# Analysis of Combined Stresses

## 1. CONCEPT OF CUBICAL UNIT

Structural members are often subject to combined loading, such as axial tension and transverse bending. These external forces induce internal stresses as forces of resistance. Even without combined loading, there may be combined stress at points within the member.

The analysis of combined stresses is based on the concept of a cubic unit taken at any point of intersection of three planes perpendicular to each other. The total forces in play against these planes result in proportionate forces of the same nature acting against faces of the cube, tending to hold it in equilibrium. Since any member is made up of a multitude of such cubes, the analysis of stresses at a critical point is the key to analysis of the member's resistance to combined external forces.

## 2. COMBINING STRESSES

*Biaxial and triaxial stresses* are tensile and compressive stresses combined together.

*Combined stresses* are tensile and compressive stresses combined together.

*Principal planes* are planes of no shear stress.

*Principal stresses* are normal stresses (tensile or compressive) acting on these principal planes. These are the greatest and smallest of all the normal stresses in the element.

Normal stresses, either tensile or compressive, act normal or at right angles to their reference planes. Shear stresses act parallel to their reference planes.

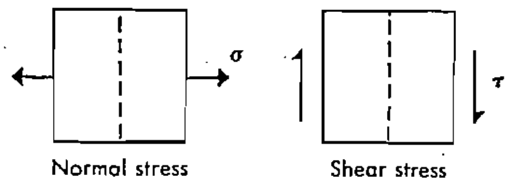



FIGURE 1

These stresses may be represented graphically on Mohr's circle of stress. By locating the points $(\sigma_3, \tau_1)$ and $(\sigma_2, \tau_1)$ on a graph, Figure 2, and drawing a circle through these two points, the other stresses at various planes may be determined.

By observation of Mohr's circle of stress, it is found that—

Stress in Member | Mohr's Circle of Stress

FIGURE 2

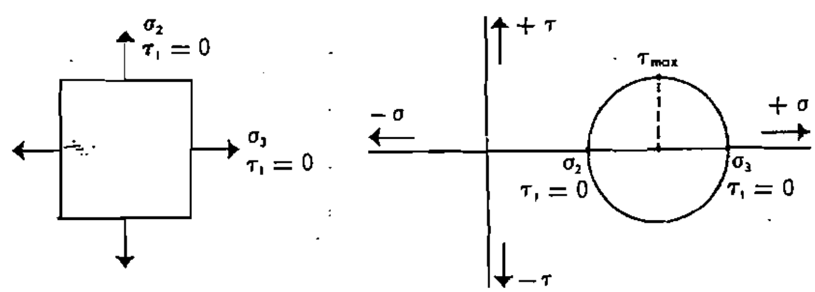
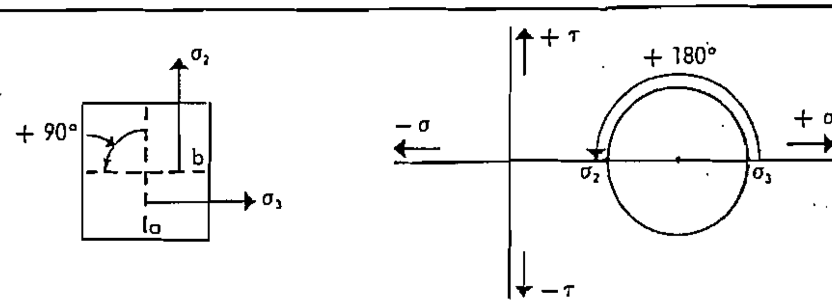



FIGURE 3



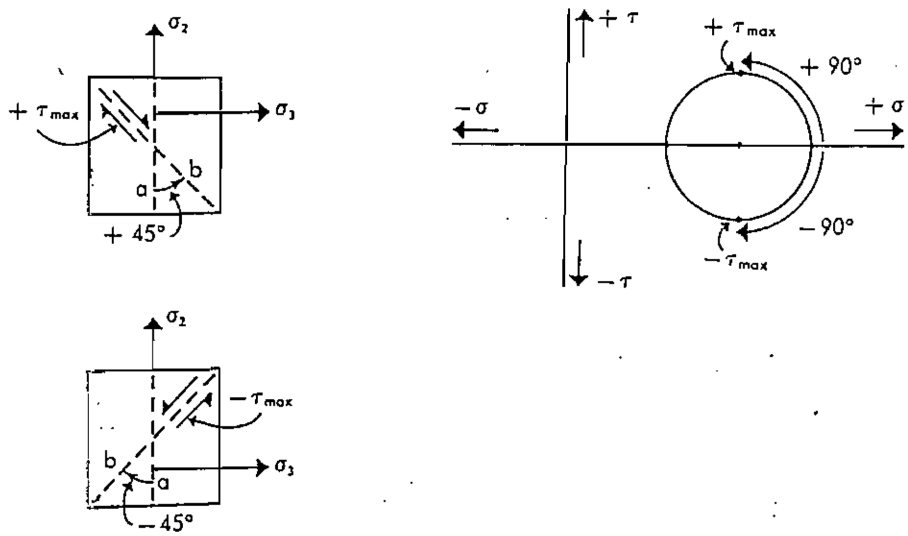



FIGURE 4

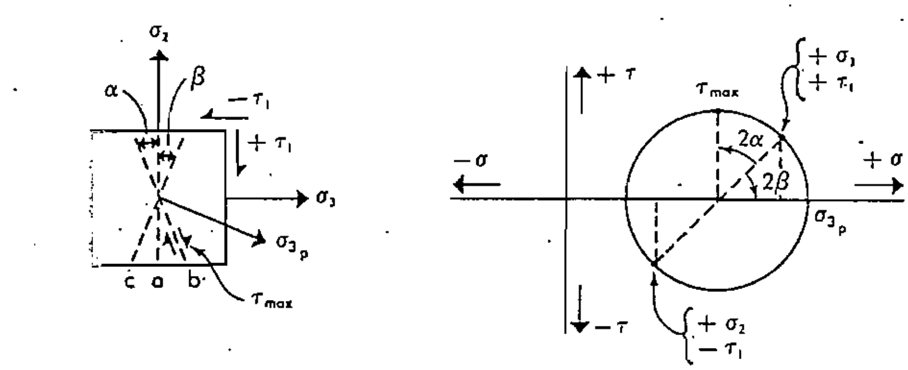



FIGURE 5

$$\boxed{\tau_{max} = \frac{\sigma_3 - \sigma_2}{2}} \quad \text{.......................(1)}$$

In this case, $\sigma_3$ and $\sigma_2$ are principal stresses $\sigma_{3p}$ and $\sigma_{2p}$ since they act on planes of zero shear stress.

For any angle of rotation on Mohr's circle of stress, the corresponding planes on which these stresses act in the member rotate through just half this angle and in the same direction.

Notice in Figure 3, $\sigma_2$ lies at $+180°$ from $\sigma_3$ in Mohr's circle of stress, and the plane (b) on which $\sigma_2$ acts in the member lies at $+90°$ from the plane (a) on which $\sigma_3$ acts.

Notice in Figure 4, $\tau_{max}$ lies at $+90°$ from $\sigma_3$ and the plane (b) on which $\tau_{max}$ acts in the member lies at $+45°$ from the plane (a) on which $\sigma_3$ acts. In this case $\sigma_2$ and $\sigma_3$ are principal stresses because there is no applied shear on these planes.

This is a simple method to graphically show how stresses within a member combine; see Figure 5. On the graph, right, locate the two stress points $(+\sigma_3, +\tau_1)$ and $(+\sigma_2, -\tau_1)$ and draw a circle through these points. Now determine maximum normal and shear stresses.

By observation of Mohr's circle of stress, it is found that—

$$\boxed{\sigma_{3p}\ (\max) = \frac{\sigma_3 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_3 - \sigma_2}{2}\right)^2 + \tau_1^{\,2}}} \quad \text{.......................(2)}$$

$$\boxed{\tau_{max} = \sqrt{\left(\frac{\sigma_3 - \sigma_2}{2}\right)^2 + \tau_1^{\,2}}} \quad \text{.............(3)}$$

The above formula for the maximum shear stress ($\tau_{max}$) is true for the flat plane considered; however, there are really two other planes not yet considered and their maximum shear stress could possibly be greater than this value.

This is a very common mistake among engineers. To be absolutely sure, when dealing with biaxial

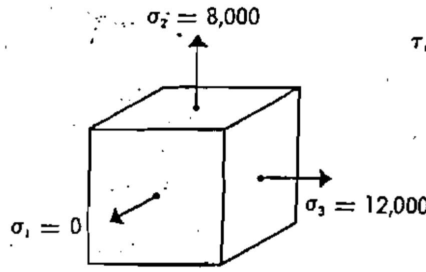
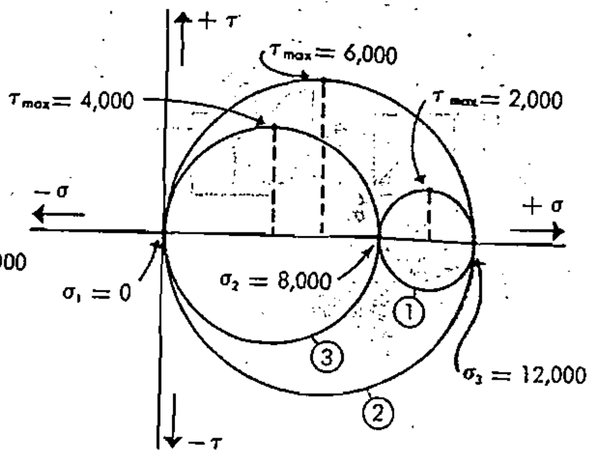



FIGURE 6




stresses, always let the third normal stress be zero instead of ignoring it, and treat the problem as a triaxial stress problem.

The example in Figure 2 will now be reworked, Figure 6, and the third normal stress ($\sigma_1$) will be set equal to zero.

Here,

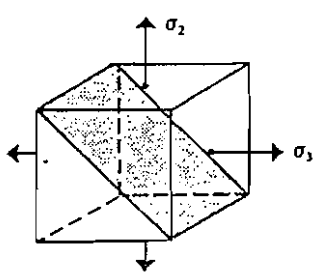
$$\sigma_3 = +\ 12,000 \text{ psi} \qquad \tau_3 = 0$$
$$\sigma_2 = +\ 8,000 \text{ psi} \qquad \tau_2 = 0$$
$$\sigma_1 = 0 \qquad \tau_1 = 0$$

On graph, right: Locate stress points ($\sigma_1$) ($\sigma_2$), ($\sigma_3$) and draw three circles through these points. Now determine the three maximum shear stresses.

There are three values for the maximum shear stress, each equal to half of the difference between two principal (normal) stresses. The plane of maximum shear stress (shaded in the following sketches) is always at 45° to the planes of principal stress.

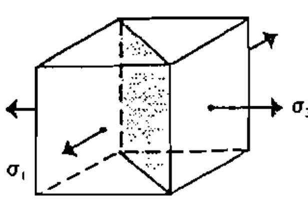
*Circle 1*

$$\tau_{max} = \frac{\sigma_3 - \sigma_2}{2} = \frac{12{,}000\text{-}8{,}000}{2} = 2{,}000 \text{ psi}$$



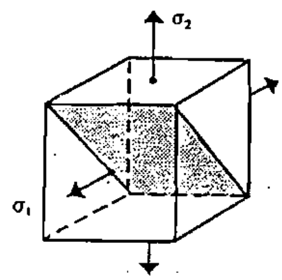

*Circle 2*

$$\tau_{max} = \frac{\sigma_3 - \sigma_1}{2} = \frac{12{,}000\text{–}0}{2} = 6{,}000 \text{ psi}$$




*Circle 3*

$$\tau_{max} = \frac{\sigma_2 - \sigma_1}{2} = \frac{8{,}000\text{–}0}{2} = 4{,}000 \text{ psi}$$




It is seen that, in this example, the maximum shear stress is 6,000 psi, and not the 2,000 psi value that would usually be found from the conventional formulas for biaxial stress.

## 3. TRIAXIAL STRESS COMBINED WITH SHEAR STRESS (See Figure 7)

The three principal stresses ($\sigma_{1p}$, $\sigma_{2p}$, $\sigma_{3p}$) are given by the three roots ($\sigma_p$) of this cubic equation:

$$\boxed{\begin{aligned}&\sigma_{p3} - (\sigma_1 + \sigma_2 + \sigma_3)\sigma_p{}^2\\ &\quad + (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3 - \tau_1{}^2 - \tau_2{}^2 - \tau_3{}^2)\sigma_p\\ &\quad - (\sigma_1\sigma_2\sigma_3 + 2\,\tau_1\tau_2\tau_3 - \sigma_1\tau_1{}^2 - \sigma_2\tau_2{}^2 - \sigma_3\tau_3{}^2) = 0\end{aligned}} \quad (4)$$

For maximum shear stress, use the two principal stresses ($\sigma_p$) whose algebraic difference is the greatest. The maximum shear stress ($\tau_{max}$) is equal to half of this difference.

*Since a, b, and c are coefficients of this equation:

$$a = -(\sigma_1 + \sigma_2 + \sigma_3)$$
$$b = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3 - \tau_1{}^2 - \tau_2{}^2 - \tau_3{}^2$$
$$c = \sigma_1\tau_1{}^2 + \sigma_2\tau_2{}^2 + \sigma_3\tau_3{}^2 - \sigma_1\sigma_2\sigma_3 - 2\,\tau_1\tau_2\tau_3$$

*Solution of Cubic Equation from "Practical Solution of Cubic Equations", G. L. Sullivan, MACHINE DESIGN, Feb. 21, 1957.

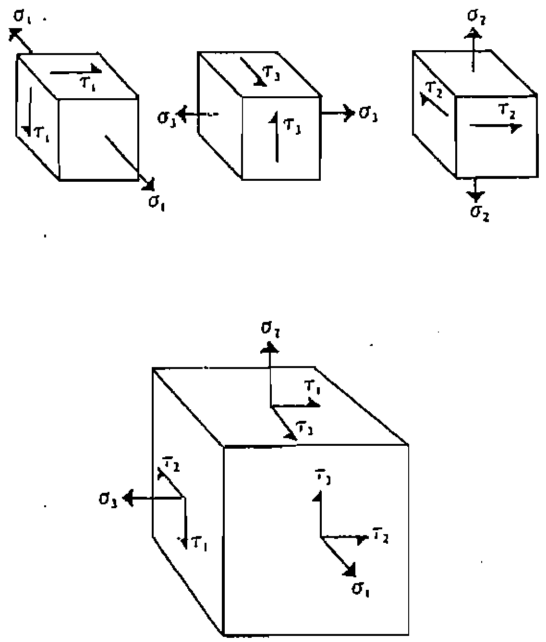


FIGURE 7

Let $N = \frac{b}{3} - \left(\frac{a}{3}\right)^2$

and $Q = \frac{c}{2} - \frac{a\,b}{6} + \left(\frac{a}{3}\right)^3$

Then calculate—

$$K = \frac{N^3}{Q^2} \text{ as a test ratio.}$$

*Case 1*

When $(1 + K)$ is positive (one real root) or when $(1 + K)$ is zero (three real roots, two of which are equal)
calculate—

$$S = \sqrt[3]{Q\left[1 + \sqrt{1 + K}\right]}$$

and compute the root—

$$\sigma_{1p} = \frac{N}{S} - S - \frac{a}{3}$$

*Case 2*

When $(1 + K)$ is negative (three real and unequal roots)
calculate—

$$T = \sqrt{-K}$$

and compute the root—

$$\sigma_{1p} = \mp\sqrt{-3N}\left(\frac{T + 0.386}{T + 0.2}\right) - \frac{a}{3}$$

The ambiguous sign is opposite to the sign of Q (approximate, but very accurate).

*For either Case 1 or Case 2*

The additional two roots ($\sigma_{2p}$, $\sigma_{3p}$) of the general cubic equation are calculated by solving for $\sigma_p$ using the exact quadratic:

$$\sigma_p^2 + (a + \sigma_{1p})\sigma_p - \frac{c}{\sigma_{1p}} = 0$$

$$\text{or } \sigma_p = \frac{-(a + \sigma_{1p}) \pm \sqrt{(a + \sigma_{1p})^2 + \frac{4c}{\sigma_{1p}}}}{2}$$

**Problem 1**

Determine the maximum normal and shear stress in this web section, Figure 8:

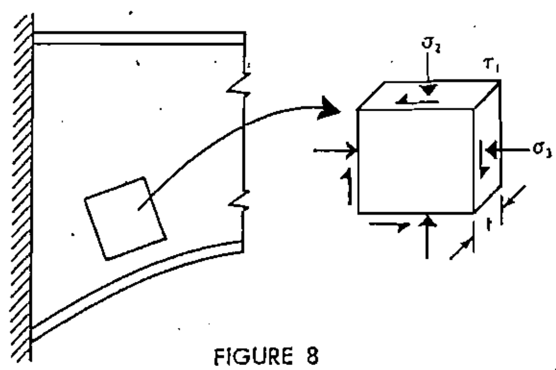


FIGURE 8

where:

$\sigma_1 = 0$ $\quad\tau_1 = 11{,}000$ psi

$\sigma_2 = -13{,}650$ psi $\quad\tau_2 = 0$

$\sigma_3 = -14{,}500$ psi $\quad\tau_3 = 0$

Substituting these values into the general cubic equation:

$$\sigma_p^3 - (-13{,}650 - 14{,}500)\sigma_p^2 + [(-13{,}650)(-14{,}500) - (11{,}000)^2]\,\sigma_p = 0$$

$$\sigma_p^2 + 28{,}150\,\sigma_p + 76{,}925{,}000 = 0$$

the three principal normal stresses are—

$\sigma_{1p} = 0$

$\sigma_{2p} = -25{,}075$ psi $\quad\sigma_{3p} = -3{,}075$ psi

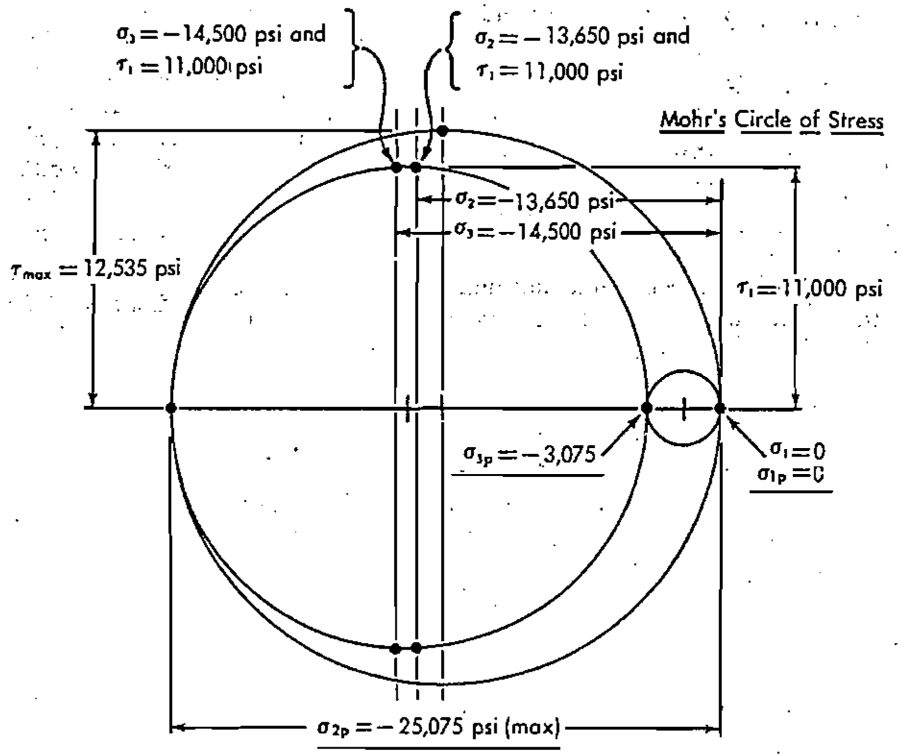



FIGURE 9

and taking one-half of the greatest difference of two principal stresses:

$$\tau_{max} = \frac{25{,}075 - 0}{2} = \underline{12{,}535 \text{ psi}}$$

These various values are shown diagramed on Mohr's Circle of Stress, Figure 9.

### *Checking Effect of Applied Stresses*

The Huber-Mises formula is convenient for checking the effect of applied stresses on the yielding of the plate. If a certain combination of normal stresses ($\sigma_x$ and $\sigma_y$) and shear stress ($\tau_{xy}$) results in a critical stress ($\sigma_{cr}$) equal to the yield strength ($\sigma_y$) of the steel when tested in uniaxial tension, this combination of stresses is assumed to just produce yielding in the steel.

$$\sigma_{cr} = \sqrt{\sigma_x^2 - \sigma_x\,\sigma_y + \sigma_y^2 + 3\,\tau_{xy}}$$

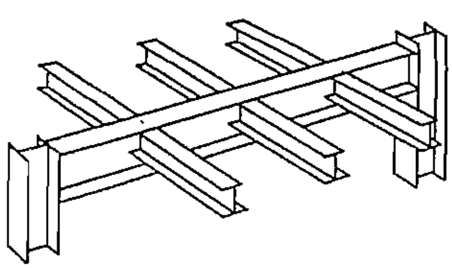


FIGURE 10

**Problem 2**

For the beam-to-girder network represented by Figure 10, assume the combination of stresses represented by Figure 11.

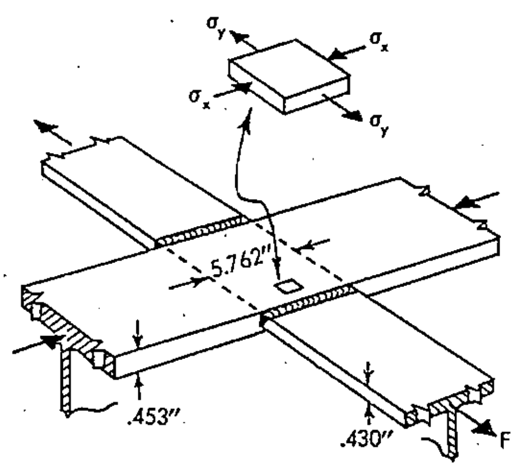



FIGURE 11

Here:

$$\sigma_{cr} = \sqrt{\sigma_x^2 - \sigma_x\,\sigma_y + \sigma_y^2 + 3\,\tau_{xy}}$$
$$= \sqrt{(-14{,}350)^2 - (-14{,}350(15{,}900) + (15{,}900)^2 + 0}$$
$$= 21{,}600 \text{ psi}$$

The apparent factor of yielding is

$$k = \frac{\sigma_y}{\sigma_{cr}}$$
$$= \frac{(36{,}000)}{(21{,}600)}$$
$$= 1.67$$

This seems reasonable and under these conditions, the beam flange could be groove welded directly to the edge of the girder flange without trying to isolate the two intersecting flanges.

## 4. STRENGTH UNDER COMBINED LOADING

A very convenient method of treating combined loadings is the interaction method. Here each type of load is expressed as a ratio of the actual load (P,M,T) to the ultimate load ($P_u$,$M_u$,$T_u$) which would cause failure if acting alone.

| *axial load* | *bending load* | *torsional load* |
|---|---|---|
| $R_a = \frac{P}{P_u}$ | $R_b = \frac{M}{M_u}$ | $R_t = \frac{T}{T_u}$ |

In the general example shown in Figure 12, the effect of two types of loads (x) and (y) upon each other is illustrated.

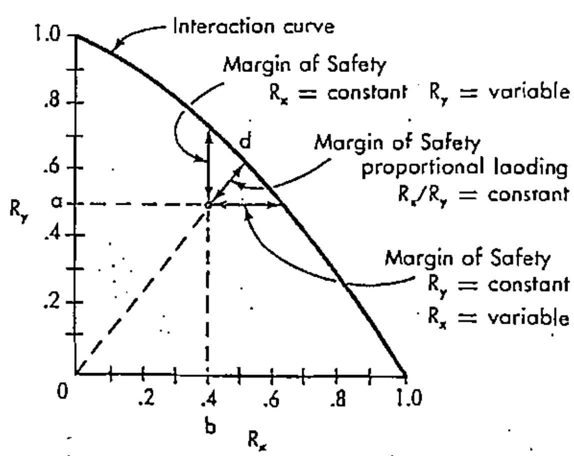



FIGURE 12

The value of $R_y = 1$ at the upper end of the vertical axis is the ultimate value for this type of load on the member. The value $R_x = 1$ at the extreme right end of the horizontal axis is the ultimate value for this type of load on the member. These values are determined by experiment; or when this data is not available, suitable calculations may be made to estimate them.

The interaction curve is usually determined by actual testing of members under various combined-load conditions, and from this a simple formula is derived to express this relationship.

If points *a* and *b* are the ratios produced by the actual loads, point *c* represents the combination of these conditions, and the margin of safety is indicated by how close point *c* lies to the interaction curve. A suitable factor of safety is then applied to these values.

### *Combined Bending and Torsion*

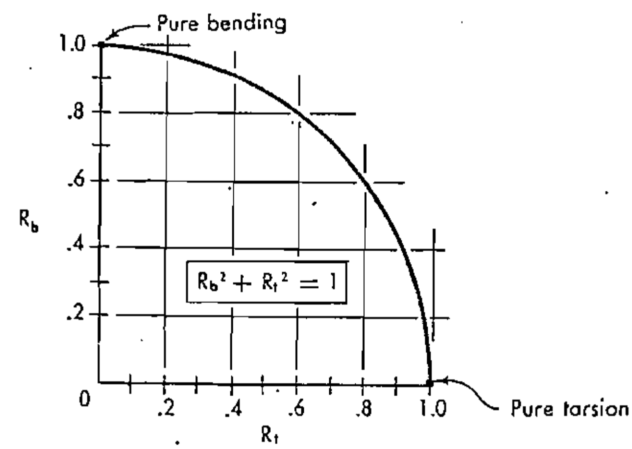



FIGURE 13

### *Combined Axial Loading and Torsion*

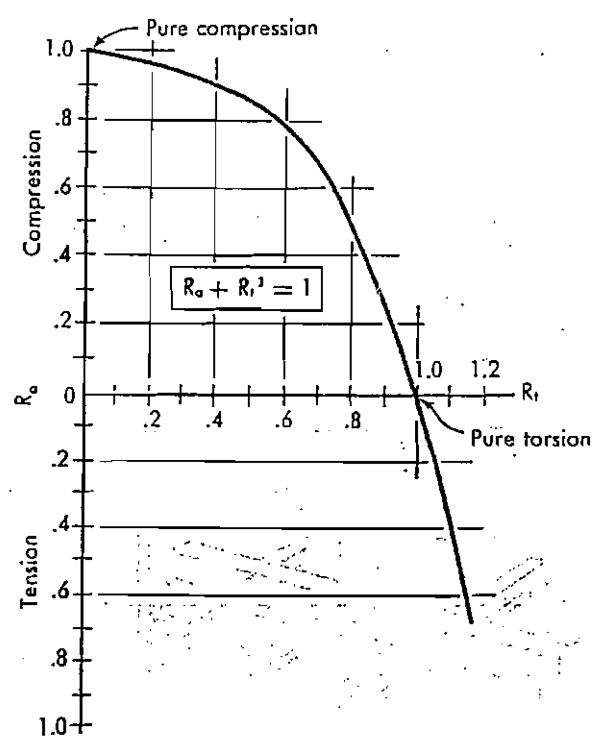
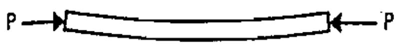



FIGURE 14

## *Combined Axial Compression and Bending*

In this case, the axial compression will cause additional deflection, which in turn increases the moment of the bending load. This increase can easily be taken care of by an amplification factor (k). See Figures 15 and 16.

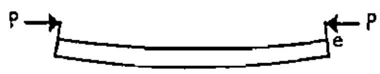
For sinusoidal initial bending moment curve




$$k = \frac{1}{1 - P/P_{cr}}$$

FIGURE 15

For constant bending moment




$$k = \frac{1}{\cos \frac{\pi}{2} \sqrt{P/P_{cr}}}$$

FIGURE 16

Here:

$$P_{cr} = \frac{\pi^2 E I}{L^2}$$

The bending moment applied to the member (chosen at the cross-section where it is maximum) is then multiplied by this amplification factor (k), and this value is then used as the applied moment (M) in the ratio:

$$R_b = \frac{M}{M_a}$$

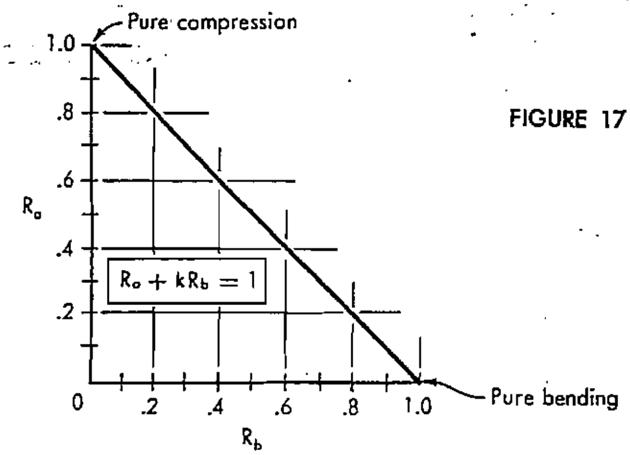



FIGURE 17

The chart in Figure 18 is used to determine the amplification factor (k) for the bending moment

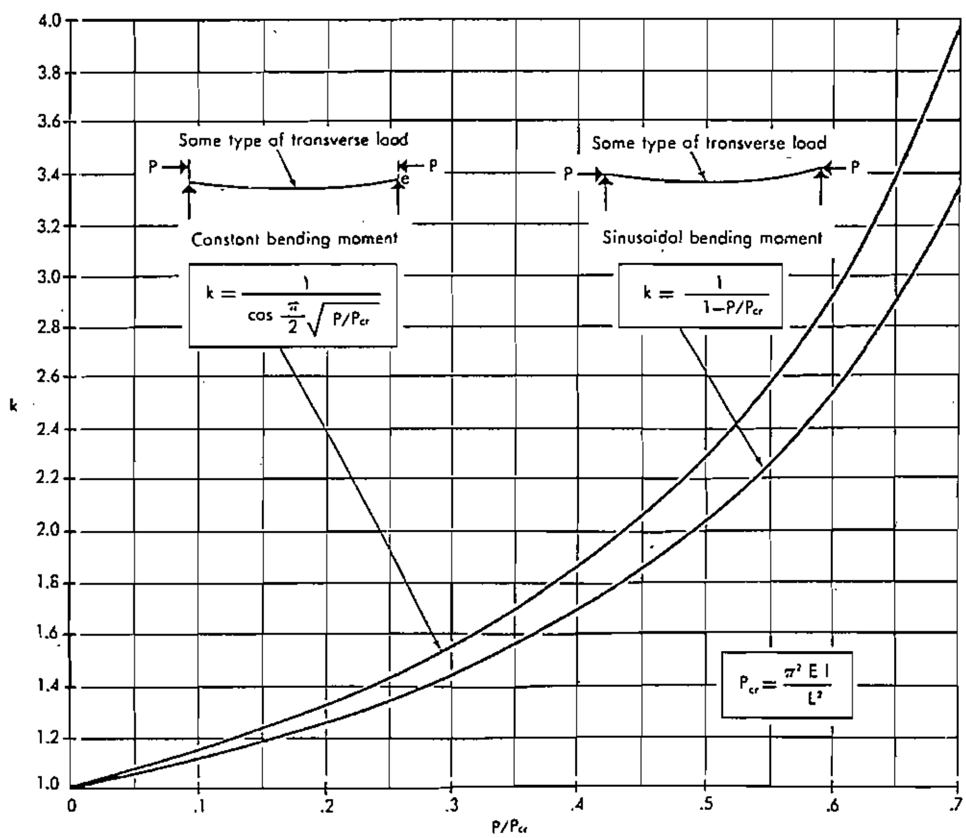



FIG. 18 Amplification factor (k) for bending moment on beam also subject to axial compression.

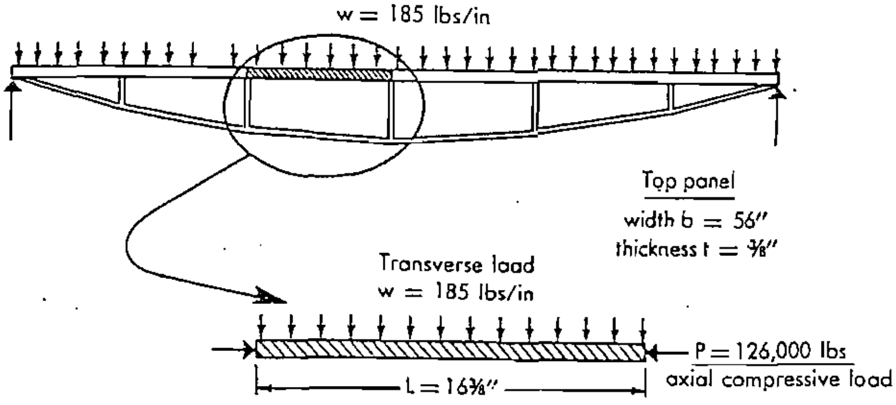



FIGURE 19

applied to a beam when it is also subject to axial compression.

The resulting combined stress is found from the following formula:

$$\sigma = \frac{P}{A} \pm \frac{k\ M\ c}{I}$$

**Problem 3**

A loading platform is made of a ⅜″ top plate and a 10-gage bottom sheet. The whole structure is in the form of a truss, Figure 19.

***Determination of combined stress (axial compression and bending) in top compression panel:***

With $L = 16⅜''$

$A = 21 \text{ in.}^2$

$I = .247 \text{ in.}^4$

First the critical load—

$$P_{cr} = \frac{\pi^2\ E\ I}{L^2}$$

$$= \frac{\pi^2\ (30 \times 10^6)\ (.247)}{(16⅜)^2}$$

$$= 272{,}000 \text{ lbs}$$

Then the ratio—

$$P/P_{cr} = \frac{126{,}000}{272{,}000}$$

$$= .464$$

The bending moment—

$$M = \frac{w\ L^2}{8}$$

$$= \frac{(185)\ (16⅜)^2}{8}$$

$$= 6200 \text{ in.-lbs}$$

Obtaining the amplification factor (k) for the sinusoidal bending moment from the curve, Figure 18—

$$k = 1.87$$

The actual applied moment due to extra deflection is found to be—

$$k\ M = (1.87)(6200)$$

$$= 11{,}600 \text{ in.-lbs.}$$

The resulting combined stress formula being—

$$\sigma = \frac{P}{A} \pm \frac{k\ M\ c}{I}$$

of which there are two components:

(a) the compressive stress above the neutral axis of the top panel being—

$$\sigma_c = \frac{126{,}000}{21} + \frac{11{,}600(\tfrac{3}{16})}{.247}$$

$$= \underline{14{,}800 \text{ psi}}$$

(b) and the tensile stress below the neutral axis of the top panel being—

$$\sigma_t = \frac{126{,}000}{21} - \frac{11{,}600(\tfrac{3}{16})}{.247}$$

$$= 2{,}800 \text{ psi}$$

### *Determination of Factor of Safety*

The ultimate load values for this member in compression alone and in bending alone are unknown, so the following are used.

For *compression* alone —

*Since $\frac{L}{r} = 150$ (where $r$ = radius of gyration)

assume $P_u = P_{cr} = 272{,}000$ lbs

For *bending* alone—

The plastic or ultimate bending moment is—

$$M_u = \left(b\ \sigma_y\ \frac{t}{2}\right)\frac{t}{2} = \frac{b\ t^2\ \sigma_y}{4}$$

$$= \frac{(56)\ (3/8)^2\ (33{,}000)}{4}$$

$$= 64{,}900 \text{ in.-lbs}$$

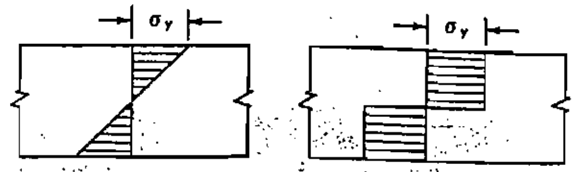



FIGURE 20

These ultimate values are represented on the following interaction curve, Figure 21. Plotting the present load values at *a* against the curve, indicates there is about a *2:1 factor of safety* before the top compression panel will buckle.

*This L/r ratio of 150 is high enough so we can assume the ultimate load carrying capacity of the column ($P_u$) is about equal to the critical value ($P_{cr}$). If this had been an extremely short column (very low L/r ratio), the critical value ($P_{cr}$) could be quite a bit higher than the actual ultimate value ($P_u$).

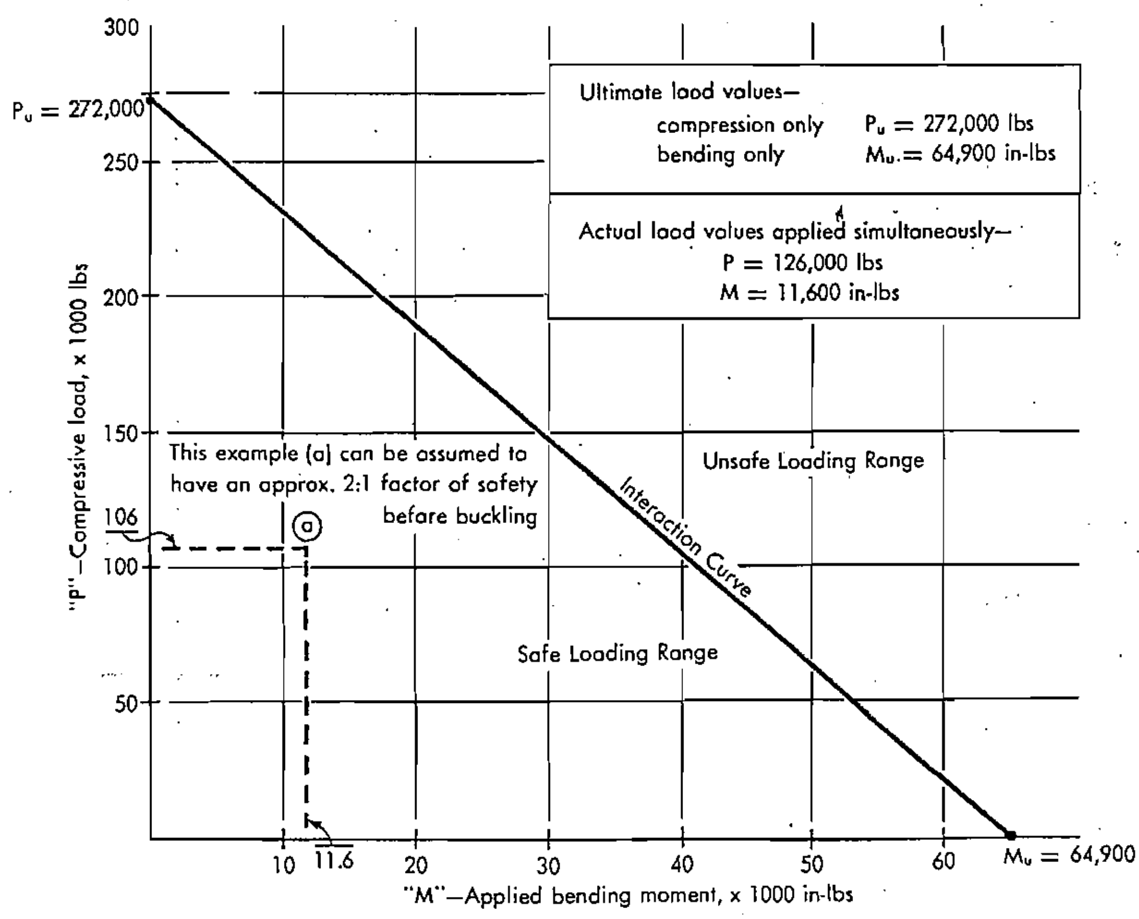
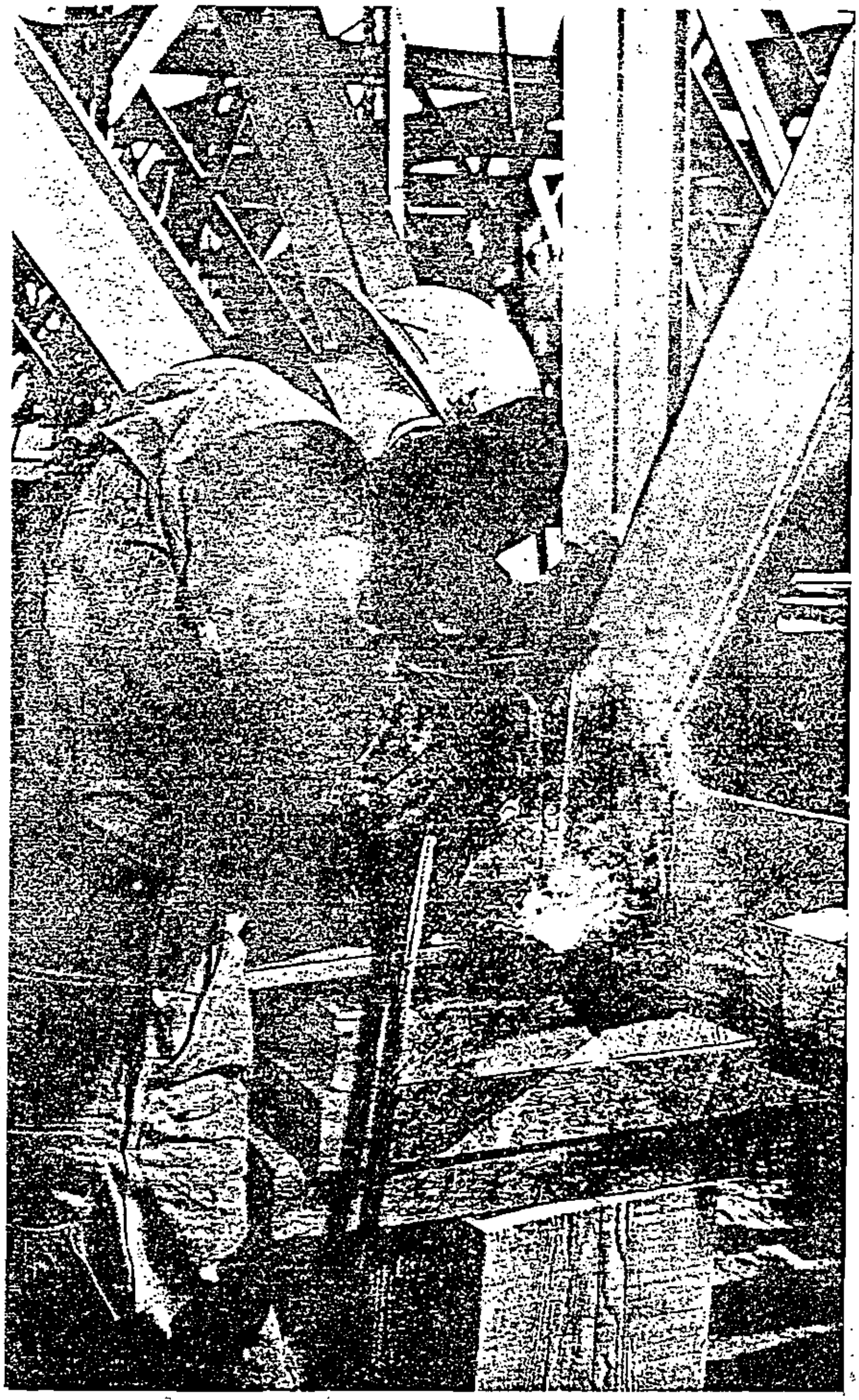



FIG. 21 Interaction Curve for Problem 3



The Air Force Academy Dining Hall (seating the entire student body) at Colorado Springs was built on the ground and jacked into position atop columns. The complexity of joints, the heavy cantilevered construction and large lateral forces offered unique problems in combined stresses. Welding was the only practical approach to the complex connections required to join members of this three-dimensional truss system.