\,dx\,dy = \frac{3}{2}.$$

<u>Surface Integral</u>:$\longrightarrow$ Let $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be differentiable vector point funⁿ defined over the surface S then, its surface integral is $\int_S \vec{F}\cdot d\vec{s} = \int_S \vec{F}\cdot\vec{n}\,ds$.

where, $\vec{n}$ – unit outward drawn normal to the surface.

In cartesian form,

$$\int_S \vec{F}\cdot d\vec{s} = \int_S \vec{F}\cdot\vec{n}\,ds = \int(F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy)$$

<u>Methods of Evaluation</u> :$\longrightarrow$

(i) If R_1 is the projection of 'S' on to x-y plane then,

$$\int_S \vec{F}\cdot\vec{n}\,ds = \iint_{R_1} \vec{F}\cdot\vec{n}\,\frac{dx\,dy}{|\vec{n}\cdot\hat{k}|}$$

(ii) If $R_2 \rightarrow$ y-z plane then,

$$\int_S \vec{F}\cdot\vec{n}\,ds = \iint_{R_2} \vec{F}\cdot\vec{n}\,\frac{dy\,dz}{|\vec{n}\cdot\hat{i}|}$$

(iii) If $R_3 \rightarrow$ x-z plane then,

$$\int_S \vec{F}\cdot\vec{n}\,ds = \iint_{R_3} \vec{F}\cdot\vec{n}\,\frac{dx\,dz}{|\vec{n}\cdot\hat{j}|}$$

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Questions:→

1. The value of $\int_S \vec{F}\cdot\vec{n}\, ds$, where, $\vec{F} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the 1st octent between $z=0$ & $z=5$.

Solⁿ:→ Let $\phi = x^2+y^2$

$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j}$$

$$\vec{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}}$$

$$\therefore \vec{n} = \frac{x\hat{i}+y\hat{j}}{4}$$

$$\vec{F}\cdot\vec{n} = \frac{xz}{4} + \frac{xy}{4} = \frac{x}{4}(y+z)$$

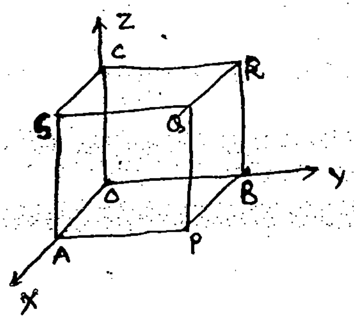
Let, $R \to$ y-z plane,

$$\int_S \vec{F}\cdot\vec{n}\, ds = \iint_R \vec{F}\cdot\vec{n}\, \frac{dy\,dz}{|\vec{n}\cdot\hat{i}|} = \iint_R \frac{x}{4}(y+z)\frac{dy\,dz}{(x/4)}$$

$$= \int_{z=0}^{5}\int_{y=0}^{4} (y+z)\, dy\, dz$$

$$= \int_0^5 \left(\frac{y^2}{2} + yz\right)_0^4 dz = 90.$$

2. The value of $\int_S \vec{F}\cdot\vec{n}\, ds$, where, $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ & S is surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ & $z=1$, is ______



Solⁿ:→ Method I:—

$$\int \vec{F}\cdot\vec{n}\, ds = \int_{S_1} + \int_{S_2} + \cdots + \int_{S_6}$$

Over S_1:- In X-Y plane (OAPR),

$$z=0,\ \vec{n} = -\hat{k}$$

$$\vec{F}\cdot\vec{n} = -yz = 0$$

$$\therefore \int_{S_1} \vec{F}\cdot\vec{n}\, ds = 0$$

Over S_2:- parallel to xy plane (SQRC)

$$z=1,\ \vec{n}=\hat{k}\ ,\quad \vec{F}\cdot\vec{n} = yz = y$$

$$\therefore \int_{S_2} \vec{F}\cdot\vec{n}\, ds = \iint_R \vec{F}\cdot\vec{n}\ \frac{dx\,dy}{|\vec{n}\cdot\hat{k}|} = \int_0^1\int_0^1 y\,dx\,dy = \frac{1}{2}.$$

Over S_3:- in y-z plane (OBRC)

$$x=0,\ \vec{n}=-\hat{i}\ ,\quad \vec{F}\cdot\vec{n} = -4xz = 0$$

$$\therefore \int_{S_3} \vec{F}\cdot\vec{n}\, ds = 0$$

Over S_4:- parallel to y-z plane (AP

$$x=1,\ \vec{n}=\hat{i}\ ,\quad \vec{F}\cdot\vec{n} = 4xz = 4z$$

$$\therefore \int_{S_4} \vec{F}\cdot\vec{n}\, ds = \iint_R \vec{F}\cdot\vec{n}\ \frac{dy\,dz}{|\vec{n}\cdot\hat{i}|} = \int_0^1\int_0^1 4z\,dy\,dz = 2$$

Over S_5:- In x-z plane (OCSA)

$$y=0,\ \vec{n}=-\hat{j}\ ,\quad \vec{F}\cdot\vec{n} = y^2 = 0$$

$$\therefore \int_{S_5} \vec{F}\cdot\vec{n}\, ds = 0$$

Over S_6:- parallel to x-z plane (BRQP)

$$y=1,\ \vec{n}=\hat{j}\ ,\quad \vec{F}\cdot\vec{n} = -y^2 = -1$$

$$\therefore \int_{S_6} \vec{F}\cdot\vec{n}\, ds = \iint_R \vec{F}\cdot\vec{n}\ \frac{dx\,dz}{|\vec{n}\cdot\hat{j}|} = \int_0^1\int_0^1 (-1)\,dx\,dz = -1$$

~~Method 2~~ $$\therefore \int_S \vec{F}\cdot\vec{n}\, ds = 0+\frac{1}{2}+0+2+0-1 = \frac{3}{2}.$$

Method 2:-

$$\int_S \vec{F}\cdot\vec{n}\, ds = \int_v \vec{\nabla}\cdot\vec{F}\, dv \qquad \text{(Divergence Theorem)}$$

$$\vec{\nabla}\cdot\vec{F} = 4z-2y+y = 4z-y$$

$$\therefore \int_S \vec{F}\cdot\vec{n}\, ds = \int_0^1\int_0^1\int_0^1 (4z-y)\,dz\,dy\,dx = \frac{3}{2}.$$

Volume Integral:→ Let $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be the differentiable vector point fun defined in volume V, then, its volume integral is $\int_V \vec{F}\cdot dV$

Similarly, if $\phi(x,y,z)$ is scalar point fun, then, $\int_V \phi \, dV$

Questions:→

1. Evaluate $\int_V \nabla\vec{F}\cdot dV$ where, $\vec{F} = (2x^2-3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ and V is the region bounded by the planes $x=0,\ y=0,\ z=0$ & $2x+2y+z=4$.

Solⁿ:→ $\vec{\nabla}\cdot\vec{F} = 4x - 2x = 2x$

$$\int_V 2x \, dV = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x \, dz\, dy\, dx = \frac{8}{3}$$

2. The volume of an object expressed in spherical coordinate, $V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1} r^2\sin\theta \, dr\, d\phi\, d\theta$, then V is ______.

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

Solⁿ:→ $V = \int_0^{2\pi}\int_0^{\pi/2} \sin\phi \left.\frac{r^2}{2}\right|_0^1 d\phi\, d\theta = \frac{1}{2}\int_0^{2\pi} (-\cos\phi)_0^{\pi/2} \, d\theta = \frac{\pi}{3}$

Gauss-Divergence Theorem:→ Let S be a closed surface enclosing a volume V & $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be the differentiable vector point fun defined over S, then,

$$\int_S \vec{F}\cdot d\vec{s} = \int_V \text{div}\,\vec{F}\, dV$$

Questions:→

1. Evaluate $\int_S \vec{r}\cdot\vec{n}\, ds$, where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ & S is a closed surface enclosing a volume V is ____.

(a) V (b) 2V (c) 3V (d) 4V.

2. The value of $\int_S (x\,dy\,dz + y\,dx\,dz + z\,dx\,dy)$, where S is a surface,

(i) cylinder $x^2+z^2=16$ & $y=0$ to $y=3$,

(ii) is a sphere $x^2+y^2+z^2=9$,

is ________.

Sol^n:→ $\vec{F} = x\hat{i}+y\hat{j}+z\hat{k}$ $\left(\because \int_S \vec{r}\cdot\vec{n}\,ds = \int_S \vec{F}\cdot d\vec{S} = \int_S (F_1\hat{i}+F_2\hat{j}+F_3\hat{k})\,d\vec{S}\right)$

$$\int_V \vec{r}\cdot\vec{n}\,ds = 3V$$

(i) $\int_V \vec{r}\cdot\vec{n}\,ds = 3\pi r^2 h = 3\pi\times 4^2\times 3 =$

(ii) $\int_V \vec{r}\cdot\vec{n}\,ds = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\times 3^3 =$

3. Evaluate $\int_S (x^2+2y^2+3z^2)\,ds$, where, S is the surface $x^2+y^2+z^2=1$.

Sol^n:→ $\vec{F}\cdot\vec{n} = x^2+2y^2+3z^2$ ---(1)

$\phi = x^2+y^2+z^2$

$\vec{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = x\hat{i}+y\hat{j}+z\hat{k}$

Let, $F_1\hat{i}+F_2\hat{j}+F_3\hat{k} = \vec{F}$, then,

$\vec{F}\cdot\vec{n} = F_1x+F_2y+F_3z = x^2+2y^2+3z^2$

$\Rightarrow F_1 = x$, $F_2 = 2y$, $F_3 = 3z$

$\therefore$ $\vec{F} = x\hat{i}+2y\hat{j}+3z\hat{k}$

$\therefore$ $\int_S \vec{F}\,ds = \int_V \vec{\nabla}\cdot\vec{F}\,dV = \int_V (1+2+3)\,dV = 6V = 6\times\frac{4}{3}\pi\times 1 = 8\pi$.

4. The value of $\int_S \vec{F}\cdot\vec{n}\,ds$, where $\vec{F} = 4x^2\hat{i}-3y\hat{j}+8xz\hat{k}$ & S is a surface $0\le x\le 1$, $0\le y\le 2$, & $0\le z\le 3$, is ________.

Sol^n:→ $\vec{\nabla}\cdot\vec{F} = 8x-3+8x = 16x-3$

$\therefore$ $\int_S \vec{F}\cdot\vec{n}\,ds = \int_V \vec{\nabla}\cdot\vec{F}\,dV = \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} (16x-3)\,dx\,dy\,dz = 30.$

5. Evaluate $\int_S \vec{F}\cdot\vec{n}\,ds$, where, $F = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$, taken over the regio bounded by $x^2+y^2=4$ & $z=0$ to $z=3$.

Solⁿ:→ $\vec{\nabla}\cdot\vec{F} = 4 - 4y + 2z$

$$\therefore \int_S \vec{F}\cdot\vec{n}\,ds = \int_V \vec{\nabla}\cdot\vec{F}\,dV = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\,dx\,dy\,dz$$

$$= 2\int_0^3\int_{-2}^{2} (4+2z)\sqrt{4-x^2}\,dx\,dz$$

$$= 4\int_0^3\int_0^2 (4+2z)\sqrt{4-x^2}\,dx\,dz$$

$$= 4\int_0^3 (4+2z)\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 dz$$

$$= 4\pi\int_0^3 (4+2z)\,dz = 84\pi$$

Note:→ we can also solve the above problem in polar form.

6. Evaluate $\int_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,ds$, where, $\vec{F} = 4x^2z\,\hat{i} - (yz-7)\hat{j} + xy^2z\,\hat{k}$, and S is the surface bounded by $y^2+z^2=25$ & $x=0$ to $x=2$.

Solⁿ:→ $\int_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,ds = \int_V \operatorname{div}(\vec{\nabla}\times\vec{F})\,dV = 0.$

Stoke's Theorem:→ Let S be an open surface bounded by a closed curve 'C' & $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a differentiable vector funⁿ defined over 'S', then $\oint_C \vec{F}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{F}\cdot d\vec{s} = \int_S \vec{\nabla}\times\vec{F}\cdot\vec{n}\,ds$

i.e. $\oint_C (F_1\,dx + F_2\,dy + F_3\,dz) = \int_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,ds$

Questions: $\rightarrow$ 1. The value of $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$, & C is a curve $x^2+y^2=4$ in XY plane is ______.

(a) 0 (b) $\frac{1}{2}$ (c) 2 (d) 3

Soln: $\rightarrow$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z)$$

$$= \vec{0}$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \int_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, ds = \int_S \vec{0}\cdot\vec{n}\, ds = 0.$$

2. The value of $\int_C \vec{F}\cdot d\vec{r}$; where, $\vec{F} = -y^3\hat{i} + x^3\hat{j}$ & 'C' is the circular disc $x^2+y^2 \leq 1$, $z=0$ is ______.

Soln: $\rightarrow$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(3x^2+3y^2)$$

$$= 3(x^2+y^2)\,\hat{k}$$

$$\therefore \vec{\nabla}\times\vec{F}\cdot\vec{n} = 3(x^2+y^2) \qquad \{\because \vec{n} = \hat{k}\}$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, ds$$

$$= \int_S 3(x^2+y^2)\, ds$$

Let, $R \rightarrow xy$ plane

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \iint_R 3(x^2+y^2)\,\frac{dx\,dy}{|\vec{n}\cdot\hat{k}|} = \iint_R 3(x^2+y^2)\, dx\, dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2$, $|J| = r$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\int_0^1 3r^2\, r\, dr\, d\theta = \frac{3\pi}{2}.$$

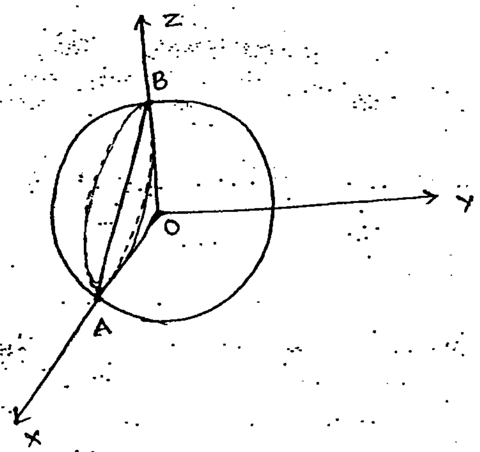
3. Evaluate $\oint_C (y\,dx + z\,dy + x\,dz)$, where C is the curve of intersection of $x^2+y^2+z^2 = a^2$ & $x+z = a$.

Solⁿ:→ $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \hat{i}(0-1) - \hat{j}(1-0) + \hat{k}(0-1)$$

$$= -\hat{i} - \hat{j} - \hat{k}$$

The intersection of a sphere $x^2+y^2+z^2 = a^2$ with the plane $x+z=a$ is a circle in the plane $x+z=a$, with AB as diameter, where $A(a,0,0)$ & $B(0,0,a)$



$AB = \sqrt{a^2+a^2} = a\sqrt{2}$

$\Rightarrow$ radius $= \frac{a}{\sqrt{2}}$

Let, $\phi = x+z$

$\vec{\nabla}\phi = \hat{i}+\hat{k}$

$\vec{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{\hat{i}+\hat{k}}{\sqrt{2}}$

$(\vec{\nabla}\times\vec{F})\cdot\vec{n} = (-\hat{i}-\hat{j}-\hat{k})\cdot\frac{(\hat{i}+\hat{k})}{\sqrt{2}} = -\sqrt{2}$

$\oint_C (y\,dx + z\,dy + x\,dz) = \int_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS = \int_S -\sqrt{2}\,dS = -\sqrt{2}\,S$

$$= -\sqrt{2}\,\pi \times \left(\frac{a}{\sqrt{2}}\right)^2 = -\frac{\pi a^2}{\sqrt{2}}.$$

8. Fourier Series :→ Let $f(x)$ be a periodic fu^n defined in $(c, c+2l)$ with period $2l$, then, the fourier series of $f(x)$ is given by,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right]$$

where, a_0, a_n & b_n are fourier coeff. given by,

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Note:→ $[-l, l]$, $[0, 2l]$, $[-\pi, \pi]$ (or) $[0, 2\pi]$

Dirchilet's Conditions :→ A fu^n is said to satisfy dirchilet's $cond^n$ if

(i) $f(x)$ and its integrals are finite & single valued.

(ii) $f(x)$ has finite no. of finite discontinuities.

(iii) $f(x)$ has finite no. of maxima & minima.

Note:→ If $f(x)$ ~~is~~ satisfied Dirichlet's $cond^n$ then the fourier series is convergent.

Convergence:→ (i) If $f(x)$ is continuous at $x=c \in (a,b)$ then fourier series of $f(x)$ at $x=c$ converges to $F(c)$.

(ii) If $f(x)$ is discontinuous at $x=c \in (a,b)$ then fourier series of $f(x)$ at $x=c$ converges to $\frac{1}{2}\left[\lim_{x\to c^-} F(x) + \lim_{x\to c^+} F(x)\right]$

(iii) fourier Series of $f(x)$ at the end ~~pot~~ pts i.e $x=a$ (or) b converges to $\frac{1}{2}\left[\lim_{x\to a^+} F(x) + \lim_{x\to b^-} F(x)\right]$

Fourier Series of Even & Odd Function in $[-l, l]$ (or) $[-\pi, \pi]$: → 40

(i) Fourier series of an even function :→

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where, $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

(ii) Fourier series of an Odd function :→

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Half-Range Series :→

(i) Half-Range cosine series in $[0, l]$:→

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where, $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

(ii) Half-Range sine series in $[0, l]$:→

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Questions:→

1. The coeff. of $\sin x$ in the fourier series expansion of $f(x) = x^2$ in $(-\pi, \pi)$ is ____.

(a) $\sum \frac{(-1)^n}{n^2}$ (b) $\sum \frac{1}{n^2}$ (c) $\frac{\pi^2}{6}$ (d)✓ 0

Solⁿ:→ Even funⁿ hence, coeff. of $\sin x = 0$.

2. If $f(x) = \begin{cases} 0 ; & -2 < x < 0 \\ 1 ; & 0 < x < 2 \end{cases}$

then the term independent of x in the fourier series of f(x) is ____

(a) 0 (b) 1 (c)✓ ½ (d) 2

Solⁿ:→ Here, $(-2, 2) \Rightarrow l = 2$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx = \frac{1}{2} \int_0^2 1 \cdot dx = 1$$

$\therefore$ Independent term $= \frac{a_0}{2} = \frac{1}{2}$

3. The funⁿ $f(x) = \begin{cases} -x+1 ; & -\pi \le x \le 0 \\ x+1 ; & 0 \le x \le \pi \end{cases}$

then f(x) has following terms in its expansion

(a)✓ cosine (b) sine (c) both (d) cannot be determined.

Solⁿ:→ $f(-x) = \begin{cases} x+1 ; & -\pi \le -x \le 0 \\ -x+1 ; & 0 \le -x \le \pi \end{cases}$

$$= \begin{cases} x+1 ; & \pi \ge x \ge 0 \\ -x+1 ; & 0 \ge x \ge -\pi \end{cases}$$

$$= f(x)$$

$\Rightarrow$ Even funⁿ.

4. If $f(x) = \begin{cases} 0 & ; \quad -2 < x < 0 \\ 1 & ; \quad 0 < x < 2 \end{cases}$

then the coeff. of $\cos \frac{n\pi x}{2}$ is ____.

(a) ✓ 0 (b) $\frac{1}{n}$ (c) $\frac{1}{n^2}$ (d) $-\frac{1}{n}$

Solⁿ:→ $(-2, 2) \Rightarrow l = 2$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx$$

$$= \frac{1}{2} \int_0^2 \cos \frac{n\pi x}{2} \, dx$$

$$= \frac{1}{2} \left[\frac{\sin \frac{n\pi x}{2}}{n\pi/2} \right]_0^2$$

$$= \frac{1}{2} \times \frac{2}{n\pi} [0 - 0] = 0$$

5. If $f(x) = x^2$ in $[-\pi, \pi]$ has its fourier expansion as $f(x) = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots \infty\right]$ then the value of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ is ____.

(a) ✓ $\frac{\pi^2}{6}$ (b) $\frac{\pi^2}{12}$ (c) π^2 (d) None

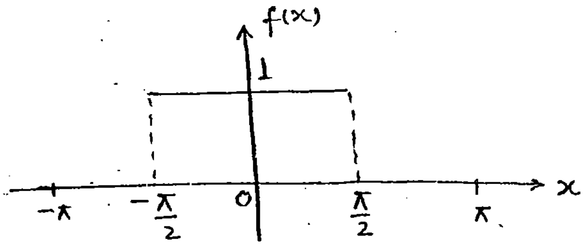
Solⁿ:→ At $x = \pi$, we have

$$\frac{\pi^2}{3} - 4\left[\frac{-1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \cdots\right] = \frac{1}{2}\left[\lim_{x \to \pi^+} f(x) + \lim_{x \to \pi^-} f(x)\right]$$

$$= \frac{1}{2}[\pi^2 + \pi^2] = \pi^2$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{1}{4}\left[\pi^2 - \frac{\pi^2}{3}\right] = \frac{\pi^2}{6}$$

6. A fu^n with period 2π is shown below



then fourier series is ______.

(a) $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\frac{n\pi}{2} \cos nx$

(b) $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\frac{n\pi}{2} \sin nx$

✓(c) $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\frac{n\pi}{2} \cos nx$

(d) $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\frac{n\pi}{2} \sin nx$

Sol^n:→

$$f(x) = \begin{cases} 0 \ ; & -\pi < x < -\pi/2 \\ 1 \ ; & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 \ ; & \frac{\pi}{2} < x < \pi \end{cases}$$

$f(-x) = f(x) \Rightarrow$ Even fu^n $\Rightarrow$ cosine series

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2} 1.\,dx = 1$$

$$\therefore \ \frac{a_0}{2} = 1$$

7. In $[0,\pi]$ the constant term in the cosine series of $f(x) = x^2+2x$ is

(a) $\pi\left(\frac{\pi}{3}-1\right)$ ✓(b) $\pi\left(\frac{\pi}{3}+1\right)$ (c) $\pi\left(\frac{\pi}{2}+1\right)$ (d) None

Sol^n:→ $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} [x^2+2x]\,dx = \frac{2}{\pi}\left[\frac{\pi^3}{3} + \pi^2\right]$

$$= \cancel{\frac{2\pi^2}{3} + \pi}$$

$$= \cancel{\pi\left(\frac{2\pi}{3}+1\right)}$$

$$\therefore \ \frac{a_0}{2} = \frac{1}{\pi}\left(\frac{\pi^3}{3}+\pi^2\right) = \pi\left(\frac{\pi}{3}+1\right).$$

8. If $f(x) = x$ is expressed on a half range cosine series in $[0, 2]$ then the coeff. of $\cos \pi x$ is ____. Q8

(a) $\frac{4}{\pi^2}$ (b) $\frac{2}{\pi^2}$ (c) 0 (d) None.

9. In the interval $[0, \pi]$ if a const. c is expressed as a half range sine series then coeff. of $\sin 5x$ is ____.

(a) $\frac{2c}{5\pi}$ (b) 0 (c) $\frac{4c}{5\pi}$ (d) $\frac{c}{5\pi}$

Soln:→ $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, $f(x) = c$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cdot \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} c \sin 5x \, dx$$

$$= \frac{2c}{\pi}\left[-\frac{\cos 5x}{5}\right]_0^{\pi}$$

$$= \frac{4c}{5\pi}$$

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