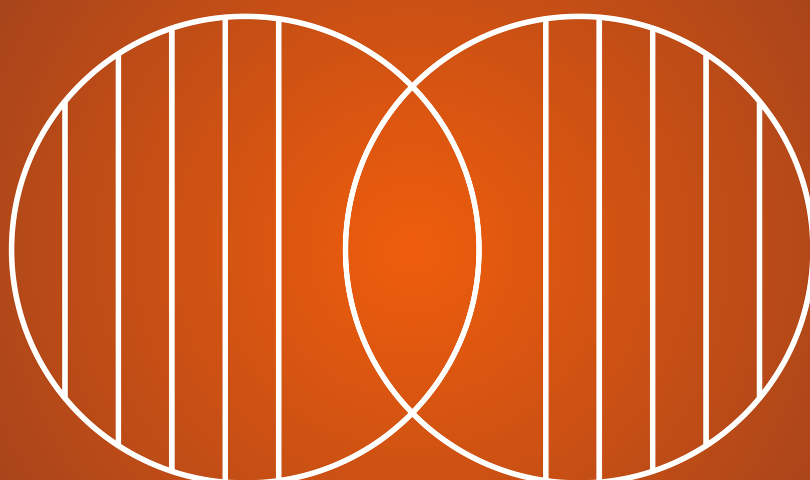
Vibration Modal Analysis Challenge Quiz Part 2

Solution



Modal Decoupling of the System Equations of Motion

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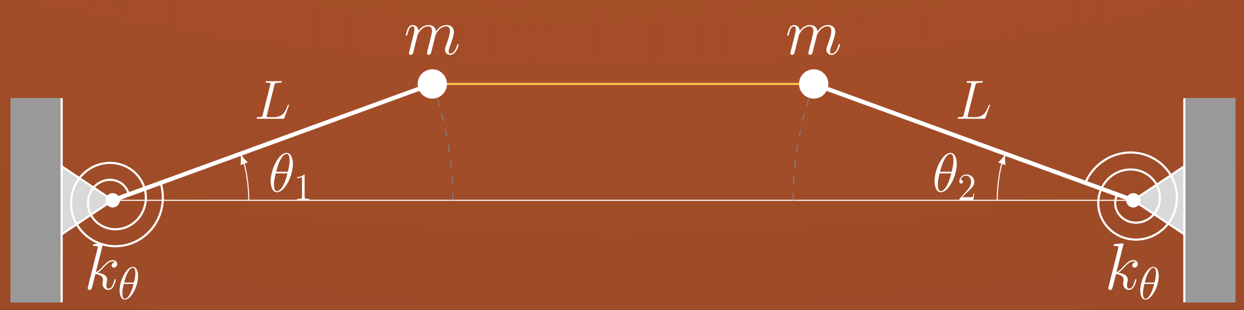
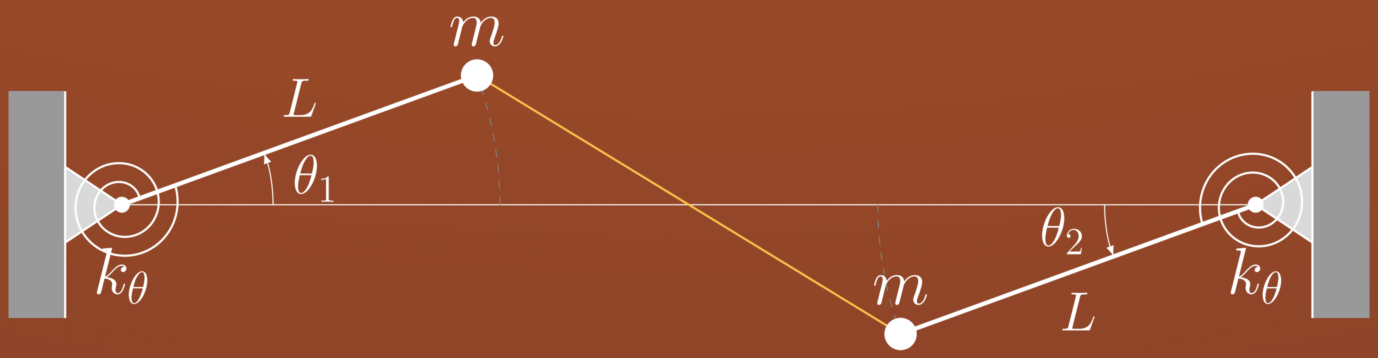
Problem Statement

- The natural frequencies and mode shapes for a rigid bar and cable structure have been computed from the two coupled equations of motion for discrete rotations in terms of mass inertia, rotational spring stiffness, and cable tension. This analysis identified two modes: One symmetric and the other antisymmetric.



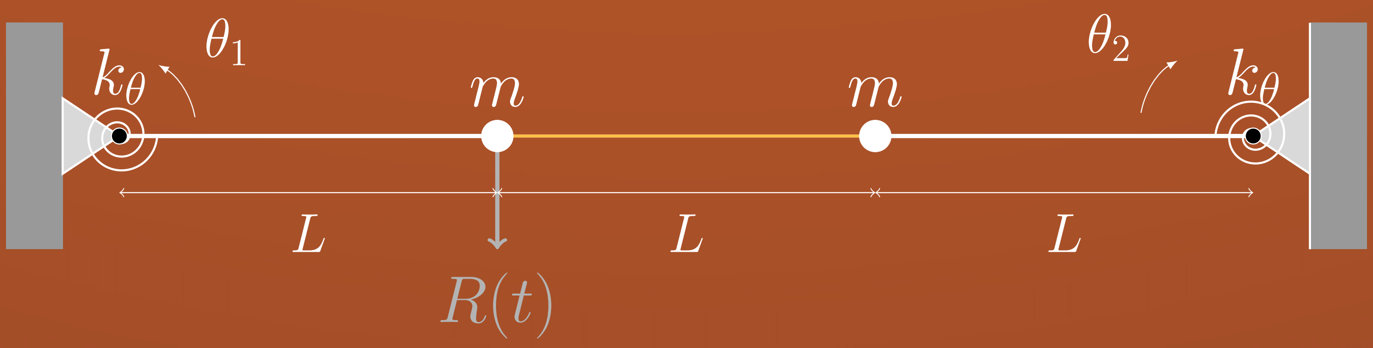
Symmetric Mode $\theta_2 = \theta_1$.



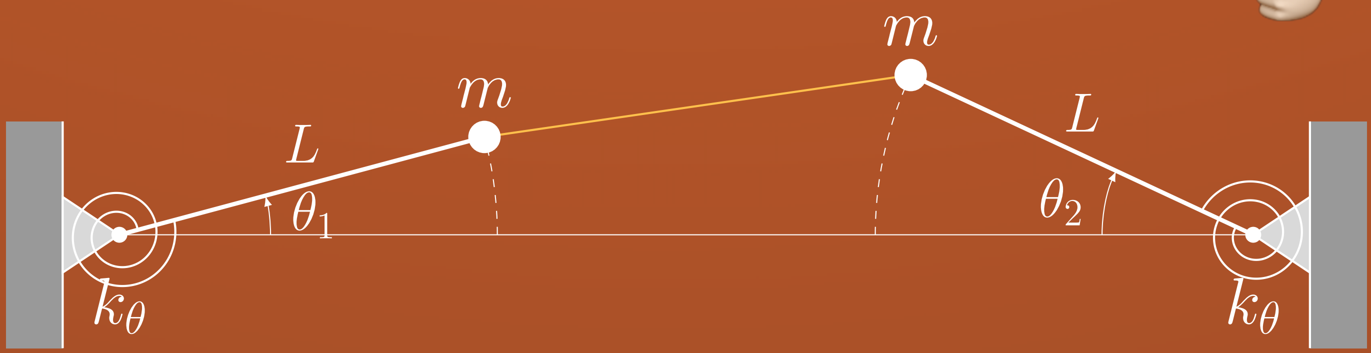
Antisymmetric Mode $\theta_2 = -\theta_1$.

- **Your Task:** Show how to use the orthogonal eigenvector mode shapes to define modal co-ordinates and decouple the two coupled system equations of motion, transforming to two independent transformed modal equations with modal generalized forces.

System Description and Assumptions



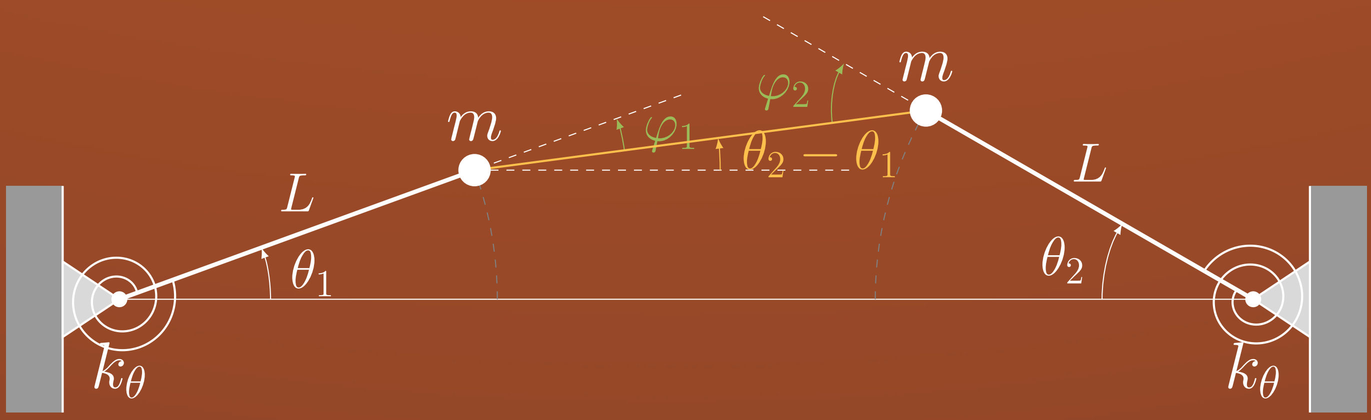
- Two massless rigid bars of length L are connected by a massless cable that is kept under a constant tension T.
- An external time-dependent load $R(t)$ is applied at the end of the left bar.
- The cable is massless with an initial length L but not assumed inextensible, yet we are neglecting the axial stiffness of the cable.
- The bars are pinned at their outer ends by fixed supports equipped with rotational springs of stiffness k_θ.



- Lumped masses m are located at the cable attachment points (the free ends of the bars).
- Gravity is acting perpendicular to the planar bar rotation motion, and therefore, the weight of the masses does not contribute to the equations of motion in the plane. Or the gravitational force acting on the masses is negligible compared to the restoring forces of cable tension and rotational spring stiffness.
- Small rotation angles (θ) are assumed so that the linearized kinematic relationships apply.
- Each bar rotates about its pin support. Let θ_1 and θ_2 denote the rotation angles of the left and right bars, respectively.

Geometry used to draw Free-Body Diagram (FBD)

Newton's equations of motion for each rigid bar with end mass and cable tension force are written by examining the forces and moments on a Free-Body Diagram (FBD) equated to mass inertia forces and moments due to accelerations on a Kinetic Diagram (KD) for a constrained dynamic deformed position with finite angles θ_1 and θ_2.



Angle of cable relative to left and right bars:
$\varphi_1 = \theta_1 - (\theta_2 - \theta_1) = 2\theta_1 - \theta_2,$
$\varphi_2 = \theta_2 + (\theta_2 - \theta_1) = 2\theta_2 - \theta_1.$

Newton's force balance equations for rotational motion about the fixed pivots

For small angles, after linearizing $\sin\theta \approx \theta$, and $\cos\theta \approx 1$.

- For the left rigid bar with lumped mass at its end, sum moments about the fixed point of rotation gives,

  $$\circlearrowleft \sum M_1 = RL - T(2\theta_1 - \theta_2) - k_\theta\, \theta_1 = I_1 \alpha_1$$

  where $\alpha_1 = \ddot{\theta}_1$ is the counter-clockwise positive angular acceleration of the bar and $I_1 = m, L^2$ is the mass moment of inertia for the lumped mass about the pivot.

- Similarly, for the right bar,

$$\circlearrowleft \sum M_2 = -T(2\theta_2 - \theta_1) - k_\theta\,\theta_2 = I_2\alpha_2$$

- Rearranging the equations of motion for the left and right bars in terms of bar rotation angles $\theta_1(t)$ and $\theta_2(t)$,

$$\begin{aligned} mL^2\ddot{\theta}_1 + (k_\theta + 2TL)\theta_1 - TL\theta_2 &= -RL \\ mL^2\ddot{\theta}_2 + (k_\theta + 2TL)\theta_2 - TL\theta_1 &= 0 \end{aligned}$$

System Equations of Motion

- For small angles, after linearizing the direct moment balance about the two pivots for the two bars, the equations of motion for the two lumped masses are:

$$
\begin{aligned}
mL^2\ddot{\theta}_1 + (k_\theta + 2TL)\theta_1 - TL\theta_2 &= -RL \\
mL^2\ddot{\theta}_2 + (k_\theta + 2TL)\theta_2 - TL\theta_1 &= 0
\end{aligned}
$$

- Expressing these two coupled second-order differential equations in (θ_1, θ_2), in matrix form we have,

$$
\begin{bmatrix} mL^2 & 0 \\ 0 & mL^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} (k_\theta + 2TL) & -TL \\ -TL & (k_\theta + 2TL) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -RL \\ 0 \end{Bmatrix}
$$

or in abstract form,

$$[\boldsymbol{M}]\{\ddot{\boldsymbol{\theta}}(t)\} + [\boldsymbol{K}]\{\boldsymbol{\theta}(t)\} = \boldsymbol{R(t)}$$

where

$$[\boldsymbol{M}] = \begin{bmatrix} mL^2 & 0 \\ 0 & mL^2 \end{bmatrix}$$

$$[\boldsymbol{K}] = \begin{bmatrix} (k_\theta + 2TL) & -TL \\ -TL & (k_\theta + 2TL) \end{bmatrix}$$

$$\boldsymbol{R}(t) = \begin{Bmatrix} -R(t)L \\ 0 \end{Bmatrix}$$

Solving the Eigenproblem for Natural Frequencies and Mode Shapes

For harmonic solutions of the form

$$\theta(t) = \Theta e^{i\omega t}$$

we have the generalized eigenproblem:

$$\left[\boldsymbol{K} - \omega^2 \boldsymbol{M}\right]\{\boldsymbol{\Theta}\} = \{\boldsymbol{0}\}$$

where the eigenvector mode shapes are orthogonal, such that

$$\boldsymbol{\Theta}_1^T \boldsymbol{\Theta}_2 = 0$$

Solving the eigenproblem, we find the two natural frequencies and mode shapes for the system.

1. **Symmetric Mode,** $\theta_2 = \theta_1$

$$\omega_1 = \frac{1}{L}\sqrt{\frac{k_1}{m}} \quad \text{with mode shape } [1, 1]$$

   where $k_1 = k_\theta + TL$ is the effective rotational stiffness for the symmetric mode.

2. **Antisymmetric Mode,** $\theta_2 = -\theta_1$

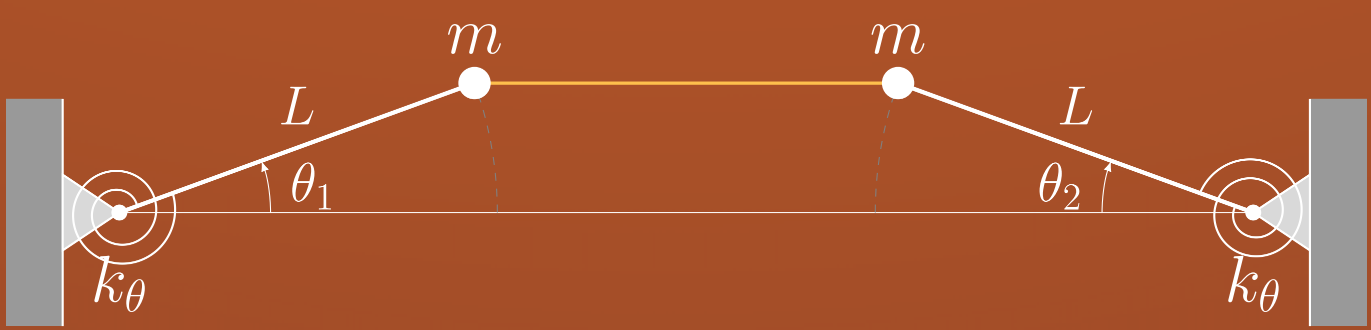
$$\omega_2 = \frac{1}{L}\sqrt{\frac{k_2}{m}} \quad \text{with mode shape } [1, -1]$$

   where $k_2 = k_\theta + 3TL$ is the effective rotational stiffness for the antisymmetric mode.

- Verifying positive real eigenvalues and mode orthogonality:

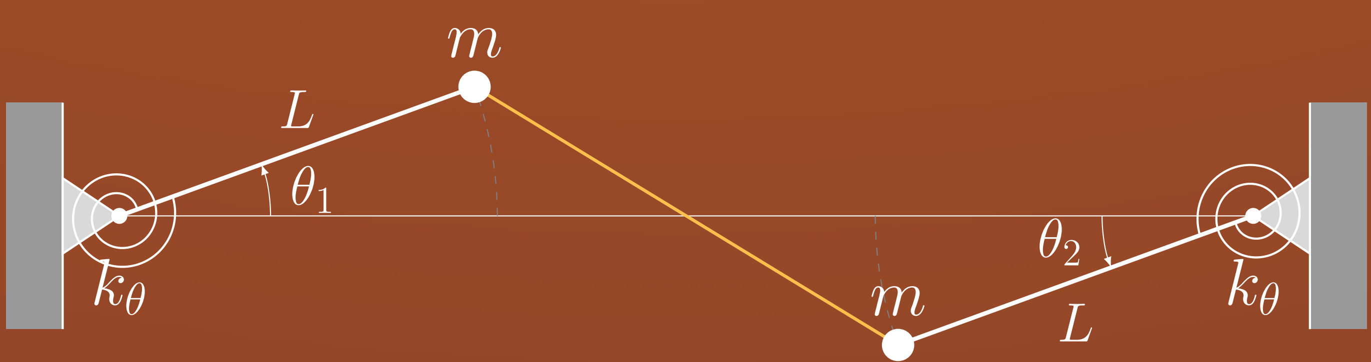
$$0 < \omega_1 < \omega_2, \qquad [1, 1]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0 \checkmark$$

Symmetric and Antisymmetric Mode Shapes



Symmetric Mode $\theta_2 = \theta_1$,

with natural frequency scaled with L: $\omega_1 L = \sqrt{\frac{k_\theta + TL}{m}}$.



Antisymmetric Mode $\theta_2 = -\theta_1$,

with natural frequency scaled with L: $\omega_2 L = \sqrt{\frac{k_\theta + 3TL}{m}}$.

What is left for you to finish?

- Given this information, your task is to show how to define modal coordinates using the orthogonal eigenvector mode shapes so that the two coupled equations decouple into two independent modal equations corresponding to the symmetric and antisymmetric modes with generalized modal forces.

Solution

1. Start with the Coupled Equations in Matrix Form in the coordinate vector $\boldsymbol{\theta} = (\theta_1, \theta_2)$:

$$[\boldsymbol{M}]\ddot{\boldsymbol{\theta}} + [\boldsymbol{K}]\boldsymbol{\theta} = \boldsymbol{R}(t),$$

with

$$[\boldsymbol{M}] = mL^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [\boldsymbol{K}] = \begin{bmatrix} k_\theta + 2TL & -TL \\ -TL & k_\theta + 2TL \end{bmatrix}$$

and the forcing vector

$$\boldsymbol{R}(t) = \left\{ \begin{array}{c} -R(t)L \\ 0 \end{array} \right\}$$

2. **Eigenvector Basis and Modal Coordinates**

   Earlier, the eigenanalysis of the stiffness matrix (when combined with the diagonal mass matrix) yielded two orthogonal mode shapes:

   (a) **Symmetric Mode:** $\boldsymbol{\Phi}_1 = [1, 1]^T$ with effective stiffness $k_1 = k_\theta + TL$ and natural frequency

   $$\omega_1 = \frac{1}{L}\sqrt{\frac{k_1}{m}}$$

   (b) **Antisymmetric Mode:** $\boldsymbol{\Phi}_2 = [1, -1]^T$ with effective stiffness $k_2 = k_\theta + 3TL$ and natural frequency

   $$\omega_2 = \frac{1}{L}\sqrt{\frac{k_2}{m}}$$

- **Normalize Eigenvector Modes**

  For convenience, let's scale the eigenvectors such that their magnitude equals unity.

  - For the first mode vector

$$\mathbf{\Phi}_1^T \mathbf{\Phi}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1)(1) + (1)(1) = 2$$

  Similarly, for the second mode.

  - To normalize to unit magnitude, we scale the modes by $1/\sqrt{2}$ and define the orthonormal vectors.

$$\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

  such that

$$\boldsymbol{v}_1^T \boldsymbol{v} = 1, \quad \boldsymbol{v}_2^T \boldsymbol{v}_2 = 1, \quad \boldsymbol{v}_1^T \boldsymbol{v}_2 = 0$$

- Since these eigenvectors are orthogonal, we can use them as a vector basis and introduce normalized modal coordinates $q_1(t)$ and $q_2(t)$ using the linear combination:

$$\boldsymbol{\theta} = q_1(t)\boldsymbol{v}_1 + q_2(t)\boldsymbol{v}_2 = \left[\, \boldsymbol{v}_1 \,\middle|\, \boldsymbol{v}_2 \,\right] \left\{ \begin{array}{c} q_1(t) \\ q_2(t) \end{array} \right\} = [\boldsymbol{T}]\, \boldsymbol{q}(t)$$

Here, the columns of the transformation matrix $[\boldsymbol{T}]$ defined by the the normalized modal basis vectors

$$[\boldsymbol{T}] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array} \right]$$

and

$$\boldsymbol{q}(t) = \left\{ \begin{array}{c} q_1(t) \\ q_2(t) \end{array} \right\}.$$

and the generalized coordinates on a modal basis.

- Inverting the transformation,

$$\boldsymbol{q}(t) = [\boldsymbol{T}]^T \boldsymbol{\theta}(t) = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array} \right] \left\{ \begin{array}{c} \theta_1(t) \\ \theta_2(t) \end{array} \right\}$$

- Since the orthogonal basis is normalized to unity, the inverse of the transformation matrix is equal to the transpose of the matrix, $\boldsymbol{T}^{-1} = \boldsymbol{T}^T$. For our problem, since the transformation matrix is also symmetric, we observe $\boldsymbol{T}^T = \boldsymbol{T}$, so that $\boldsymbol{T}^{-1} = \boldsymbol{T}$.

  Explicitly,

$$q_1 = \frac{1}{\sqrt{2}} (\theta_1 + \theta_2) \,, \quad q_2 = \frac{1}{\sqrt{2}} (\theta_1 - \theta_2) \,.$$

3. Transforming the Equations of Motion to Modal Coordinates

- To decouple the equations, we substitute the new basis $\boldsymbol{\theta}(t) = [\boldsymbol{T}]\boldsymbol{q}(t)$ into our original equation. First, note that because the mass matrix is diagonal,

$$[\boldsymbol{M}]\ddot{\boldsymbol{\theta}}(t) = mL^2\boldsymbol{T}\ddot{\boldsymbol{q}}(t).$$

- The stiffness term transforms as:

$$[\boldsymbol{K}]\boldsymbol{\theta}(t) = [\boldsymbol{K}]\boldsymbol{T}\boldsymbol{q}(t)$$

- Multiplying the entire equation from the left by $\boldsymbol{T}^T$ (which diagonalizes the system due to the orthogonality of the eigenvectors) gives

$$\left(\boldsymbol{T}^T[\boldsymbol{M}]\boldsymbol{T}\right)\ddot{\boldsymbol{q}}(t) + \left(\boldsymbol{T}^T[\boldsymbol{K}]\boldsymbol{T}\right)\boldsymbol{q}(t) = \boldsymbol{T}^T\boldsymbol{R}(t).$$

- Since $\boldsymbol{T}$ is orthonormal,

$$\boldsymbol{T}^T[\boldsymbol{M}]\boldsymbol{T} = mL^2\boldsymbol{I}$$

  where $\boldsymbol{I} = \text{diag}(1, 1)$ is the unit diagonal identity matrix.

- In the transformed coordinates, the stiffness matrix becomes diagonal with elements corresponding to the effective modal stiffnesses:

$$\boldsymbol{T}^T[\boldsymbol{K}]\boldsymbol{T} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

  where

$$k_1 = k_\theta + TL, \quad k_2 = k_\theta + 3TL$$

- Thus, the modal equation becomes

$$\begin{bmatrix} mL^2 & 0 \\ 0 & mL^2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array}\right] \begin{Bmatrix} -R(t)L \\ 0 \end{Bmatrix} = \frac{(-R(t)L)}{\sqrt{2}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

and the decoupled single-degree-of-freedom equations are

$$mL^2\ddot{q}_1 + k_1 q_1 = r_1(t)$$
$$mL^2\ddot{q}_2 + k_2 q_2 = r_2(t)$$

where, in modal coordinates, the forcing on both modes is

$$r_1(t) = -\frac{R(t)L}{\sqrt{2}}, \quad r_2(t) = -\frac{R(t)L}{\sqrt{2}}.$$

Summary of the Decoupled Modal Equations

- After scaling with the mass mL^2, the final decoupled equations in modal coordinates are:

$$\ddot{q}_1(t) + \omega_1^2 q_1(t) = -\frac{R(t)L}{\sqrt{2}}$$
$$\ddot{q}_2(t) + \omega_2^2 q_2(t) = -\frac{R(t)L}{\sqrt{2}}.$$

where for $j = 1, 2$

$$\omega_j^2 = \frac{k_j}{mL^2}$$

- Each equation represents a single-degree-of-freedom oscillator with its effective stiffness and natural frequency:

$$\omega_1 = \frac{1}{L}\sqrt{\frac{k_\theta + TL}{m}}, \quad \omega_2 = \frac{1}{L}\sqrt{\frac{k_\theta + 3TL}{m}}.$$

1. The first equation for the modal coordinate q_1 is the symmetric mode.
2. The second equation for the modal coordinate q_2 is the antisymmetric mode.

- Once these equations are solved for the modal coordinates $\boldsymbol{q} = (q_1, q_2)$, the time histories for physical coordinates for the bar rotation angles can be reconstructed from the transformation equations $\boldsymbol{\theta} = [\boldsymbol{T}]\boldsymbol{q}$, explicitly

$$\theta_1 = \frac{1}{\sqrt{2}}\left(q_1 + q_2\right), \quad \theta_2 = \frac{1}{\sqrt{2}}\left(q_1 - q_2\right).$$

Conclusion

- By introducing modal coordinates based on the orthogonal eigenvectors $[1, 1]^T$ and $[1, -1]^T$, we decoupled the original two-degree-of-freedom system into two independent equations.
- This decoupling simplifies the analysis and allows for separate solutions of the symmetric and antisymmetric modes of vibration.

References

- Any book on Mechanical Vibration or Structural Dynamics

Thanks for Reading

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