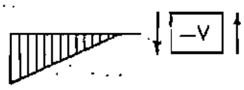
Beam Diagrams and Formulas

The following beam diagrams and formulas have been found useful in the design of welded steel structures.

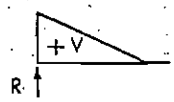
Proper signs, positive (+) and negative (—), are not necessarily indicated in the formulas. The following are suggested:

Shear diagram above reference line is (+)

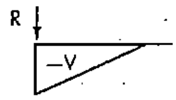
Shear diagram below reference line is (—)



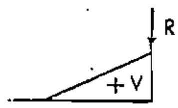
Reaction to left of (+) shear is upward (+)



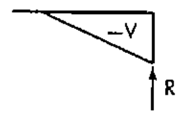
Reaction to left of (—) shear is downward (—)



Reaction to right of (+) shear is downward (—)



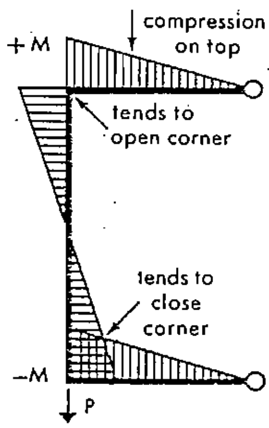
Reaction to right of (—) shear is upward (+)



Moment above reference line is (+)
Compressive bending stresses on top fibers
also tends to open up a corner connection

Moment diagram on same side as compressive stress

Moment below reference line is (—)
Compressive bending stresses on bottom fibers
also tends to close up a corner connection



Angle of slope, θ
clockwise rotation (—), counter-clockwise rotation (+)

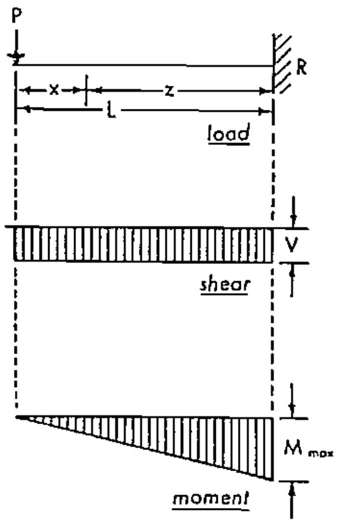
On the next page is a visual index to the various beam diagrams and formulas. As indicated, these are keyed by number to the type of beam and by capital letter to the type of load.

For some conditions, influence curves are included to illustrate the effect of an important variable. These are keyed to the basic beam diagram and are positioned as close as practical to the diagram.

VISUAL INDEX TO FORMULAS ON FOLLOWING PAGES FOR VARIOUS BEAM-LOAD CONDITIONS

Type of LOAD / Type of BEAM	P Concentrated force (A)	w Uniform load entire span (B)	w Uniform load partial span (C)	W Varying load (D)	M_o Couple (E)
(1) Cantilever — free, fixed	1Aa, 1Ab	1B	1C	1Da, 1Db	1E
(2) guided, fixed	2A	2B			
(3) Simply supported — supported	3Aa, 3Ab, 3Ac, 3Ad	3B	3C	3Da, 3Db, 3Dc	3Ea, 3Eb, 3Ec
(4) fixed	4Aa, 4Ab, 4Ac	4Ba, 4Bb	4C	4D	4E
(5) supported, fixed	5Aa, 5Ab	5B	5C	5Da, 5Db	5E
(6) Single span with overhang	6Aa, 6Ab	6Ba, 6Bb	6Ca, 6Cb		
(7) Continuous two span	7Aa, 7Ab	7B		7D — See adjacent to (3D)	For other multi-span load conditions, see discussion under (7)

1Aa Beam fixed at one end only (cantilever)
Concentrated load at free end



At support, $R = V = P$

$M_{max} = P\,L$

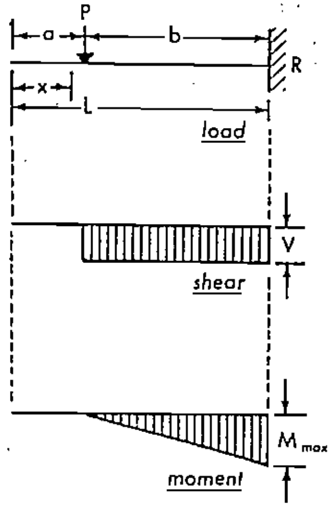
$M_x = P\,x$

At free end, $\Delta_{max} = \dfrac{P\,L^3}{3\,E\,I}$

$\Delta_x = \dfrac{P}{6EI}(2L^3 - 3L^2x + x^3)$

$= \dfrac{P\,z^2}{6\,E\,I}(3\,L - z)$

1Ab Beam fixed at one end only (cantilever)
Concentrated load at any point



$R = V = P$

At support, $M_{max} = P\,b$

When $x > a$ $\quad M_x = P\,(x - a)$

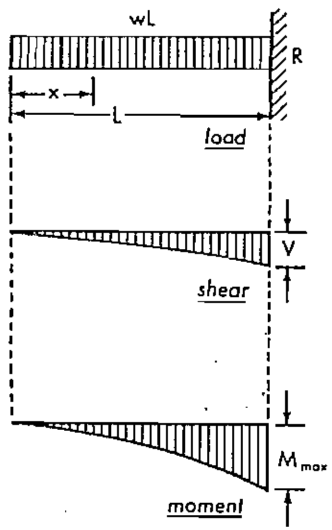
At free end, $\Delta_{max} = \dfrac{P\,b^2}{6\,E\,I}(3\,L - b)$

At load, $\Delta = \dfrac{P\,b^3}{3\,E\,I}$

When $x < a$ $\quad \Delta_x = \dfrac{Pb^2}{6EI}(3L - 3x - b)$

When $x > a$ $\quad \Delta_x = \dfrac{P(L - x)^2}{6EI}(3b - L + x)$

1B Beam fixed at one end only (cantilever)
Uniform load over entire span



$R = V = w\,L$

$V_x = w\,x$

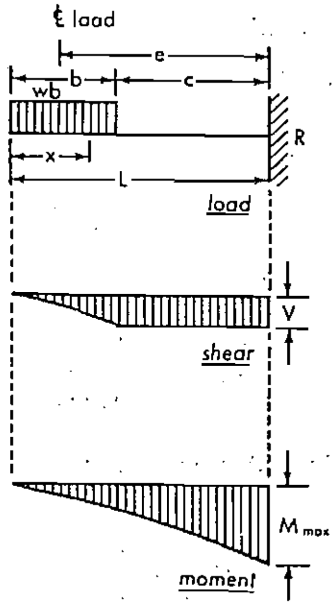
At support, $M_{max} = \dfrac{w\,L^2}{2}$

$M_x = \dfrac{w\,x^2}{2}$

At free end, $\Delta_{max} = \dfrac{w\,L^4}{8\,E\,I}$

$\Delta_x = \dfrac{w}{24EI}(x^4 - 4L^3x + 3L^4)$

1C Beam fixed at one end only (cantilever)
Uniform load partially distributed at free end



$R = V = w\,b$

At support, $M_{max} = w\,b\,e$

When $x < b$ $\quad M_x = \dfrac{w\,x^2}{2}$

When $x > b$ $\quad M_x = \dfrac{w\,b}{2}(b - 2\,x)$

At free end, $\Delta_{max} = \dfrac{wb}{48EI}(8e^3 - 24e^2L - b^3)$

When $x < b$

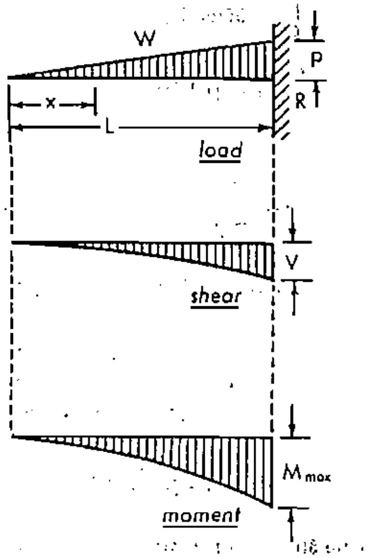
$\Delta_x = \dfrac{w}{48EI}[8be^3 - 24be^2(L - x) + 2b^3x - b^4 - 2x^4]$

When $x > b$

$\Delta_x = \dfrac{wb}{48EI}[8e^3 - 24e^2(L - x) - (2\,x - b)^3]$

At free end, $\theta = + \dfrac{w\,b}{24\,E\,I}(b^2 + 12\,e^2)$

(1Da) Beam fixed at one end only (cantilever)
Varying load increasing uniformly from free end to support



$$W = \frac{pL}{2}$$

$$R = V = W$$

$$V_x = W\frac{x^2}{L^2}$$

At support, $$M_{max} = \frac{WL}{3}$$

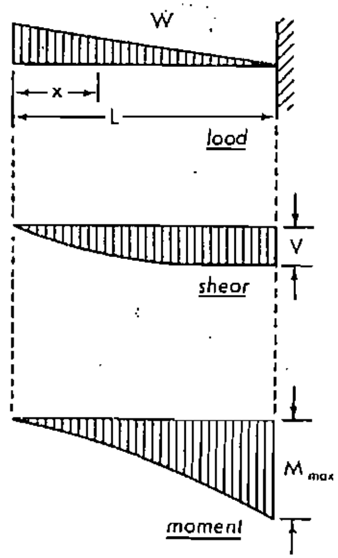
$$M_x = \frac{Wx^3}{3L^2}$$

At free end, $$\Delta_{max} = \frac{WL^3}{15EI}$$

$$\Delta_x = \frac{W}{60EIL^2}(x^5 - 5L^4x + 4L^5)$$

At free end, $$\theta = +\frac{WL^2}{12EI}$$

(1Db) Beam fixed at one end only (cantilever)
Varying load increasing uniformly from support to free end



$$R = V = W$$

$$V_x = \frac{2Wx}{L^2}\left(L - \frac{x}{2}\right)$$

At support, $$M_{max} = \frac{2WL}{3}$$

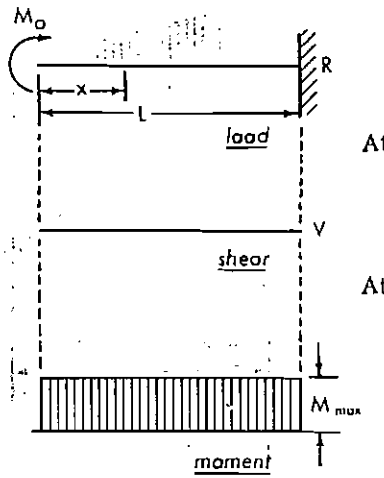
$$M_x = \frac{Wx^2}{3L^2}(x - 3L)$$

At free end, $$\Delta_{max} = \frac{11WL^3}{60EI}$$

$$\Delta_x = \frac{W}{60EIL^2}[L^4(15x - 11L) - x^4(5L - x)]$$

At free end, $$\theta = +\frac{WL^2}{4EI}$$

(1E) Beam fixed at one end only (cantilever)
Moment applied at free end



$$R = V = 0$$

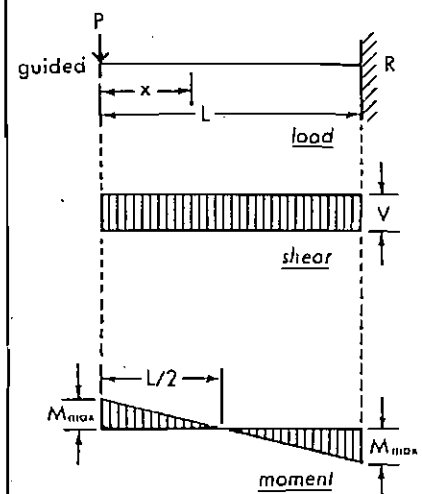
$$M_x = M_o$$

At free end, $$\Delta_{max} = \frac{M_o L^2}{2EI}$$

$$\Delta x = \frac{M_o}{2EI}(L - x)^2$$

At free end, $$\theta = -\frac{M_o L}{EI}$$

(2A) Beam fixed at one end and free but guided at the other end
Concentrated load at guided end



$$R = V = P$$

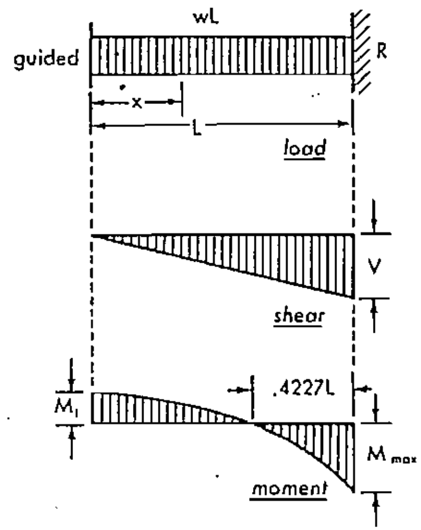
At both ends, $$M_{max} = \frac{PL}{2}$$

$$M_x = P\left(\frac{L}{2} - x\right)$$

At guided end, $$\Delta_{max} = \frac{PL^3}{12EI}$$

$$\Delta_x = \frac{P(L - x)^2}{12EI}(L + 2x)$$

(2B) Beam fixed at one end and free but guided at the other end
Uniform load over entire span



$$R = V = w\,L$$

$$V_x = w\,x$$

At support, $M_{max} = \dfrac{w\,L^2}{3}$

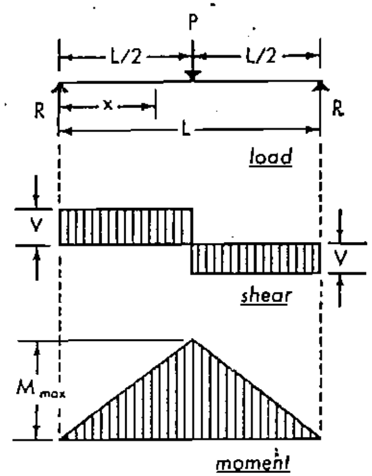
At guided end, $M_1 = \dfrac{w\,L^2}{6}$

$$M_x = \frac{w}{6}(L^2 - 3\,x^2)$$

At guided end, $\Delta_{max} = \dfrac{w\,L^4}{24\,E\,I}$

$$\Delta_x = \frac{w\,(L^2 - x^2)^2}{24\,E\,I}$$

(3Aa) Beam supported at both ends
Concentrated load at mid-span



$$R_1 = R_2 = V = P/2$$

At load, $M_{max} = \dfrac{P\,L}{4}$

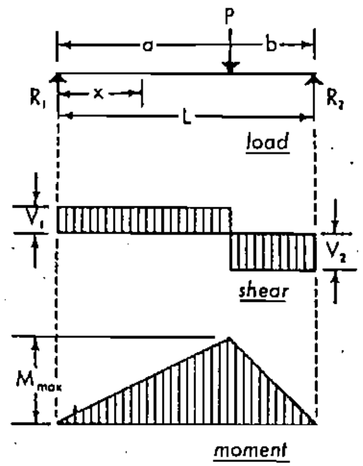
When $x < L/2$ $\quad M_x = \dfrac{P\,x}{2}$

At load, $\Delta_{max} = \dfrac{P\,L^3}{48\,E\,I}$

When $x < L/2$ $\quad \Delta_x = \dfrac{P\,x}{48\,E\,I}(3\,L^2 - 4\,x^2)$

At end, $\theta_1 = -\dfrac{P\,L^2}{16\,E\,I} = -\theta_2$

(3Ab) Beam supported at both ends
Concentrated load at any point



Max when $a < b$ $\quad R_1 = V_1 = \dfrac{P\,b}{L}$

Max when $a > b$ $\quad R_2 = V_2 = \dfrac{P\,a}{L}$

At load, $M_{max} = \dfrac{P\,a\,b}{L}$

When $x < a$ $\quad M_x = \dfrac{P\,b\,x}{L}$

At $x = \sqrt{\dfrac{L^2 - b^2}{3}}$

when $a > b$ $\quad \Delta_{max} = \dfrac{Pb}{3EIL}\sqrt{\left(\dfrac{L^2 - b^2}{3}\right)^3}$

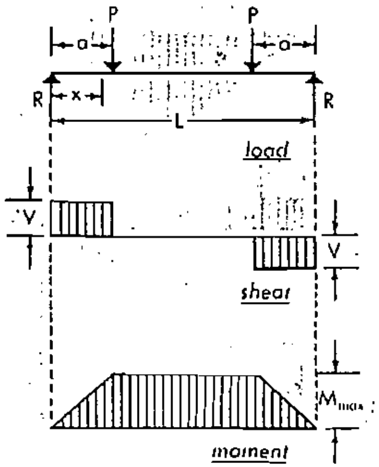
At load, $\Delta = \dfrac{P\,a^2\,b^2}{3\,E\,I\,L}$

When $x < a$ $\quad \Delta_x = \dfrac{Pbx}{6EIL}(L^2 - b^2 - x^2)$

When $a < b$ $\quad \Delta_{c} = \dfrac{Pa}{48EI}(3L^2 - 4a^2)$

At ends,
$$\left\{\begin{aligned} \theta_1 &= -\frac{P}{6EI}\left(2aL + \frac{a^3}{L} - 3a^2\right) \\ \theta_2 &= +\frac{P}{6EI}\left(aL - \frac{a^3}{L}\right) \end{aligned}\right.$$

(3Ac) Beam supported at both ends
Two equal concentrated loads, equally spaced from ends



$R = V = P$

$M_{max} = P\,a$

When $x < a$ $\quad M_x = P\,x$

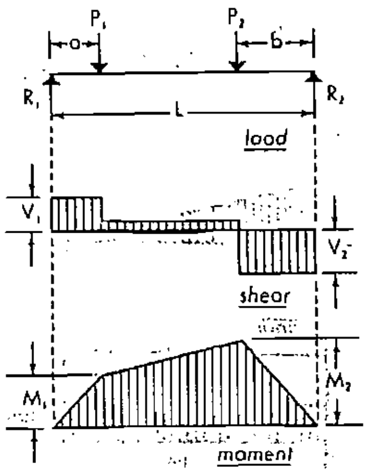
At center, $\quad \Delta_{max} = \dfrac{P\,a}{24\,E\,I}(3\,L^2 - 4\,a^2)$

When $x < a$ $\quad \Delta_x = \dfrac{Px}{6EI}(3La - 3a^2 - x^2)$

When $x > a$ but $x < (L - a)$ $\quad \Delta_x = \dfrac{Pa}{6EI}(3Lx - 3x^2 - a^2)$

At ends, $\quad \theta = \dfrac{P\,a}{2\,E\,I}(L - a)$

(3Ad) Beam supported at both ends
Two unequal concentrated loads, unequally spaced from ends



$R_1 = V_1 = \dfrac{P_1(L - a) + P_2 b}{L}$

$R_2 = V_2 = \dfrac{P_1 a + P_2(L - b)}{L}$

When $x > a$ but $x < (L - b)$ $\quad V_x = R_1 - P_1$

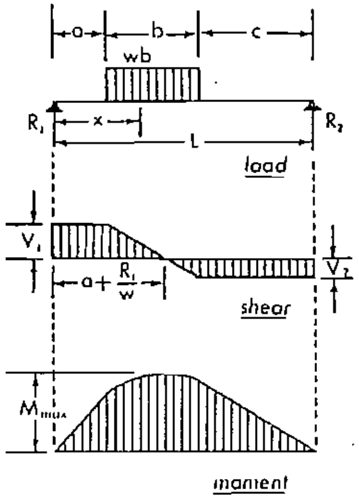
Max when $R_1 < P_1$ $\quad M_1 = R_1\,a$

Max when $R_2 < P_2$ $\quad M_2 = R_2\,b$

When $x < a$ $\quad M_x = R_1\,x$

When $x > a$ but $x < (L - b)$ $\quad M_x = R_1\,x - P_1\,(x - a)$

(3C) Beam supported at both ends
Uniform load partially distributed over span



Max when $a < c$ $\quad R_1 = V_1 = \dfrac{wb}{2L}(2c + b)$

Max when $a > c$ $\quad R_2 = V_2 = \dfrac{wb}{2L}(2a + b)$

When $x > a$ but $x < (a + b)$ $\quad V_x = R_1 - w\,(x - a)$

At $x = a + \dfrac{R_1}{w}$ $\quad M_{max} = R_1\left(a + \dfrac{R_1}{2\,w}\right)$

When $x < a$ $\quad M_x = R_1\,x$

When $x > a$ but $x < (a + b)$ $\quad M_x = R_1 x - \dfrac{w}{2}(x - a)^2$

When $x > (a + b)$ $\quad M_x = R_2\,(L - x)$

<u>When $a = c$</u>

$R = V = \dfrac{w\,b}{2}$

$V_x = w\left(a + \dfrac{b}{2} - x\right)$

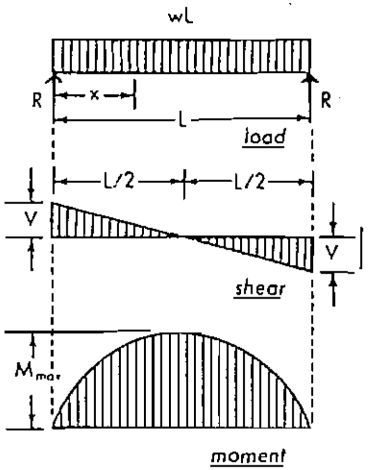
At center, $\quad M_{max} = \dfrac{w\,b}{2}\left(a + \dfrac{b}{4}\right)$

When $x < a$ $\quad M_x = \dfrac{w\,b\,x}{2}$

When $x > a$ but $x < (a + b)$ $\quad M_x = \dfrac{wbx}{2} - \dfrac{w}{2}(x - a)^2$

At center, $\quad \Delta_c = \dfrac{wb}{384EI}(+8L^3 - 4b^2L + b^3)$

(3B) Beam supported at both ends
Uniform load over entire span



$$R = V = \frac{wL}{2}$$

$$V_x = w\left(\frac{L}{2} - x\right)$$

At center, $$M_{max} = \frac{wL^2}{8}$$

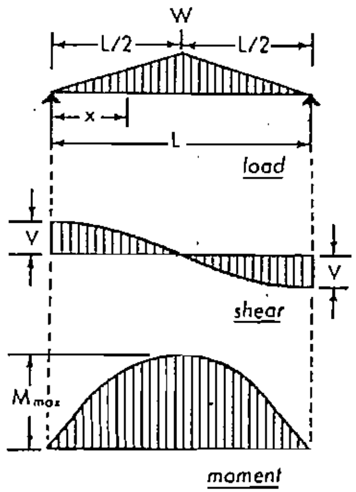
$$M_x = \frac{wx}{2}(L - x)$$

At center, $$\Delta_{max} = \frac{5\,wL^4}{384\,EI}$$

$$\Delta_x = \frac{wx}{24EI}(L^3 - 2Lx^2 + x^3)$$

At ends, $$\theta = \frac{wL^3}{24\,EI}$$

(3Db) Beam supported at both ends
Varying load, increasing uniformly to center



$$R_1 = R_2 = V = \frac{W}{2}$$

When $x < L/2$ $$V_x = \frac{W}{2L^2}(L^2 - 4x^2)$$

At center, $$M_{max} = \frac{WL}{6}$$

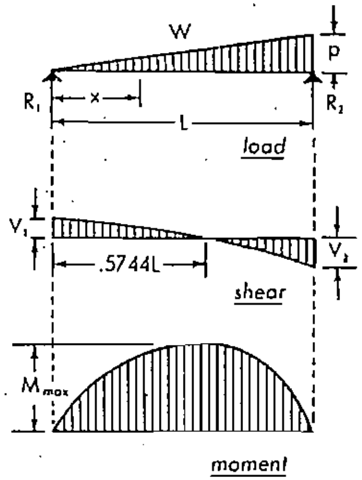
When $x < L/2$ $$M_x = Wx\left(\frac{1}{2} - \frac{2x^2}{3L^2}\right)$$

At center, $$\Delta_{max} = \frac{WL^3}{60\,EI}$$

$$\Delta_x = \frac{Wx}{480EIL^2}(5L^2 - 4x^2)^2$$

At ends, $$\theta = \frac{5\,WL^2}{96\,EI}$$

(3Da) Beam supported at both ends
Varying load, increasing uniformly to one end



$$W = \frac{PL}{2}$$

$$R_1 = V_1 = \frac{W}{3}$$

$$R_2 = V_{2\,(max)} = \tfrac{2}{3}W$$

$$V_x = \frac{W}{3} - \frac{Wx^2}{L^2}$$

At $x = L/\sqrt{3} = .5744\,L$

$$M_{max} = \frac{2\,WL}{9\sqrt{3}} = .1283\,WL$$

$$M_x = \frac{Wx}{3L^2}(L^2 - x^2)$$

At $x = L\sqrt{1 - \sqrt{8/15}} = .5193\,L$

$$\Delta_{max} = .01304\,\frac{WL^3}{EI}$$

$$\Delta_x = \frac{Wx}{180\,EIL^2}(3x^4 - 10L^2x^2 + 7L^4)$$

At center, $$\Delta_{\text{¢}} = \frac{5\,WL^3}{384\,EI}$$

At ends, $$\theta_1 = -\frac{7\,WL^2}{180\,EI}$$

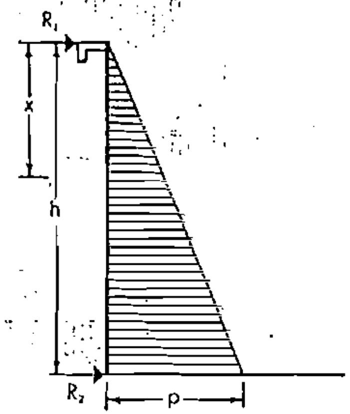
$$\theta_2 = +\frac{8\,WL^2}{180\,EI}$$

Also see formulas on page 8

BEAM FORMULAS APPLIED TO SIDE OF TANK, BIN OR HOPPER

(p = pressure, psi; m = width of panel considered)

3Da



$$R_1 = \frac{p\,h\,m}{6} \qquad R_2 = \frac{p\,h\,m}{3} = V_{max}$$

$$M_{max} = \frac{p\,h^2\,m}{9\sqrt{3}} = .0642\,p\,h^2\,m$$

$$M_x = \frac{p\,x\,m}{6\,h}(h^2 - x^2)$$

$$\Delta_{℄} = \frac{5\,p\,h^4\,m}{768\,E\,I}$$

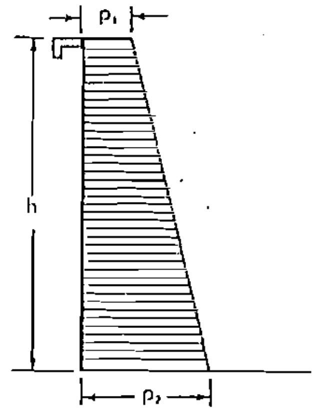
$$\Delta x = \frac{p\,x\,m}{360\,E\,I\,h}(3\,x^4 - 10\,h^2\,x^2 + 7\,h^4)$$

$$\Delta_{max} = .00652\,\frac{p\,h^4\,m}{E\,I}$$

(at $x = .5193\,h$)

Also see formulas on page 7

3Dc



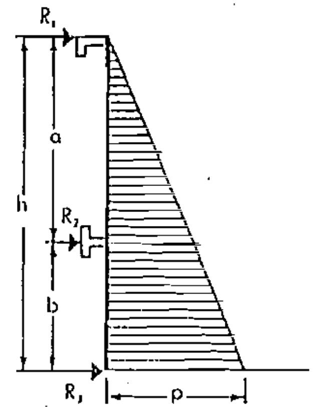
$$M_{℄} = \frac{h^2\,m}{16}(p_1 + p_2)\ ^*$$

$$\Delta_{℄} = \frac{5\,h^4\,m}{768\,E\,I}(p_1 + p_2)\ ^*$$

$$V_{max} = \frac{m\,h}{6}\ (p_1 + 2\,p_2)$$

(* These values are within 98% of maximum.)

7D



Maximum bending moment is least when

$a = .57\,h$

$b = .43\,h$

$M_{max} = .0147\,p\,h^2\,m$

(negative moment at middle support, 2)

$R_1 = +\ .030\,p\,h\,m$

$R_2 = +\ .320\,p\,h\,m$

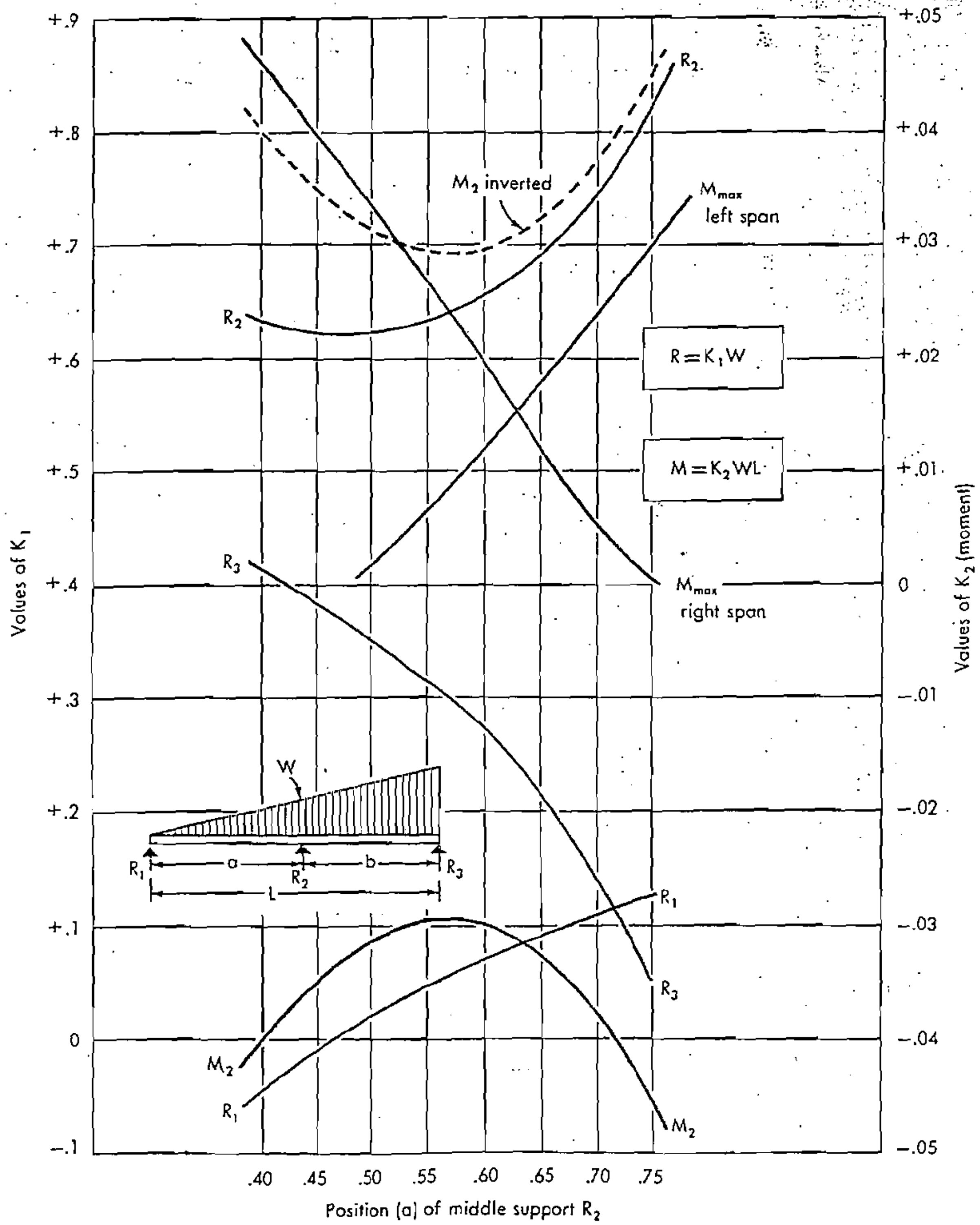
$R_3 = +\ .150\,p\,h\,m$

$V_{max} = +\ .188\,p\,h\,m$

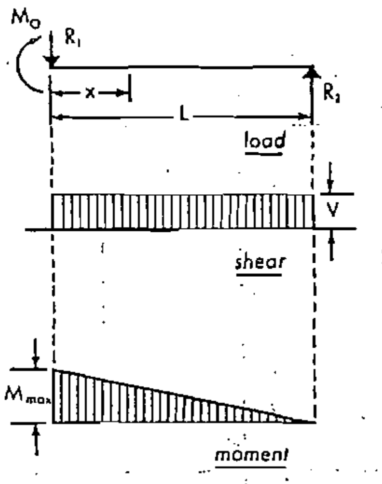
(at middle support, 2)

7D Influence Lines

Effect of location of middle support (2) upon reactions (R) and moments (M)



3Ea Beam supported at both ends — Moment applied at one end



$$R_1 = R_2 = V = \frac{M_o}{L}$$

At R_1, $M_{max} = M_o$

$$M_x = M_o + R_1 x = M_o\left(1 - \frac{x}{L}\right)$$

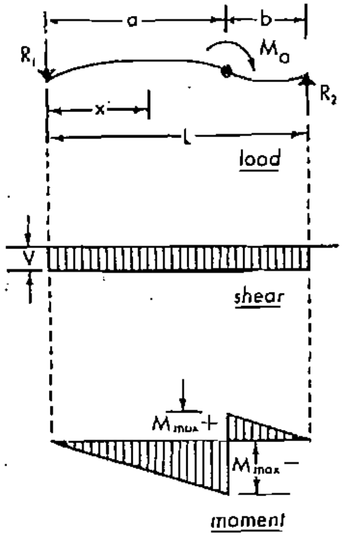
When $x = .422\,L$, $\Delta_{max} = .0642\,\frac{M_o L^2}{E I}$

$$\Delta_x = \frac{M_o}{6EI}\left(3x^2 - \frac{x^3}{L} - 2Lx\right)$$

At R_1, $\theta_1 = -\frac{M_o L}{3 E I}$

At R_2, $\theta_2 = \frac{M_o L}{6 E I}$

3Eb Beam supported at both ends — Moment applied at any point



When $a > b$ $R_1 = -\frac{M_o}{L} = V$ $R_2 = \frac{M_o}{L}$

At $x = a$, $M_{max\,(-)} = -\frac{M_o a}{L}$

At $x = a$, $M_{max\,(+)} = M_o\left(1 - \frac{a}{L}\right)$

When $x < a$ $M_x = -\frac{M_o x}{L}$

When $x > a$ $M_x = M_o\left(1 - \frac{x}{L}\right)$

When $x < a$ $\Delta_x = +\frac{M_o x}{6EIL}(L^2 - 3b^2 - x^2)$

When $x > a$ $\Delta_x = \frac{M_o(L - x)}{6EIL}(3a^2 - 2Lx + x^2)$

At $x = \sqrt{\frac{L^2 - 3b^2}{3}}$ if $a > .4226\,L$,

$$\Delta_{max} = \frac{M_o}{3EIL}\left(\frac{L^2 - 3b^2}{3}\right)^{3/2}$$

At $x = L - \sqrt{\frac{L^2 - 3a^2}{3}}$ if $a < .5774\,L$,

$$\Delta_{max} = -\frac{M_o}{3EIL}\left(\frac{L^2 - 3a^2}{3}\right)^{3/2}$$

At center, $M_{\mathrm{¢}} = -\frac{M_o}{2}$

At center, $\Delta_{\mathrm{¢}} = +\frac{M_o}{16EI}(L^2 - 4b^2)$

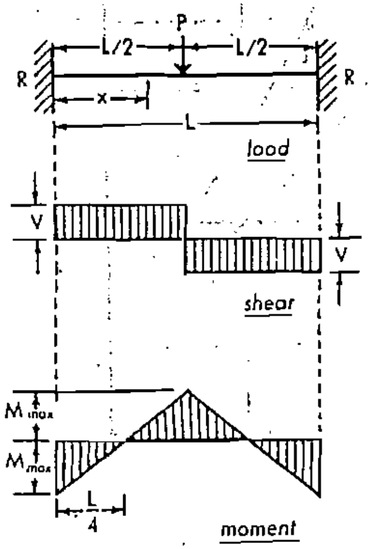
When $a = b = L/2$

At $x = \frac{\sqrt{3}}{6}L = .28867\,L$,

$$\Delta_{max} = \frac{M_o L^2}{124.71\,E I}$$

At center, $\theta_{\mathrm{¢}} = \frac{M_o L}{12 E I}$

4Aa Beam fixed at both ends — Concentrated load at mid-span



$$R = V = \frac{P}{2}$$

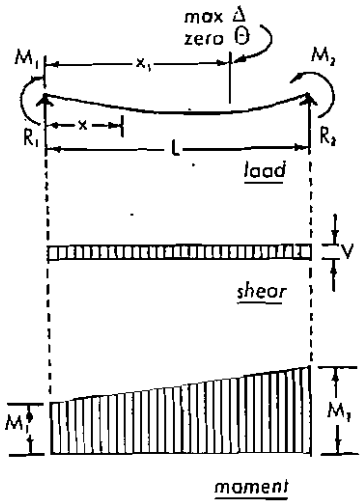
At center and at ends, $M_{max} = \frac{PL}{8}$

When $x < L/2$ $M_x = \frac{P}{8}(4x - L)$

At center, $\Delta_{max} = \frac{PL^3}{192 E I}$

$$\Delta_x = \frac{P x^2}{48 E I}(3L - 4x)$$

(3Ec) Beam supported at both ends
Moments applied at each end



$$R_1 = -R_2 = V = \frac{M_2 - M_1}{L}$$

$$M_x = (M_2 - M_1)\frac{x}{L} + M_1$$

$$\Delta_x = \frac{x(L-x)}{6EIL}[M_1(2L - x) + M_2(L + x)]$$

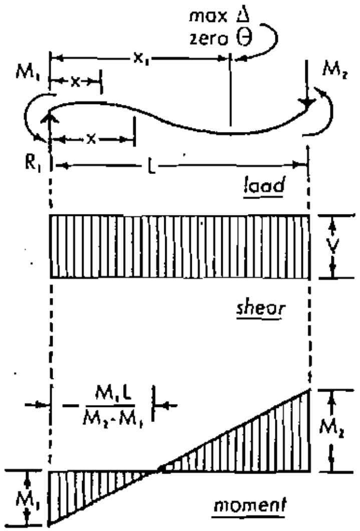
Max Δ and θ = zero at $x_1 = \dfrac{6M_1L \pm \sqrt{36M_1^2L^2 - 12(M_1 - M_2)L^2(2M_1 + M_2)}}{6(M_1 - M_2)}$

At ends $\begin{cases} \theta_1 = -\dfrac{L}{6EI}(2M_1 + M_2) \\ \theta_2 = +\dfrac{L}{6EI}(M_1 + 2M_2) \end{cases}$

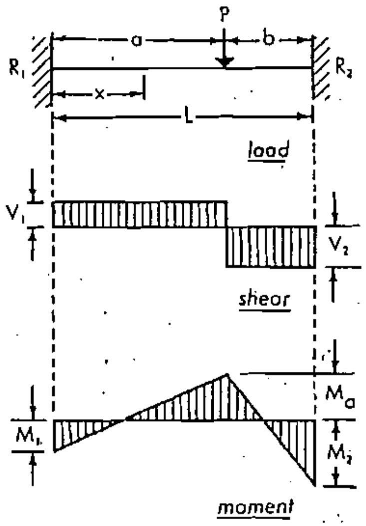
If M_1 and M_2 are of opposite signs, the above formulas hold; just use actual sign of moment

At point of contraflexure, M_x = zero and

$$x = -\frac{M_1 L}{M_2 - M_1}$$



(4Ab) Beam fixed at both ends
Concentrated load at any point



Max when a<b $\quad R_1 = V_1 = \dfrac{Pb^2}{L^3}(3a + b)$

Max when a>b $\quad R_2 = V_2 = \dfrac{Pa^2}{L^3}(a + 3b)$

Max when a<b $\quad M_1 = \dfrac{Pab^2}{L^2}$

Max, when a = ⅓ L, then $\quad M_1 = \dfrac{4PL}{27}$

Max when a>b $\quad M_2 = \dfrac{Pa^2b}{L^2}$

Max when a = ⅔ L, then $\quad M_2 = \dfrac{4PL}{27}$

When $x < a$ $\quad M_x = R_1 x - \dfrac{Pab^2}{L^2}$

At load, $\quad M_a = \dfrac{2Pa^2b^2}{L^3}$

At $x = \dfrac{2aL}{3a + b}$ $\quad \Delta_{max} = \dfrac{2Pa^3b^2}{3EI(3a + b)^2}$

(when a > b and greatest when a = L/2)

At load, $\quad \Delta = \dfrac{Pa^3b^3}{3EIL^3}$

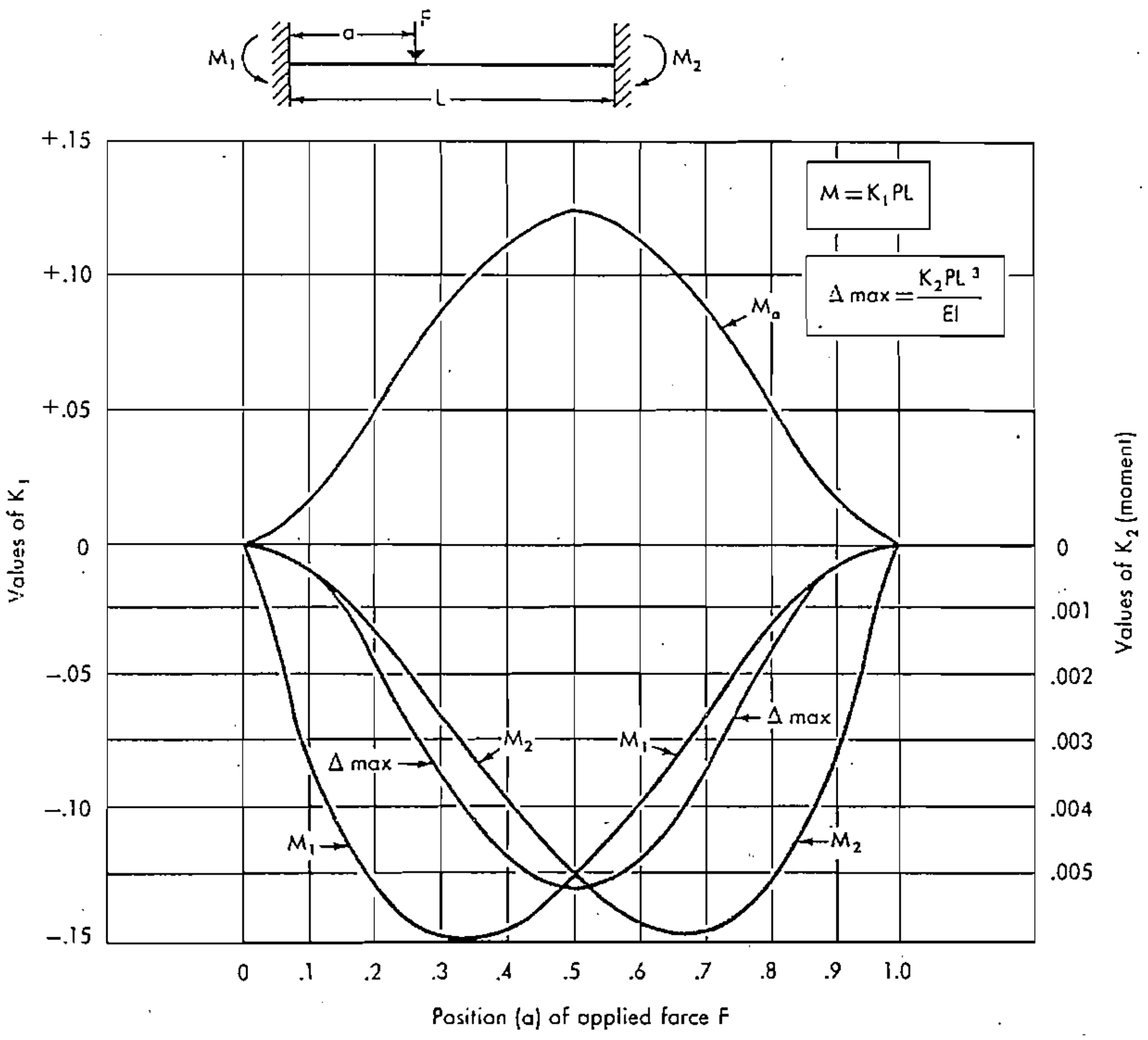
When $x < a$ $\quad \Delta_x = \dfrac{Pb^2x^2}{6EIL^3}(3aL - 3ax - bx)$

At center, $\quad \Delta_{\mathrm{ℓ}} = \dfrac{PL^3}{48EI}(3K - 4K^3)$

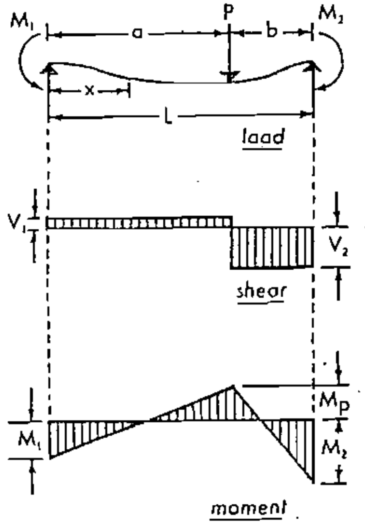
(where K = a/L and a < L/2)

(4Ab) Influence Lines

Effect of position of force (F) upon moments M_a, M_1, M_2 and upon Δ_{max}



(4Ac) Beam supported and partially restrained at both ends
Concentrated load at any point



Here $a > b$

$$R_1 = \frac{Pb}{L} + \frac{M_2 - M_1}{L} = V_1$$

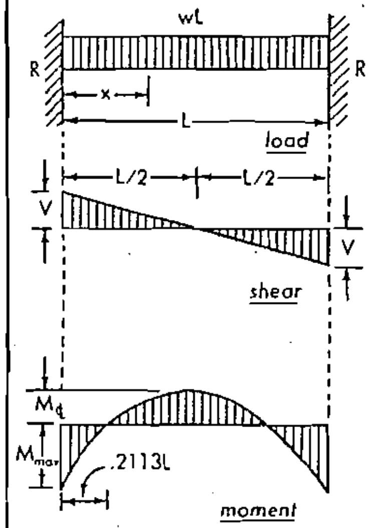
$$R_2 = \frac{Pa}{L} + \frac{M_1 - M_2}{L} = V_2$$

When $x < a$ $\quad M_x = \frac{x}{L}(Pb + M_2 - M_1) + M_1$

At load, $\quad M_p = \frac{a}{L}(Pb + M_2 - M_1) + M_1$

When $x < a$ $\quad \Delta_x = \frac{-x}{6EIL}\left\{Pb^3 - (L-x)[M_1(2L-x) + (Pb+M_2)(L+x)]\right\}$

(4Ba) Beam fixed at both ends
Uniform load over entire span



$$R = V = \frac{wL}{2}$$

$$V_x = w\left(\frac{L}{2} - x\right)$$

At ends, $\quad M_{max} = \frac{wL^2}{12}$

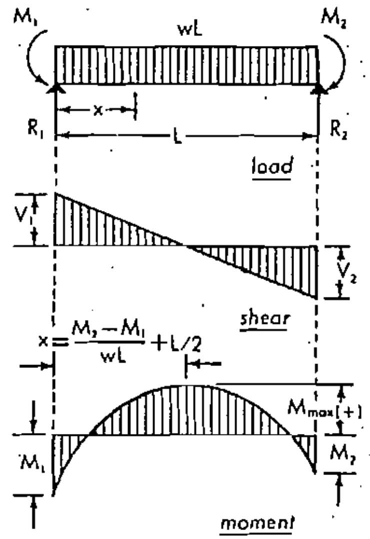
At center, $\quad M_{℄} = \frac{wL^2}{24}$

$$M_x = \frac{w}{12}(6Lx - L^2 - 6x^2)$$

At center, $\quad \Delta_{max} = \frac{wL^4}{384EI}$

$$\Delta_x = \frac{wx^2}{24EI}(L-x)^2$$

(4Bb) Beam supported and partially restrained at both ends
A portion of a continuous beam
Uniform load over entire span



(Use actual signs of moments)

$$V_1 = \frac{M_2 - M_1}{L} + \frac{wL}{2}$$

$$V_2 = \frac{M_1 - M_2}{L} + \frac{wL}{2}$$

$$M_x = \frac{wx}{2}(L-x) + M_1\left(\frac{L-x}{L}\right) + M_2\frac{x}{L}$$

When $x = \frac{M_2 - M_1}{wL} + \frac{L}{2}$

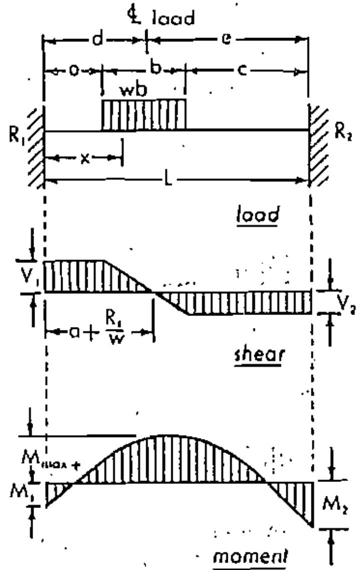
$$M_{max\,(+)} = \frac{wL^2}{8} + \frac{M_1 + M_2}{2} + \frac{(M_2 - M_1)^2}{2wL^2}$$

To find value of x for a given value of M_x:

$$\underbrace{x^2}_{a} - \underbrace{x\left[\frac{2(M_2 - M_1)}{wL} + L\right]}_{b} + \underbrace{\frac{2}{w}(M_x - M_1)}_{c} = 0$$

and $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

4C Beam fixed at both ends
Uniform load partially distributed over span



$$R_1 = V_1 = \frac{w\,b}{4L^3}\left[4e^2(L + 2d) - b^2(c - a)\right]$$

$$R_2 = V_2 = w\,b - R_1$$

$$M_1 = \frac{wb}{24L^2}\left\{b^2[L + 3(c - a)] - 24\,e^2\,d\right\}$$

$$M_2 = R_1\,L - w\,b\,e + M_1$$

At $x = a + \dfrac{R_1}{w}$

$$M_{max\,(+)} = M_1 + R_1\left(a + \frac{R_1}{2w}\right)$$

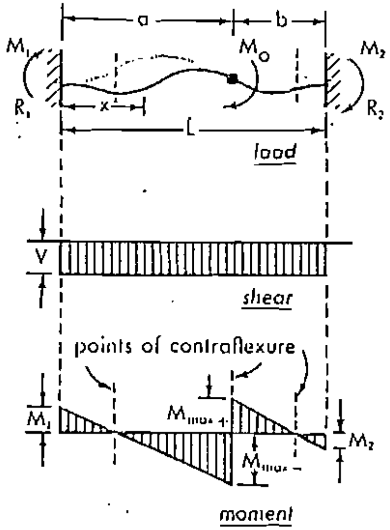
When $x < a$ $\qquad M_x = M_1 + R_1\,x$

When $x > a$ but $x < (a + b)$ $\qquad M_x = M_1 + R_1 x - \dfrac{w}{2}(x - a)^2$

When $x < a$ $\qquad \Delta_x = \dfrac{1}{6EI}(3M_1x^2 + R_1x^3)$

When $x > a$ but $x < (a + b)$ $\qquad \Delta_x = \dfrac{1}{24EI}\left[12M_1x^2 + 4R_1x^3 - w(x - a)^4\right]$

4E Beam fixed at both ends
Moment applied at any point



$$R_1 = -\frac{6\,M_o\,a\,b}{L^3} = V$$

$$R_2 = +\frac{6\,M_o\,a\,b}{L^3}$$

$$M_1 = -\frac{M_o\,b}{L^2}(L - 3a)$$

$$M_2 = -\frac{M_o\,a}{L^2}(2\,L - 3\,a)$$

When $x < a$

$$M_x = -\frac{M_o}{L^2}\left|\frac{6abx}{L} + b(L - 3a)\right|$$

When $x > a$

$$M_x = \frac{M_o a}{L^2}\left(6b - \frac{6bx}{L} - 2L + 3a\right)$$

At $x = a$ (left side),

$$M_{max\,(-)} = M_{max\,(+)} - M_o$$

At $x = a$ (right side),

$$M_{max\,(+)} = M_o\left|-\frac{6a^2b}{L^3} - \frac{b}{L^2}(L - 3a) + 1\right|$$

$$\text{At } x = \frac{-2\,M_1}{R_1} = -\frac{L\,(L - 3\,a)}{3\,a}$$

if $a > L/3$ $\qquad \Delta_{max\,(+)} = +\dfrac{M_o\,b\,(L - 3\,a)^3}{54\,E\,I\,a^2}$

At $x = L/3\,b$ if $a < 2\,L/3$ $\qquad \Delta_{max\,(-)} = -\dfrac{M_o\,a\,(2\,L - 3\,a)^3}{54\,E\,I\,b^2}$

When $x < a$ $\qquad \Delta_x = -\dfrac{M_o\,b\,x^2}{2\,E\,I\,L^2}\left(L - 3\,a + \dfrac{2\,a\,x}{L}\right)$

When $x > a$ $\qquad \Delta_x = \dfrac{M_o\,a\,(L - x)^2}{2\,E\,I\,L^2}\left(3\,a - 2\,L + 2\,b - \dfrac{2\,b\,x}{L}\right)$

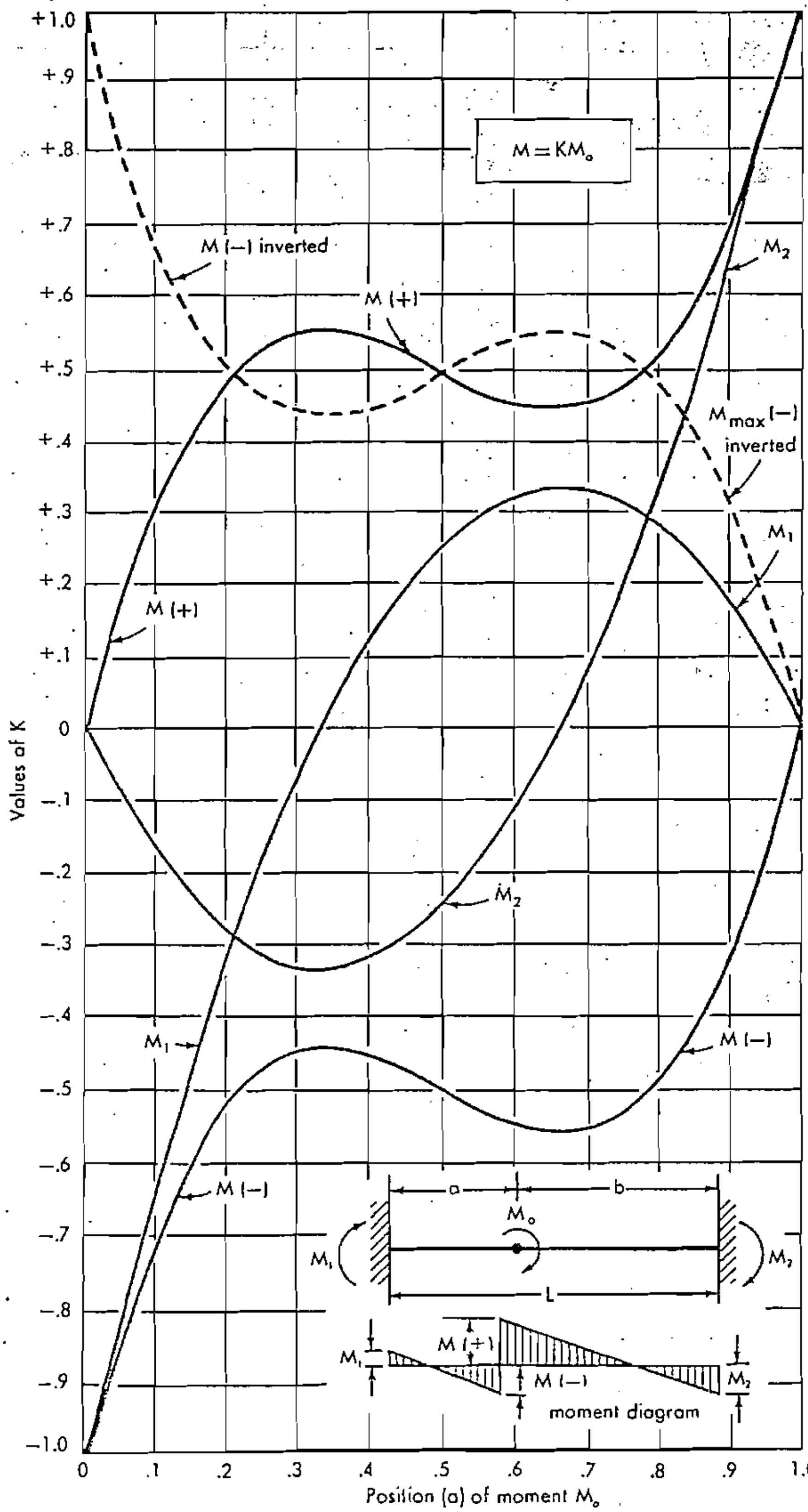
At center, $\qquad M_{\text{℄}} = -\dfrac{M_o}{L^2}\left[3\,a\,b + b\,(L - 3\,a)\right]$

At center, $\qquad \Delta_{\text{℄}} = -\dfrac{M_o\,b}{8\,E\,I}(L - 2\,a)$

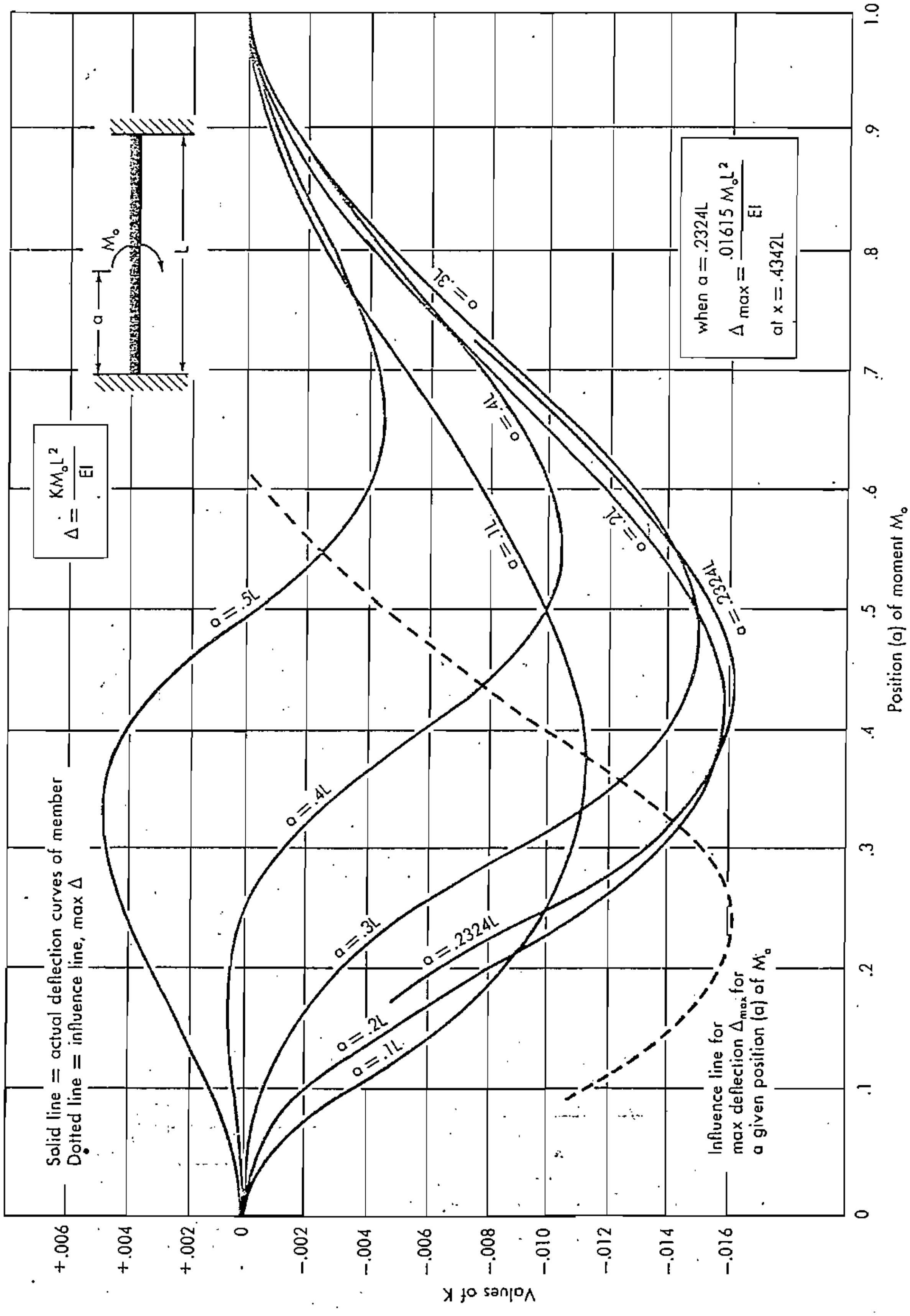
Greatest maximum deflection Δ
when $a = .2324\,L$ $\qquad \Delta_{max} = -\dfrac{.01615\,M_o\,L^2}{E\,I}$

(4E) Influence Lines

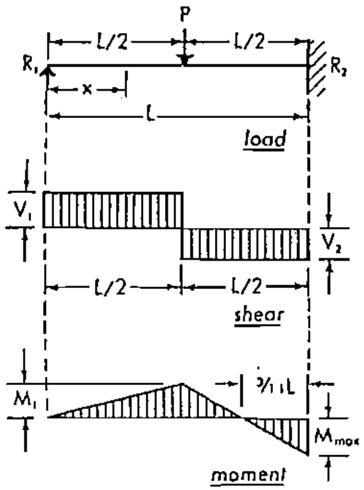
Effect of position of moment (M_0) upon M_1, M_2, M+ and M-



4E Influence Line for Maximum Deflection



(5Aa) Beam fixed at one end and supported at the other end
Concentrated load at mid-span



$$R_1 = V_1 = \frac{5\,P}{16}$$

$$R_2 = V_{2\,max} = \frac{11\,P}{16}$$

At fixed end, $\quad M_{max} = \frac{3\,P\,L}{16}$

At load, $\quad M_1 = \frac{5\,P\,L}{32}$

When $x < L/2$ $\quad M_x = \frac{5\,P\,x}{16}$

When $x > L/2$ $\quad M_x = P\left(\frac{L}{2} - \frac{11\,x}{16}\right)$

At $x = L\sqrt{.2} = .4472\,L$,

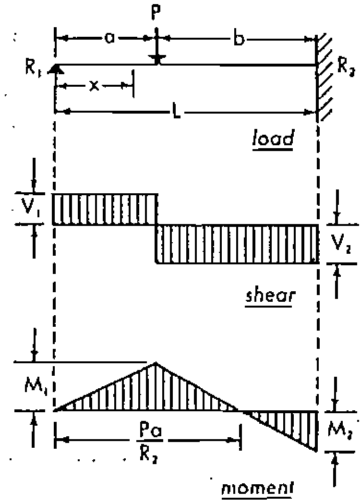
$$\Delta_{max} = \frac{PL^3}{48EI\sqrt{5}} = .009317\,\frac{PL^3}{EI}$$

At load, $\quad \Delta = \frac{7\,P\,L^3}{768\,E\,I}$

When $x < L/2$ $\quad \Delta_x = \frac{P\,x}{96\,E\,I}(3\,L^2 - 5\,x^2)$

When $x > L/2$ $\quad \Delta_x = \frac{P}{96EI}(x - L)^2\,(11x - 2L)$

(5Ab) Beam fixed at one end and supported at the other end
Concentrated load at any point



$$R_1 = V_1 = \frac{P\,b^2}{2\,L^3}\,(a + 2\,L)$$

$$R_2 = V_2 = \frac{P\,a}{2\,L^3}\,(3\,L^2 - a^2)$$

At load, $\quad M_1 = R_1\,a$

At fixed end, $\quad M_2 = \frac{P\,a\,b}{2\,L^2}\,(a + L)$

When $x < a$ $\quad M_x = R_1\,x$

When $x > a$ $\quad M_x = R_1\,x - P\,(x - a)$

At $x = L\,\frac{L^2 + a^2}{3\,L^2 - a^2}$

$$\Delta_{max} = \frac{Pa}{3EI}\,\frac{(L^2 - a^2)^3}{(3L^2 - a^2)^2} \quad \text{when } a < .414\,L$$

At $x = L\sqrt{\frac{a}{2\,L + a}}$

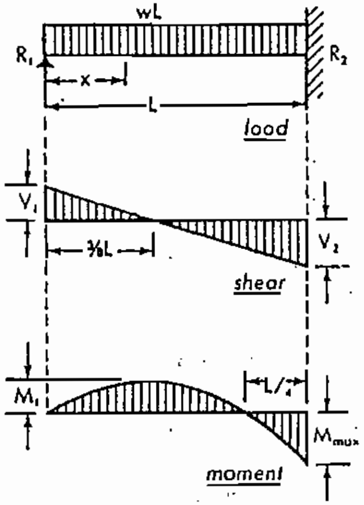
$$\Delta_{max} = \frac{Pab^2}{6EI}\sqrt{\frac{a}{2L + a}} \quad \text{when } a > .414\,L$$

At load, $\quad \Delta = \frac{Pa^2b^3}{12EIL^3}\,(3L + a)$

When $x < a$ $\quad \Delta_x = \frac{Pb^2x}{12EIL^3}\,(3aL^2 - 2Lx^2 - ax^2)$

When $x > a$ $\quad \Delta_x = \frac{Pa}{12EIL^3}\,(L - x)^2\,(3L^2x - a^2x - 2a^2L)$

5B Beam fixed at one end and supported at the other end
Uniform load over entire span



$$R_1 = V_1 = \frac{3\,w\,L}{8}$$

$$R_2 = V_2 = \frac{5\,w\,L}{8}$$

$$V_x = R_1 - w\,x$$

$$M_{max} = \frac{w\,L^2}{8}$$

At $x = 3/8\ L$, $\quad M_1 = \frac{9}{128}\,w\,L^2$

$$M_x = R_1\,x - \frac{w\,x^2}{2}$$

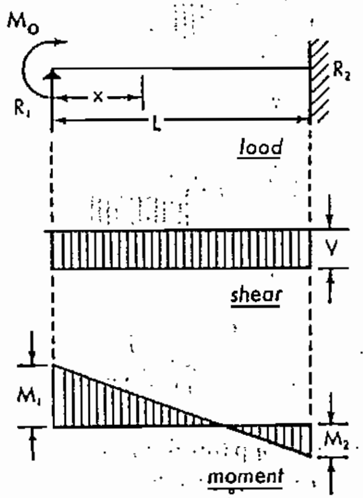
At $x = \frac{L}{16}(1 + \sqrt{33}) = .4215\ L$,

$$\Delta_{max} = \frac{w\,L^4}{185\,E\,I}$$

$$\Delta_x = \frac{wx}{48EI}(L^3 - 3Lx^2 + 2x^3)$$

$$\theta_1 = \frac{w\,L^3}{48\,E\,I}$$

5E Beam fixed at one end and supported at the other end
Moment applied at the flexible end



$$R_1 = R_2 = V = \frac{3\,M_o}{2\,L}$$

$$M_1 = M_o$$

$$M_2 = 1/2\,M_o$$

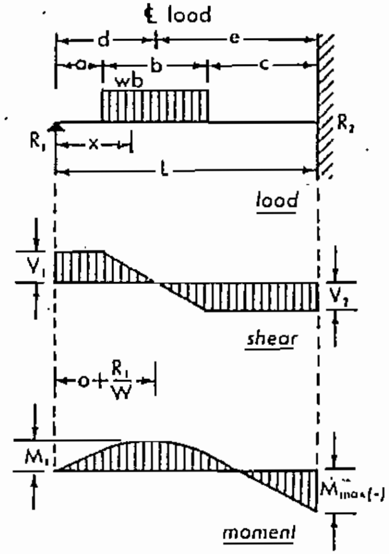
$$M_x = \frac{M_o}{2\,L}(2\,L - 3x)$$

At $x = L/3$, $\quad \Delta_{max} = \frac{M_o\,L^2}{27\,E\,I}$

$$\Delta_x = \frac{M_o\,x}{4\,E\,I\,L}(L - x)^2$$

At supported end, $\quad \theta = -\frac{M_o\,L}{4\,E\,I}$

5C Beam fixed at one end and supported at the other end
Uniform load partially distributed over span



$$R_1 = V_1 = \frac{wb}{8L^3}(12e^2L - 4e^3 + b^2d)$$

$$R_2 = V_2 = w\,b - R_1$$

$$M_{max\,(-)} = \frac{wb}{8L^2}(12e^2L - 4e^3 + b^2d - 8eL^2)$$

$$M_1 = R_1\left(a + \frac{R_1}{2\,w}\right)$$

When $x < a$ $\quad M_x = R_1\,x$

When $x > a$ but $x < (a+b)$ $\quad M_x = R_1\,x - \frac{w}{2}(x - a)^2$

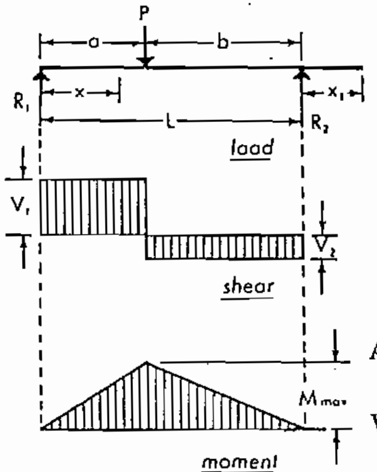
When $x > (a + b)$ but $x < L$ $\quad M_x = R_1\,x - w\,b\,(x - d)$

When $x < a$ $\quad \Delta_x = \frac{x}{24EI}[4R_1(x^2 - 3L^2) + wb\,(b^2 + 12e^2)]$

When $x > a$ but $x < (a + b)$ $\quad \Delta_x = \frac{1}{24EI}[4R_1x\,(x^2 - 3L^2) + wbx\,(b^2 + 12e^2) - w\,(x - a)^4]$

When $x > (a + b)$ but $x < L$ $\quad \Delta_x = \frac{1}{6EI}[3M_{max}(L - x)^2 + R_2(L - x)^3]$

(6Aa) Single span, simply supported beam, with overhang
Concentrated load at any point between supports



$$R_1 = V_1 \begin{pmatrix}\text{max}\\ \text{when}\\ a < b\end{pmatrix} = \frac{P\,b}{L}$$

$$R_2 = V_2 \begin{pmatrix}\text{max}\\ \text{when}\\ a > b\end{pmatrix} = \frac{P\,a}{L}$$

At load, $M_{max} = \dfrac{P\,a\,b}{L}$

When $x < a$ $\quad M_x = \dfrac{P\,b\,x}{L}$

At $x = \sqrt{\dfrac{a\,(a + 2\,b)}{3}}$

$$\Delta_{max} = \frac{P a b\,(a + 2b)\sqrt{3a\,(a + 2b)}}{27 E I L} \quad \text{when } a > b$$

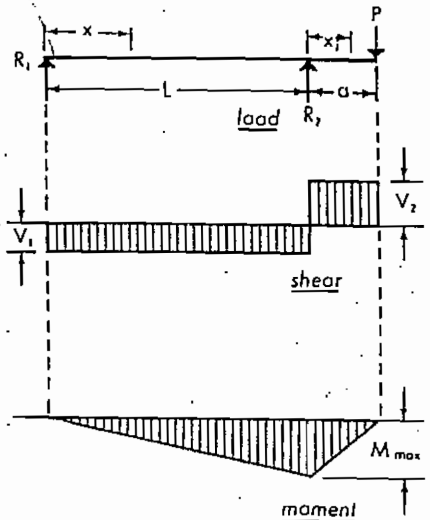
At load, $\quad \Delta = \dfrac{P\,a^2\,b^2}{3\,E\,I\,L}$

When $x < a$ $\quad \Delta_x = \dfrac{P\,b\,x}{6\,E\,I\,L}\,(L^2 - b^2 - x^2)$

When $x > a$ $\quad \Delta_x = \dfrac{P a\,(L - x)}{6 E I L}\,(2Lx - x^2 - a^2)$

For overhang, $\quad \Delta_{x1} = -\dfrac{P\,a\,b\,x_1}{6\,E\,I\,L}\,(L + a)$

(6Ab) Single span, simply supported beam, with overhang
Concentrated load at outer end



$$R_1 = V_1 = \frac{P\,a}{L}$$

$$R_2 = V_1 + V_2 = \frac{P}{L}\,(L + a)$$

$$V_2 = P$$

At R_2, $\quad M_{max} = P\,a$

Between supports, $M_x = \dfrac{P\,a\,x}{L}$

For overhang, $\quad M_{x1} = P\,(a - x_1)$

Between supports at $x = L/\sqrt{3}$, = 0.57 L

$$\Delta_{max} = -\frac{P\,a\,L^2}{9\sqrt{3}\,E\,I}$$

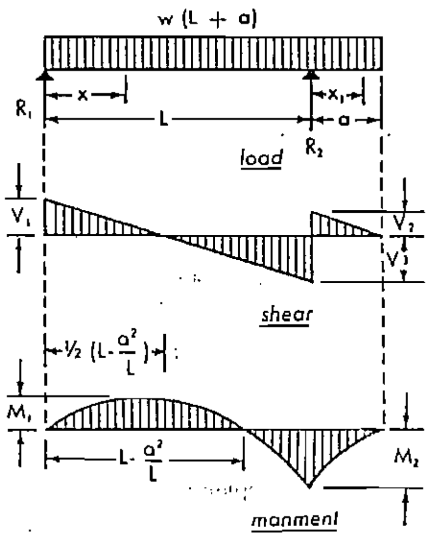
For overhang $x_1 = a$,

$$\Delta_{max} = \frac{P\,a^2}{3\,E\,I}\,(L + a)$$

Between supports, $\quad \Delta_x = -\dfrac{P\,a\,x}{6\,E\,I\,L}\,(L^2 - x^2)$

For overhang, $\quad \Delta_{x1} = \dfrac{P x_1}{6EI}\,(2aL + 3ax_1 - x_1^2)$

(6Ba) Single span, simply supported beam, with overhang
Uniform load over entire beam



$$R_1 = V_1 = \frac{w}{2L}(L^2 - a^2)$$

$$R_2 = V_2 + V_3 = \frac{w}{2L}(L + a)^2$$

$$V_2 = wa$$

$$V_3 = \frac{w}{2L}(L^2 + a^2)$$

Between supports, $V_x = R_1 - wx$

For overhang, $V_{x1} = w(a - x_1)$

At $x = \frac{1}{2}\left(L - \frac{a^2}{L}\right)$

$$M_1 = \frac{w}{8L^2}(L^2 - a^2)^2$$

At R_2, $M_2 = \frac{wa^2}{2}$

Between supports, $M_x = \frac{wx}{2L}(L^2 - a^2 - xL)$

For overhang, $M_{x1} = \frac{w}{2}(a - x_1)^2$

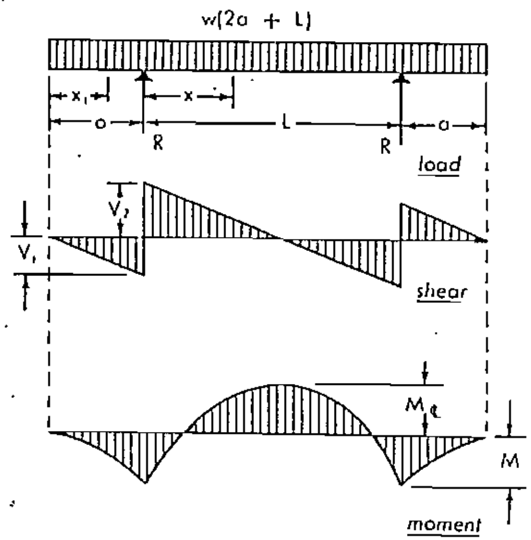
Between supports, $\Delta_x = \frac{wx}{24EIL}(L^4 - 2L^2x^2 + Lx^3 - 2a^2L^2 + 2a^2x^2)$

For overhang, $\Delta_{x1} = \frac{wx_1}{24EI}(4a^2L - L^3 + 6a^2x_1 - 4ax_1^2 + x_1^3)$

At free end, $\Delta = \frac{wa}{24EI}(3a^3 + 4a^2L - L^3)$

When $a = .414\,L$, $M_1 = M_2 = .08579\,wL^2$

(6Bb) Single span beam, overhanging at both ends
Uniform load over entire beam



$$R = V_1 + V_2 = w(a + L/2)$$

$$V_{x1} = wx_1$$

$$V_x = w(x - L/2)$$

For overhang, $M_{x1} = \frac{wx_1^2}{2}$

At support, $M = \frac{wa^2}{2}$

Between supports, $M_x = \frac{w}{2}(Lx - x^2 - a^2)$

At center, $M_{℄} = \frac{w}{8}(L^2 - 4a^2)$

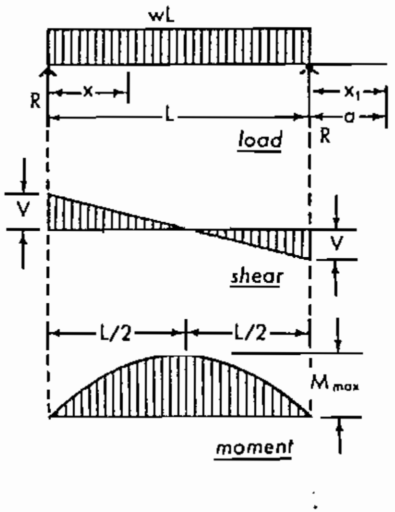
At ends, $\Delta = \frac{wa}{24EI}(L^3 - 6a^2L - 3a^3)$

At center, $\Delta_{℄} = \frac{wL^2}{384EI}(5L^2 - 24a^2)$

When $a = .207 \times$ total length
or $a = .354\,L$

$$M = M_{℄} = \frac{wL^2}{16}$$

6Ca Single span, simple supported beam, with overhang
Uniform load over entire span



$$R = V = \frac{wL}{2}$$

$$V_x = w\left(\frac{L}{2} - x\right)$$

At center, $$M_{max} = \frac{wL^2}{8}$$

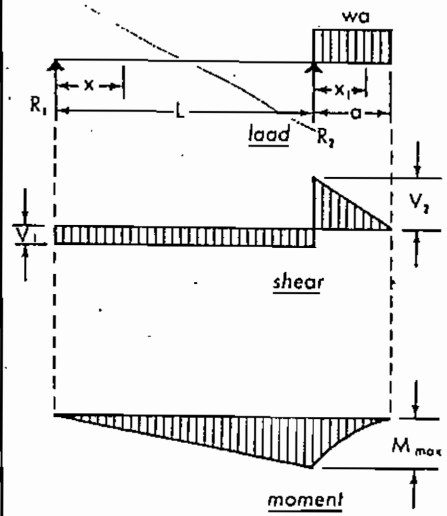
$$M_x = \frac{wx}{2}(L - x)$$

At center, $$\Delta_{max} = \frac{5wL^4}{384EI}$$

$$\Delta_x = \frac{wx}{24EI}(L^3 - 2Lx^2 + x^3)$$

$$\Delta_{x1} = \frac{wL^3x_1}{24EI}$$

6Cb Single span, simply supported beam, with overhang
Uniform load on overhang



$$R_1 = V_1 = \frac{wa^2}{2L}$$

$$R_2 = V_1 + V_2 = \frac{wa}{2L}(2L + a)$$

$$V_2 = wa$$

For overhang, $$V_{x1} = w(a - x_1)$$

At R_2, $$M_{max} = \frac{wa^2}{2}$$

Between supports, $$M_x = \frac{wa^2x}{2L}$$

For overhang, $$M_{x1} = \frac{w}{2}(a - x_1)^2$$

At $x = L/\sqrt{3}$, $$\Delta_{max} = -\frac{wa^2L^2}{18\sqrt{3}EI}$$

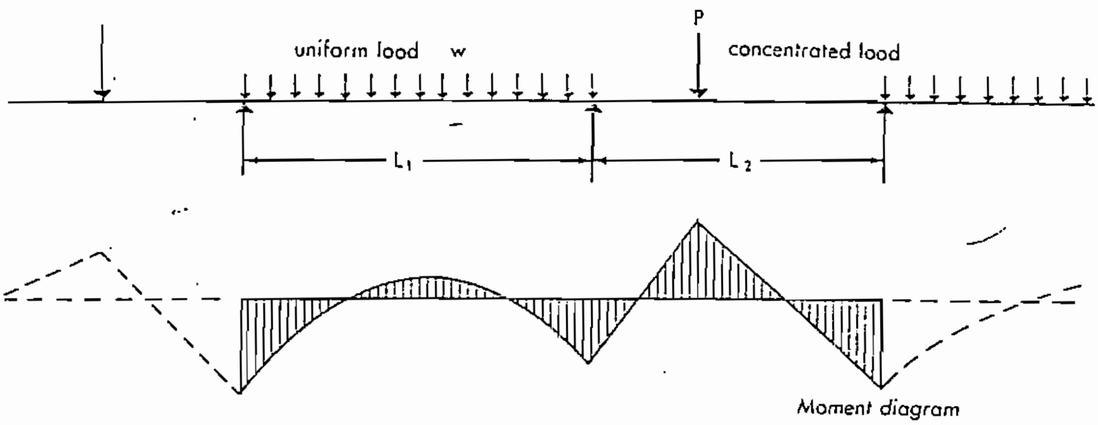
At free end, $$\Delta_{max} = \frac{wa^3}{24EI}(4L + 3a)$$

Between supports, $$\Delta_x = -\frac{wa^2x}{12EIL}(L^2 - x^2)$$

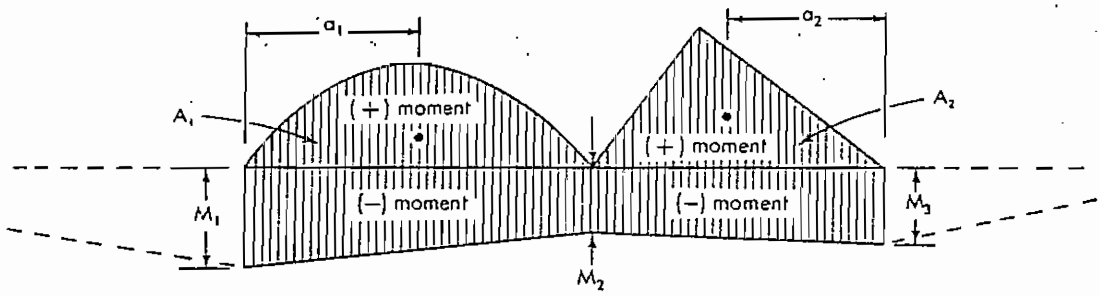
For overhang, $$\Delta_{x1} = \frac{wx_1}{24EI}(4a^2L + 6a^2x_1 - 4ax_1^2 + x_1^3)$$

7 THEORY OF THREE MOMENTS

Consider the following continuous beam:



The above moment diagram may be considered as made up of two parts: the positive moment due to the applied loads, and the negative moment due to the restraining end moments over the supports.



For any two adjacent spans, the following relationship is true:

$$\boxed{+\frac{M_1 L_1}{6\,E I_1} + \frac{M_2}{3\,E}\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + \frac{M_3 L_2}{6\,E I_2} + \frac{A_1 a_1}{E I_1 L_1} + \frac{A_2 a_2}{E I_2 L_2} = 0}$$

where:

M_1, M_2, and M_3 are the end moments at the 1st, 2nd, and 3rd supports.

L_1 and L_2 are the lengths of the 1st and 2nd span.

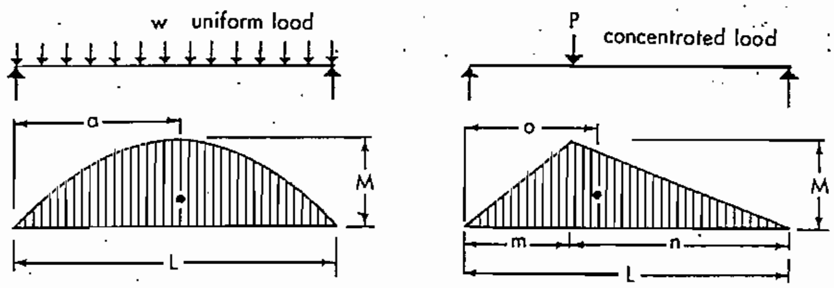
I_1 and I_2 are the moments of inertia of the 1st and 2nd span.

A_1 and A_2 are the areas under the positive moment diagrams of the 1st and 2nd span.

a_1 and a_2 are the distance of the centroids of the areas of the positive moment diagrams to the 1st and 3rd outer supports.

By writing this equation for each successive pair of spans, all of the moments may be found.

The moment diagram for a simply supported, uniformly loaded beam is a parabola; and a concentrated load produces a triangular moment diagram. The following shows the area and distance to the centroid of these areas.



Area

$A = 2/3\,M\,L$

Distance to centroid

$a = L/2$

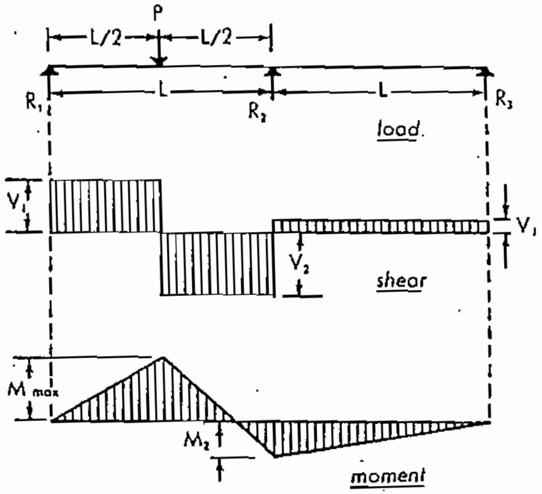
Area

$A = 1/2\,M\,L$

Distance to centroid

$$a = \frac{m + L}{3}$$

7Aa Two span, continuous beam — Concentrated load at center of one span only



$$R_1 = V_1 = \frac{13}{32} P$$

$$R_2 = V_2 + V_3 = \frac{11}{16} P$$

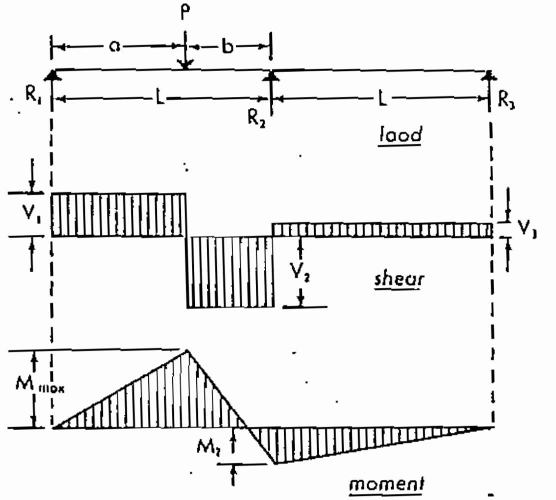
$$R_3 = V_3 = \frac{3}{32} P$$

$$V_2 = \frac{19}{32} P$$

At load, $M_{max} = \frac{13}{64} P L$

At R_2, $M_2 = \frac{3}{32} P L$

7Ab Two span, continuous beam — Concentrated load at any point of one span only



$$R_1 = V_1 = \frac{Pb}{4L^3} [4L^2 - a(L+a)]$$

$$R_2 = V_2 + V_3 = \frac{Pa}{2L^3} [2L^2 + b(L+a)]$$

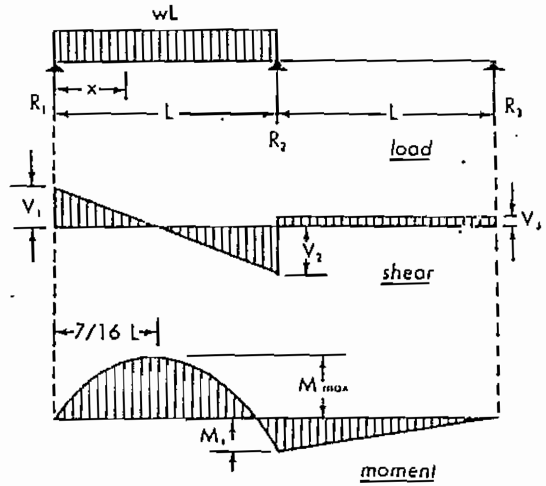
$$R_3 = V_3 = \frac{P a b}{4 L^3} (L + a)$$

$$V_2 = \frac{P a}{4 L^3} [4 L^2 - b (L + a)]$$

At load, $M_{max} = \frac{P a b}{4 L^3} [4 L^2 - a (L + a)]$

At R_2, $M_2 = \frac{P a b}{4 L^2} (L + a)$

7B Two span, continuous beam — Uniform load over one span only



$$R_1 = V_1 = \frac{7}{16} w L$$

$$R_2 = V_2 + V_3 = \frac{5}{8} w L$$

$$R_3 = V_3 = \frac{1}{16} w L$$

$$V_2 = \frac{9}{16} w L$$

At $x = 7/16\,L$, $M_{max} = \frac{49}{512} w L^2$

At R_2, $M_1 = \frac{w L^2}{16}$

When $x < L$, $M_x = \frac{w x}{16} (7 L - 8 x)$

See pages 8 and 9 for beam-load condition 7D