Design of Rigid Frames (Elastic Design)

1. METHODS OF ANALYSIS

There is no single best method to analyze statically indeterminate structures. There are many methods, and many combinations and adaptations of these methods. One method may be simple and quick, but can only be used to a limited extent. Another method may have wide application, but be so laborious that it is not used much.

Most texts on statically indeterminate structures start out with the various methods of determining deflections of the structure. They then consider the analysis of these structures. The methods of finding deflections are simple tools which may be used in the analysis of the structure.

There are actually about five basic, well used methods for the analysis of statically indeterminate structures encountered in rigid frame designing:

1. Least Work Method
2. General Method
3. Slope Deflection Method
4. Moment and Shear Distribution Method
5. Column Analogy Method

All of these methods, when applied to continuous beams and frames, give the resulting bending moments at various points along the structure. In order to proceed this far to get the resulting moments on the structure, it is first necessary to assume the moments of inertia of the members. This is usually a good guess or approximation. Then, from these resulting bending moments, the member is built up. If the final required moment of inertia is more than that which was started with, the work must be repeated, or adjusted, using this newer value. In some methods only the ratios of the various moments of inertia need be used.

Method of Least Work

The method of least work depends on the following. It is considered that a structure will deform under the application of a load, in such a manner that the internal work of deformation will be held to a minimum. This method may be outlined as follows:

1. Cut the structure so that it becomes statically determinate.
2. The unknown moments or forces become the redundants or unknown quantities.
3. Set up an equation for the internal work of the structure in terms of these redundants.
4. A derivative of this is then set equal to zero, and this will give the minimum value of this redundant force.

General Method

The general method consists of the following:

1. Cut the structure at the redundant or unknown force.
2. Determine the opening of this gap caused by the given load (while cut). Several methods may be used to find this deflection.
3. Apply a redundant force to close this gap.
4. From the given loads and this redundant force, make up a moment diagram and design the structure from this.

For more than one redundant force, cut all members at these redundant forces and close the gaps simultaneously.

To use the general method, the designer must be able to find deflections in Step 2. Some of the methods for finding deflections are as follows:

(a) Real Work
(b) Castigliano's Theorem
(c) Virtual Work
(d) Area Moment
(e) Conjugate Beam
(f) Angle Weights
(g) Willot-Mohr Diagram

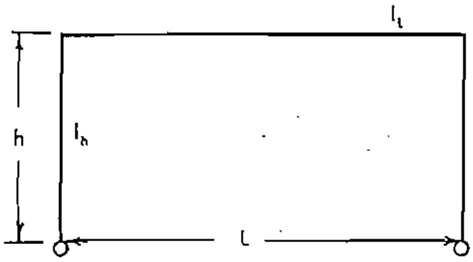
Several of these methods are described in Section 2.5 on Deflection by Bending and will not be discussed here.

Slope Deflection Method

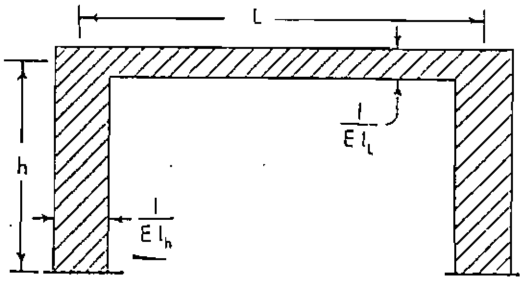
In the general method just outlined, the redundant or unknown forces and moments are found. In a similar manner, it is possible to solve for the unknown joint rotations and deflections. As soon as these are found, the end moments may be determined and these combined with the original moments from the applied load.

Moment and Shear Distribution Method

The moment distribution or Hardy Cross method consists of holding the joints in a frame fixed so that they cannot rotate. The end moments of each loaded member are found from standard beam diagrams in handbooks. Then, one at a time, a joint is released, allowed



(a) Actual Frame



(b) Elastic area of analogous column

FIGURE 1

to rotate, and then fixed again. This release causes a new distribution of the moment about this point, and some of this change is carried over into the next joint. This procedure is followed for each joint in the entire frame, and then the whole process is repeated over all the joints as often as required until these corrections become very small.

This method is outlined as follows:

1. Fix the joints from rotation and find the moments, treating the member as a simple structure.

2. Remove the joint restraints one at a time, and balance moments about the joint. This unbalanced moment is then distributed about the joint.

3. Some of this distributed moment is then carried over into the other end of the member.

4. This is repeated until the unbalanced moments become very small. The final moments are then used to design the structure.

Column Analogy Method

The outline or over-all shape of the given frame is considered as a column cross-section, called an elastic area. The length of each portion of this elastic area is equal to the actual length of the corresponding member of the frame. The width of each portion of this elastic area is equal to the 1/EI of the corresponding member of the frame.

The properties of this elastic area are determined: area, center of gravity or elastic center, and moments of inertia about the two axes (x-x and y-y).

The statically indeterminate frame must be cut, usually at one of the supports, so that it becomes statically determinate. Under this condition, the moment diagram caused by the applied loads is constructed and then treated as a load (M/EI) applied to the elastic area of the analogous column.

Just as an eccentrically loaded column has an axial compressive stress and bending stresses about the two axes (x-x and y-y), so the analogous column has "stresses" at any point equal to the axial compressive "stress" and the two bending "stresses". These resulting "stresses" of the analogous column are the corrective moments which must be added to the statically determinate moments of the "cut" frame in order to bring the frame back to its original shape and condition before it was "cut".

This is outlined as follows:

1. Determine properties of the elastic area: area, center of gravity or elastic center, and moments of inertia about the two axes (x-x and y-y).

2. Cut the frame to make it statically determinate. Use moment diagram from applied loads as a load (M/EI) on the elastic area of the analogous column.

3. Determine axial "stress" and the two bending "stresses" of the analogous column. These become corrective moments which must be added to the statically determinate moment of Step 2 to give the final moments of the statically indeterminate frame.

4. From these moments, find the redundant forces at the cut portion of the frame.

2. COLUMN ANALOGY METHOD

The outline of the given frame is considered to be a column cross-section, called an elastic area; Figure 1.

The length of each member in the elastic area is considered equal to the actual length of the corresponding member of the actual frame.

The width of each member in the elastic area is equal to 1/EI of the corresponding member of the frame.

It is seen by Figure 1 that for a pinned-end frame the moment of inertia of the flexible pin is zero. Hence the width of the elastic area at this point is

$$\frac{1}{E\,I} = \frac{1}{0} = \infty$$

and the elastic area at this pinned end would equal ∞.

For a fixed end, the moment of inertia at this rigid support is assumed to be ∞. The resulting width of the elastic area at this point is—

$$\frac{1}{E\,I} = \frac{1}{\infty} = 0$$

and the elastic area at this fixed end would be zero.

The elastic area, with its dimensions now known

Length $= L$

Height $= h$

Width $= \frac{1}{E\,I}$

is now treated like any other cross-section, and its properties determined.

In this example of pinned ends:

Area

$$A = 2\left(\frac{1}{E\,I_h}\right) h + \left(\frac{1}{E\,I_L}\right) L + 2\left(\frac{1}{0}\right) = \infty$$

(2 columns) (beam) (pinned ends)

Elastic Center

The elastic center is found as though it were the center of gravity of the elastic area.

axis x-x

Taking moments about the base line, it is seen that the elastic axis x-x of the elastic area must pass through the frame base since, in the analogous column, the pinned ends have infinite (∞) area.

This may be proved by mathematically determining the elastic center of gravity:

$$C.G. = \frac{\Sigma M}{\Sigma A}$$

$$= \frac{2\left(\frac{1}{E\,I_h}\right) h \left(\frac{h}{2}\right) + \left(\frac{1}{E\,I_L}\right) L\,(h) + 2\,(\infty)\,(0)}{\infty}$$

$$= 0$$

axis y-y

By observation, it is seen that the y-y axis would pass through the center of this elastic area because of section symmetry.

Moment of Inertia

$$I_{x\text{-}x} = 2\left(\frac{1}{E\,I_h}\right)\frac{h^3}{3} + \left(\frac{1}{E\,I_L}\right) L\,h^2 + 2\left(\frac{1}{0}\right) 0$$

(2 columns) (beams) (pinned ends)

$$= \frac{h^2}{3\,E}\left(\frac{2\,h}{I_h} + \frac{3\,L}{I_L}\right)$$

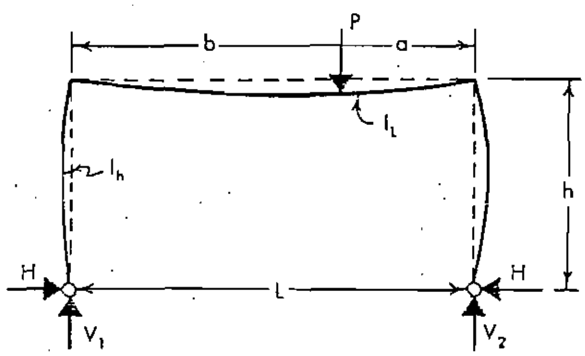
Since the infinite elastic area at the pin lies along the elastic axis x-x, it will have no effect upon $I_{x\text{-}x}$.

$I_{y\text{-}y} = \infty$, since there is an infinite elastic area at the two pinned ends and these lie at the extreme ends of the section about axis x-x.

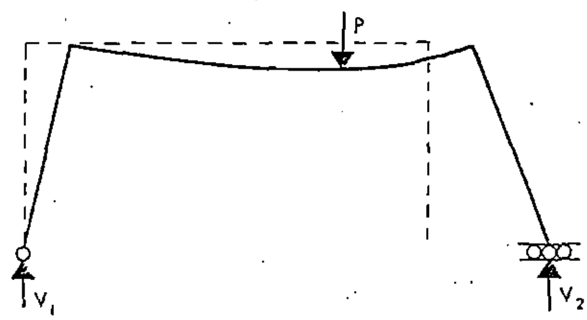
Apply Load to Elastic Area

The statically indeterminate frame, Figure 2(a), must have some portion cut, usually at one of the supports, so that it becomes statically determinate, Figure 2(b). Under this condition, the bending moment diagram caused by the applied loads is constructed, Figure 2(c). This is then treated as a load (M_s/EI) applied to the elastic area of the analogous column, Figure 3(a).

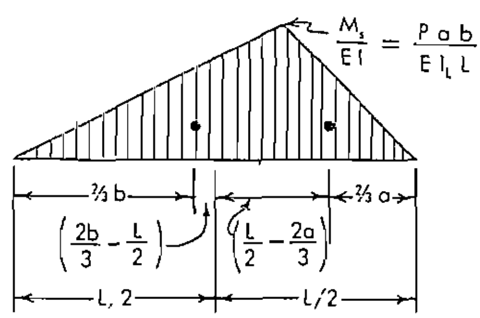
Just as an eccentrically loaded column has an axial load and tilting moments ($M_x = P\,y$, and $M_y = P\,x$),



(a) Statically indeterminate frame

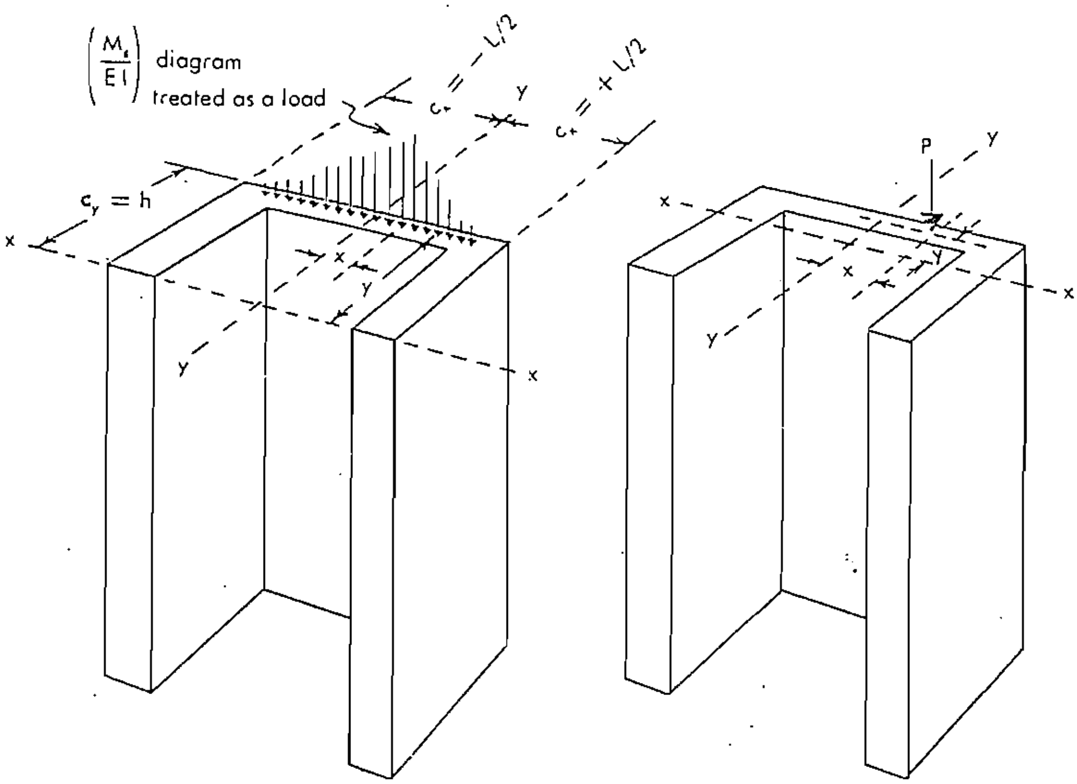


(b) One support cut to make frame statically determinate



(c) Moment diagram for the statically determinate frame

FIGURE 2



(a) Analogous column loaded with $\frac{M_x}{E \cdot I}$ (b) Actual column with eccentric load (P)

FIGURE 3

so the analogous column has an axial load and tilting moments. Consider the moment diagram divided by EI as the load about the two axes (x-x and y-y) through the elastic center:

axial load on analogous column

$$P = \frac{1}{2}\left(\frac{P\,a\,b}{E\,I_L\,L}\right) L$$

$$= \frac{P\,a\,b}{2\,E\,I_L}$$

moment about axis x-x on analogous column

$$M_{x\text{-}x} = \frac{1}{2}\left(\frac{P\,a\,b}{E\,I_L\,L}\right) L\,h$$

$$= \frac{P\,a\,b\,h}{2\,E\,I_L}$$

moment about axis y-y on analogous column

$$M_{y\text{-}y} = \frac{1}{2}\left(\frac{P\,a\,b}{E\,I_L\,L}\right) b\left(\frac{2b}{3}-\frac{L}{2}\right) + \frac{1}{2}\left(\frac{P\,a\,b}{E\,I_L\,L}\right) a\left(\frac{L}{2}-\frac{2a}{3}\right)$$

$$= \frac{P\,a\,b}{12\,E\,I_L\,L}\,(b-a)\,(4b+4a-3L)$$

Just as the eccentrically loaded column has stresses at any point equal to the axial compressive stress plus the two bending stresses—

$$\sigma = \sigma_a\,(\text{axial}) \pm \sigma_x\,(\text{bending}_{x\text{-}x}) \pm \sigma_y\,(\text{bending}_{y\text{-}y})$$

$$\text{or } \sigma = \frac{P}{A} \pm \frac{M_{x\text{-}x}\,c_y}{I_{x\text{-}x}} \pm \frac{M_{y\text{-}y}\,c_x}{I_{y\text{-}y}}$$

so the analogous column has "stresses" at any point equal to the axial "stress" (σ_a) plus the two bending "stresses" (σ_y & σ_x). These are the corresponding corrective moments (M_a, M_x, & M_y) which must be applied to the statically determinate moments of the "cut" frame in Figure 2(b) to bring the frame back to its original shape and condition, Figure 2(a).

$$\boxed{\sigma_a = -M_a = -\frac{P}{a}} \quad \text{(1)}$$

$$M_a = -\frac{\frac{P\,a\,b}{2\,E\,I_L}}{\infty}$$

$$= 0 \qquad \text{(See Figure 4.)}$$

$$\boxed{\sigma_x = -M_x = -\frac{M_{x\text{-}x}\,c_y}{I_{x\text{-}x}}} \quad \text{(2)}$$

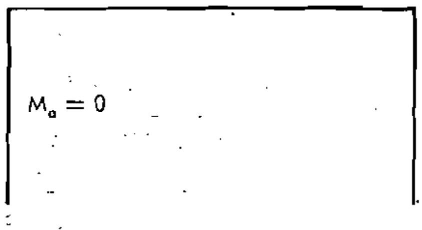


FIG. 4 No corrective moment to be added here.

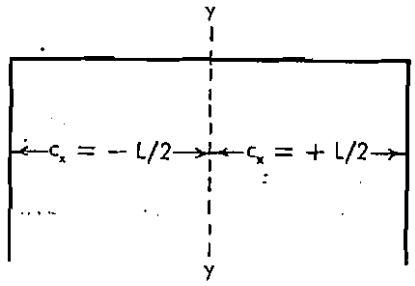


FIG. 6 No corrective moment to be added here.

when $c_y = +h$

$$M_x = -\frac{\left(\frac{P\ a\ b\ h}{2\ E\ I_L}\right) h}{\frac{h^2}{3\ E}\left(\frac{2\ h}{I_h} + \frac{3\ L}{I_L}\right)}$$

$$= -\frac{3\ P\ a\ b}{\frac{4\ h\ I_L}{I_h} + 6\ L}$$

when $c_y = 0$

$M_x = 0$ (See Figure 5.)

$$\boxed{\sigma_y = -M_y = -\frac{M_{y\text{-}y}\ c_x}{I_{y\text{-}y}}} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

Since $I_{y\text{-}y} = \infty$

$M_y = 0$ (See Figure 6.)

The final moment on the frame will be as given in Figure 7.

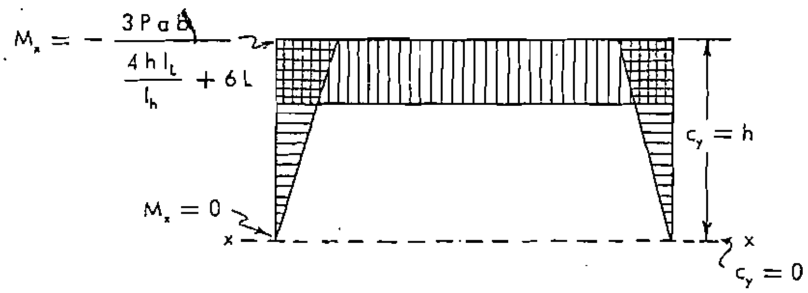


FIG. 5 Corrective moment to be added here.

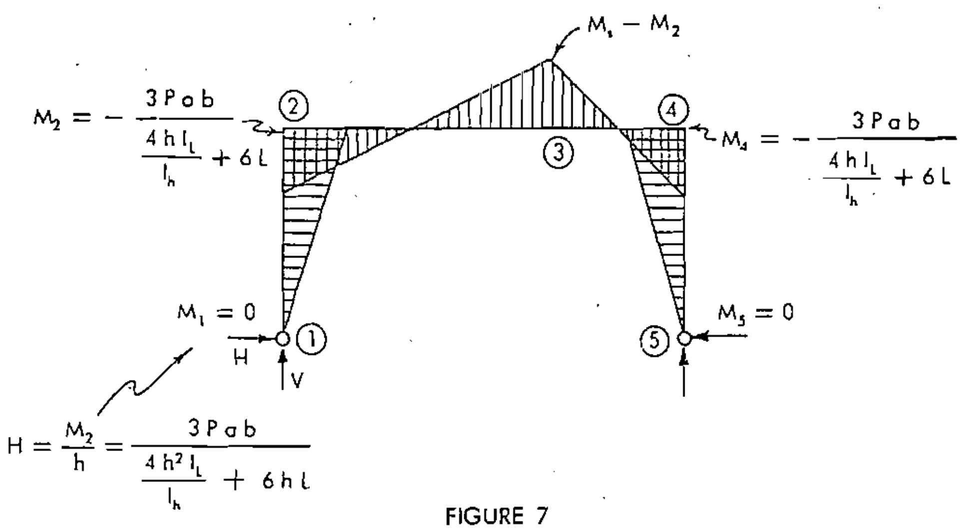


FIGURE 7

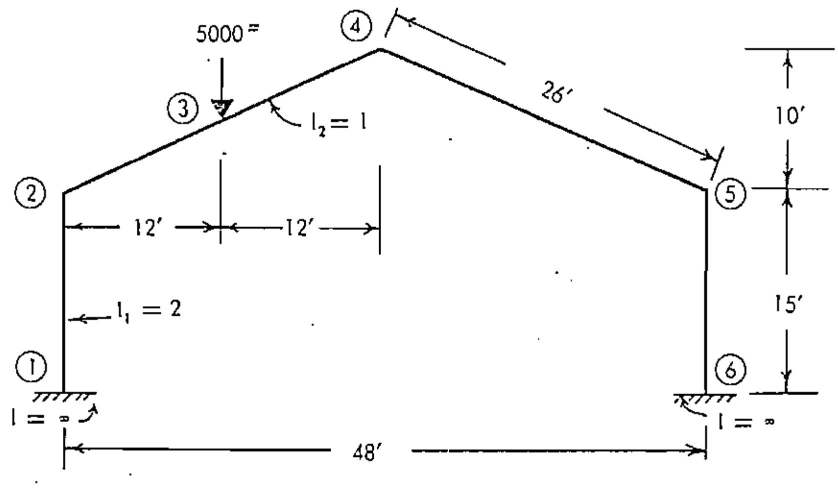


FIGURE 8

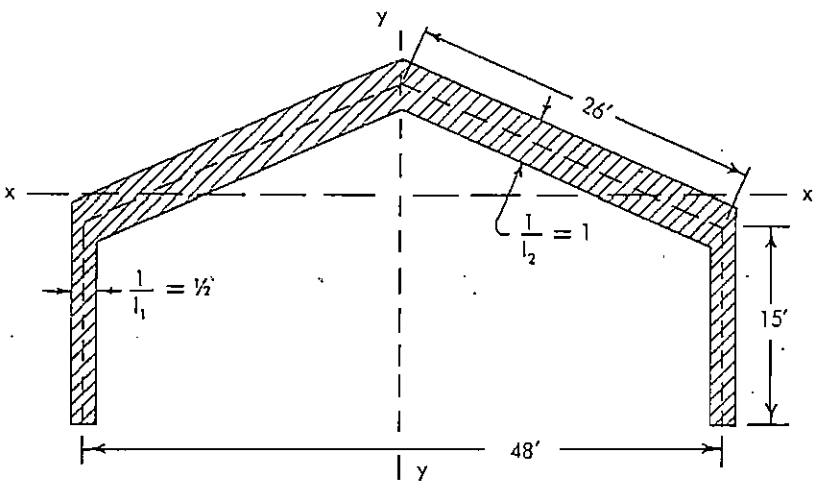


FIGURE 9

Problem 1

Find the moments (M) and the other redundant forces (H and V) of the following frame, having fixed ends, by means of the Column Analogy Method; Figure 8.

This frame must be transformed into the analogous column, and the properties of this equivalent elastic area determined; Figure 9.

Here:

$$A = 2\,\frac{15'}{I_1 = 2} + 2\,\frac{26''}{I_2 = 1} = 67$$

axis x-x of elastic center (See Figure 10.)

Use a reference axis (x′-x′) through the top of the column.

Member	A	y′	M = Ay′	$I_x' = M \cdot y'$	I_g
Columns	15	−7.5	−112.5	+ 843.75	281
Rafters	52	+5.0	+260	+1300	433
Total	67		+147.5	2856	

$$\text{elastic center} = \frac{M}{A} = \frac{+147.5}{67}$$

$= +\ 2.2'$ measured from reference axis (x′-x′)

$$\therefore I_{x\text{-}x} = I_x' + I_g - \frac{M^2}{A}$$

$$= 2856 - \frac{+147.5^2}{67}$$

$$= 2856 - 325$$

$$= 2531 \text{ in.}^4$$

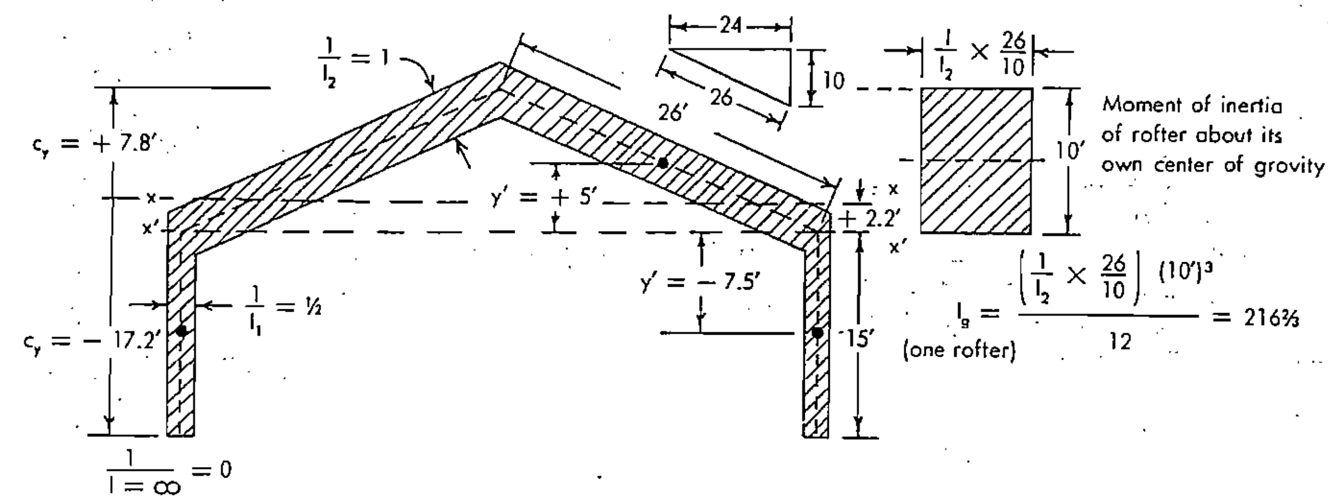


FIGURE 10

distance from elastic center (x-x) to outer fiber

(bottom) $c_y = -15 - 2.2 = -17.2'$

(top) $c_y = +10 - 2.2 = +7.8'$

axis y-y of elastic center (See Figure 11.)

By observation it is seen that this passes through the centerline of the frame:

$$I_{y\text{-}y} = \underset{(2\text{ columns})}{2(7.5)(24)^2} + \underset{(2\text{ rafters})}{2(26)(12^2) + 2(1248)}$$

$$= 18{,}624 \text{ in.}^4$$

distance from elastic center (y-y) to outer fiber

(right side) $c_x = +24$

(left side) $c_x = -24$

Cutting Frame So It Becomes Statically Determinate

The frame is now cut so that it becomes statically determinate. The resulting moment diagram, divided by the real moment of inertia (I), is treated as a load upon the analogous column or elastic area. (We don't divide by E here because E is constant; for steel, E = 30 x 10^6.) This may be done in several ways, principally:

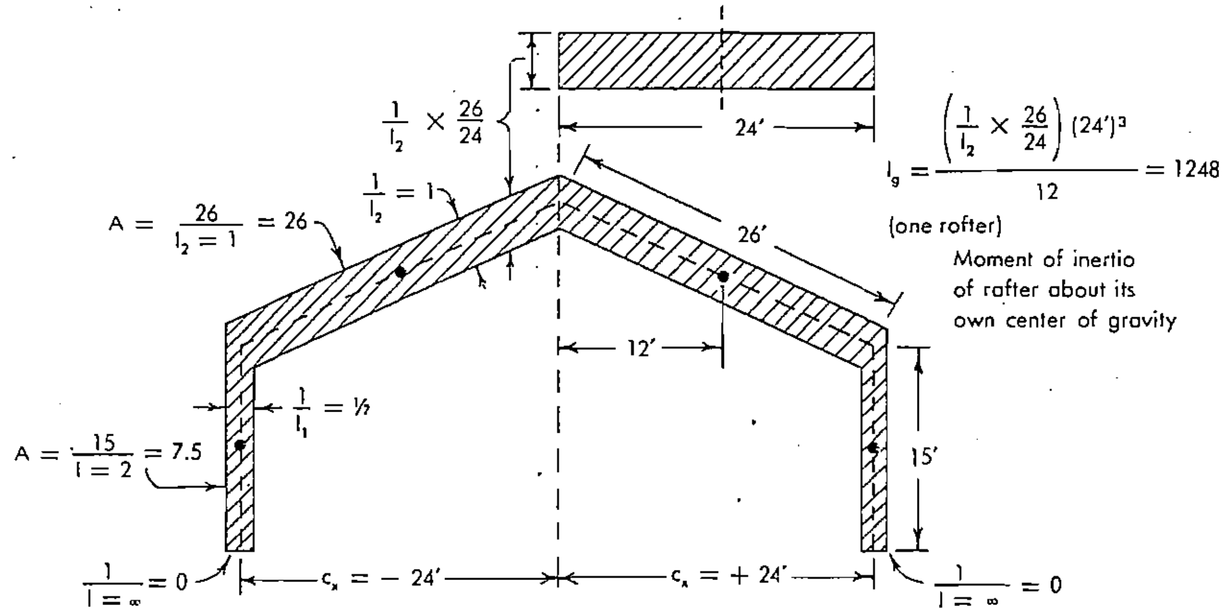


FIGURE 11

A. Cut the right fixed end support at ⑥. The portion of the rafter to the left of the applied load becomes a statically determinate cantilever beam.

B. Release the ends of the rafters at ② and ⑤. This becomes a statically determinate simply supported haunched beam.

Method A: Cut the frame at ⑥. With the load applied at ③, the rafter cantilevers out from ②. The end moment at ②, M = —60,000 ft-lbs, is also applied to the left column ①—②. (See Figure 15.)

the three loads on elastic area

$$P = \frac{-60{,}000 \text{ x } 15'}{I_1 = 2} + \frac{\tfrac{1}{2}(-60{,}000 \text{ x } 13')}{I_2 = 1}$$

$$= -450{,}000 - 390{,}000$$

$$= -840{,}000$$

$$M_{x\text{-}x} = (-450{,}000)(-9.7) + (-390{,}000)(-.53)$$

$$= +\ 4{,}571{,}700$$

$$M_{y\text{-}y} = (-450{,}000)(-24) + (-390{,}000)(-20)$$

$$= +\ 18{,}600{,}000$$

correction moment at ①

$$c_y = -17.2' \qquad c_x = -24'$$

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\ c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\ c_x}{I_{y\text{-}y}}$$

$$= \frac{(-840{,}000)}{67} + \frac{(+4{,}571{,}700)(-17.2)}{2531} + \frac{(+18{,}600{,}000)(-24)}{18{,}624}$$

$$= -\ 67{,}570 \text{ ft-lbs}$$

HOW TO DETERMINE CORRECTIVE MOMENTS

(Diagrams Apply to Option A)

The moment diagram divided by the moment of inertia of the statically determinant frame is considered to be the load on the elastic area of the analogous column. (E is constant.)

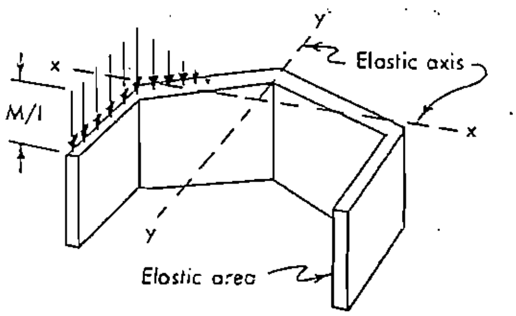


FIGURE 12

This total load on the elastic area may be broken down into 3 loads:

a. Axial load, P

b. Moment, $M_{x\text{-}x}$, about axis x-x

c. Moment, $M_{y\text{-}y}$, about axis y-y

These loads, in turn, result in 3 types of resisting "stresses":

$$\sigma_a = \frac{P}{a} \qquad \sigma_x = \frac{M_{x\text{-}x}\ c_y}{I_{x\text{-}x}} \qquad \sigma_y = \frac{M_{y\text{-}y}\ c_x}{I_{y\text{-}y}}$$

The resultant "stress" at any point of the elastic area may be found from the conventional stress in an eccentrically-loaded column:

$$\sigma = \frac{P}{a} + \frac{M_{y\text{-}y}\ c_y}{I_{y\text{-}y}} + \frac{M_{x\text{-}x}\ c_x}{I_{x\text{-}x}}$$

These "stresses" are the correcting moments, which must be applied to the original moments of the statically determinate frame to produce the final moments of the statically indeterminate frame.

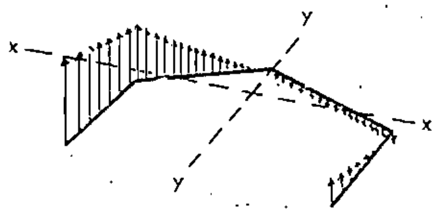


FIG. 14 Correcting moments

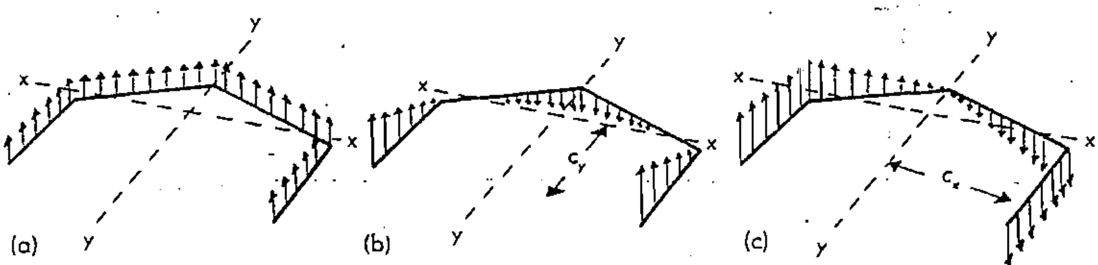


FIGURE 13

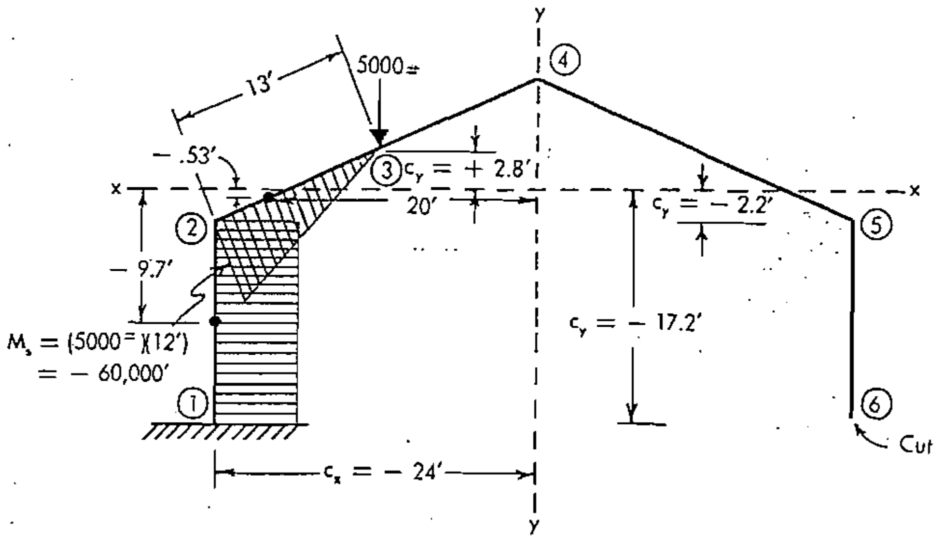


FIGURE 15

final moment = original moment − correction moment

$$M_1 = -60{,}000 + 67{,}620$$
$$= +7570 \text{ ft-lbs}$$

correction moment at ②

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\, c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\, c_x}{I_{y\text{-}y}} \quad \begin{array}{|l} c_y = -2.2' \\ c_x = -24' \end{array}$$
$$= \frac{-840{,}000}{67} + \frac{(+4{,}571{,}700)(-2.2)}{2531} + \frac{(+18{,}600{,}000)(-24)}{18{,}624}$$
$$= -40{,}480 \text{ ft-lbs}$$

final moment = original moment − correction moment

$$M_2 = -60{,}000 + 40{,}480$$
$$= -19{,}520 \text{ ft-lbs}$$

correction moment at ③

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\, c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\, c_x}{I_{y\text{-}y}} \quad \begin{array}{|l} c_y = +2.8' \\ c_x = -12' \end{array}$$
$$= \frac{-840{,}000}{67} + \frac{(+4{,}571{,}700)(+2.8)}{2531} + \frac{(+18{,}600{,}000)(-12)}{18{,}624}$$
$$= -19{,}460 \text{ ft-lbs}$$

final moment = original moment − correction moment

$$M_3 = 0 + 19{,}460$$
$$= +19{,}460 \text{ ft-lbs}$$

correction moment at ④

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\, c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\, c_x}{I_{y\text{-}y}} \quad \begin{array}{|l} c_y = +7.8' \\ c_x = 0 \end{array}$$
$$= \frac{-840{,}000}{67} + \frac{(+4{,}571{,}700)(+7.8)}{2531} + \frac{(+18{,}600{,}000)(0)}{18{,}624}$$
$$= +1550 \text{ ft-lbs}$$

final moment

$$M_4 = -1550 \text{ ft-lbs}$$

correction moment at ⑤

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\, c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\, c_x}{I_{y\text{-}y}} \quad \begin{array}{|l} c_y = -2.2' \\ c_x = +24' \end{array}$$
$$= \frac{-840{,}000}{67} + \frac{(+4{,}571{,}700)(-2.2)}{2531} + \frac{(+18{,}600{,}000)(+24)}{18{,}624}$$
$$= +7460 \text{ ft-lbs}$$

final moment

$$M_5 = -7460 \text{ ft-lbs}$$

correction moment at ⑥

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\, c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\, c_x}{I_{y\text{-}y}} \quad \begin{array}{|l} c_y = -17.2' \\ c_x = +24' \end{array}$$
$$= \frac{-840{,}000}{67} + \frac{(+4{,}571{,}700)(-17.2)}{2531} + \frac{(+18{,}600{,}000)(+24)}{18{,}624}$$
$$= -19{,}640 \text{ ft-lbs}$$

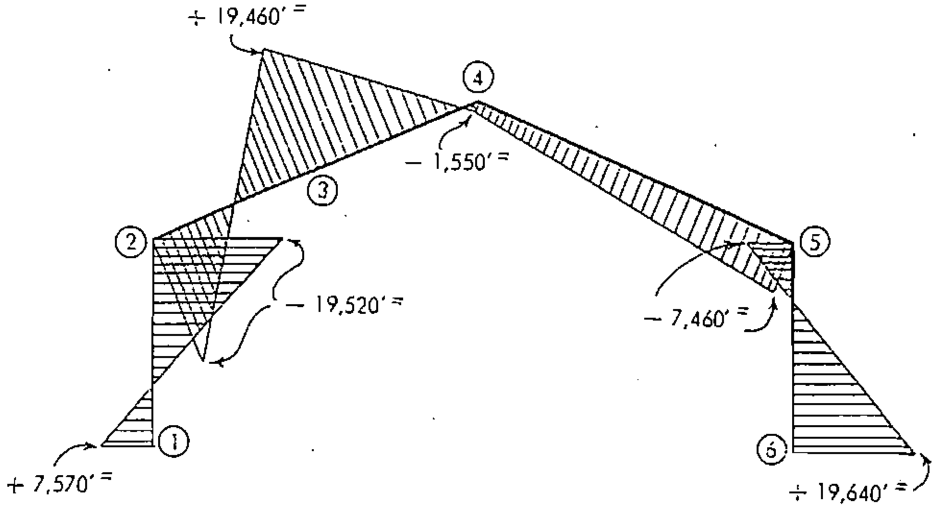


FIGURE 16

final moment

$M_6 = +\ 19{,}640$ ft-lbs

The final moments of the statically indeterminate frame are diagrammed in Figure 16.

Horizontal Redundant Force

To find the horizontal redundant force (H) at the base of the column, first find the point of inflection (zero moment) in the column. Then find the horizontal force required at this point to equal the end moment at the base of the column.

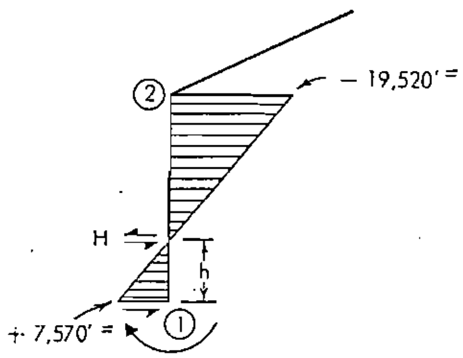


FIGURE 17

$$\frac{h}{15} = \frac{7570}{7570 + 19{,}520}$$
$$= .2794$$

$$\therefore\ h = 4.191'$$

$$H\ h = M_1$$

$$H = \frac{M_1}{h}$$
$$= \frac{7570 \text{ ft-lbs}}{4.191}$$
$$= 1806 \text{ lbs}$$

Vertical Reaction

To find the vertical reaction (V) at the base of the column, take the moments about the base of the opposite column and set them equal to zero. (See Figure 18.)

$$\Sigma\ M_6 = 0 \text{ or}$$
$$+ M_1 + V_1\ (48) - 5000(36) - M_6 = 0$$
$$V_1(48) = M_6 - M_1 + 5000(36)$$
$$= 19{,}640 - 7570 + 180{,}000$$
$$= 192{,}070$$
$$V_1 = \frac{192{,}070}{48}$$
$$= 4000 \text{ lbs}$$

Method B: Release ends of the rafters at ② and ⑤, so that the rafter becomes simply supported and statically indeterminate. (See Figure 19.)

the three loads on elastic area

$$P = \frac{\tfrac{1}{2}\ (+45{,}000)(13)}{I_2 = 1} + \frac{\tfrac{1}{2}\ (+15{,}000)(13)}{I_2 = 1} + \frac{(+30{,}000)(13)}{I_2 = 1} + \frac{\tfrac{1}{2}\ (+30{,}000)(26)}{I_2 = 1}$$
$$= +\ 292{,}500 + 97{,}500 + 390{,}000 + 390{,}000$$
$$= +\ 1{,}170{,}000$$

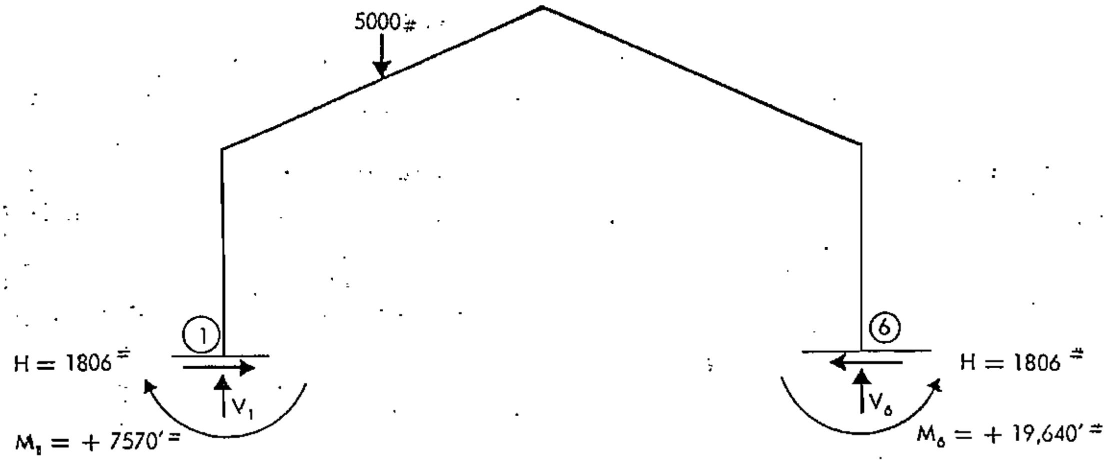


FIGURE 18

$$M_{x\text{-}x} = (+292{,}500)(+1.13) + (97{,}500)(+4.47) + (390{,}000)(+5.3) + (390{,}000)(+4.47)$$
$$= +4{,}576{,}650$$

$$M_{y\text{-}y} = (+292{,}500)(-16) + (97{,}500)(-8) + (390{,}000)(-6) + (390{,}000)(+8)$$
$$= -4{,}680{,}000$$

correction moment at ①

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\ c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\ c_x}{I_{y\text{-}y}} \qquad \left| \begin{array}{l} c_y = -17.2' \\ c_x = -24' \end{array} \right.$$

$$= \frac{+1{,}170{,}000}{67} + \frac{(+4{,}576{,}650)(-17.2)}{2531} + \frac{(-4{,}680{,}000)(-24)}{18{,}624}$$

$$= -7600 \text{ ft-lbs}$$

final moment = original moment − correction moment

$$M_1 = 0 + 7600$$
$$= +7600 \text{ ft-lbs}$$

correction moment at ②

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\ c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\ c_x}{I_{y\text{-}y}} \qquad \left| \begin{array}{l} c_y = -2.2' \\ c_x = -24' \end{array} \right.$$

$$= \frac{+1{,}170{,}000}{67} + \frac{(+4{,}576{,}650)(-2.2)}{2531} + \frac{(-4{,}680{,}000)(-24)}{18{,}624}$$

$$= +19{,}520 \text{ ft-lbs}$$

final moment

$$M_2 = -19{,}520 \text{ ft-lbs}$$

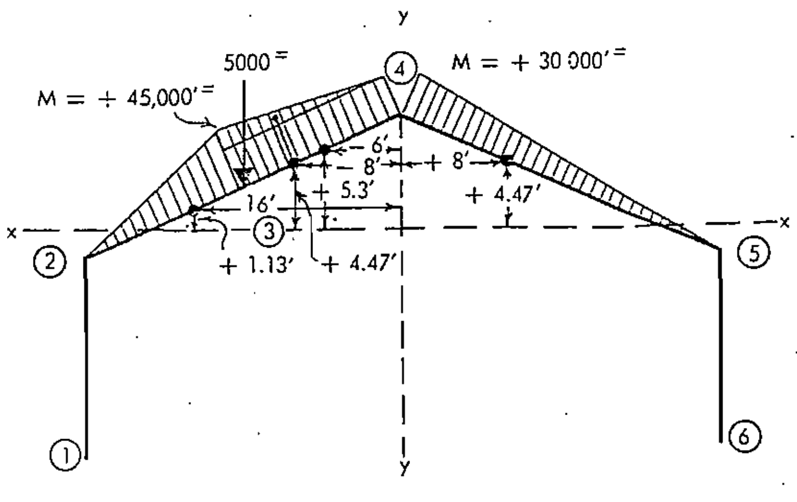


FIGURE 19

correction moment at ③

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\ c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\ c_x}{I_{y\text{-}y}} \qquad \begin{array}{|l} c_y = +\ 2.8' \\ c_x = -\ 12' \end{array}$$

$$= \frac{+1{,}170{,}000}{67} + \frac{(+4{,}576{,}650)(+2.8)}{2531} + \frac{(-4{,}680{,}000)(-12)}{18{,}624}$$

$$= +\ 25{,}540 \text{ ft-lbs}$$

final moment

$$M_3 = +\ 45{,}000 - 25{,}540$$
$$= +\ 19{,}460 \text{ ft-lbs}$$

correction moment at ④

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\ c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\ c_x}{I_{y\text{-}y}} \qquad \begin{array}{|l} c_y = +\ 7.8' \\ c_x = 0 \end{array}$$

$$= \frac{+1{,}170{,}000}{67} + \frac{(+4{,}576{,}650)(+7.8)}{2531} + \frac{(-4{,}680{,}000)(0)}{18{,}624}$$

$$= +\ 31{,}560 \text{ ft-lbs}$$

final moment

$$M_4 = +\ 30{,}000 - 31{,}560$$
$$= -\ 1560 \text{ ft-lbs}$$

correction moment at ⑤

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\ c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\ c_x}{I_{y\text{-}y}} \qquad \begin{array}{|l} c_y = -\ 2.2' \\ c_x = \quad 24' \end{array}$$

$$= \frac{+1{,}170{,}000}{67} + \frac{(+4{,}576{,}650)(-2.2)}{2531} + \frac{(-4{,}680{,}000)(+24)}{18{,}624}$$

$$= +\ 7450 \text{ ft-lbs}$$

final moment

$$M_5 = -\ 7450 \text{ ft-lbs}$$

correction moment at ⑥

$$= \frac{P}{A} + \frac{M_{x\text{-}x}\ c_y}{I_{x\text{-}x}} + \frac{M_{y\text{-}y}\ c_x}{I_{y\text{-}y}} \qquad \begin{array}{|l} c_y = -\ 17.2' \\ c_x = +\ 24' \end{array}$$

$$= \frac{+1{,}170{,}000}{67} + \frac{(+4{,}576{,}650)(-17.2)}{2531} + \frac{(-4{,}680{,}000)(+24)}{18{,}624}$$

$$= -\ 19{,}670 \text{ ft-lbs}$$

final moment

$$M_6 = 0 + 19{,}670$$
$$= +\ 19{,}670 \text{ ft-lbs}$$

Alternate Method

It is possible to work this problem in a slightly different manner. As before—

1. Determine the properties of the elastic area.
2. Cut the frame to make it statically determinate, as before.
3. Dividing the moment diagram of this cut frame by the moment of inertia of the corresponding members of the frame, treat it as the load on the elastic area. (E is constant.)
4. Find the resulting three parts of this load on the elastic area; that is,
   a. Load, P
   b. Moment, $M_{x\text{-}x}$
   c. Moment, $M_{y\text{-}y}$

Then find the three corrective actions—fixed end moment (M_{fe}), horizontal force (H), and vertical force (V)—which must be applied at the base of the frame to bring it back to the original shape and condition of the statically indeterminate frame. Find these from the following formulas:

$$M_{fe} = -\ \frac{P}{A}$$

$$H = -\ \frac{M_{x\text{-}x}}{I_{x\text{-}x}}$$

$$V = -\ \frac{M_{y\text{-}y}}{I_{y\text{-}y}}$$

Figure 20 shows their application to solution of the immediate problems.

The resulting moments about the frame for each of these corrective actions are determined and placed for convenience in table form. This facilitates totaling them to produce the final moments at any point of the statically indeterminate frame. See Figure 21.

3. FIXED END MOMENTS, STIFFNESS FACTORS, AND CARRY-OVER FACTORS

When some type of moment distribution is used for the analysis of continuous frames, it is necessary to know the following:

1. Fixed end moments (M_{fe}) of the beam.
2. Stiffness factor (K) for each end of the beam so the distribution factors may be determined.
3. Carry-over factor (C) of a moment from one end of the beam to the other end.

These items may be found from already-developed charts, or by use of the column analogy method which

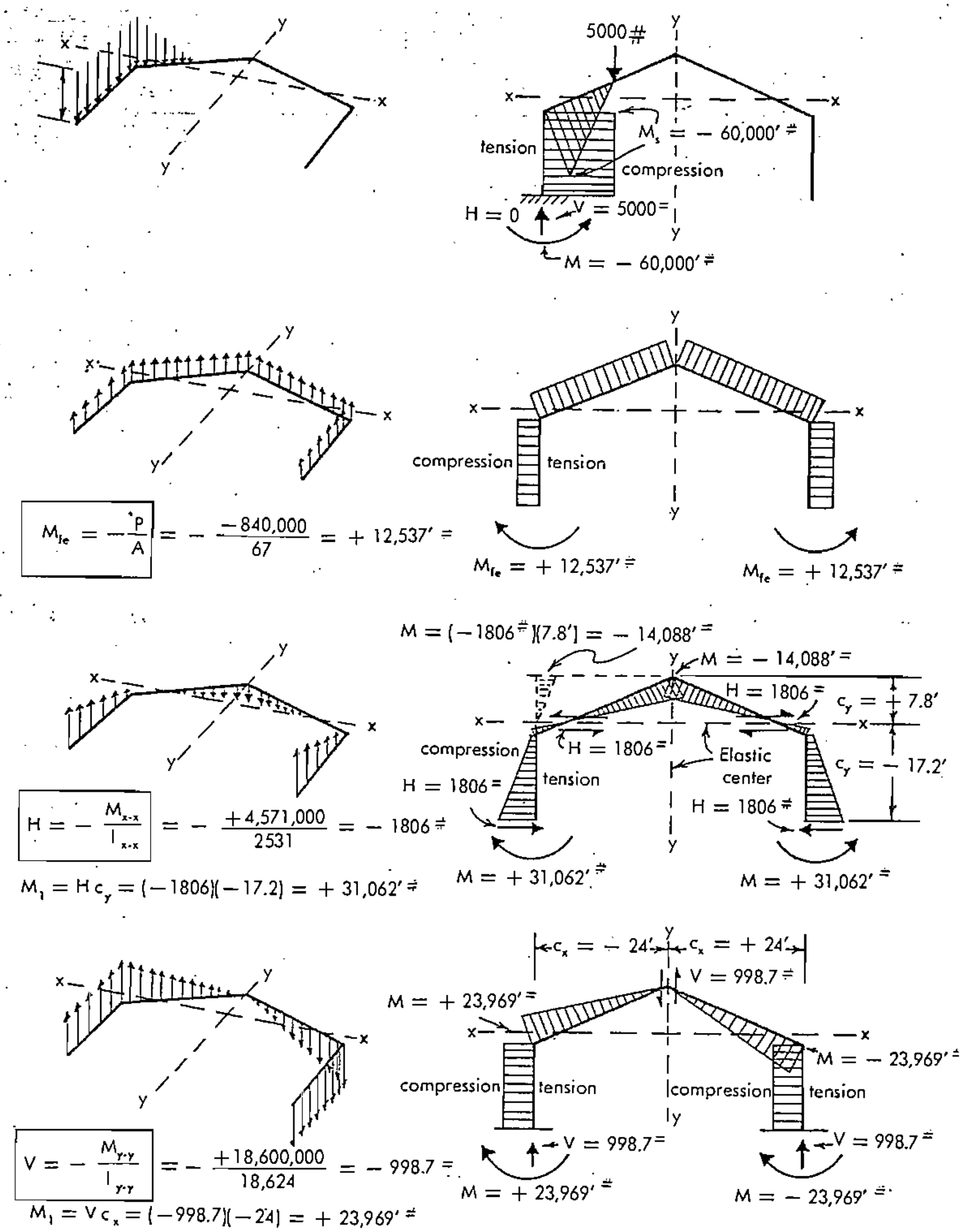

5000#
tension
$M_s = -60,000'^{\#}$
compression
$H = 0$
$V = 5000^{\#}$
$M = -60,000'^{\#}$
compression
tension
$M_{fe} = -\frac{{}^{*}P}{A} = -\frac{-840,000}{67} = +12,537'^{\#}$
$M_{fe} = +12,537'^{\#}$
$M_{fe} = +12,537'^{\#}$
$M = (-1806^{\#})(7.8') = -14,088'^{\#}$
$M = -14,088'^{\#}$
$H = 1806^{\#}$
$c_y = +7.8'$
compression
$H = 1806^{\#}$
Elastic center
$H = 1806^{\#}$
tension
$c_y = -17.2'$
$H = 1806^{\#}$
$H = -\frac{M_{x\text{-}x}}{I_{x\text{-}x}} = -\frac{+4,571,000}{2531} = -1806^{\#}$
$M_1 = H c_y = (-1806)(-17.2) = +31,062'^{\#}$
$M = +31,062'^{\#}$
$M = +31,062'^{\#}$
$c_x = -24'$
$c_x = +24'$
$V = 998.7^{\#}$
$M = +23,969'^{\#}$
$M = -23,969'^{\#}$
compression
tension
compression
tension
$V = 998.7^{\#}$
$V = 998.7^{\#}$
$V = -\frac{M_{y\text{-}y}}{I_{y\text{-}y}} = -\frac{+18,600,000}{18,624} = -998.7^{\#}$
$M_1 = V c_x = (-998.7)(-24) = +23,969'^{\#}$
$M = +23,969'^{\#}$
$M = -23,969'^{\#}$

FIGURE 20

	①	②	③	④	⑤	⑥
M_o	−60,000	−60,000	0	0	0	0
$M_{ro} = \frac{P}{A}$	+12,537	+12,537	+12,537	+12,537	+12,537	+12,537
H • y	+31,062	+ 3,974	− 5,057	−14,008	+ 3,974	+31,062
V • x	+23,969	+23,969	+11,985	0	−23,969	−23,969
Total	+ 7,568	−19,520	+19,465	− 1,551	− 7,458	+19,630

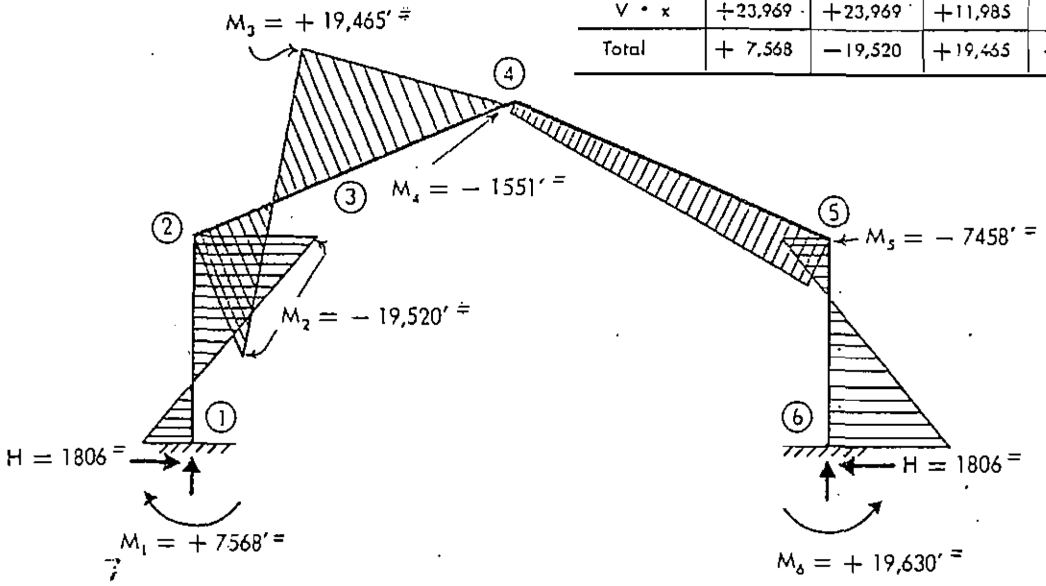


FIGURE 21

is applicable to any type of beam, Figure 22.

The cover-plated beam is representative of any beam in which there is an abrupt change of section . . . and of moment of inertia. The other two common conditions in which there is an abrupt change of section are 1) where plate of heavier thickness is used for the flanges for a short distance at the ends of the beam, and 2) where short lengths of smaller beams are used below the regular beams to reinforce them at and near the points of support.

Constants to Help Calculate Final Moments

Charts have been developed by which the designer can readily find constants to use in determining stiffness factors, carry-over factors, and fixed-end moments for beams.

Sources include:

1. Bull. 176, R. A. Caughy and R. S. Cebula; Iowa Engineering Experiment Station, Iowa State College, Ames, Iowa. 36 charts for beams with cover plates at ends.

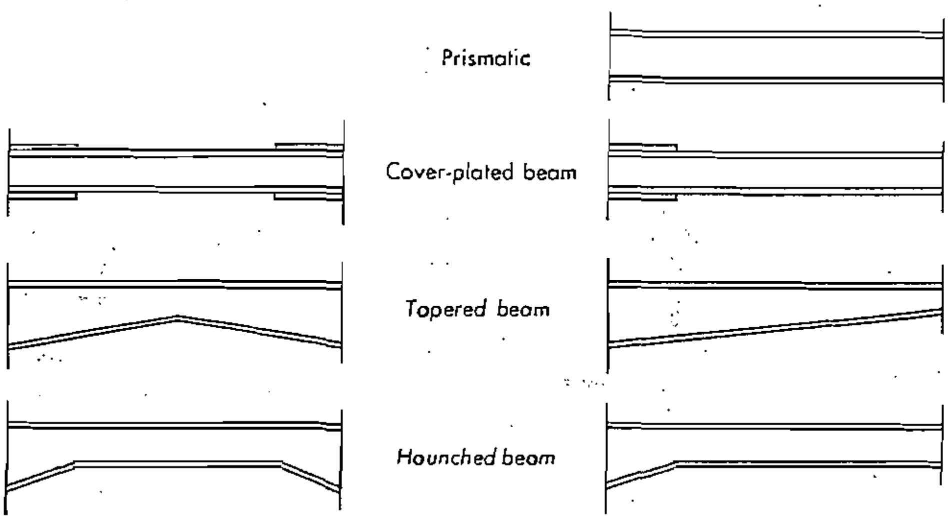


FIGURE 22

2. "Moment Distribution," J. M. Gere, 1963; D. Van Nostrand Co., 378 pages; 29 charts for beams with cover plates at ends; 42 charts for tapered beams.

4. FINDING FIXED END MOMENTS BY COLUMN ANALOGY

Referring back to Topic 2, The Column Analogy Method, the outline of the beam is considered to be the cross-section of a column (or elastic area). See Figure 23.

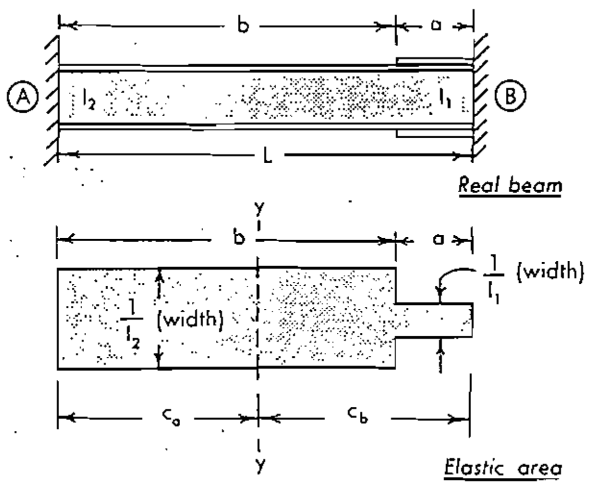


FIGURE 23

The length of the elastic area is equal to the length of the real beam, and the width at any point of the elastic area is equal to the 1/EI of the real beam at the corresponding point. Since we are dealing with steel, the modulus of elasticity (E) is constant and will drop out of the calculations. As the depth and moment of inertia of the real beam increases, the elastic area decreases.

The following design procedure may then be followed.

1. Determine the properties of the elastic area:
   a. Area of the elastic area (A).
   b. Location of axis y-y through the elastic center of the elastic area.
   c. Distance from the elastic center (y-y) to the outer fibers of the elastic area (c_A) and (c_B).
   d. Moment of inertia of the elastic area (I_{y-y}).

2. Release both ends (A) and (B) of the fixed-end beam and draw the moment diagram of this "simply-supported" beam. Use this moment diagram, divided by EI, as the load upon the elastic area (analogous column).

3. The resulting "stresses" at the ends (A) and (B)

$$\sigma = -\frac{P}{A} \pm \frac{M_{y-y}\, c}{I_{y-y}}$$

become the correction moments which must be added to the moment of the "simply-supported" beam to transform it back to the original fixed-end, statically indeterminate beam. Since in this case we started out with zero end moments for the "simply-supported" beam, these correction moments then become the fixed end moments of the final rigid beam:

M_{fe} at end (A)
M_{fe} at end (B)

Stiffness Factor by Column Analogy

The stiffness factor (K) is a measure of the resistance of the member against end rotation. It may be defined as the moment necessary to produce a unit end rotation at the same end, while the opposite end is held fixed:

$$K_A = M_A$$

Carry-Over Factor by Column Analogy

For any applied moment (M_A) at (A), the resulting moment (M_{AB}) at the other end (B) is determined. The carry-over factor is the ratio of these two moments:

$$C_A = -\frac{M_{AB}}{M_A}$$

In both of these two cases, Stiffness Factor and Carry-Over Factor, the fixed-end beam is released at one end (A) and rotated through a unit angle change (ϕ). The resulting end moments (M_A) at (A) and (M_{AB}) at (B) are found.

FIGURE 24

This unit angle rotation is applied as a single load at the outer edge of the elastic area (analogous column), just as an eccentric load might be applied to a real column. See Table 1.

For a uniformly loaded, simply supported beam, the bending moment has the shape of a parabola. It will be helpful to know the loads (P) and distances (e) at the center of gravity of these areas. See Table 2.

5. COLUMN ANALOGY METHOD APPLIED TO BEAMS HAVING ABRUPT CHANGE OF SECTION

The Column Analogy Method will now be used to find the fixed end moments, stiffness factor, and the carry-over factors for a fixed-end beam with cover plates at one end, supporting a uniform load (w). The technique would be applied similarly to any beam having abrupt change of section.

Figure 25 diagrams the real loaded beam, at top, and the elastic area of an analogous column, below. On this elastic area,

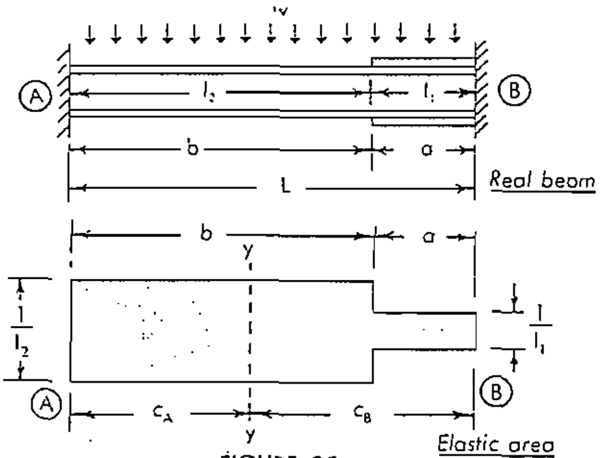
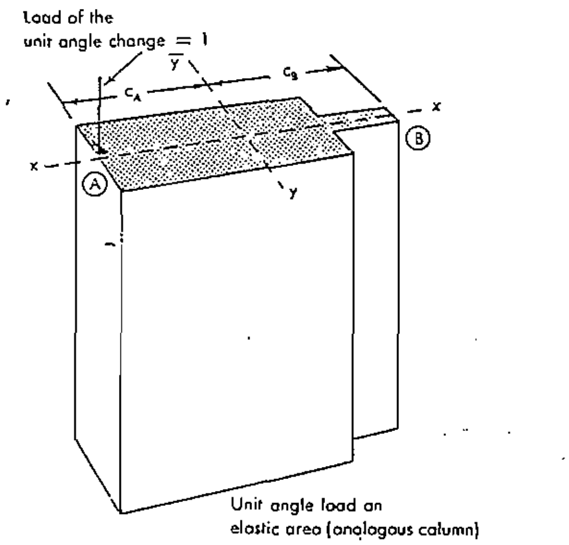


FIGURE 25

TABLE 1—Column Analogy: Unit Angle Rotation



Unit angle change (load) placed at (A)

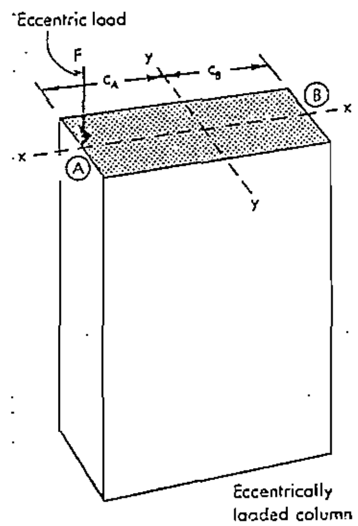
$$M_{y-y} = 1\ c_A$$

at (A)

$$M_A = \text{“}\sigma_A\text{”} = \frac{1}{A} + \frac{M_{y-y}\ c_A}{I_{y-y}} = \boxed{\frac{1}{A} + \frac{1\ c_A^2}{I_{y-y}}}$$

at (B)

$$M_{AB} = \text{“}\sigma_B\text{”} = \frac{1}{A} - \frac{M_{y-y}\ c_B}{I_{y-y}} = \boxed{\frac{1}{A} - \frac{1\ c_A c_B}{I_{y-y}}}$$



Load F placed at (A)

$$M_{y-y} = F\ c_A$$

at (A)

$$\sigma_A = \frac{F}{A} + \frac{M_{y-y}\ c_A}{I_{y-y}} = \frac{F}{A} + \frac{F\ c_A^2}{I_{y-y}}$$

at (B)

$$\sigma_B = \frac{F}{A} - \frac{M_{y-y}\ c_B}{I_{y-y}} = \frac{F}{A} - \frac{F\ c_A c_B}{I_{y-y}}$$

In same manner—
Unit angle change (load) placed at (B)

$$M_{y-y} = 1\ c_B$$

at (B)

$$M_B = \text{“}\sigma_B\text{”} = \boxed{\frac{1}{A} + \frac{1\ c_B^2}{I_{y-y}}}$$

at (A)

$$M_{BA} = \text{“}\sigma_A\text{”} = \boxed{\frac{1}{A} - \frac{1\ c_A c_B}{I_{y-y}}}$$

Load F placed at (B)

$$M_{y-y} = F\ c_B$$

at (B)

$$\sigma_B = \frac{F}{A} + \frac{F\ c_B^2}{I_{y-y}}$$

at (A)

$$\sigma_A = \frac{F}{A} - \frac{F\ c_A c_B}{I_{y-y}}$$

length = actual length of beam

$$\text{width} = \frac{1}{I_{beam}}$$

STEP 1: Determine Properties of this Elastic Area

area

$$A = \frac{a}{I_1} + \frac{b}{I_2}$$

elastic center (y-y)

Take moments about (A).

$$c_A = \frac{\Sigma\ M}{\Sigma\ A}$$

$$= \frac{\frac{a}{I_1}\left(b + \frac{a}{2}\right) + \frac{b}{I_2}\left(\frac{b}{2}\right)}{A}$$

$$c_A = \frac{\frac{a}{I_1}\left(a + 2\ b\right) + \frac{b^2}{I_2}}{2\ A}$$

$$c_B = L - c_A$$

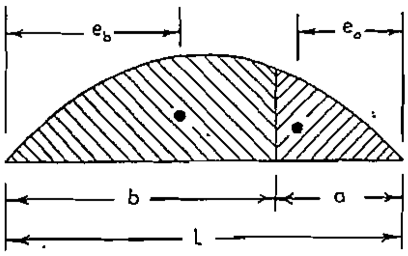
moment of inertia ($I_{y\text{-}y}$)

$$I_{y\text{-}y} = \frac{1}{12}\left(\frac{1}{I_2}\right)b^3 + \frac{1}{12}\left(\frac{1}{I_1}\right)a^3 + \frac{b}{I_2}\left(c_A - \frac{b}{2}\right)^2 + \frac{a}{I_1}\left(c_B - \frac{a}{2}\right)^2$$

STEP 2: Determine the Fixed End Moments

Both ends (A) and (B) of the beam are released so that it now becomes simply supported. This moment diagram now becomes the load on the elastic area, Figure 26.

TABLE 2—Loads and Their Eccentricity



$L = a + b$

Load (P_a) or (P_b) of portion of moment diagram

$$P_b = \frac{w\ b^2}{12\ I_2}\ (3\ a + b)$$

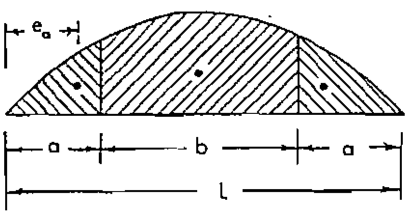
$$P_a = \frac{w\ a^2}{12\ I_1}\ (a + 3\ b)$$

Distance to cg of this load

$$e_a = \frac{a}{2}\left(\frac{a + 4\ b}{a + 3\ b}\right)$$

$$e_b = \frac{b}{2}\left(\frac{4\ a + b}{3\ a + b}\right)$$

w = unit uniform load (lbs/in.)



$L = 2a + b$

Load (P_a) or (P_b) of portion of moment diagram

$$P_b = \frac{w\ b}{12\ I_2}\ (6\ a^2 + 6\ a\ b + b^2)$$

$$P_a = \frac{w\ a^2}{12\ I_1}\ (4\ a + 3\ b)$$

Distance to CG of this load

$$e_a = \frac{a}{2}\left(\frac{5\ a + 4\ b}{4\ a + 3\ b}\right)$$

w = unit uniform load (lbs/in.)

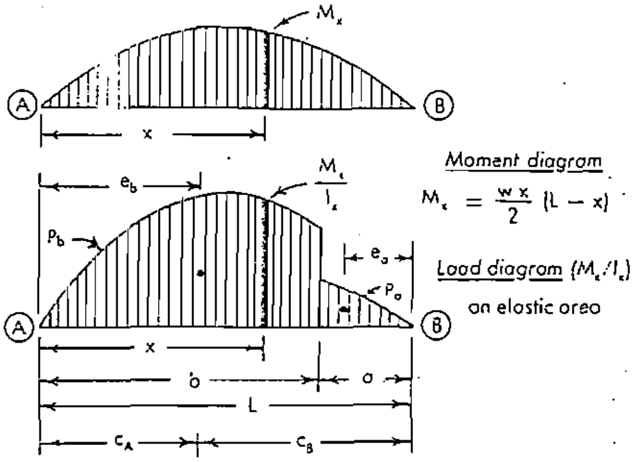


FIGURE 26

"axial" load (P)

$$P_a = \frac{w\ a^2}{12\ I_1}\ (a + 3\ b)$$

$$P_b = \frac{w\ b^2}{12\ I_2}\ (3a + b)$$

$$P = P_a + P_b$$

moment about elastic axis

$$M_{y\text{-}y} = P_a(c_B - e_a) + P_b(e_b - c_A)$$

fixed end moments

This load (P) and ($M_{y\text{-}y}$) on the elastic area causes "stresses" similar to those on an eccentrically loaded column. These "stresses" become the correction moments, or in this case the end moments of the fixed-end beam.

$$\sigma = \frac{P}{A} \pm \frac{M\ c}{I}$$

at A

$$M_{fe} = \frac{P_a + P_b}{A} - \frac{M_{y\text{-}y}\ c_A}{I_{y\text{-}y}}$$

at B

$$M_{fe} = \frac{P_a + P_b}{A} + \frac{M_{y\text{-}y}\ c_B}{I_{y\text{-}y}}$$

STEP 3: Determine Stiffness and Carry-Over Factors

A load of a unit angle change (ϕ) is applied to the elastic area at the outer edge (A), and the resulting end moments (M_A) at (A) and (M_{AB}) at (B) are found.

$$M_{y\text{-}y} = 1\ c_A$$

at (A)

$$M_A = \frac{1}{A} + \frac{1\ c_A^2}{I_{y\text{-}y}}$$

at (B)

$$M_{AB} = \frac{1}{A} - \frac{1\ c_A\ c_B}{I_{y\text{-}y}}$$

Now the load of a unit angle change (ϕ) is applied to the elastic area at the other outer edge (B), and the resulting end moment (M_B) at (B) is found. Notice that the end moment (M_{BA}) at (A) is equal to (M_{AB}) at (B) which is already found.

$$M_{y\text{-}y} = 1\ c_B$$

at (B)

$$M_B = \frac{1}{A} + \frac{1\ c_B^2}{I_{y\text{-}y}}$$

From these three values (M_A), (M_{AB}) and (M_B), the following may be found:

stiffness factor at (A)

$$K_A = M_A$$

stiffness factor at (B)

$$K_B = M_B$$

carry-over factor, (A) *to* (B)

$$C_{AB} = -\frac{M_{AB}}{M_A}$$

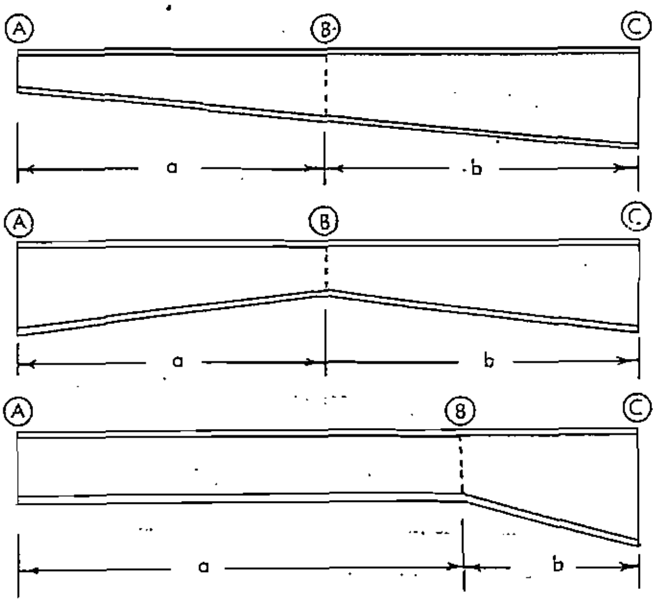
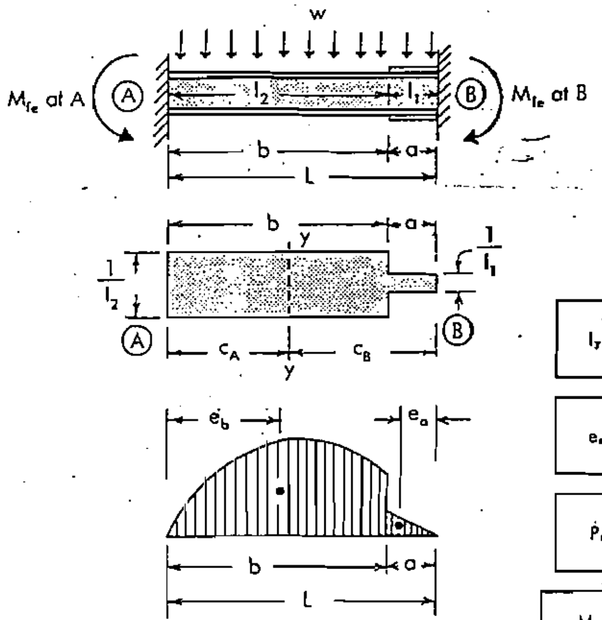


FIGURE 27

TABLE 3—Design Summary: Beam Cover Plated At One End



$$A = \frac{a}{I_1} + \frac{b}{I_2}$$

$$c_A = \frac{\frac{a(a + 2\,b)}{I_1} + \frac{b^2}{I_2}}{2 \cdot A}$$

$$c_B = L - c_A$$

$$I_{y-y} = \frac{a^3}{12\,I_1} + \frac{b^3}{12\,I_2} + \frac{a}{I_1}\left(c_B - \frac{a}{2}\right)^2 + \frac{b}{I_2}\left(c_A - \frac{b}{2}\right)^2$$

$$e_a = \frac{a}{2}\left(\frac{a + 4\,b}{a + 3\,b}\right) \qquad e_b = \frac{b}{2}\left(\frac{4\,a + b}{3\,a + b}\right)$$

$$P_a = \frac{w\,a^2}{12\,I_1}(a + 3\,b) \qquad P_b = \frac{w\,b^2}{12\,I_2}(3a + b)$$

$$M_{y-y} = P_a\,(C_B - e_a) + P_b\,(e_b - c_A)$$

C_a, C_b, e_a + e_b are considered to be (+)

Fixed End Moments

At (A) $$M_{fe} = \frac{P_a + P_b}{A} - \frac{M_{y-y}\,c_A}{I_{y-y}}$$

at (B) $$M_{fe} = \frac{P_a + P_b}{A} + \frac{M_{y-y}\,c_B}{I_{y-y}}$$

End Moments Resulting from Treating Angular Rotation as a Load

$$M_A = \frac{1}{A} + \frac{c_A^2}{I_{y-y}} \qquad M_{AB} = \frac{1}{A} - \frac{c_A\,c_B}{I_{y-y}} \qquad M_B = \frac{1}{A} + \frac{c_B^2}{I_{y-y}}$$

Stiffness Factors

at (A) $K_A = M_A$ at (B) $K_B = M_B$

Carry-Over Factors

from (A) to (B) $$C_{BA} = -\frac{M_{AB}}{M_A}$$

from (B) to (A) $$C_{BA} = -\frac{M_{AB}}{M_B}$$

carry-over factor, (B) *to* (A)

$$C_{BA} = -\frac{M_{AB}}{M_B}$$

Summary

This example of the uniformly-loaded, fixed-end beam with cover plates at *one* end may be summarized as in Table 3.

Modified Example

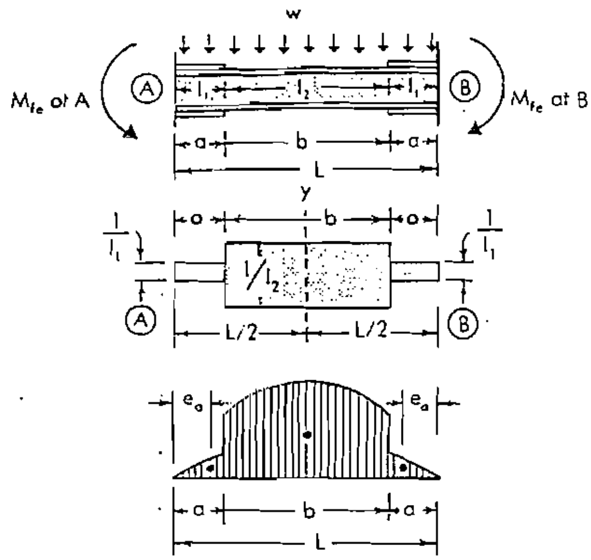
Although the work is not shown, the same fixed-end beam with cover plates at *both* ends, uniformly loaded, may be summarized as in Table 4. (See next page)

6. COLUMN ANALOGY METHOD APPLIED TO BEAMS HAVING GRADUALLY VARYING SECTION

The following method may be used to find the fixed end moments, stiffness factors, and carry-over factors of beams which have constantly varying moments of inertia, such as haunched and tapered beams, Figure 27.

A beam which tapers along a straight line (in other words, its depth increases linearly along the length of the beam, see Fig. 28, top) will have a moment of inertia (I) which does not increase linearly

TABLE 4—Design Summary: Beam Cover Plated At Both Ends



$$A = \frac{2\,a}{I_1} + \frac{b}{I_2}$$

$$I_{y-y} = \frac{L^3 - b^3}{12\,I_1} + \frac{b^3}{12\,I_2}$$

$$P_a = \frac{w\,a^2}{12\,I_2}\,(4\,a + 3\,b)$$

$$P_b = \frac{w\,b}{12\,I_2}\,(6\,a^2 + 6\,a\,b + b^2)$$

here $M_{y-y} = 0$

Fixed End Moments

at (A) and (B) $\quad M_{fe} = \dfrac{2\,P_a + P_b}{A}$

End Moments Resulting from Treating Angular Rotation as a Load

$$M_A = M_B = \frac{1}{A} + \frac{L^2}{4\,I_{y-y}} \qquad M_{AB} = \frac{1}{A} - \frac{L^2}{4\,I_{y-y}}$$

Stiffness Factors

at (A) and (B) $\quad K_A = K_B = M_A$

Carry-Over Factors

from (A) to (B) or from (B) to (A) $\quad c_{AB} = c_{BA} = -\dfrac{M_{AB}}{M_A}$

---

but will have a slight curve (see Fig. 28, center, solid line). This curve approaches a straight line as the beam becomes less tapered.

Although a slight error will be introduced, it will greatly simplify the analysis if we assume this moment of inertia distribution to be a straight (dotted) line. However, this slight error may be reduced by breaking the beam into two parts (see Fig. 28, bottom) and assuming a straight line variation of the moment of inertia between the three points (A), (B), and (C). This is represented by the dashed line in Figure 28, center.

STEP 1: Determine Properties of the Elastic Area

area of elastic area

$$A_x = \frac{a}{I_B - I_A}\log_e\frac{I_B}{I_A}$$

$$A_z = \frac{b}{I_C - I_B}\log_e\frac{I_C}{I_B}$$

moment of elastic area A_x about axis A-A

$$M_{Ax}/_{A-A} = \left(\frac{a}{I_B - I_A}\right)^2 \left(I_B - I_A - I_A \log_e\frac{I_B}{I_A}\right)$$

moment of elastic area A_z about axis B-B

$$M_{Az}/_{B-B} = \left(\frac{b}{I_C - I_B}\right)^2 \left(I_C - I_B - I_B \log_e\frac{I_C}{I_B}\right)$$

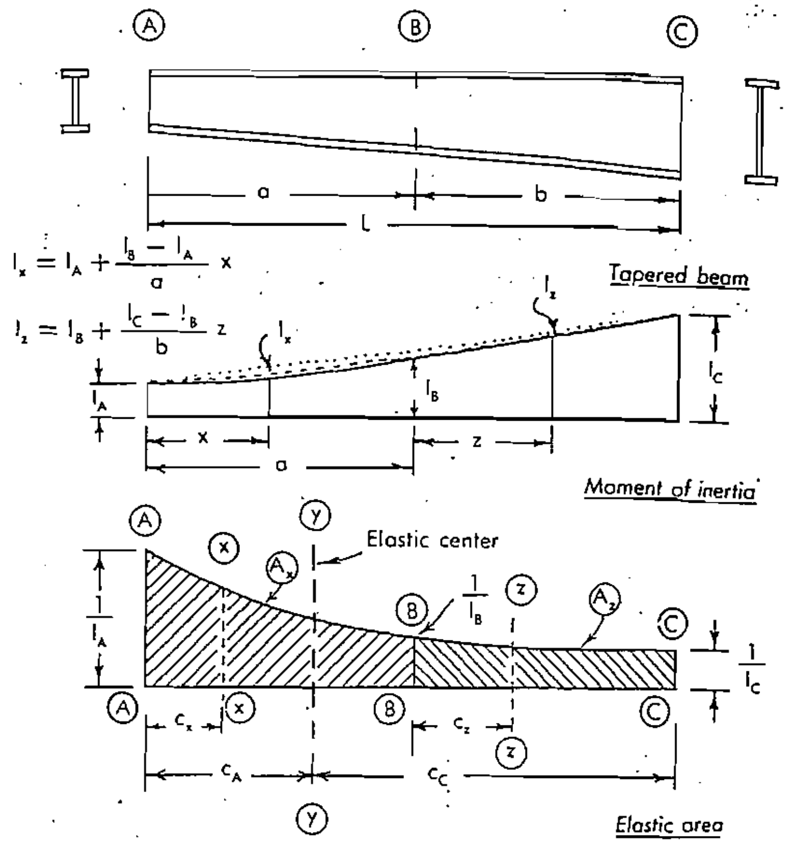


FIGURE 28

distance from C.G. of elastic area A_x to axis A-A

$$c_x = \frac{M_{Ax/A\text{-}A}}{A_x}$$

distance from C.G. of elastic area A_z to axis B-B

$$c_z = \frac{M_{Az/B\text{-}B}}{A_z}$$

moment of elastic area A_z about axis A-A

$$M_{Az/A\text{-}A} = a\ A_z + M_{Az/B\text{-}B}$$

total moment of elastic area about axis A-A

$$M_{A\text{-}A} = M_{Ax/A\text{-}A} + M_{Az/A\text{-}A}$$

elastic center (y-y)

$$c_A = \frac{M_{A\text{-}A}}{A}$$

$$c_C = L - c_A$$

moment of inertia of elastic area A_x about axis A-A

$$I_{Ax/A\text{-}A} = \left(\frac{a}{I_B - I_A}\right)^3 \left[\frac{(I_B - I_A)(I_B - 3\ I_A)}{2} + I_A^2 \log_e \frac{I_B}{I_A}\right]$$

moment of inertia of elastic area A_z about axis B-B

$$I_{Az/B\text{-}B} = \left(\frac{b}{I_C - I_B}\right)^3 \left[\frac{(I_C - I_B)(I_C - 3\ I_B)}{2} + I_B^2 \log_e \frac{I_C}{I_B}\right]$$

Since these moments of inertia can't be added, not being taken about the same axis, it will be necessary to shift axis B-B and axis A-A to the elastic center y-y. If axis A-A is always taken at the shallow end of the tapered beam, negative signs will be avoided in the calculations.

moment of inertia of elastic area A_x about axis y-y

Using the parallel axis theorem:

$$I_{Ax}/_{A\text{-}A} = I_{Ax}/_{x\text{-}x} + A_x c_x^2$$

$$\therefore \quad I_{Ax}/_{x\text{-}x} = I_{Ax}/_{A\text{-}A} - A_x c_x^2$$

Now we wish moments of inertia of A_x about the elastic axis y-y, and again using parallel axis theorem—

$$I_{Ax}/_{y\text{-}y} = I_{Ax}/_{x\text{-}x} + A_x(c_A - c_x)^2$$

or $$I_{Ax}/_{y\text{-}y} = I_{Ax}/_{A\text{-}A} - A_x c_x^2 + A_x(c_A - c_x)^2$$

and $$I_{Ax}/_{y\text{-}y} = I_{Ax}/_{A\text{-}A} + A_x c_A(c_A - 2 c_x)$$

moment of inertia of elastic area A_z about axis y-y

in same manner—

$$I_{Az}/_{y\text{-}y} = I_{Az}/_{B\text{-}B} + A_z\left[(c_z + b - c_A)^2 - c_z^2\right]$$

total moment of inertia of elastic area

$$I_{y\text{-}y} = I_{Ax}/_{y\text{-}y} + I_{Az}/_{y\text{-}y}$$

STEP 2: Determine the Fixed End Moments

The moment diagram from the applied load on the real beam is divided by the moment of inertia (I) of the real beam, and becomes the load (M/I) on the elastic area which is treated as a column.

The axial load (P) applied to the elastic area is equal to the total M/I. This axial load applied at some distance from the elastic center of the elastic area causes a moment (M) on the elastic area.

Both of these loads cause "stresses" on the elastic area.

The following applies if the designer can assume a uniform load (w):

axial load (P) applied to elastic area

$$P_x = \frac{w}{2}\left(L\, M_{Ax}/_{A\text{-}A} - I_{Ax}/_{A\text{-}A}\right)$$

$$P_z = \frac{w}{2}\Big[- I_{Az}/_{B\text{-}B} + (L - 2\,a)\, M_{Az}/_{B\text{-}B} + a\,(L - a)\, A_y\Big]$$

and,

$$P = P_x + P_z$$

momen (M) applied to elastic area about its elastic center

$$M_x/_{y\text{-}y} = \frac{w}{2}\Big[- Q_x + (L + c_A)\, I_{Ax}/_{A\text{-}A} - c_A\, L\, M_{Ax}/_{A\text{-}A}\Big]$$

where:

$$Q_x = \left(\frac{a}{I_B - I_A}\right)^4 \left[\left(\frac{I_B - I_A}{6}\right)(2\, I_B^2 - 7\, I_B\, I_A + 11\, I_A^2) - I_A^3 \log_e \frac{I_B}{I_A}\right]$$

$$M_z/_{y\text{-}y} = \frac{w}{2}\Big[- Q_z + (L - 3\,a + c_A)\, I_{Az}/_{B\text{-}B} + [a(2\,L - 3a) - c_A(L - 2a)]\, M_{Az}/_{B\text{-}B} + a\,(L - a)(a - c_A)\, A_y\Big]$$

where:

$$Q_z = \left(\frac{b}{I_C - I_B}\right)^4 \left[\left(\frac{I_C - I_B}{6}\right)(2\, I_C^2 - 7\, I_C\, I_B + 11\, I_B^2) - I_B^3 \log_e \frac{I_C}{I_B}\right]$$

and the total moment—

$$M_{y\text{-}y} = M_x/_{y\text{-}y} + M_z/_{y\text{-}y}$$

fixed end moments

at (A)

$$M_{fe} = \frac{P}{A} - \frac{M_{y\text{-}y}\, c_A}{I_{y\text{-}y}}$$

at (C)

$$M_{fe} = \frac{P}{A} + \frac{M_{y\text{-}y}\, c_C}{I_{y\text{-}y}}$$

STEP 3: Determine Stiffness and Carry-Over Factors

$$M_A = \frac{1}{A} + \frac{c_A^2}{I_{y\text{-}y}}$$

$$M_{AC} = \frac{1}{A} - \frac{c_A\, c_C}{I_{y\text{-}y}}$$

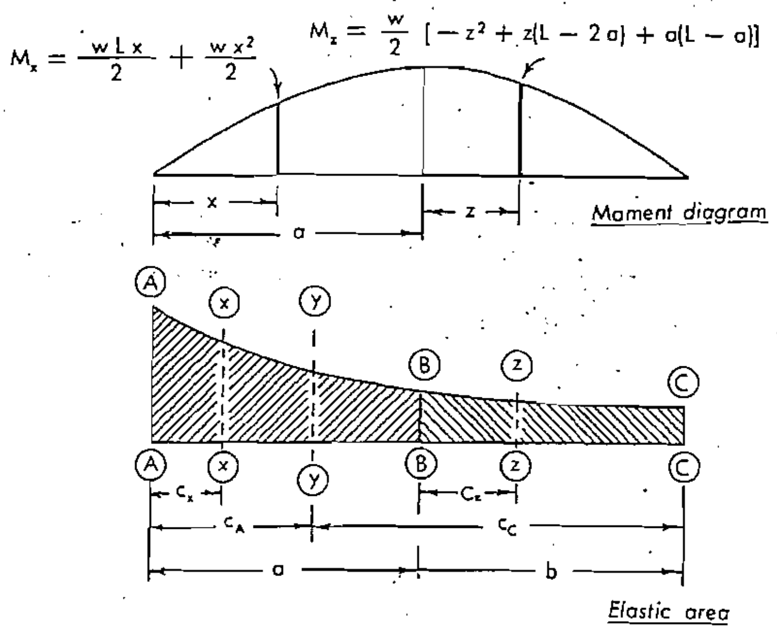


FIGURE 29

$$M_C = \frac{1}{A} + \frac{c_C^2}{I_{y\text{-}y}}$$

stiffness factor at (A)

$$K_A = M_A$$

stiffness factor at (C)

$$K_C = M_C$$

carry-over factor, (A) *to* (C)

$$C_{AC} = -\frac{M_{AC}}{M_A}$$

carry-over factor, (C) *to* (A)

$$C_{CA} = -\frac{M_{AC}}{M_C}$$

Problem 2

For the uniformly-loaded beam shown at top in Figure 30, having fixed ends, find the fixed end moments, stiffness factor, and carry-over factors.

At center in Figure 30, the solid curve is the actual moment of inertia (I) as it varies along the length of the beam. The dashed line is the assumed straight-line variation in moment of inertia along the two halves of the tapered beam.

The following properties are established:

$a = 100''$ $\quad I_B^2 = 6.4516 \times 10^6$

$I_A = 646.7$ $\quad I_B^3 = 1.6386 \times 10^{10}$

$I_A^2 = 4.1822 \times 10^5$ $\quad L = 100''$

$I_A^3 = 3.678 \times 10^8$ $\quad I_C = 5930$

$b = 100''$ $\quad I_C^2 = 3.5163 \times 10^7$

$I_B = 2540$

Then proceed first to find formula elements made up of these properties:

$$\log_e n = 2.3026 \quad \log_{10} n$$

$$\log_e \frac{I_B}{I_A} = \log_e \frac{(2540)}{(6467)} = \log_e 3.9276$$

$$= 1.3680$$

$$\log_e \frac{I_C}{I_B} = \log_e \frac{(5930)}{(2540)} = \log_e 2.3346$$

$$= .84780$$

$$I_B - I_A = (2540) - (646.7)$$

$$= 1893.3$$

$$I_C - I_B = (5930) - (2540)$$

$$= 3390$$

$$\frac{a}{(I_B - I_A)} = \frac{(100)}{(1893.3)}$$

$$= .052813$$

$$\left(\frac{a}{(I_B - I_A)}\right)^2 = 2.7892 \times 10^{-3}$$

$$\left(\frac{a}{(I_B - I_A)}\right)^3 = 1.4731 \times 10^{-4}$$

$$\left(\frac{a}{(I_B - I_A)}\right)^4 = .77800 \times 10^{-5}$$

$$\frac{b}{(I_C - I_B)} = \frac{(100)}{(3390)} = .029499$$

$$\left(\frac{b}{(I_C - I_B)}\right)^2 = .8722 \times 10^{-3}$$

$$\left(\frac{b}{(I_C - I_B)}\right)^3 = 2.5728 \times 10^{-5}$$

$$\left(\frac{b}{(I_C - I_B)}\right)^4 = .75895 \times 10^{-6}$$

STEP 1: Determine Properties of the Elastic Area

area of elastic area

$$A_x = \frac{a}{(I_B - I_A)} \log_e \frac{I_B}{I_A}$$

$$= (.052813)(1.3680)$$

$$= .072252$$

$$A_z = \frac{b}{(I_C - I_B)} \log_e \frac{I_C}{I_B}$$

$$= (.029499)(.84780)$$

$$= .025008$$

$$A = A_x + A_z$$

$$= (.072252) + (.025008)$$

$$= .097260$$

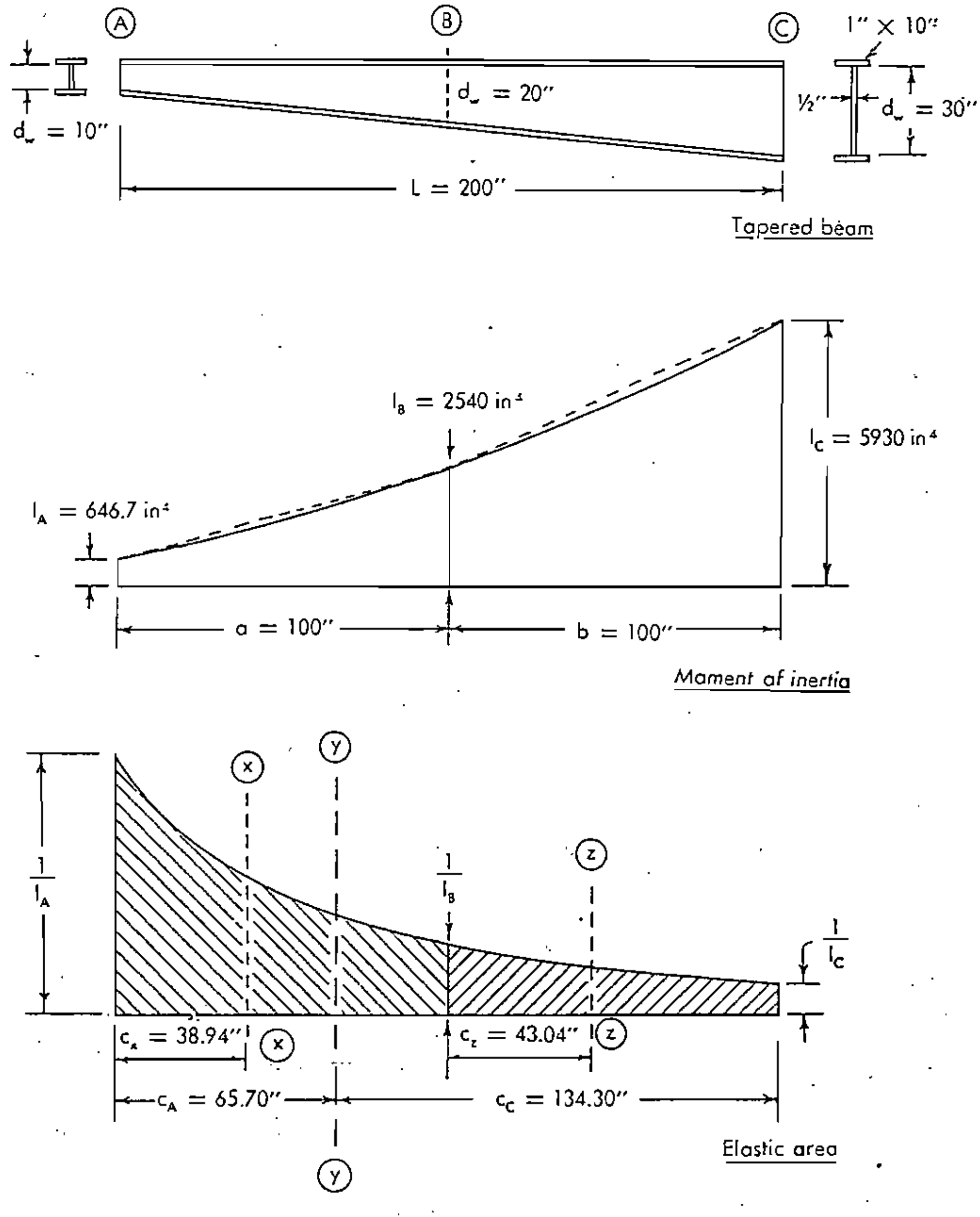


FIGURE 30

moments of elastic area

$$M_{Ax}/_{A\text{-}A} = \left(\frac{a}{(I_B - I_A)}\right)^2 \left(I_B - I_A - I_A \log_e \frac{I_B}{I_A}\right)$$

$$= (2.7892 \times 10^{-3})(1893.3 - 646.7 \times 1.3680)$$

$$= 2.8132$$

$$M_{Az}/_{B\text{-}B} = \left(\frac{b}{(I_C - I_B)}\right)^2 \left(I_C - I_B - I_B \log_e \frac{I_C}{I_B}\right)$$

$$= (.8722 \times 10^{-3})(3390 - 2540 \times .84780)$$

$$= 1.0762$$

$$c_x = \frac{M_{Ax}/_{A\text{-}A}}{A_x}$$

$$= \frac{(2.8132)}{(.072252)}$$

$$= 38.937''$$

$$c_z = \frac{M_{Az}/_{B\text{-}B}}{A_z}$$

$$= \frac{(1.0762)}{(.025008)}$$

$$= 43.038''$$

$$M_{Az}/_{A\text{-}A} = A_z a + M_{Az}/_{B\text{-}B}$$

$$= (.025008)(100) + (1.0762)$$

$$= 3.5570$$

$$M_{A\text{-}A} = M_{Ax}/_{A\text{-}A} + M_{Az}/_{A\text{-}A}$$

$$= (2.8132) + (3.5570)$$

$$= 6.3902$$

$$c_A = \frac{M_{A\text{-}A}}{A}$$

$$= \frac{(6.3903)}{(.097260)}$$

$$= 65.70''$$

$$c_C = L - c_A$$

$$= (200) - (65.70)$$

$$= 134.30''$$

$$I_{Ax}/_{A\text{-}A} = \left(\frac{a}{(I_B - I_A)}\right)^3 \left(\frac{(I_B - I_A)(I_B - 3\, I_A)}{2} + I_A^2 \log_e \frac{I_B}{I_A}\right)$$

$$= \left(1.4731 \times 10^{-4}\right)\left(\frac{1893.3 \times 599.9}{2} + 4.1822 \times 10^5 \times 1.3680\right)$$

$$= 167.93$$

$$I_{Az}/_{B\text{-}B} = \left(\frac{b}{(I_C - I_B)}\right)^3 \left(\frac{(I_C - I_B)(I_C - 3\, I_B)}{2} + I_B^2 \log_e \frac{I_C}{I_B}\right)$$

$$= \left(2.5728 \times 10^{-5}\right)\left(\frac{(3390)(-1690)}{2} + 6.4516 \times 10^6 \times .84790\right)$$

$$= 67.02$$

$$I_{Ax}/_{y\text{-}y} = I_{Ax}/_{A\text{-}A} + A_x c_A (c_A - 2\, c_x)$$

$$= (167.93) + (.072252)(65.70)(65.70 - 2 \times 38.937)$$

$$= 110.16$$

$$I_{Az}/_{y\text{-}y} = I_{Az}/_{B\text{-}B} + A_z \left[(c_z + b - c_A)^2 - c_z^2\right]$$

$$= (67.02 + .025008)\left[(43.04 + \frac{200}{2} - 65.70)^2 - 43.04^2\right]$$

$$= 170.28$$

$$I_{y\text{-}y} = I_{Ax}/_{y\text{-}y} + I_{Az}/_{y\text{-}y}$$

$$= (110.16) + (170.28)$$

$$= 280.44$$

$$P_x = \frac{w}{2}\left(L\, M_{Ax}/_{A\text{-}A} - I_{Ax}/_{A\text{-}A}\right)$$

$$= \frac{w}{2}\,(200 \times 2.8132 - 167.93)$$

$$= 197.36\ w$$

$$P_z = \frac{w}{2}\left(\frac{L^2}{4}\, A_z - I_{Az}/_{B\text{-}B}\right)$$

$$= \frac{w}{2}\left(\frac{200^2}{4}\, .025008 - 67.02\right)$$

$$= 91.53\ w$$

$$P = P_x + P_z$$

$$= (197.36\ w) + (91.53\ w)$$

$$= 288.89\ w$$

$$Q_x = \left(\frac{a}{(I_B - I_A)}\right)^4 \left[\left(\frac{(I_B - I_A)}{6}\right)(2\ I_B^2 - 7\ I_B\ I_A + 11\ I_A^2) - I_A^3 \log_e \frac{I_B}{I_A}\right]$$

$$= (.7780 \times 10^{-5}) \left[\frac{1893.3}{6}[2 \times 6.4516 \times 10^6 - 7(2450)(646.7) + 11(4.1822 \times 10^5] - 3.678 \times 10^8(1.3680)\right]$$

$$= 11{,}878$$

$$Q_z = \left(\frac{b}{(I_C - I_B)}\right)^4 \left[\left(\frac{(I_C - I_B)}{6}\right)(2\ I_C^2 - 7\ I_C\ I_B + 11\ I_B^2) - I_B^3 \log_e \frac{I_C}{I_B}\right]$$

$$= (.75895 \times 10^{-6}) \left[\left(\frac{3390}{6}\right)[2 \times 3.5163 \times 10^7 - 7(5930)(2540) - 11(6.4516 \times 10^6)] - 1.6386 \times 10^{10}(.84780)\right]$$

$$= 4827$$

$$M_x/_{y\text{-}y} = \frac{w}{2}\left[- Q_x + (L + c_A)\ I_{Ax}/_{A\text{-}A} - c_A\ L\ M_{AX}/_{A\text{-}A}\right]$$

$$= \frac{w}{2}\left[- 11{,}878 + (265.70)167.93 - 65.70(200)(2.8132)\right]$$

$$= - 2113\ w$$

$$M_z/_{y\text{-}y} = \frac{w}{2}\left[- Q_z + (L - 3\ a + c_A)\ I_{Az}/_{B\text{-}B} + [a(2\ L - 3\ a) - c_A(L - 2\ a)]\ M_{Az}/_{B\text{-}B} + a\ (L - a)(a - c_A)\ A_z\right]$$

$$= \frac{w}{2}\left[- 4827 + (-34.30)67.02 + 10{,}000(1.0762) + 343{,}000(0.25008)\right]$$

$$= 6107\ w$$

$$M_{y\text{-}y} = M_x/_{y\text{-}y} + M_z/_{y\text{-}y}$$

$$= (- 2113\ w) + (6107\ w)$$

$$= + 3994\ w$$

STEP 2: Determine the Fixed End Moments

at (A)

$$M_{te} = \frac{P}{A} - \frac{M_{y\text{-}y}\ c_A}{I_{y\text{-}y}}$$

$$= \frac{(288.89\ w)}{(.09726)} - \frac{(3994\ w)(65.70)}{(280.44)}$$

$$= \underline{2034.5\ w}$$

at (C)

$$M_{te} = \frac{P}{A} + \frac{M_{y\text{-}y}\ c_C}{I_{y\text{-}y}}$$

$$= \frac{(288.89\ w)}{(.09726)} + \frac{(3994\ w)(134.30)}{(280.44)}$$

$$= \underline{4882.8\ w}$$

STEP 3: Determine Stiffness and Carry-Over Factors

$$M_A = \frac{1}{A} + \frac{c_A^2}{I_{y\text{-}y}}$$

$$= \frac{1}{(.09726)} + \frac{(65.70^2)}{(280.44)}$$

$$= 25.67$$

$$M_{AC} = \frac{1}{A} - \frac{c_A\ c_C}{I_{y\text{-}y}}$$

$$= \frac{1}{(.09726)} - \frac{(65.70)(134.30)}{(280.44)}$$

$$= - 21.18$$

$$M_C = \frac{1}{A} + \frac{c_C^2}{I_{y\text{-}y}}$$

$$= \frac{1}{(.09726)} + \frac{(134.30^2)}{(280.44)}$$

$$= 74.59$$

stiffness factor at (A)

$$K_A = M_A = \underline{25.67}$$

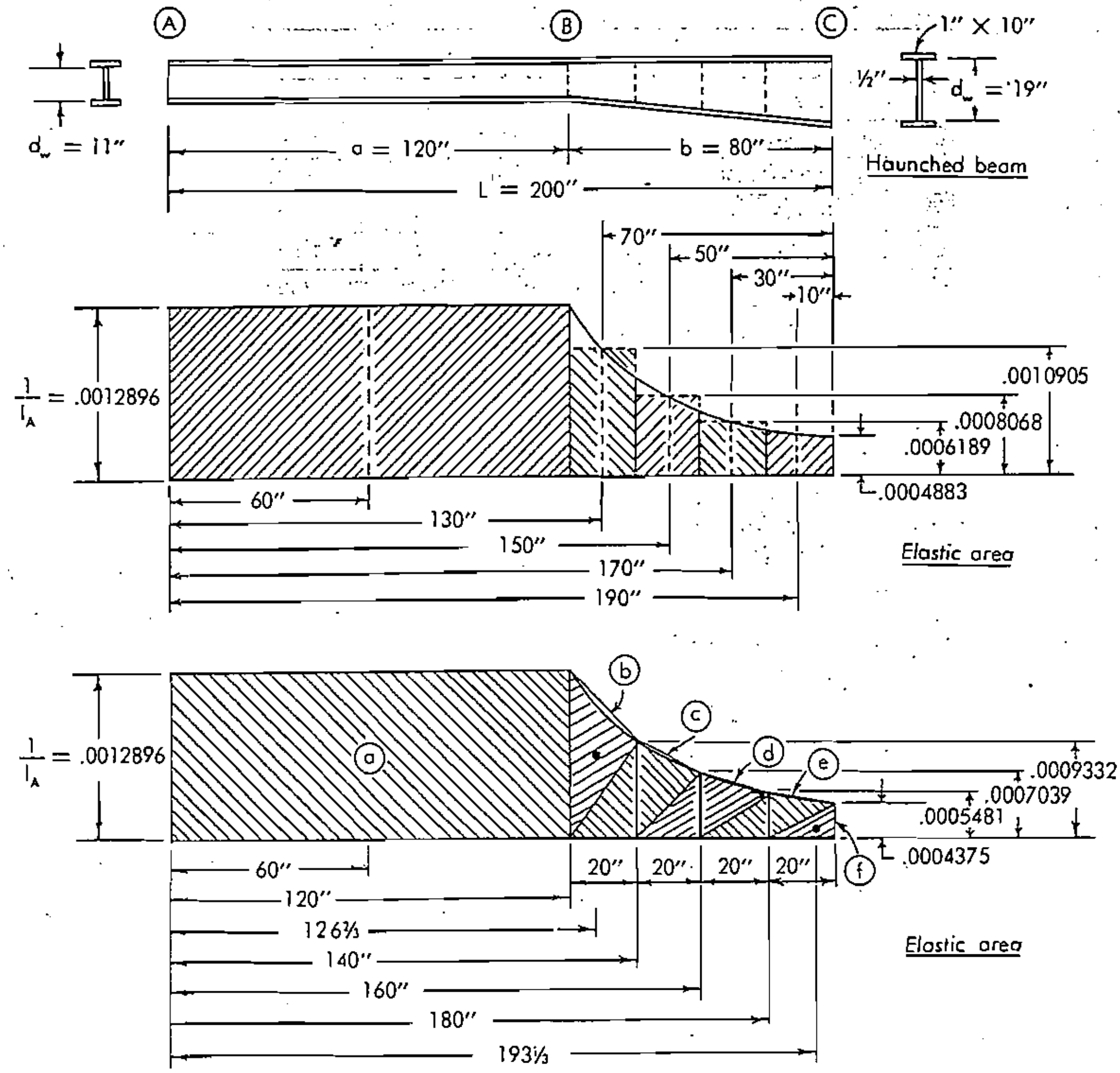


FIGURE 31

stiffness factor at C

$$K_C = M_C = \underline{74.59}$$

carry-over factor, A *to* C

$$C_{AC} = -\frac{M_{AC}}{M_A} = -\frac{(-21.18)}{(25.67)} = \underline{.825}$$

carry-over factor, C *to* A

$$C_{CA} = -\frac{M_{AC}}{M_C} = -\frac{(-21.18)}{(74.59)} = \underline{.284}$$

Problem 3

For the haunched beam at top in Figure 31, having fixed ends, find the fixed end moments (uniformly loaded), stiffness factors, and carry-over factors. Break beam into sections and use numerical integration.

The elastic area could be divided into rectangular areas, as at center in Figure 31, and the resulting properties of the elastic area found in this manner. Of course some error will be introduced because these rectangular areas do not quite equal the actual curve of the elastic area. However, as the number of divisions is increased, this error will decrease.

Without any additional work, the following method will more nearly fit the outline of the elastic area and will result in less error. See lower diagram, Figure 31. The curved portion within the elastic area is divided into triangular areas. It is noticed that a pair of tri-

Section	A (area)	y'	M(moments)	$I_{y}' = M \cdot y'$	I_g
a	(.0012896) 120 = .154752	60	9.2851	557.11	185.70
b	½(.0012896) 20 = .012896	126⅔	1.6335	206.91	.29
c	½(.0009332) 40 = .018664	140	2.6130	368.38	1.24
d	½(.0007039) 40 = .014078	160	2.2525	360.40	.94
e	½(.0005481) 40 = .010962	180	1.9732	355.17	.73
f	½(.0004375) 20 = .004375	193⅓	.8458	163.52	.09
Total →	.21573		18.6036	2199.69	

angular areas share the same altitude and since the division in length (s) is the same, they will have the same area. Therefore, the center of gravity of the two triangles lies along their common altitude. (This graphical method is applicable to any beam with a non-uniform change in moment of inertia along its length).

STEP 1: Determine the Properties of the Elastic Area

elastic center

$$c_A = \frac{M}{A}$$

$$= \frac{(18.6036)}{(.21573)}$$

$$= 86.23''$$

$$c_C = L - c_A$$

$$= (200) - (86.23)$$

$$= 113.77''$$

moment of inertia

$$I_{y\text{-}y} = I_y + I_g - \frac{M^2}{A}$$

$$= (2199.7) - \frac{(18.6036)^2}{(.21573)}$$

$$= 2199.7 - 1604.2 = 595.5$$

$$= 595.5$$

area (A) of section (a) of M_x/I_x diagram

$$A = \frac{w\ a^2}{12}\ (a + 3\ b)$$

$$= \frac{w(120)^2}{12}\ (120 + 3 \times 80)$$

$$= 557.10\ w$$

center of gravity of section (a)

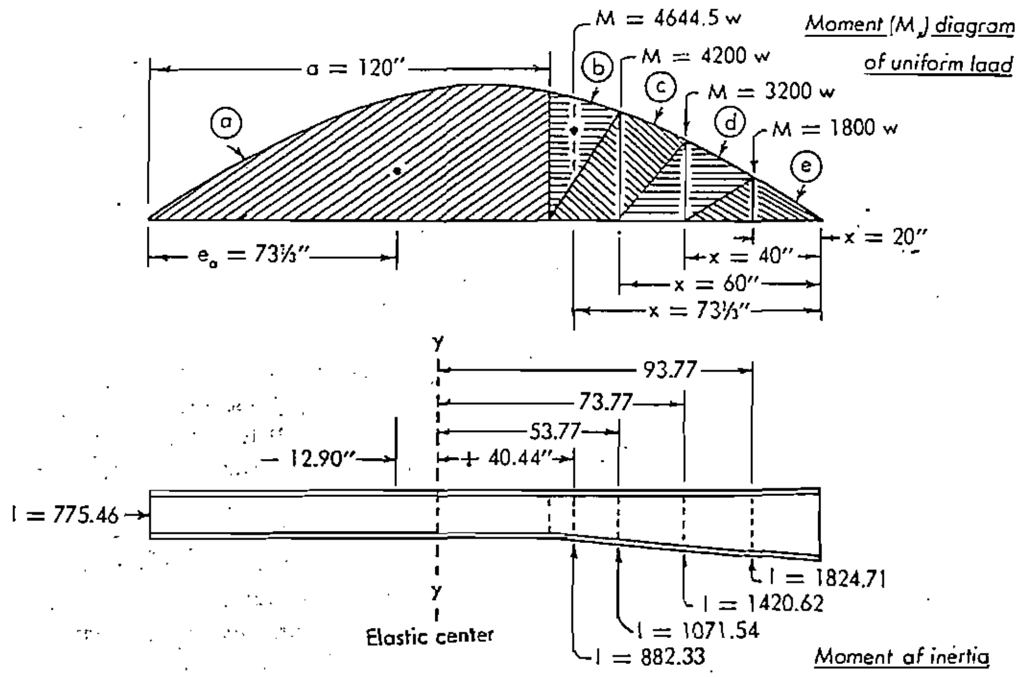


FIGURE 32

Section	C.G.	M_x	I_x	P (area of $\frac{M_x}{I_x}$ diagram)	y	M_{y-y}
a	$e_a = 73\frac{1}{3}$			see above 557.10 w	−12.90	−7,186.5 w
b	$x = 73\frac{1}{3}$	4644.5 w	882.33	$\frac{1}{2}\left(\frac{4644.5\ w}{882.33}\right) 20 =$ 52.628 w	+40.44	+2,128.7 w
c	$x = 60$	4200.0 w	1071.54	$\frac{1}{2}\left(\frac{4200.0\ w}{1071.54}\right) 40 =$ 78.392 w	+53.77	+4,215.2 w
d	$x = 40$	3200.0 w	1420.62	$\frac{1}{2}\left(\frac{3200.0\ w}{1420.62}\right) 40 =$ 45.052 w	+73.77	+3,323.4 w
e	$x = 20$	1800.0 w	1824.71	$\frac{1}{2}\left(\frac{1800.0\ w}{1824.71}\right) 40 =$ 19.728 w	+93.77	+1,849.9 w
				Total → +752.91 w		+4,321.7 w

$$e_a = \frac{a}{2}\left(\frac{a + 4\ b}{a + 3\ b}\right)$$

$$= \frac{(120)}{2}\left(\frac{120 + 4 \times 80}{120 + 3 \times 80}\right)$$

$$= 73\tfrac{1}{3}''$$

other properties of M_x/I_x diagram

These are shown in the table above.

STEP 2: Determine the Fixed End Moments

at (A)

$$M_{fe} = \frac{P}{A} - \frac{M_{y\text{-}y}\ c_A}{I_{y\text{-}y}}$$

$$= \frac{(+752.91\ w)}{(.21573)} - \frac{(+4321.7\ w)(86.23)}{(595.5)}$$

$$= \underline{+\ 2863.0\ w}$$

at (C)

$$M_{fe} = \frac{P}{A} + \frac{M_{y\text{-}y}\ c_C}{I_{y\text{-}y}}$$

$$= \frac{(+752.91\ w)}{(.21573)} + \frac{(+4321.7\ w)(113.72)}{(595.5)}$$

$$= \underline{+4{,}3141.1\ w}$$

STEP 3: Determine Stiffness and Carry-Over Factors

$$M_A = \frac{1}{A} + \frac{c_A^2}{I_{y\text{-}y}}$$

$$= \frac{1}{(.21573)} + \frac{(86.23)^2}{(595.5)}$$

$$= +\ 17.12$$

$$M_{AC} = \frac{1}{A} - \frac{c_A\ c_C}{I_{y\text{-}y}}$$

$$= \frac{1}{(.21573)} + \frac{(86.23)(113.72)}{(595.5)}$$

$$= -\ 12.10$$

$$M_C = \frac{1}{A} + \frac{c_C^2}{I_{y\text{-}y}}$$

$$= \frac{1}{(.21573)} + \frac{(113.72)^2}{(595.5)}$$

$$= +\ 26.35$$

stiffness factor at (A)

$$K_A = M_A = \underline{17.12}$$

stiffness factor at (C)

$$K_C = M_C = \underline{26.35}$$

carry-over factor, (A) *to* (C)

$$C_{AC} = -\ \frac{M_{AC}}{M_A}$$

$$= -\ \frac{(-12.10)}{(17.12)}$$

$$= \underline{.706}$$

carry-over factor, (C) *to* (A)

$$C_{CA} = -\ \frac{M_{AC}}{M_C}$$

$$= -\ \frac{(-12.10)}{(26.35)}$$

$$= \underline{.460}$$

7. READY-TO-USE DESIGN CONSTANTS

The following 36 charts—appearing on the following pages—give the fixed end moments, stiffness factors, and carry-over factors for beams with abrupt changes in moment of inertia and may be used for beams with cover plates. They were developed by R. A. Caughy, Professor of Civil Engineering, Iowa State College, and Richard S. Cebula, Head, Engineering Department, St. Martin's College. These charts appeared in Bull. 176 of the Iowa Engineering Experiment Station.

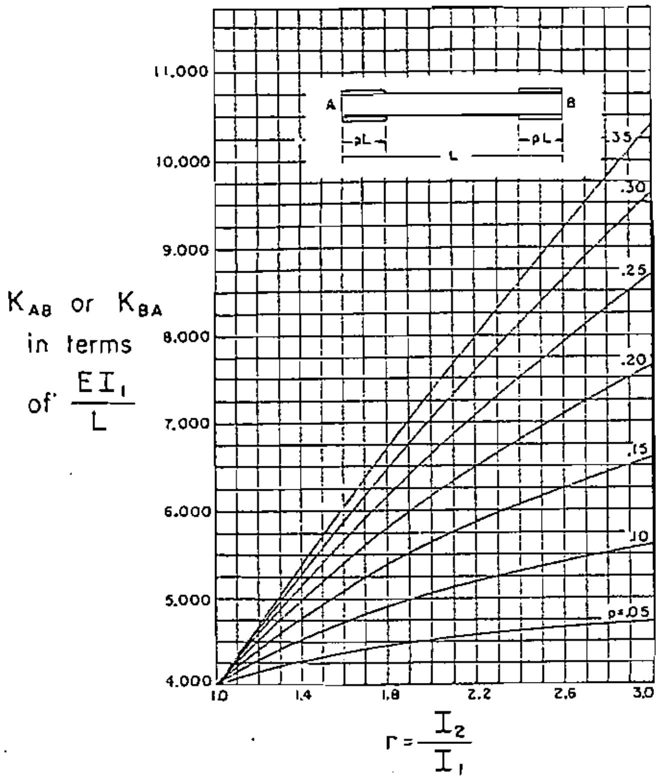


Chart 1. Stiffness factors at either end of symmetrical beam.

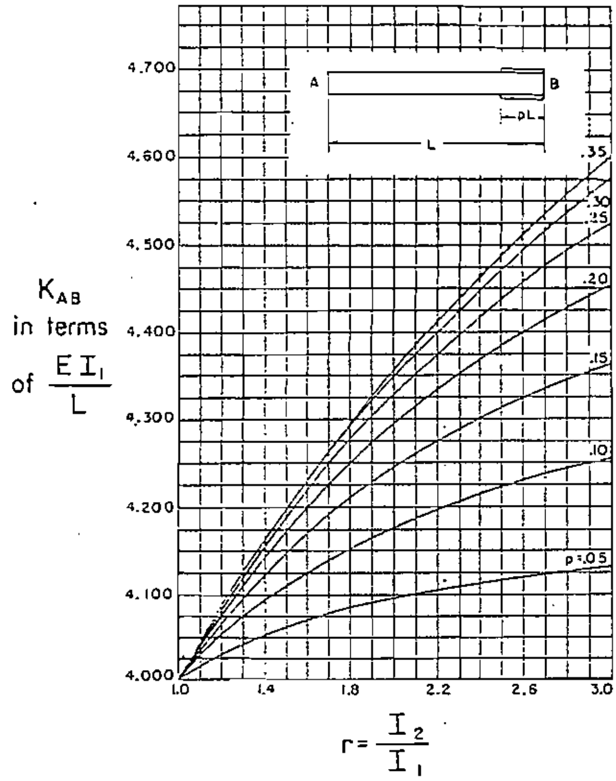


Chart 2. Stiffness factors at small end of unsymmetrical beam.

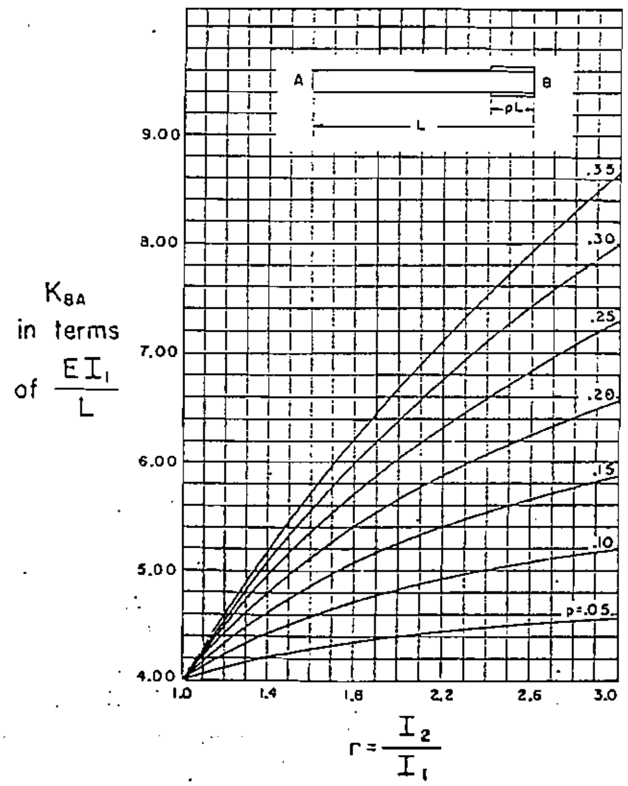


Chart 3. Stiffness factors at large end of unsymmetrical beam.

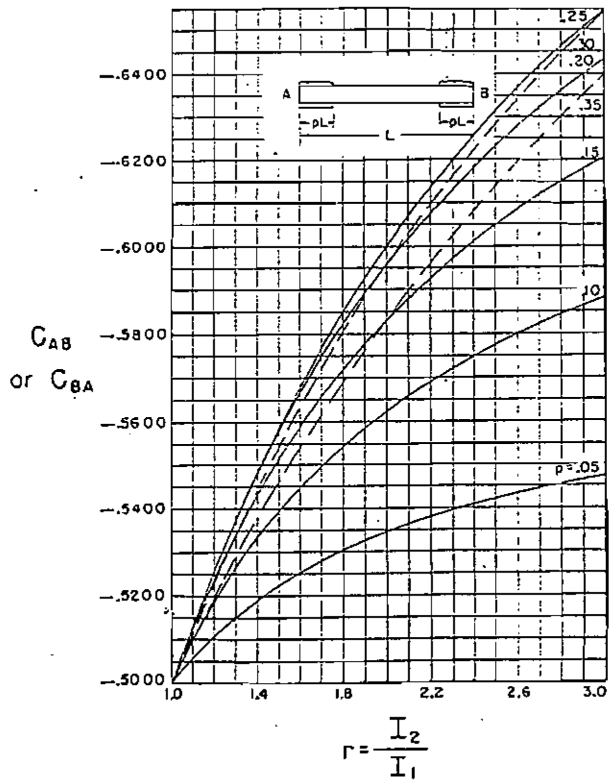


Chart 4. Carry-over factors for symmetrical beam from either end to the other.

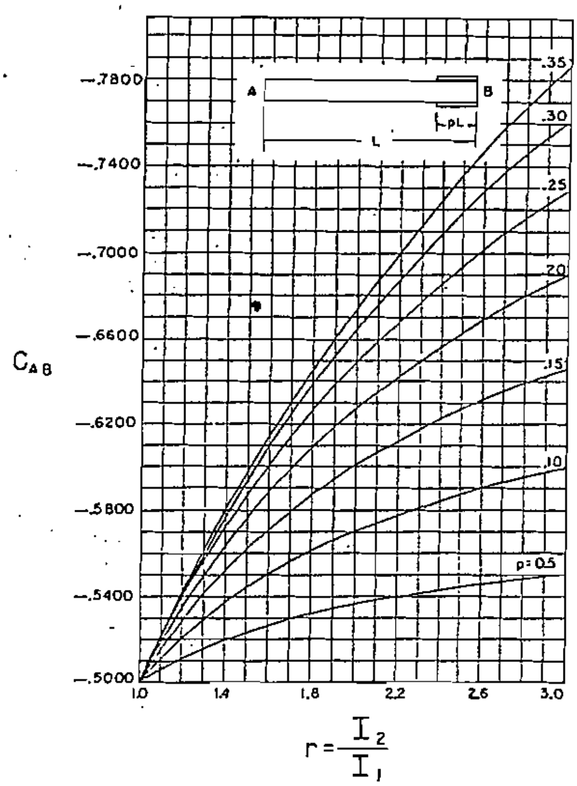


Chart 5. Carry-over factors for unsymmetrical beam from small end to large end.

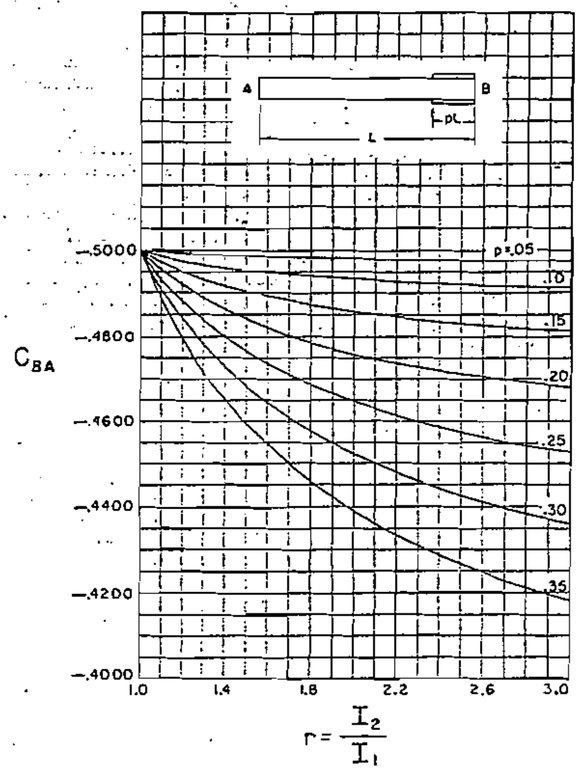


Chart 6. Carry-over factors for unsymmetrical beam from large end to small end.

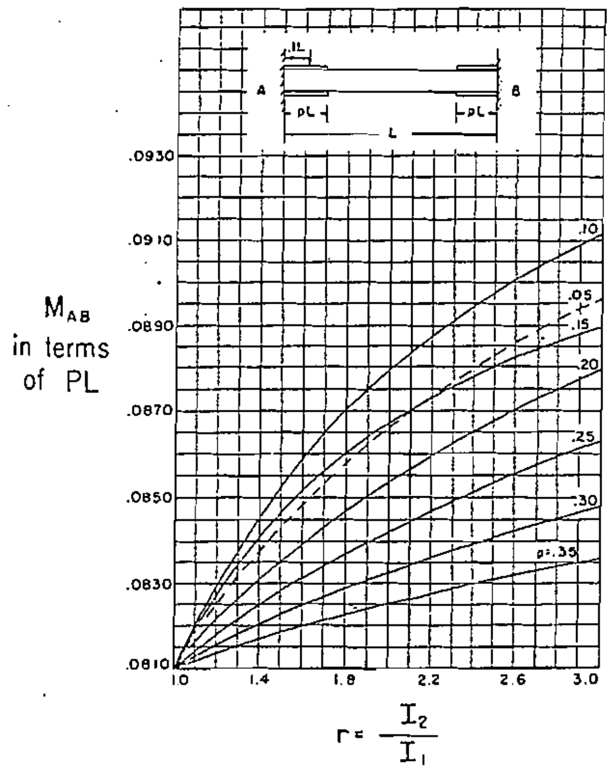


Chart 7. Fixed-end moments at left end of symmetrical beam for concentrated load at .1 point.

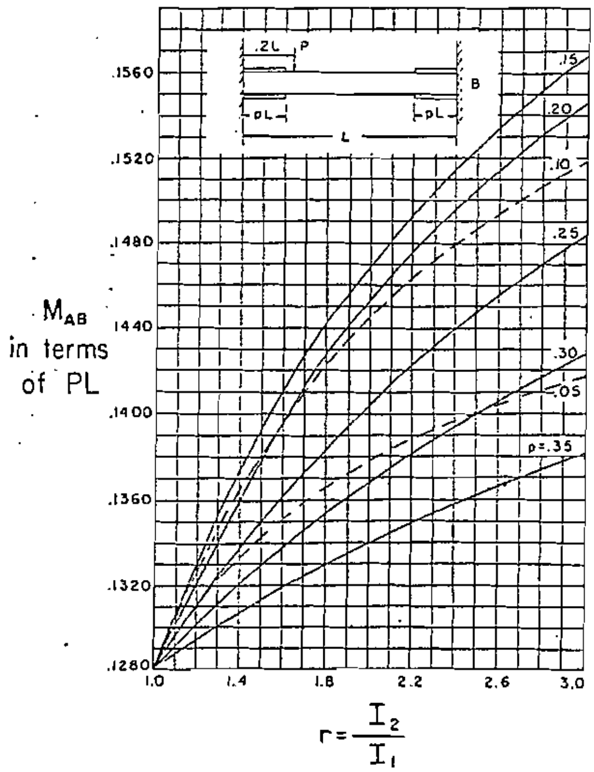


Chart 8. Fixed-end moments at left end of symmetrical beam for concentrated load at .2 point.

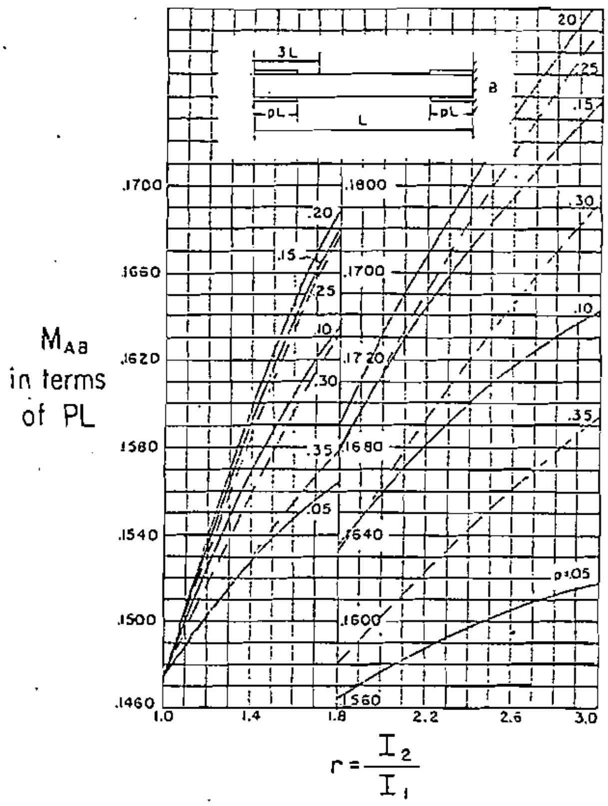


Chart 9. Fixed-end moments at left end of symmetrical beam for concentrated load at .3 point.

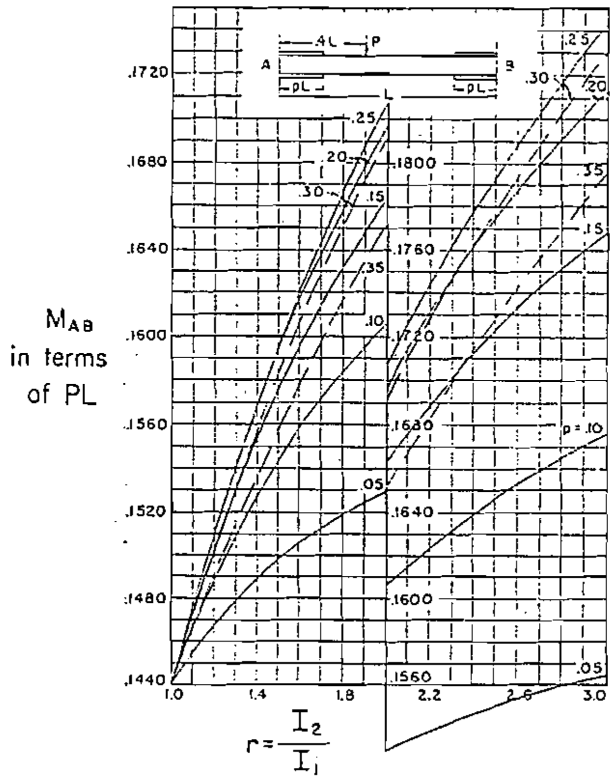


Chart 10. Fixed-end moments at left end of symmetrical beam for concentrated load at .4 point.

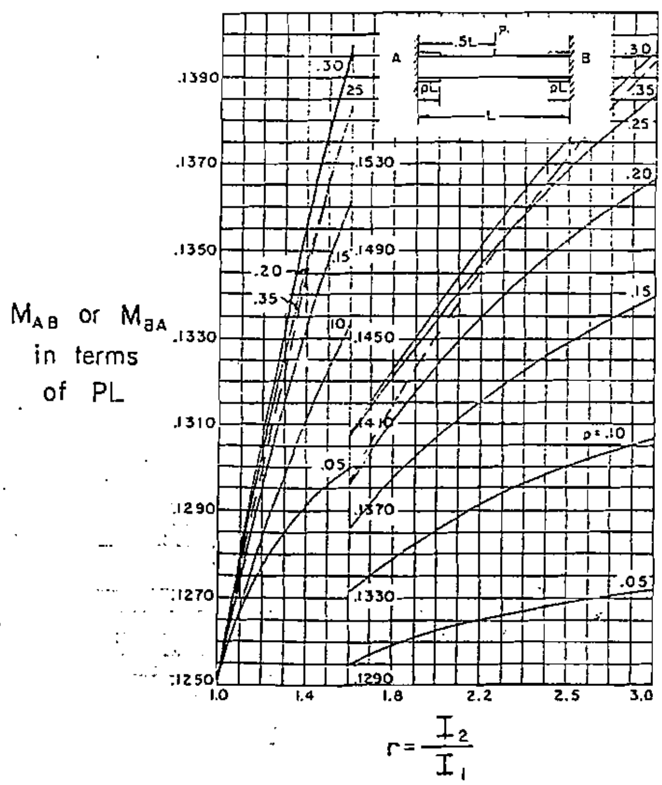


Chart 11. Fixed-end moments at left end of symmetrical beam for concentrated load at .5 point.

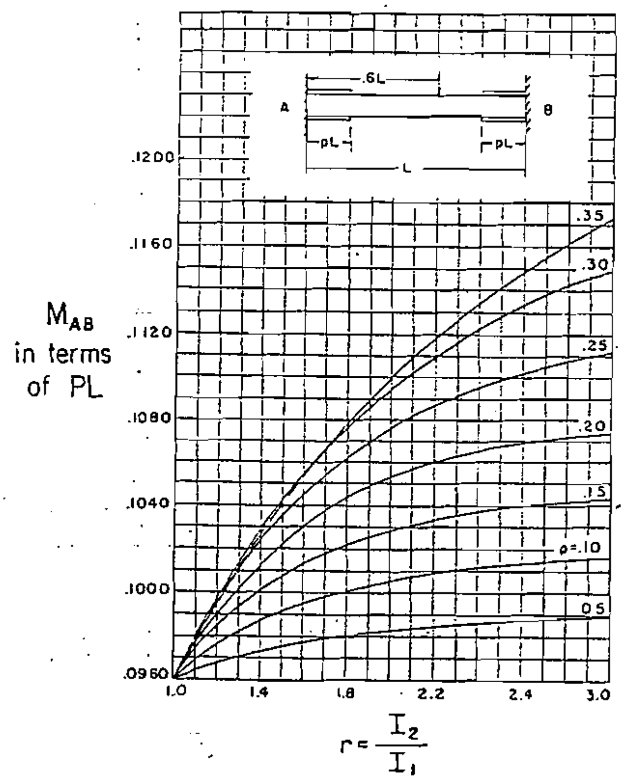


Chart 12. Fixed end moments at left end of symmetrical beam for concentrated load at .6 point.

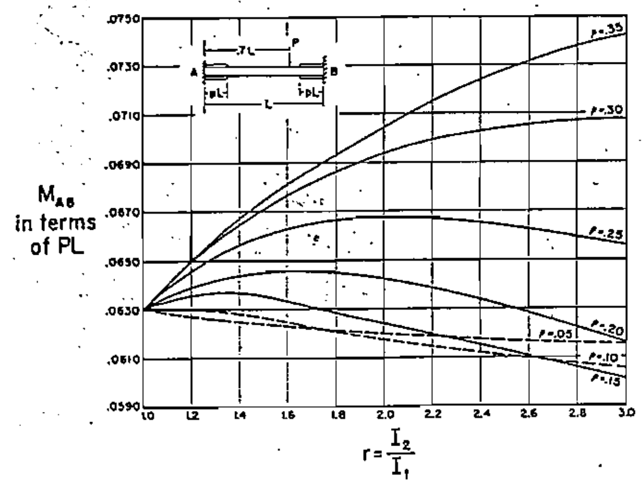


Chart 13. Fixed-end moments at left end of symmetrical beam for concentrated load at .7 point.

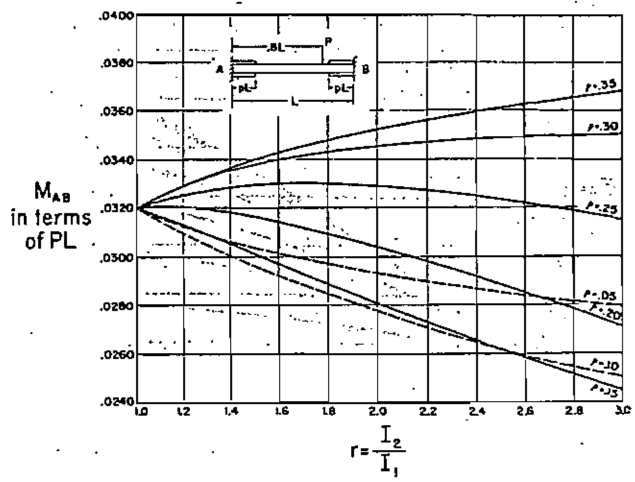


Chart 14. Fixed-end moments at left end of symmetrical beam for concentrated load at .8 point.

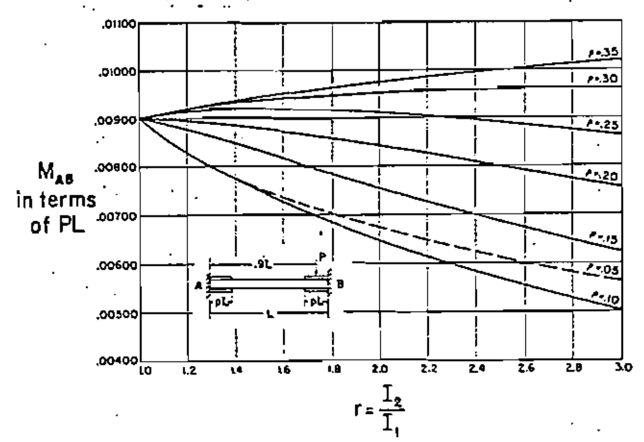


Chart 15. Fixed-end moments at left end of symmetrical beam for concentrated load at .9 point.

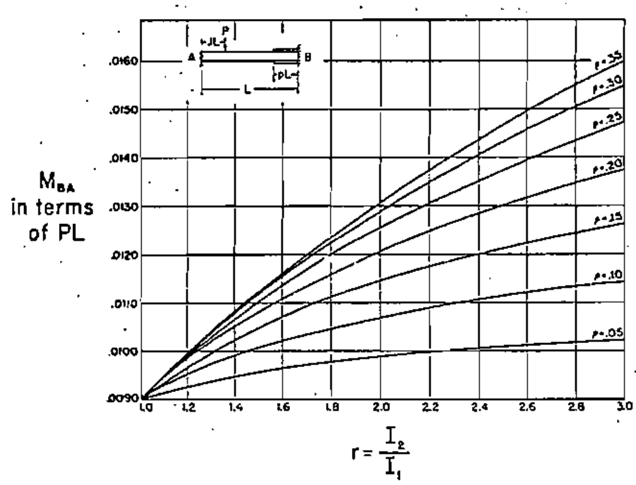


Chart 16. Fixed-end moments at large end of unsymmetrical beam for concentrated load at .1 point.

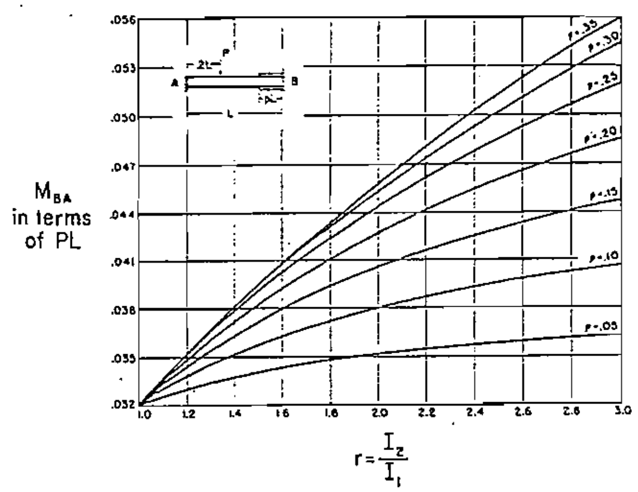


Chart 17. Fixed-end moments at large end of unsymmetrical beam for concentrated load at .2 point.

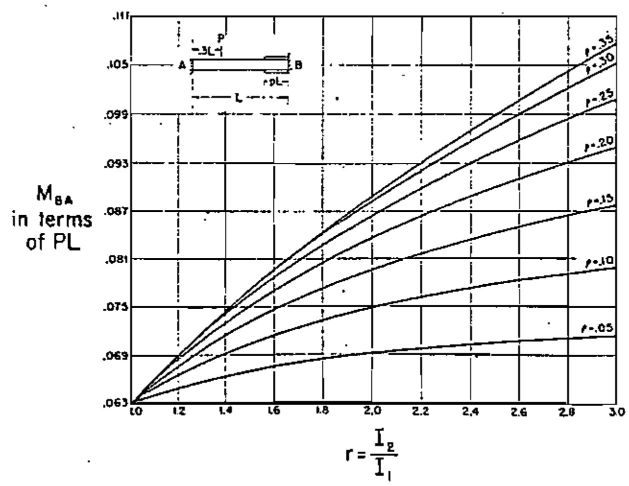


Chart 18. Fixed-end moments at large end of unsymmetrical beam for concentrated load at .3 point.

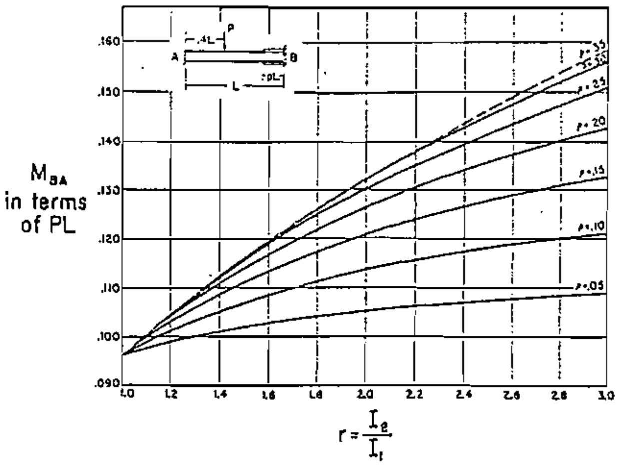


Chart 19. Fixed-end moments at large end of unsymmetrical beam for concentrated load at .4 point.

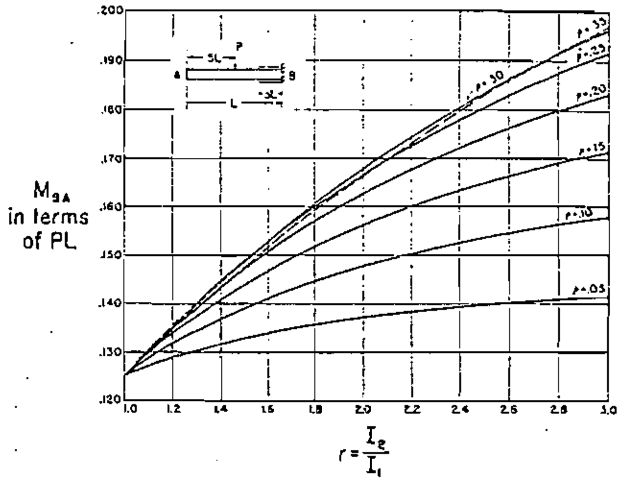


Chart 20. Fixed-end moments at large end of unsymmetrical beam for concentrated load at .5 point.

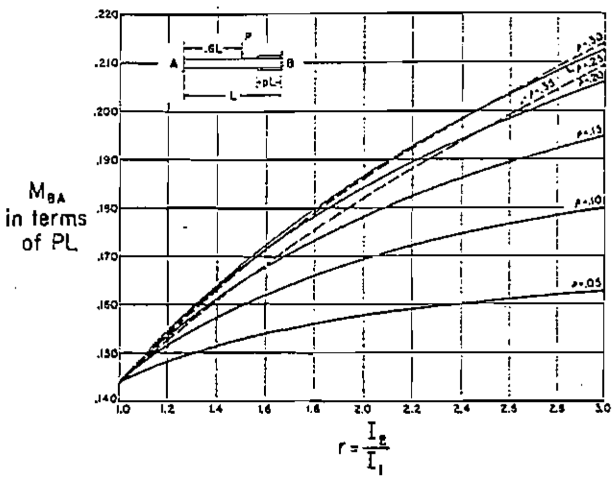


Chart 21. Fixed-end moments at large end of unsymmetrical beam for concentrated load at .6 point.

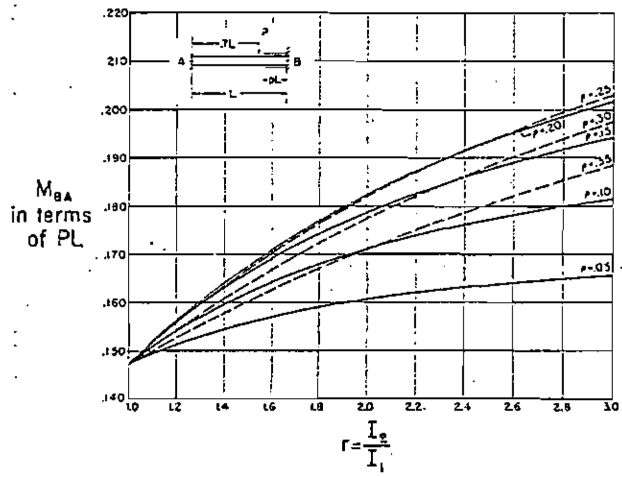


Chart 22. Fixed-end moments at large end of unsymmetrical beam for concentrated load at .7 point.

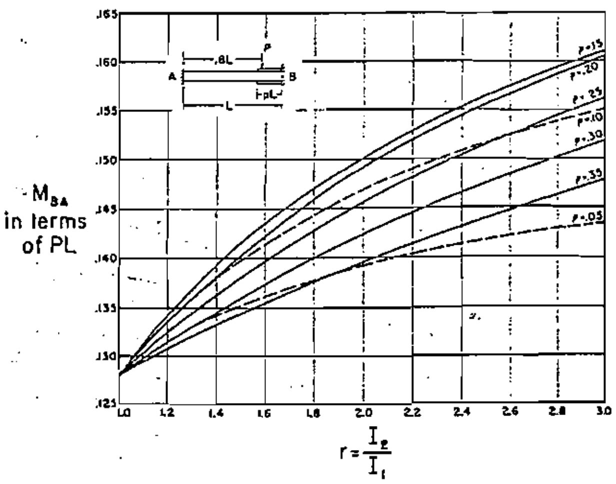


Chart 23. Fixed-end moments at large end of unsymmetrical beam for concentrated load at .8 point.

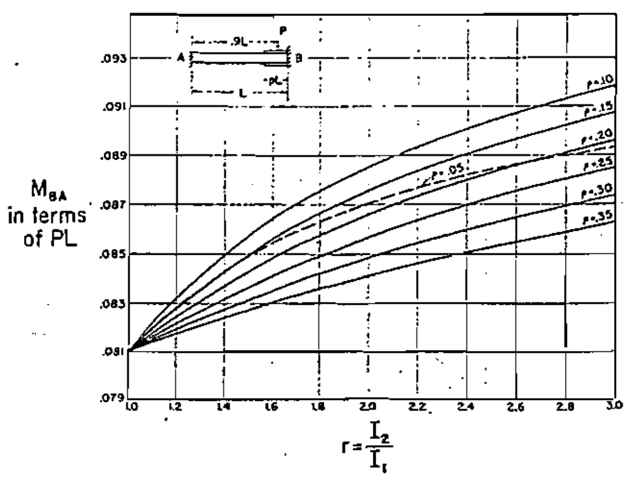


Chart 24. Fixed-end moments at large end of unsymmetrical beam for concentrated load at .9 point.

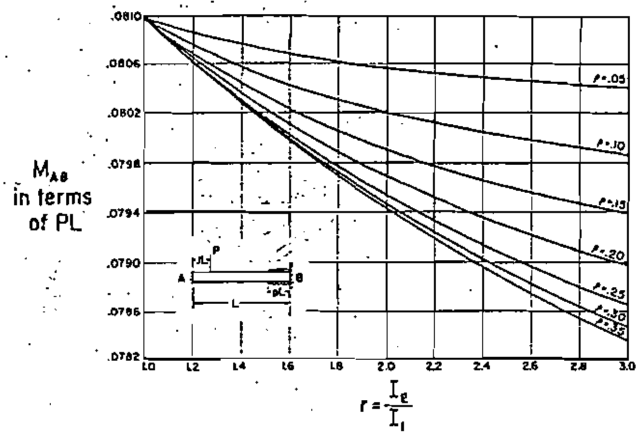


Chart 25. Fixed-end moments at small end of unsymmetrical beam for concentrated load at .1 point.

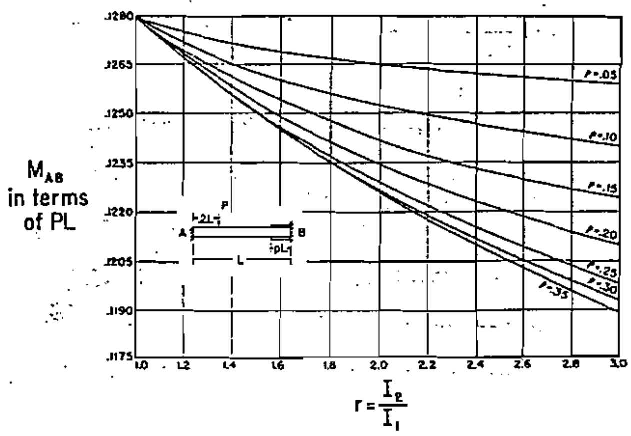


Chart 26. Fixed-end moments at small end of unsymmetrical beam for concentrated load at .2 point.

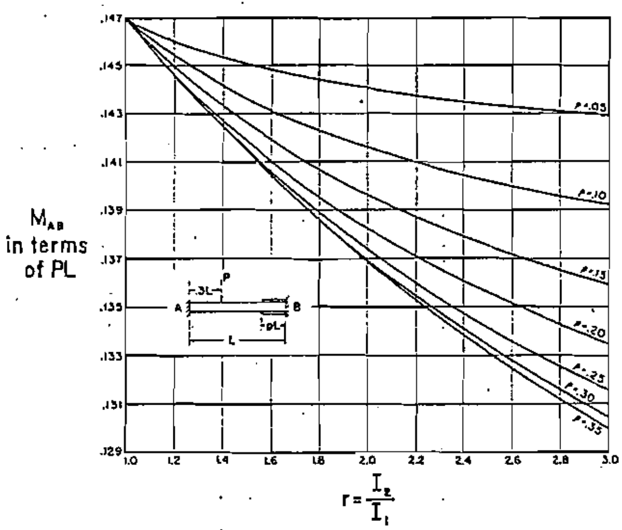


Chart 27. Fixed-end moments at small end of unsymmetrical beam for concentrated load at .3 point.

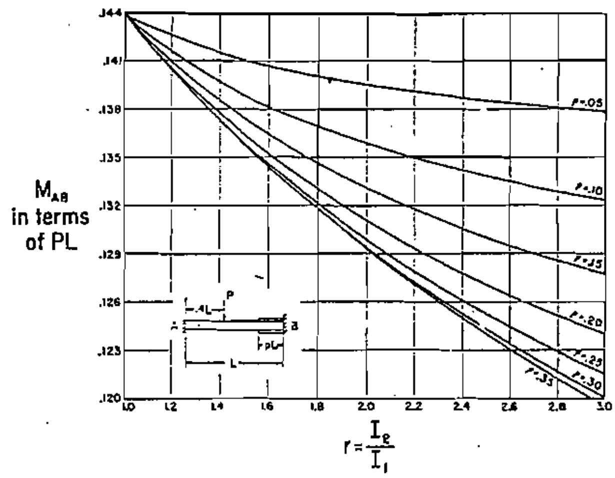


Chart 28. Fixed-end moments at small end of unsymmetrical beam for concentrated load at .4 point.

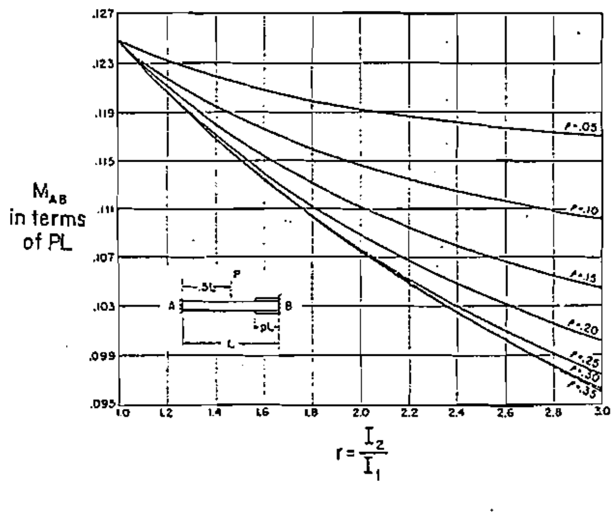


Chart 29. Fixed-end moments at small end of unsymmetrical beam for concentrated load at .5 point.

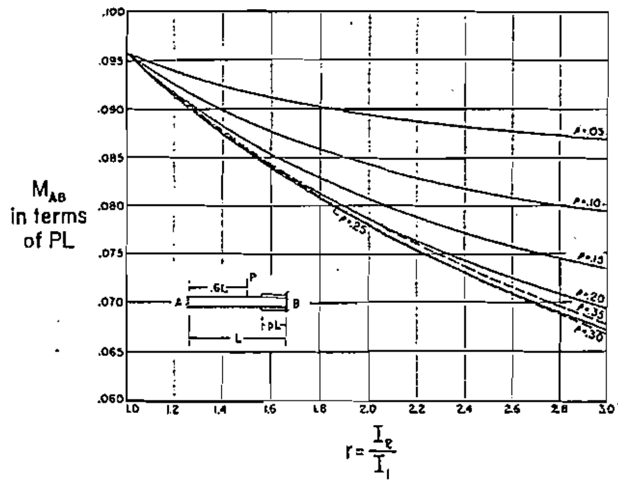


Chart 30. Fixed-end moments at small end of unsymmetrical beam for concentrated load at .6 point.

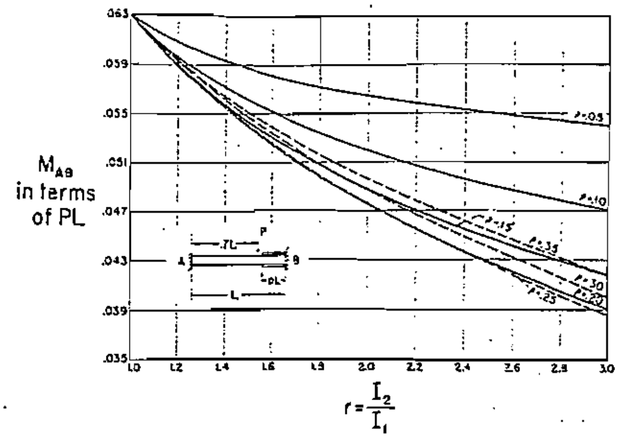


Chart 31. Fixed-end moments at small end of unsymmetrical beam for concentrated load at .7 point.

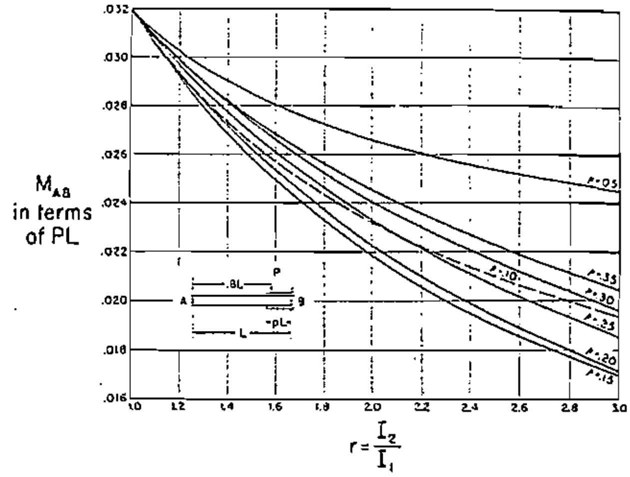


Chart 32. Fixed-end moments at small end of unsymmetrical beam for concentrated load at .8 point.

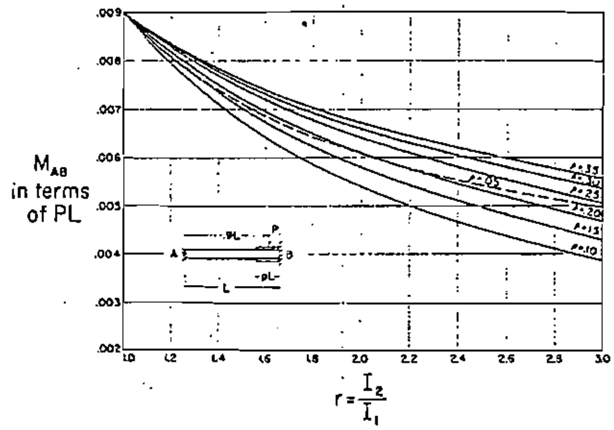


Chart 33. Fixed-end moments at small end of unsymmetrical beam for concentrated load at .9 point.

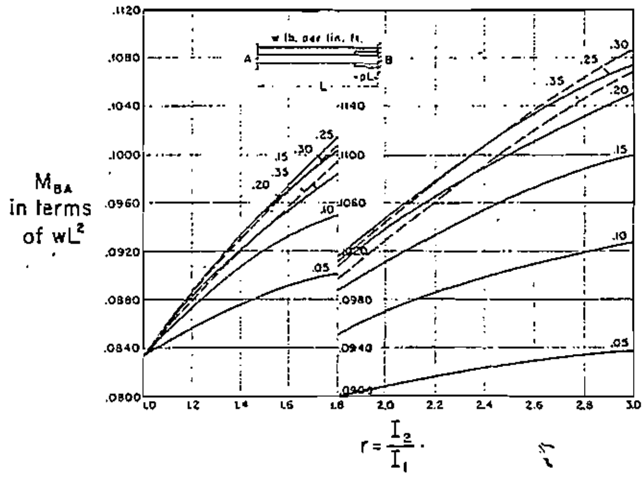


Chart 34. Fixed-end moments at large end of unsymmetrical beam for uniform load.

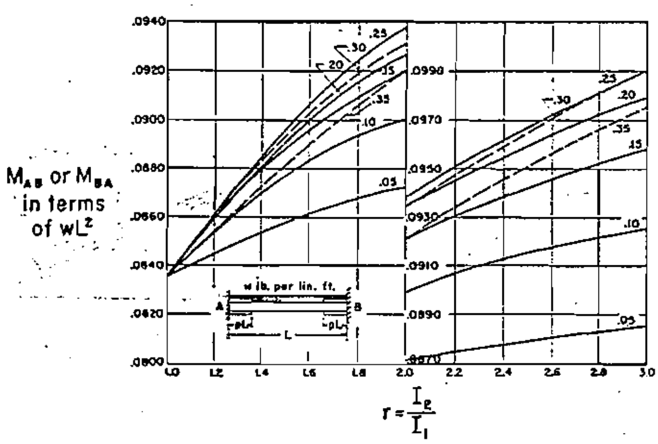


Chart 35. Fixed-end moments at either end of symmetrical beam for uniform load.

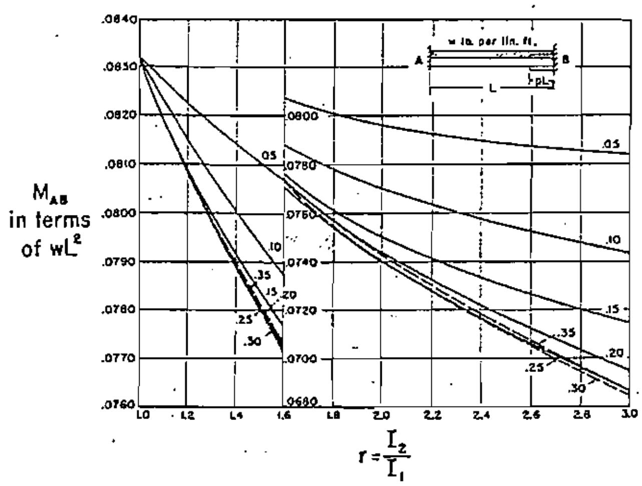


Chart 36. Fixed-end moments at small end of unsymmetrical beam for uniform load.