Deflection by Bending

1. RIGIDITY DESIGN

Under a transverse bending load, the normally straight neutral axis of a beam becomes a curved line. The deflection of interest is the linear displacement of some point on the neutral axis along a path parallel to the line of applied force. Usually it is the maximum deflection that is of value on our computations, although occasionally the deflection at a specific point is needed.

Rigidity design formulas for use when bending loads are experienced, are based on the maximum deflection being —

$$\boxed{\Delta_{max} = k \frac{P\ L^3}{E\ I}} \quad \cdots\cdots\cdots\cdots (1)$$

Two of the components in this formula have been discussed previously in detail. The critical property of the material is its modulus of elasticity (E). In the case of all steels, this has the very high value of 30,000,000 psi. The related property of the section is its moment of inertia (I), which is dependent on dimensions of the beam cross-section.

If the values for E and I are held constant, and the load (P) is a specified value, the length of the beam span (L) is one variable which will influence the deflection. The constant (k) is a function of the type of loading and also the manner in which the load is supported, and thus is subject to the designer's will. In practice "I" also is subject to the designer's will.

The several components of the basic formula are best handled by constructing a bending moment diagram from the actual beam, and then applying the appropriate standard simplified beam formula. These formulas are available in the Reference Section on Beam Diagrams included at the end of this book.

There are several methods for finding the deflection of a beam. Four of these will be shown:

1. Successive integration method
2. Virtual work method
3. Area moment method
4. Conjugate beam method

2. FUNDAMENTALS OF BEAM DEFLECTION

A transverse load placed on a beam causes bending moments along the length of the beam. These bending moments set up bending stresses (σ) across all sections of the beam. See Figure 1a, where at any given section:

$$\sigma_x = \frac{M_x\ c}{I_x}$$

It is usually assumed that the bending stress (σ) is zero at the neutral axis and then increases linearly to a maximum at the outer fibers. One surface is under compression, while the other surface is under tension. Within the elastic limit, assuming a straight-line relationship between stress and strain, the distribution of bending stress can be converted over into a distribution of strain. Correspondingly, there would be no strain (ϵ) along the neutral axis and the strain would increase linearly to a maximum at the outer fiber. See Figure 1b where at any given section:

$$\epsilon_x = \frac{\sigma_x}{E} = \frac{M_x\ c}{E\ I_x}$$

Considering a segment of the beam having only a

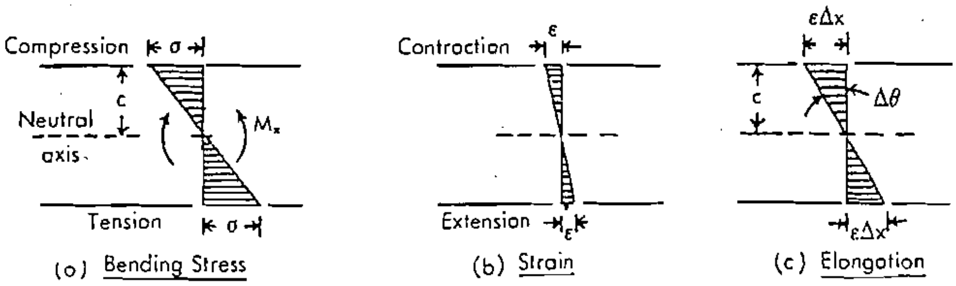


FIGURE 1

very small increment in length (Δx), Figure 1c, the elongation within this small increment would be ϵ (Δx). Also, here it can be seen that the small angular rotation ($\Delta\theta$) would be the elongation at the outer fiber divided by the distance (c) to the outer fiber from the neutral axis.

This can be expressed as —

$$\epsilon(\Delta x) = c\ (\Delta\theta)$$

$$\therefore \Delta\theta = \frac{\epsilon\ (\Delta x)}{c} = \frac{M\ c\ (\Delta x)}{E\ I\ c}$$

or:

$$(\theta\Delta)_x = \frac{M_x\ (\Delta x)}{E\ I_x}$$

In other words, the infinitesimal angle change in any section of the beam is equal to the area under the moment diagram ($M_x\ \Delta x$) divided by the ($E\ I_x$) of the section.

The angular rotation relative to stress and strain is further illustrated by Figure 2.

Figure 2a represents a straight beam under zero bending moment. Here any two given sections (a and b) would parallel each other and, in a stress-free condition, would then have a radius of curvature (R_x) equal to infinity (∞). These two sections (a and b) can be set close together to define the segment of very small increment in length (Δx).

At Figure 2b, the beam is subjected to a bending moment and this small segment (Δx) will compress on one side and will elongate on the other side where the outer fiber is in tension. This can be related to a small angular movement within this increment. It can be seen that sections *a* and *b* are no longer parallel but would converge at some point (0) in space, forming a radius of curvature (R_x).

In the sketch to the right of Figure 2b, dotted lines (a and b) represent the initial incremental segment (Δx) with zero moment, while the solid lines reflect the effect of applied load: $\Delta x\ (1 - \epsilon)$ at the surface under compression.

The total angular change (θ) between any two points (a and b) of the beam equals the sum of the incremental changes, or:

$$\boxed{\theta = \int_{x=a}^{x=b} (\Delta\theta)_x = \int_{x=a}^{x=b} \frac{M_x\ (\Delta x)}{E\ I_x}} \quad \cdots\cdots\cdots\cdots (2)$$

It is also observed from Figure 2b that —

$$(\Delta\theta)_x = \frac{\Delta x}{R_x} = \frac{M_x\ (\Delta x)}{E\ I_x}$$

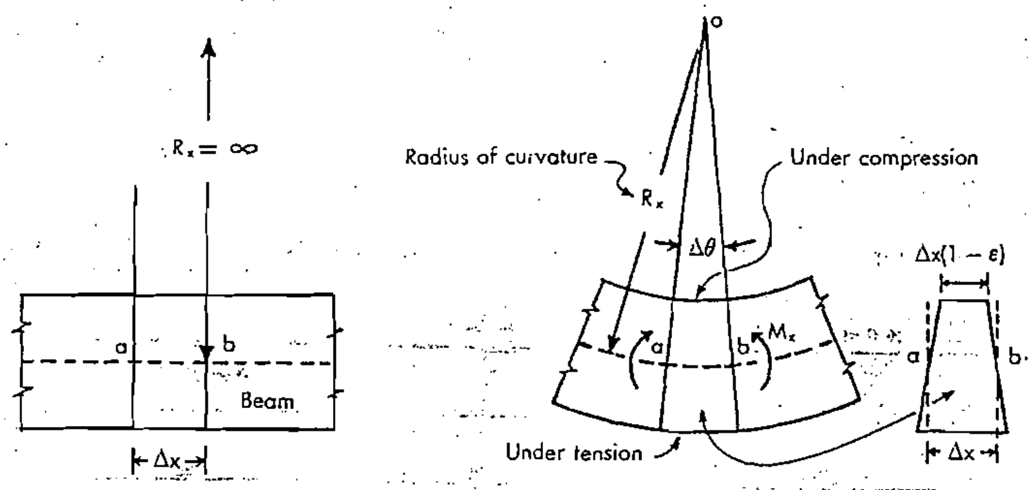
and since —

$$(\Delta\theta)_x = \frac{M_x\ (\Delta x)}{E\ I_x}$$

the reciprocal of the radius of curvature (1/R) at any given point (x) of the beam is —

$$\boxed{\frac{1}{R_x} = \frac{M_x}{E\ I_x}} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (3)$$

The next logical step would seem to be application of the Successive Integration Method to determine the beam deflection.



(a) Beam With No Load (no moment)

(b) Beam Under Load (with moment)

FIGURE 2

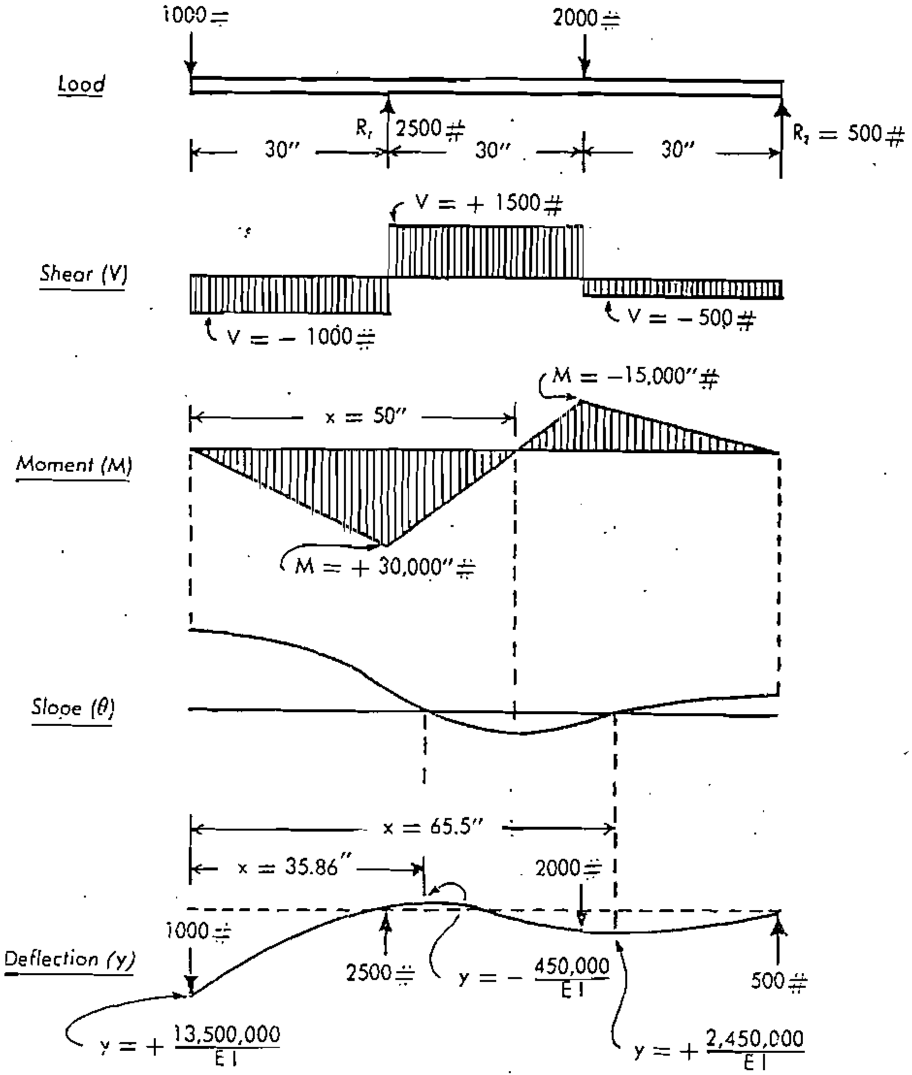


FIGURE 3

3. SUCCESSIVE INTEGRATION METHOD

For any given beam with any given load, if the load (w_x) at any point (x) can be expressed mathematically as a function of (x) and if such load condition is known for the entire beam, then:

load

$$\boxed{w_x = f_1\ (x)} \qquad (4)$$

and by successive integrations—

shear

$$\boxed{V_x = \int_{x_1}^{x_2} w_x\ (dx)} \qquad (5)$$

moment

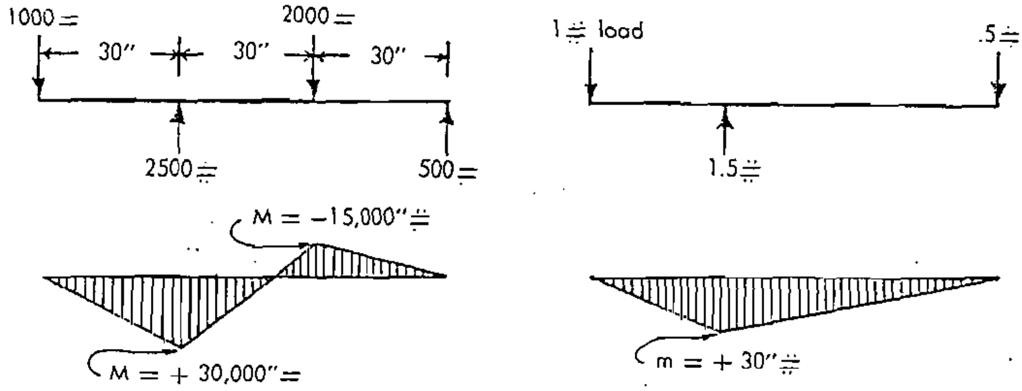
$$\boxed{M_x = \int_{x_1}^{x_2} V_x\ (dx)} \qquad (6)$$

slope

$$\boxed{\theta_x = \int_{x_1}^{x_2} \frac{M_x\ (dx)}{E\ I_x}} \qquad (7)$$

deflection

$$\boxed{y_x = \int_{x_1}^{x_2} \frac{\theta_x (dx)}{E\ I_x} = \iint_{x_1}^{x_2} \frac{M_x\ (dx)}{E\ I_x}} \qquad (8)$$



(a) Real Bending Moment (M) (b) Virtual Bending Moment (m)

FIGURE 4

Unfortunately, it is usually difficult to get a mathematical expression for the load in terms of x for the entire length of the beam for any but the simplest of beam loadings. The method is cumbersome, especially if various loads are applied, if there are various types of support, or if there are various changes in section.

For every integration, there is a constant of integration (C) which must be solved. This is done by setting up known conditions of the beam; for example, the deflection of a beam over a support is zero, the slope of a beam at a fixed end is zero, etc.

This method means several equations must be used and integrated within certain limits of x, with considerable time expended and with the possibility of compounded error.

If possible, integrate graphically rather than mathematically, this process takes on greater importance. Most of the methods in actual use for computing deflection are based on a graphical solution of the problem.

Problem 1

The example in Figure 3 will be worked through in several ways. In this case, the problem was previously worked out by longhand so it is known exactly what it looks like. Then several methods will be used in finding the deflection (y or Δ) under the conditions illustrated, to show that in each case the answer comes out the same:

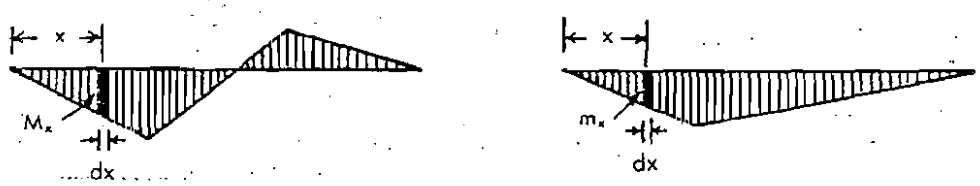
$$y = \frac{13{,}500{,}000}{E\ I} \text{ inches}$$

4. VIRTUAL WORK METHOD

This is used frequently for finding the deflection of a point on a beam in any direction, caused by the beam load. A virtual load of one pound (or one kip) is placed on the beam at the point where the amount of deflection is desired and in the same direction.

Virtual bending moments (m) caused by the 1-lb load are determined along the entire length of the beam. The internal energy of the beam after deflecting is determined by integration. This is then set equal to the external energy of the 1-lb virtual load moving a distance (y) equal to the deflection.

$$\boxed{1\# \cdot y = \int \frac{M_x\ m_x\ dx}{E\ I_x}} \quad \ldots\ldots\ldots\ldots (9)$$



(a) $\frac{M}{EI}$ curve (b) m curve

FIGURE 5

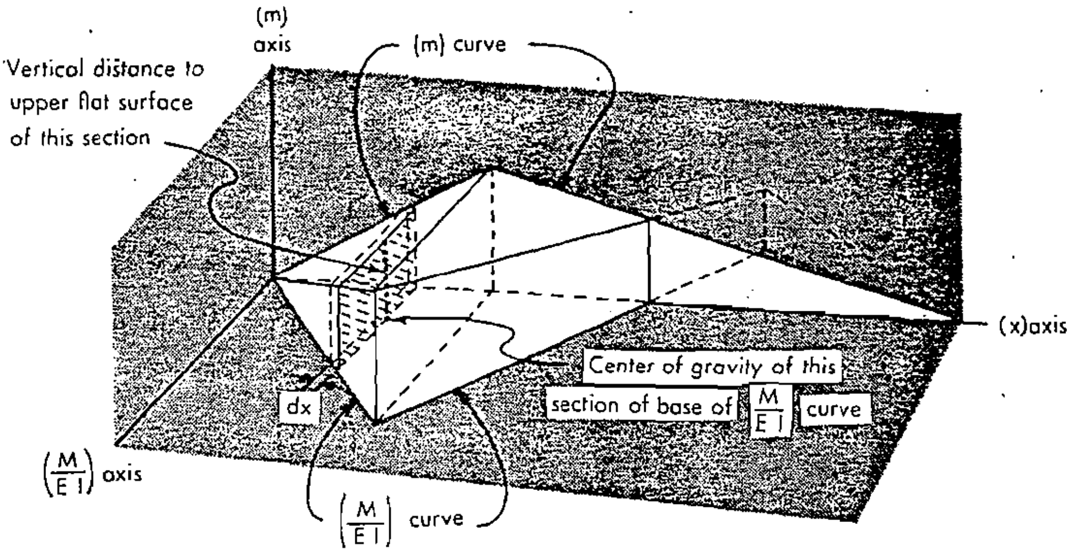


FIGURE 6

where:

m = virtual bending moment at any point caused by the 1-lb load

M = real bending moment at the same point

I = moment of inertia at this same point

dx = length of small increment of the beam

E = modulus of elasticity in tension of the material

This equation can be worked out by calculus; however, its real value is that it lends itself to a graphical approach.

The first step is to apply all of the forces (Problem 1, Fig. 3) to the member, Figure 4a, and to compute the bending diagram—the real bending moment (M) on the beam. The next step is to remove the real load and replace it with a 1-lb load at the point where the deflection is desired and also in the same direction, Figure 4b. The bending moment of this particular load is then computed; this is known as the virtual bending moment (m).

The real moment diagram can be broken down into standard geometric areas; for example, triangles and rectangles for concentrated loads, and parabolas for uniformly distributed loads. The virtual moment diagram by the very nature of the single 1-lb concentrated force is always triangular in shape.

This means that the integration of these moment diagrams to obtain the internal energy may be replaced by working directly with these areas, since their properties are known. This will greatly simplify the work.

Figure 5 separates the two moment diagrams that must be combined in the basic equation #9.

It is seen from the equation that $M_x\, m_x\, dx$ is a segment of a volume.

In the triaxial representation, Figure 6, diagrams for both the real moment (M) divided by EI and the virtual moment (m) have a common base line (the x axis). The M/EI curve for the real bending moment lies flat in the horizontal plane. The m curve for the virtual bending moment is shown in the vertical plane established by the m axis and the x axis. The solid thus defined is a series of smaller volumes with simple geometric faces.

The volume of any element of this solid equals the area of the element's base surface multiplied by the vertical distance from the center of gravity of the base surface to the upper flat surface. This vertical distance is shown by a dotted line.

Thus, in Figure 7, with the M/EI and m diagrams lined up one above the other, it is necessary to know

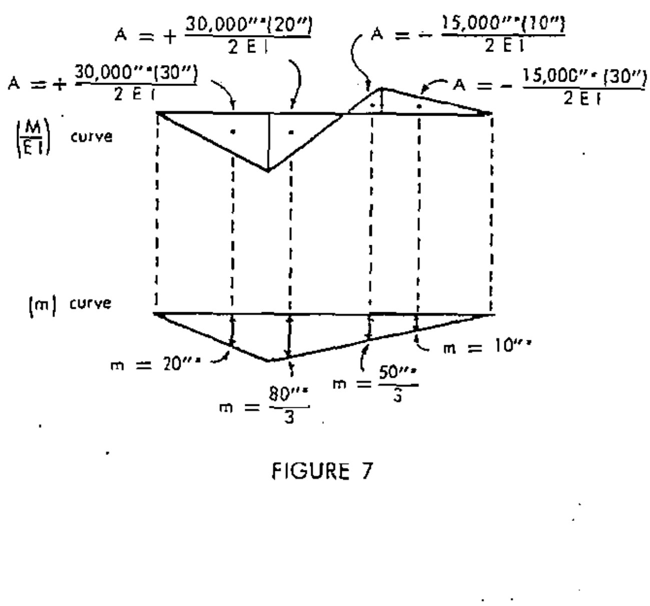


FIGURE 7

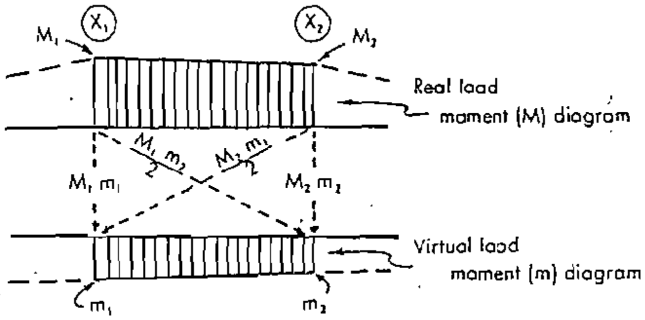


FIGURE 8

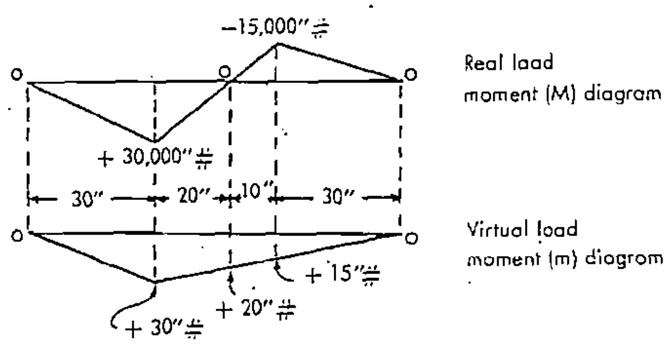


FIGURE 9

only the height of the virtual moment diagram at the same distance (x) as on the real moment diagram. The M/EI diagram is then divided into simple geometric shapes (in this case, right triangles), and the area of each is found and multiplied by the height of the m diagram along a line through the particular M/EI area's center of gravity.

From this the volume is obtained:

$$\text{Volume} = \frac{(30{,}000)(30)(20)}{2\,E\,I} + \frac{(30{,}000)(20)}{2\,E\,I}\left(\frac{80}{3}\right) - \frac{(15{,}000)(10)}{2\,E\,I}\left(\frac{50}{3}\right) - \frac{(15{,}000)(30)(10)}{2\,E\,I}$$

$$= + \frac{13{,}500{,}000}{E\,I}$$

and since:

$$\text{Volume} = 1'' \cdot y$$

the deflection in inches is —

$$y = \frac{13{,}500{,}000}{E\,I}$$

The value of I can now be inserted in this to give the deflection (y) in inches. However, if the beam has a variable section, several values of I would have to be inserted earlier in the computation—for the section taken through the center of gravity of each geometrical area of the M/EI diagram.

To simplify this further, a method of cross-multiplying has been found to give the same results. The general approach is illustrated by Figure 8, where some segment of the real moment (M) diagram between points x_1 and x_2 is at the top and a corresponding segment of the virtual moment (m) diagram is below.

The required volume can be found directly by multiplying M_1 by m_1 and M_2 by m_2 and then by cross-multiplying M_1 by m_2 and M_2 by m_1 using only ½ of the products of cross-multiplication. This is more fully related to the basic integration equation by the following:

$$\int_{x=1}^{x=2} \frac{M\,m\,dx}{E\,I} = \frac{L}{3\,E\,I}\left(M_1 m_1 + M_2 m_2 + \frac{M_1 m_2}{2} + \frac{M_2 m_1}{2}\right)$$

where L = the distance between points x_1 and x_2.

Figure 9 shows application of this method to the original Problem 1.

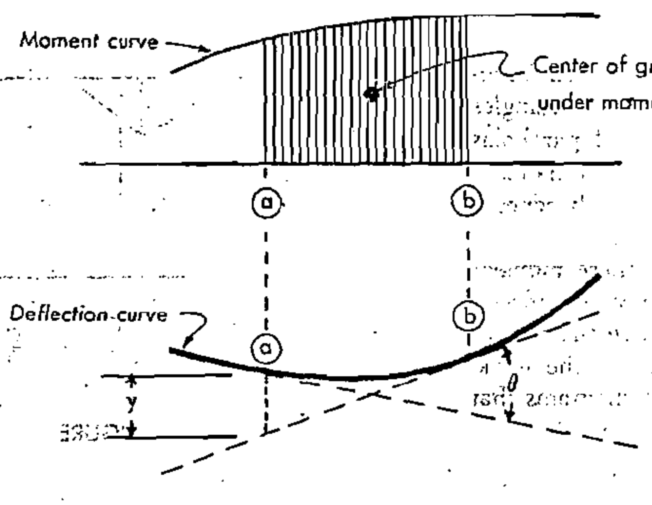


FIGURE 10

From Figure 9:

$$y = \left(\frac{30}{3}\right)\left(\frac{30 \times 30{,}000}{EI}\right) + \left(\frac{20}{3}\right)\left(\frac{30 \times 30{,}000}{EI}\right) + \left(\frac{20}{3}\right)\left(\frac{20 \times 30{,}000}{2EI}\right) - \left(\frac{10}{3}\right)\left(\frac{15 \times 15{,}000}{EI}\right) - \left(\frac{10}{3}\right)\left(\frac{20 \times 15{,}000}{2EI}\right) - \left(\frac{30}{3}\right)\left(\frac{15 \times 15{,}000}{EI}\right)$$

$$= \frac{13{,}500{,}000}{EI}$$

5. AREA MOMENT METHOD

This a very useful tool for engineers and is illustrated in Figure 10 by a general moment diagram and the corresponding deflection curve. Here points *a* and *b* represent any two points defining a simple geometric area of an actual moment diagram.

The two fundamental rules for use of this method are:

> The change in slope (radians) between two points (a and b) of a loaded beam equals the area under the moment curve, divided by E I, between these two points (a and b).

> The distance of point *a* of the beam to the tangent at point *b* of the beam equals the moment of the area under the moment diagram taken about point *a*, divided by E I.

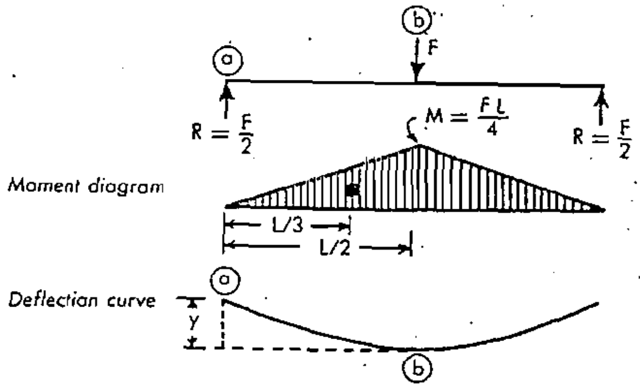


FIGURE 11

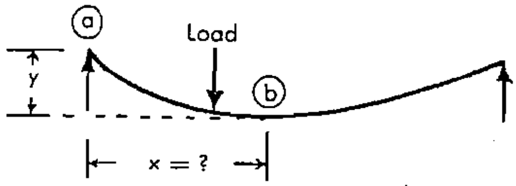
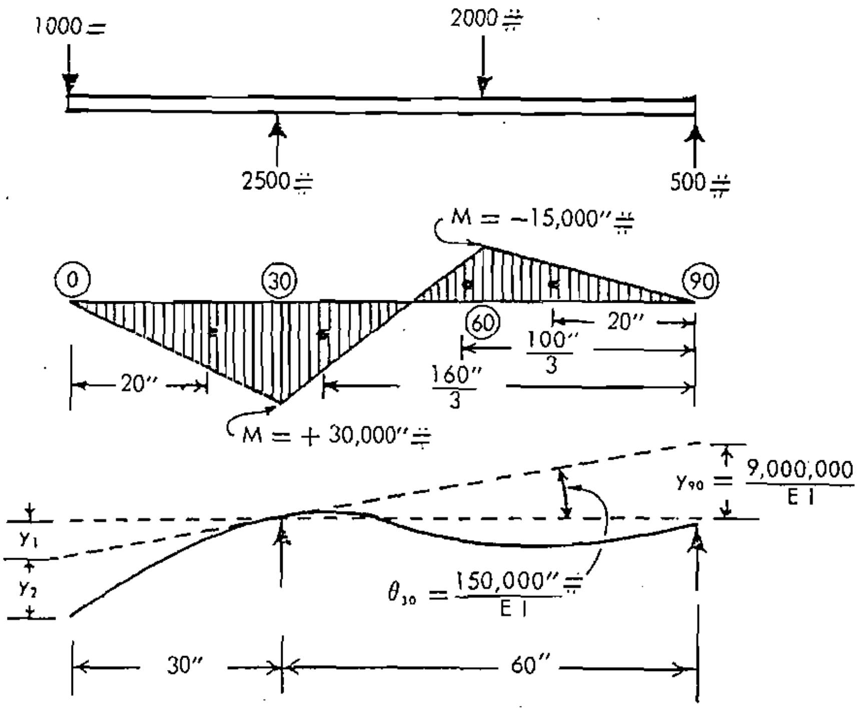


FIGURE 12

For symmetrically loaded, simply supported beams this is a convenient method with which to find the maximum deflection of the beam, because in this case the slope of the beam is zero at the mid-span (b) and the distance from *a* to the tangent at *b* equals the maximum deflection we are seeking. See Figure 11.

FIGURE 13



From Figure 11:

$$y = \frac{1}{2}\left(\frac{M}{E\ I}\right)\left(\frac{L}{2}\right)\left(\frac{2}{3}\times\frac{L}{2}\right) = \frac{1}{2}\left(\frac{F\ L}{4\ E\ I}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right)$$

$$= \frac{F\ L^3}{48\ E\ I}$$

However, for an unsymmetrically loaded beam, the point of the beam having zero slope, or maximum deflection, is unknown (Fig. 12). There are ways of getting around this.

The conditions of Problem 1 are here illustrated by Figure 13. The moments of the area under the moment curve (from point zero to point 30) is taken about point zero to give the vertical distance between point zero and the tangent to the deflection curve at point 30. This becames y_2. This is not the actual deflection, because the slope of the deflection curve at point 30 is not level. This slope is yet to be found.

First find the vertical distance between point 90 and the tangent to the deflection curve at point 30. To find this distance (y_{90}), take the moments, about point 90, of the area of the moment diagram from point 30 to point 90.

$$y_{90} = \frac{(30{,}000)(20)}{2\,E\,I}\left(\frac{160}{3}\right) - \frac{(15{,}000)(10)}{2\,E\,I}\left(\frac{100}{3}\right)$$

$$- \frac{(15{,}000)(30)(20)}{2\,E\,I}$$

$$= \frac{9{,}000{,}000}{E\ I}$$

TABLE 1—Comparative Conditions of Real and Conjugate Beams

Real Beam	Conjugate Beam
1. Simple supported ends a) zero deflection b) maximum slopes	1. Simply supported ends because — a) zero moment b) maximum shear
2. Fixed ends a) zero deflection b) zero slope	2. Free ends because — a) zero moment b) zero shear hence no support
3. Free ends a) a maximum deflection b) a maximum slope	3. Fixed ends because — a) a maximum moment b) a maximum shear hence a support
4. Interior supports of a continuous beam a) no deflection b) gradual change in slope	4. A hinge without support a) no moment b) gradual change in shear hence no support
5. Point of maximum deflection	5. Located at point of zero shear because this is a point of maximum moment
6. Either statically determinate or statically indeterminate	6. Always statically determinate

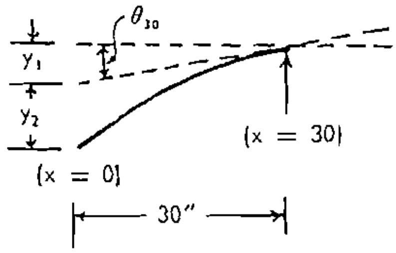


FIGURE 14

The angle of this tangent line to the horizon (θ_{30}) is then found by dividing this vertical distance (y_{90}) by the horizontal distance between point 30 and point 90.

$$\theta_{30} = \frac{y_{90}}{60''}$$

$$= \frac{9{,}000{,}000}{60\ E\ I}$$

$$= \frac{150{,}000}{E\ I}$$

This angle (θ_{30}) is the same to the left of point 30, Figure 14, and defines the vertical deflection (y_1) at point zero. This angle then, multiplied by the horizontal distance from point zero to point 30, gives the vertical displacement (y_1).

$$y_1 = \theta_{30}\ 30 = \frac{150{,}000}{E\ I}\ 30 = \frac{4{,}500{,}000}{E\ I}$$

Adding this to the initial displacement—

$$y_2 = \frac{(30{,}000)(30)(20)}{2\ E\ I} = \frac{9{,}000{,}000}{E\ I}$$

gives the total deflection at point zero of—

$$y = \frac{13{,}500{,}000}{E\ I}$$

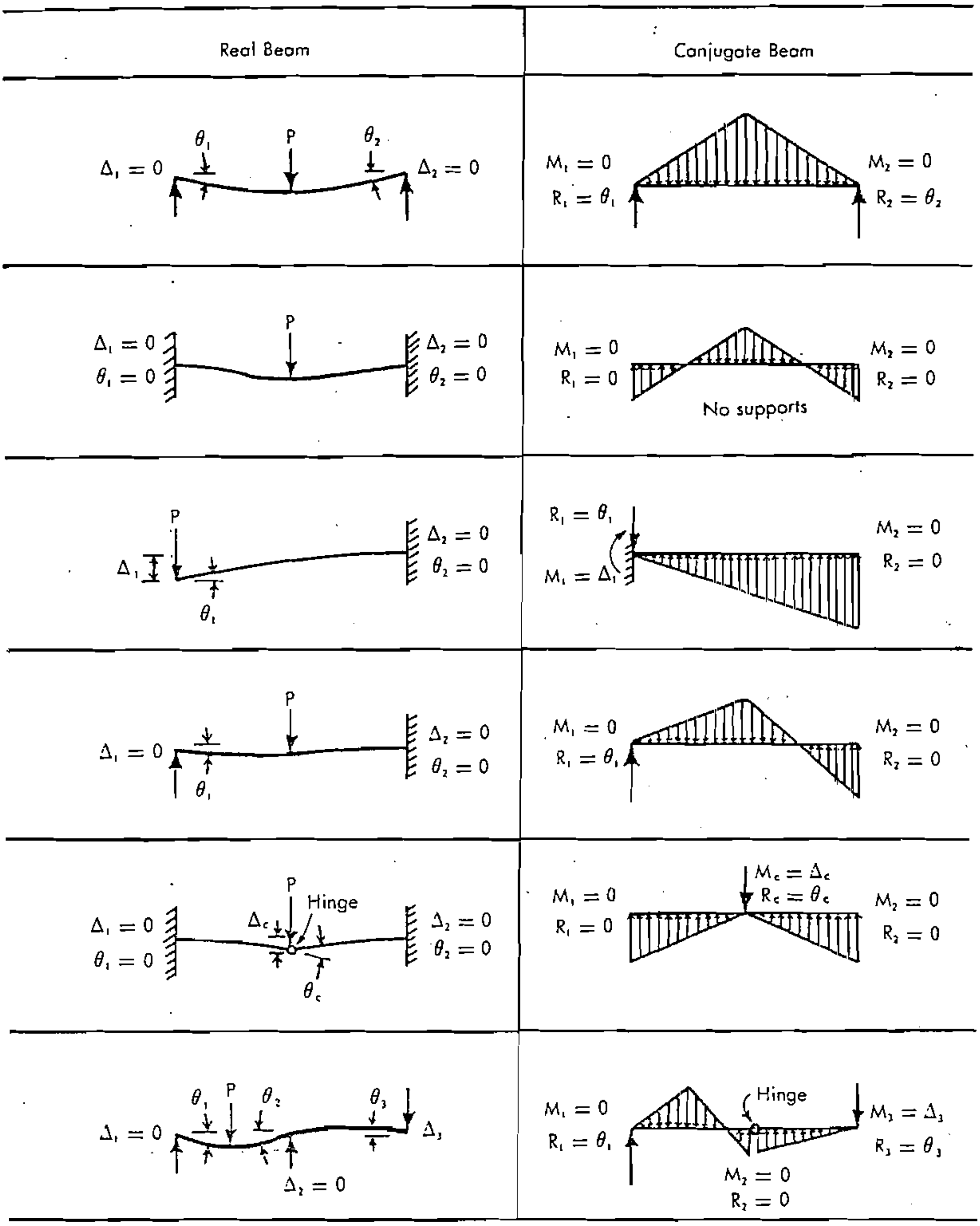
6. CONJUGATE BEAM METHOD

In using this method, the bending moment diagram of the real beam is constructed. A substitutional beam or conjugate beam is then set up; the load on this is the moment of the real beam divided by the E I of the real beam; in other words it is loaded with the M/EI of the real beam.

Five conditions must be met:

1. The length of the conjugate beam equals the length of the real beam.

TABLE 2—Typical Real Beams and Correspanding Conjugate Beams



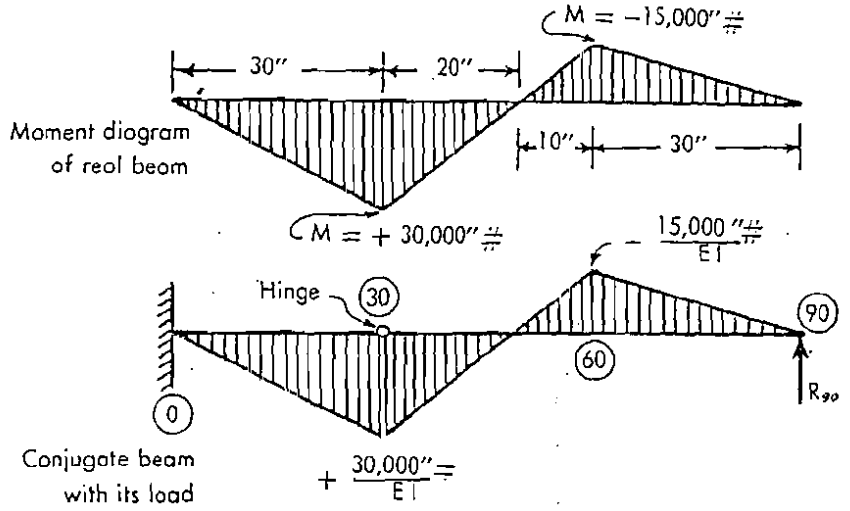


FIGURE 15

2. There are two equations of equilibrium—
   - The sum of forces acting in any one direction on the conjugate beam equals zero.
   - The sum of moments about any point of the conjugate beam equals zero.

3. The load at any point of the conjugate beam equals the moment of the real beam divided by the E I of the real beam at the same point. The real beam could have variable I.

4. The vertical shear at any point of the conjugate beam equals the slope of the real beam at the same point.

5. The bending moment at any point of the conjugate beam equals the deflection of the real beam at the same point.

The conjugate beam must be so supported that conditions 4 and 5 are satisfied. The above statements of condition may be reversed.

By knowing some of the conditions of the real beam, it will be possible to reason the nature of the support of the conjugate beam. The comparative statements of Table 1 will help in setting up the conjugate beam.

Some examples of real beams and their corresponding conjugate beams are presented in Table 2.

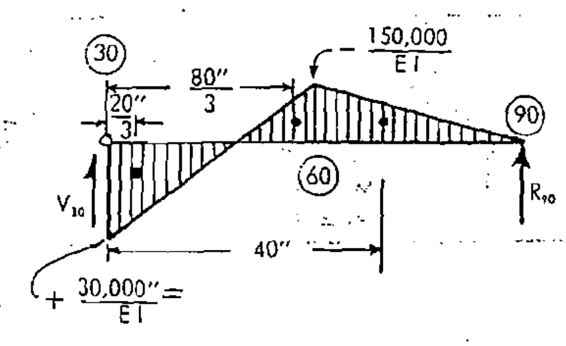


FIGURE 16

Notice that the support of the conjugate beam can be very unlike the support of the real beam.

The last example in Table 2 is similar to the Problem 1 beam to which several methods of solving deflection have already been applied. Here the conjugate beam is hinged at the point of second support of the real beam, and without this hinge the Conjugate Beam Method would not be workable.

The same Problem 1 is illustrated in Figure 15, where the real beam moment is first diagrammed. This is then divided by E I of the real beam for the load on the conjugate beam shown next.

To find the right hand reaction (R_{90}) take moments, about point 30, on the conjugate beam between points 30 and 90. See Figure 16.

Since:

$$\Sigma M_{30} = 0$$

$$\frac{1}{2}\left(\frac{+\,300{,}000}{E\,I}\right)(20)\left(\frac{20}{3}\right) + \frac{1}{2}\left(\frac{-\,15{,}000}{E\,I}\right)(10)\left(\frac{80}{3}\right) + \frac{1}{2}\left(\frac{-\,15{,}000}{E\,I}\right)(30)(40) - R_{90}(60) = 0$$

$$\therefore R_{90} = -\,\frac{150{,}000 \text{ in.}^2\text{-lbs}}{E\,I}$$

This negative sign means the reaction is directed opposite to our original assumption; hence it is directed downward.

Since the sum of vertical forces equals zero, V_{30} may be found:

$$-\overset{\text{assume upward}}{V_{30}} + \frac{1}{2}\left(\frac{+\,30{,}000}{E\,I}\right)(20) + \frac{1}{2}\left(\frac{-\,15{,}000}{E\,I}\right)(40) + \overbrace{\frac{150{,}000}{E\,I}}^{\text{downward } R_{90}} = 0$$

$$\therefore V_{90} = + \frac{150{,}000 \text{ in.}^2\text{-lbs}}{E\ I}$$

This positive sign means original assumption was correct and shear is directed upward.

The left hand moment (M_0) of the conjugate beam may be found by taking moments of the isolated element, between points zero and 30. See Figure 17.

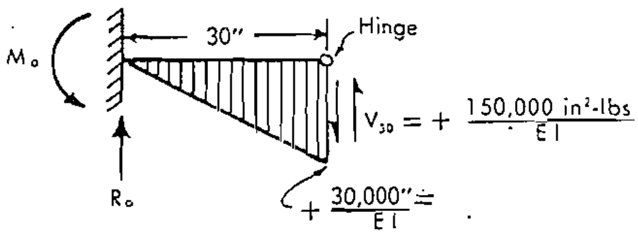


FIGURE 17

directed downward V_{30}

$$M_o = \frac{1}{2}\left(\frac{+\ 30{,}000}{E\ I}\right)(30)(20) + \left(\frac{150{,}000}{E\ I}\right)(30)$$

$$= + \frac{13{,}500{,}000 \text{ in.}^3\text{-lbs}}{E\ I}$$

The deflection of the real beam at point zero (y_o or Δ_{max}) equals the moment of the conjugate beam at this point (M_o); hence:

$$y_o = \frac{13{,}500{,}000}{E\ I} \text{ inches}$$

This would be the solution of this problem; however, to get the deflection at other points it would be necessary to continue this work and find the moment of the conjugate beam throughout its length.

The maximum deflection of the real beam on the right side occurs at the same point as zero shear of the conjugate beam. By observation this would occur somewhere between points 60 and 90, and the distance

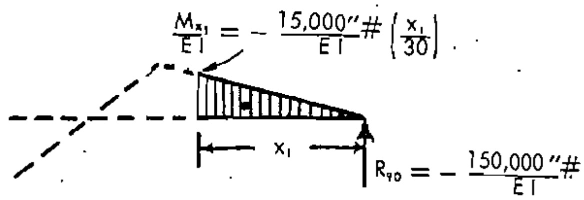


FIGURE 18

of this point of maximum deflection from point 90 is set as x_1. See Figure 18.

Since:

$$\Sigma V = 0$$

$$\frac{1}{2}\left(\frac{-\ 15{,}000}{E\ I}\right)\left(\frac{x_1}{30}\right) x_1 + \frac{150{,}000}{E\ I} = 0$$

$$250\ x_1^2 = 150{,}000$$

$$x_1^2 = 600$$

and:

$$x_1 = 24.5''$$

The moment of the conjugate beam at this point is —

$$M_x = \frac{1}{2}\left(\frac{-\ 15{,}000}{E\ I}\right)\left(\frac{x_1}{30}\right) x_1 \left(\frac{x_1}{3}\right) + \frac{150{,}000}{E\ I}\ x_1$$

$$= \frac{2{,}450{,}000}{E\ I}$$

and therefore the maximum deflection (y_{max} or Δ_{max}) of the real beam, Figure 19 —

$$y_{max} = \frac{2{,}450{,}000 \text{ in.}^3\text{-lbs}}{E\ I} \text{ inches}$$

7. DEFLECTION OF BEAM WITH VARIABLE SECTION

The area moment method may be used very nicely to find the deflection of beams in which no portion of the beam has a constant moment of inertia.

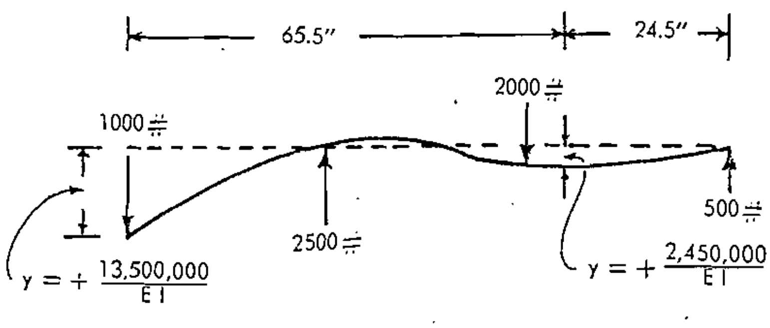


FIGURE 19

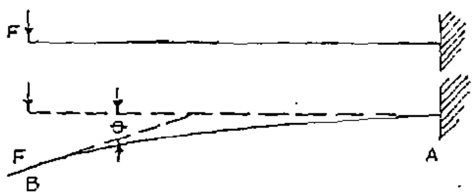


FIGURE 20

The angle between the tangents at A and B = θ = the area of the moment diagram between A and B, divided by EI.

Subdividing this beam into 10 or more segments of equal length (s):

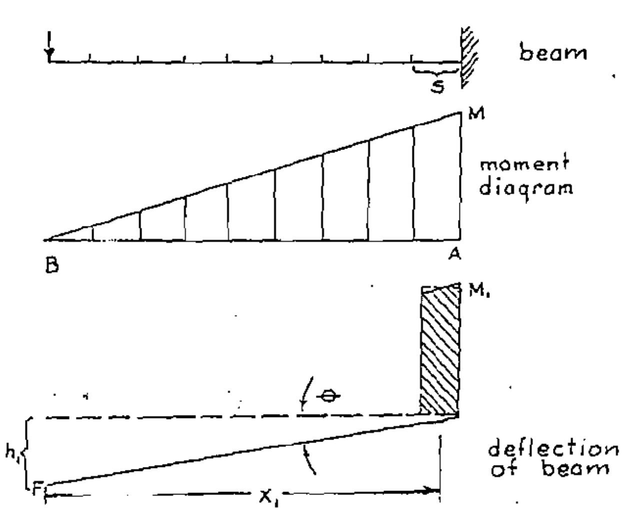


FIGURE 21

Each segment of bending moment causes the beam in this segment to bend or rotate. The angle of bend θ = area of moment diagram of this segment divided by EI, or —

$$\boxed{\theta_n = \frac{M_n\, s}{E\, I_n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

The resultant vertical momement (h_n) of the load, at the left end of the beam, is —

$$\boxed{h_n = \theta_n\, X_n = \frac{M_n\, s\, X_n}{E\, I_n}} \quad \ldots\ldots\ldots\ldots\ldots (11)$$

Each segment of the beam bends under its individual bending moment and its angle change causes the end of the beam to deflect. See Figure 22.

The total deflection at the end of the beam equals the sum of the deflections at the end of the beam caused by the angle change of each segment of the beam. See Figure 23.

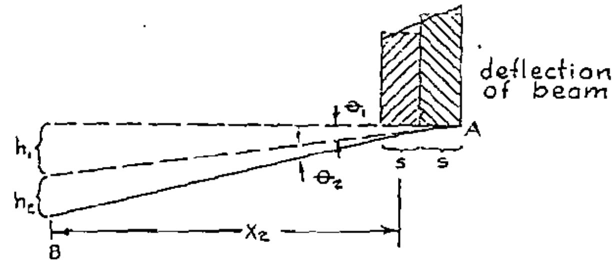


FIGURE 22

Restating the preceding, the vertical deflection of B is —

$$\boxed{\Delta = \Sigma\, \frac{M_n\, X_n\, s}{E\, I_n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

or:

$$\boxed{\Delta = \frac{s}{E} \Sigma \frac{M_n\, X_n}{I_n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Note: $\dfrac{M_n\, X_n}{I_n}$ is found for each segment. These values are added together, and this sum is multiplied by s/E to give the total deflection.

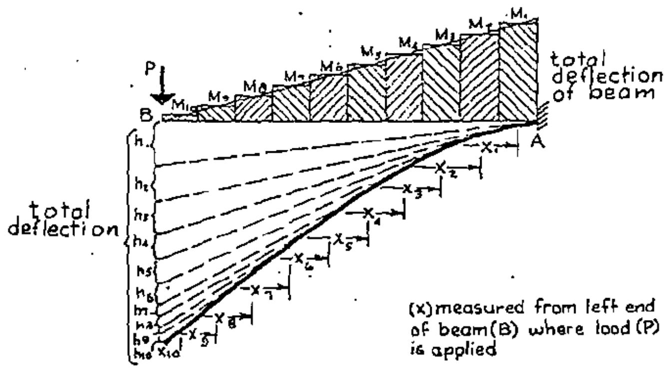


FIGURE 23

Problem 2

The following tapered beam is 30′ long. It has 1″ × 10″ flange plates and a ½″ thick web. It is 11″ deep at the ends and 33″ deep at centerline. It supports two 50-kip loads at the ⅓ points. Find the maximum deflection of the beam. See Figure 24.

Divide the length of the beam into 12 equal segments. The greater the number of segments or divisions, the more accurate will be the answer. Normally 10 divisions would give a fairly accurate result (Fig. 25).

FIGURE 24

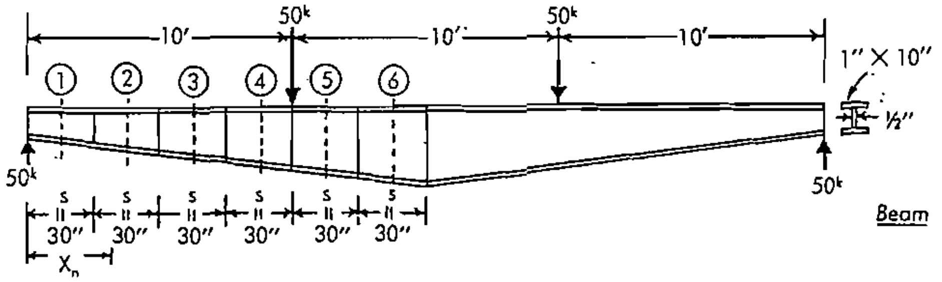
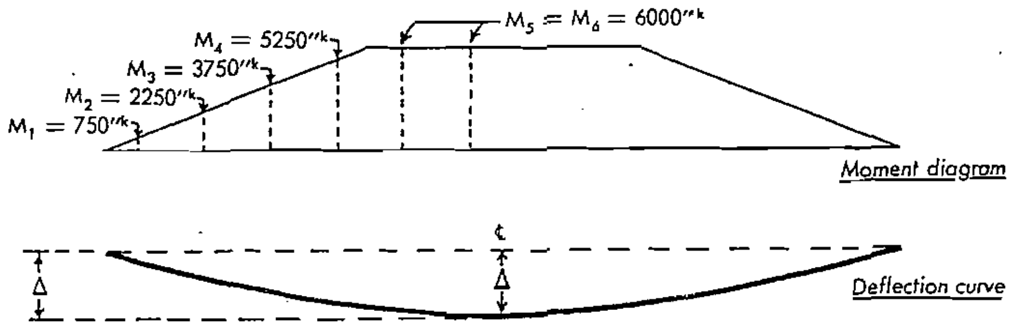


FIGURE 25



Here: $s = 30''$

and $\Delta_{total} = \frac{s}{E} \Sigma \frac{M_n X_n}{I_n}$

The moment of inertia of each segment (I_n) is taken at the sectional centroid of the segment.

The formula components M_n, X_n, and I_n are easier to handle in table form:

Segment	Depth of Web	I_n	X_n	Moment (M_n), in.-lbs	$\frac{M_n X_n}{I_n}$ psi
1	10″	646.67	15″	50^kx15″ = 750,000	17,400
2	14″	1239.33	45″	50^kx45″ = 2,250,000	81,700
3	18″	2048.00	75″	50^kx75″ = 3,750,000	137,320
4	22″	3088.67	105″	50^kx105″ = 5,250,000	178,480
5	26″	4377.33	135″	50^kx135″ — 50^kx15″ = 6,000,000	185,040
6	30″	5930.33	165″	50^kx165″ — 50^kx45″ = 6,000,000	166,940
				Total →	766,880

Total vertical deflection —

$$\Delta_{total} = \frac{s}{E} \Sigma \frac{M_n X_n}{I_n}$$

$$= \frac{(30'')(766,880 \text{ psi})}{(30,000,000 \text{ psi})}$$

$$= \underline{.77''}$$

8. DESIGNING FOR MULTIPLE LOADS

Normally, the calculation of the maximum deflection of members subjected to bending loads is very complex. The point of maximum deflection must first be found; then, from this, the maximum deflection is found. Unless there are no more than two loads of equal value and equal distance from the ends of the beam (Fig. 26), existing beam tables in handbooks do not cover this problem.

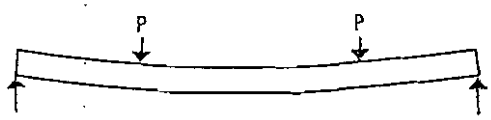


FIGURE 26

For example, most beams have more than two loads (Fig. 27). The maximum deflection usually does not occur at the middle or centerline of the beam (Fig. 28). Two things can be done to simplify this problem.

First, consider only the deflection at the middle or centerline of the member, rather than the maximum deflection at some point which is difficult to determine. This is justified, since the deflection at midpoint or centerline is almost as great as the maximum deflection,

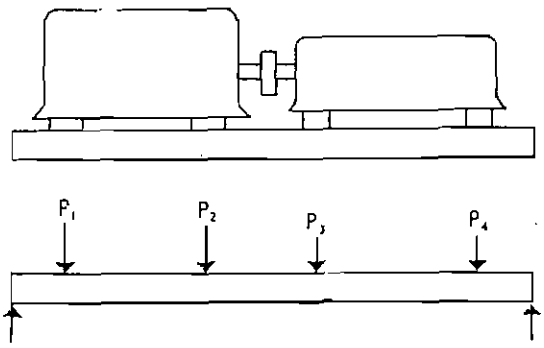


FIGURE 27

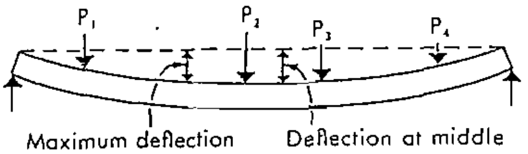


FIGURE 28

the greatest deviation coming within 1 or 2% of this value. For example, a simply supported beam with a single concentrated load at the one-quarter point has a deflection at centerline = 98.5% of the maximum deflection.

Secondly, a simple method of adding the required moments of inertia required for each individual load can be used.

For a given size member, Figure 29, it is found that each load, taken one at a time, will cause a certain amount of deflection at the middle or centerline. The total deflection at the centerline will equal the sum of these individual deflections caused by each load.

This principle of adding deflections may be used in a reverse manner to find the required section of the member (I), Figure 30. For a given allowable deflection (Δ) at the centerline, each individual load, taken one at a time, will require the member to have a certain section (I_1, I_2, etc.).

The moment of inertia (I) of the beam section required to support all of the vertical loads within this allowable vertical deflection (Δ) will equal the sum of the individual moments of inertia (I_n) required for the several loads.

Any torque or couple applied horizontal to the beam will cause it to deflect vertically. This can be handled in the same manner. The required moment of inertia of the member (I_n) for each torque acting separately is found and added into the total requirement for the property of the section (I).

The following two formulas may be used to find the individual properties of the section (I_n):

for each force

$$\boxed{I_n = \frac{P_n\, L^2}{48\, E\left(\frac{\Delta}{L}\right)}(3\, K_n - 4\, K_n^3)} \quad \ldots\ldots\ldots(14)$$

for each couple

$$\boxed{I_n = \frac{C_n\, L}{16\, E\left(\frac{\Delta}{L}\right)}(4\, K_n^2 - 1)} \quad \ldots\ldots\ldots\ldots(15)$$

where:

$$K_n = \frac{a_n}{L}$$

The two formulas have been simplified into the formulas given below in which the expression K_n now produces a constant (A or B) which is found in Table 3.

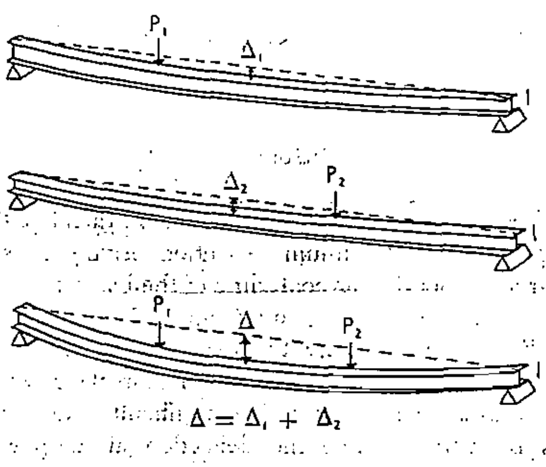


FIGURE 29

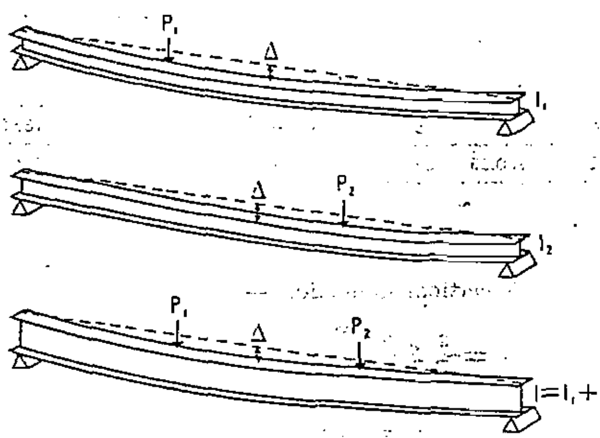


FIGURE 30

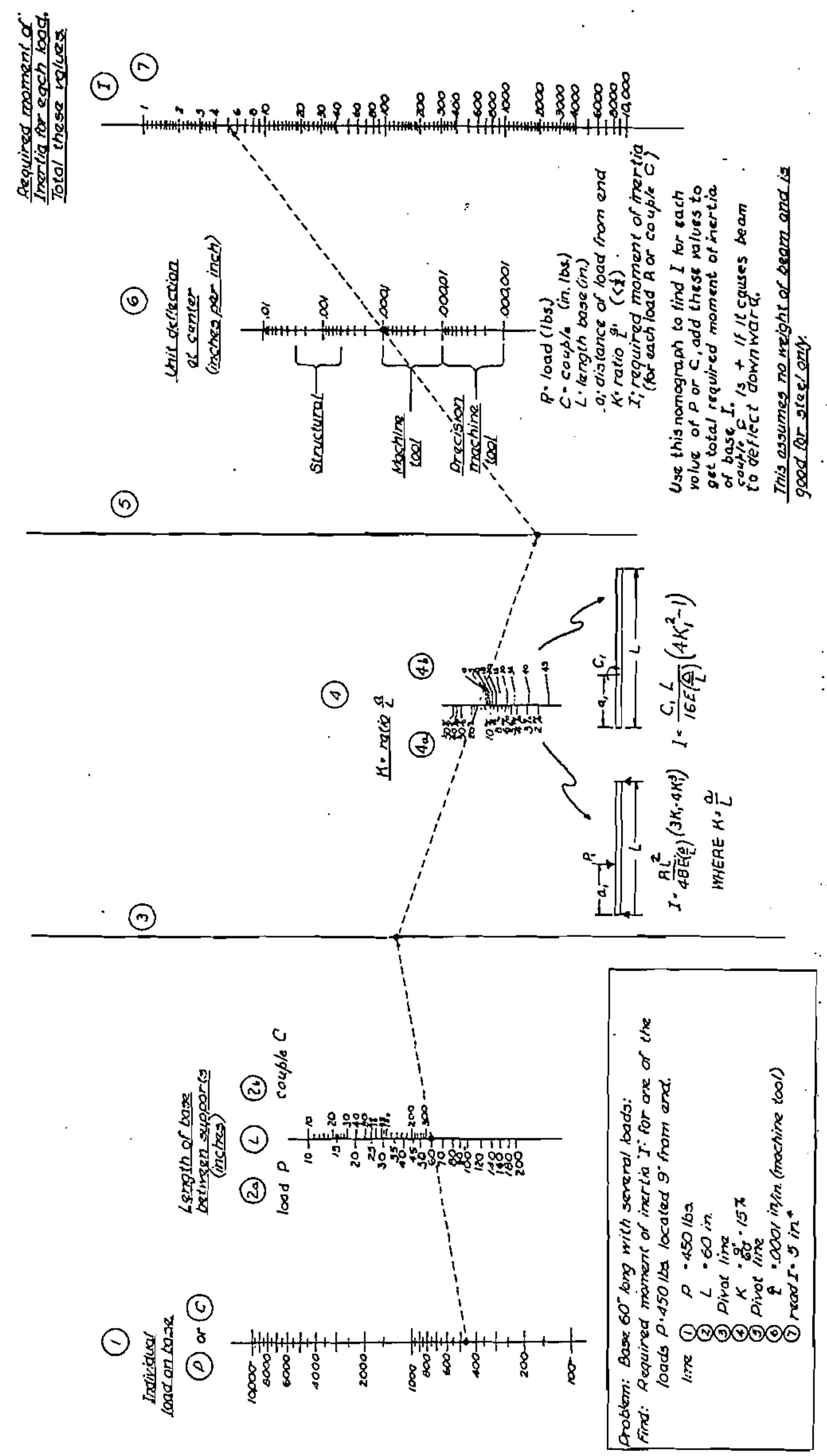


FIGURE 31—Required Moment of Inertia to Resist Bending

TABLE 3—Values of Constants (A and B) for Simplified Formulas (16 and 17)

K	A	B	K	A	B	K	A	B
0	0	2.083×10^{-9}	.17	3.045×10^{-10}	1.842×10^{-9}	.34	5.002×10^{-10}	1.120×10^{-9}
.01	$.2083 \times 10^{-10}$	2.083	.18	3.588	1.813	.35	6.101	1.063
.02	.4166	2.080	.19	3.768	1.783	.36	6.204	1.003
.03	.6243	2.076	.20	3.944	1.750	.37	6.301	.9425
.04	.8312	2.070	.21	4.118	1.715	.38	6.392	.8900
.05	1.038	2.063	.22	4.268	1.680	.39	6.477	.8158
.06	1.244	2.053	.23	4.453	1.642	.40	6.556	.7500
.07	1.449	2.043	.24	4.616	1.603	.41	6.627	.6825
.08	1.653	2.030	.25	4.774	1.563	.42	6.692	.6133
.09	1.855	2.016	.26	4.928	1.520	.43	6.750	.5425
.10	2.056	2.000	.27	5.079	1.476	.44	6.801	.4700
.11	2.355	1.983	.28	5.224	1.430	.45	6.844	.3958
.12	2.452	1.963	.29	5.364	1.381	.46	6.880	.3221
.13	2.647	1.942	.30	5.500	1.333	.47	6.898	.2425
.14	2.847	1.920	.31	5.631	1.282	.48	6.928	.1633
.15	3.031	1.896	.32	5.756	1.209	.49	6.940	.0825
.16	3.219	1.870	.33	5.876	1.176	.50	7.000	0

for each force

$$\boxed{I_n = \frac{P_n L^2 A_n}{\left(\frac{\Delta}{L}\right)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

for each couple

$$\boxed{I_n = \frac{C_n L B_n}{\left(\frac{\Delta}{L}\right)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

The value of K_n is equal to the ratio a_n/L, where a_n is the distance from the point at which the specific force or couple is applied to the nearest point of support. L is the span or length of beam between supports. From the value of K for any given load (P), the substitute constant A or B is obtained from Table 3.

When a force is applied to the member, use the constant A and substitute into the first formula. When a couple is applied to the member, use the constant B and substitute into the second formula.

A shorter method would be to make use of the nomograph in Figure 31.

9. INFLUENCE LINE FOR REACTIONS

Maxwell's Theorem of Reciprocal Deflections may be used to find the reactions of a continuous beam or frame, and is especially adaptable to model analysis.

Consider the continuous beam represented by the diagram at Figure 32a. The problem here is to find the reactions of the supports for various positions of the load (P_x).

According to Maxwell's theorem, the deflection at point 1 (Δ_b) due to the load (P_b) at point x, Figure 32b, equals the deflection at point x (Δ_c) due to the same amount of load (P_c) applied to point 1, Figure 32c. There is a similar relationship between an applied load or moment and the resulting rotation of a real beam.

Figures 32b and 32c constitute a simple reversal

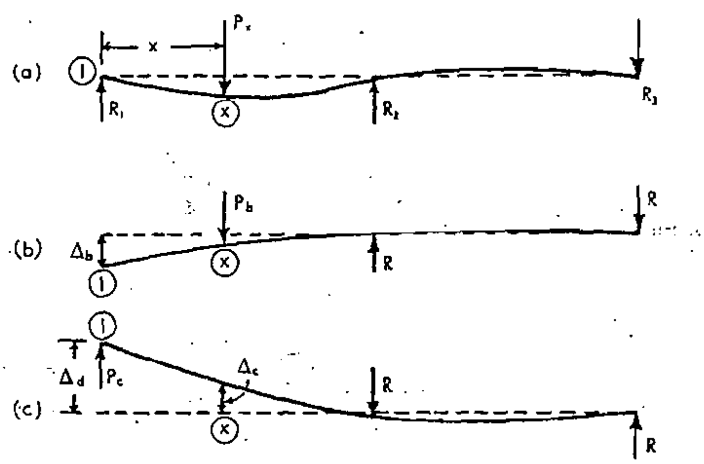


FIGURE 32

of points at which the pressure is applied. This concept supplies a very useful tool for finding influence lines for reactions, deflections, moments, or shear. In this case, the interest is in reactions.

To find the value of the reaction (R_1) at the left-hand support in Figure 32a, the support is removed; this causes the left end to deflect (Δ_b), as at Figure 32b. In order to restore the left end to its initial position, an upward reaction (P_c) must be applied, as in Figure 32c.

In extending Maxwell's theorem of reciprocal deflections to Figure 32b and Figure 32c, it is noticed:

$$\text{if } P_b = P_c \qquad \text{then } \Delta_b = \Delta_c$$

However, in order to return the beam to the initial condition of Figure 32a, Δ_d must be reduced until it equals Δ_b. To do this the upward reaction (P_c) must be reduced by the factor: Δ_b/Δ_d. And since $\Delta_b = \Delta_c$, this reduction factor becomes Δ_c/Δ_d.

$\therefore R_1 = P_b \frac{\Delta_c}{\Delta_d}$ or, using Figure 32a —

$$\boxed{R_1 = P_x \frac{\Delta_c}{\Delta_d}} \qquad (18)$$

This means that if the model beam (as in Fig. 32c) is displaced in the same direction and at the same point as the reaction in question, the resulting deflection curve becomes the plot of the reaction as the load is moved across the length of the beam.

This is called an "influence curve". Considering the conditions of the real beam represented by Figure 32a, the reaction (R_1) at point 1 due to a load (P_x) at point x will be proportional to the ratio of the two ordinates at points x and 1 of the deflection curve.

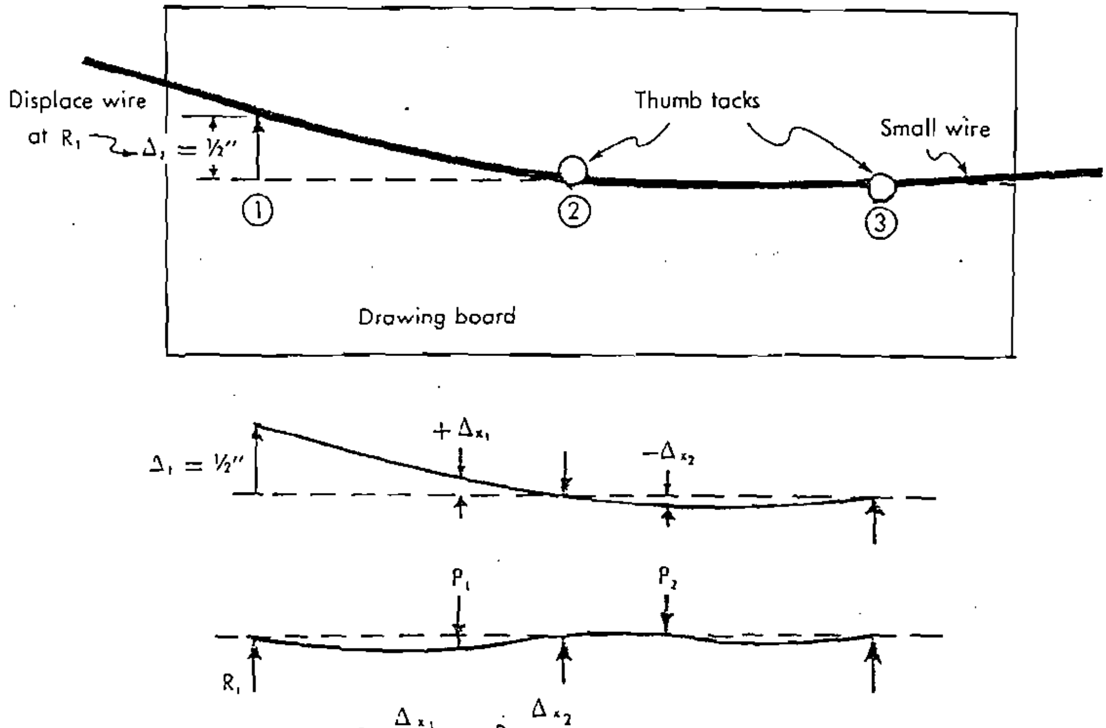
In other words:

$$\boxed{R_1 = P_x \frac{\Delta_x}{\Delta_1}} \qquad (19)$$

For continuous beams of constant cross-section, a wire model may be set up on a drawing board, with the wire beam supported by thumb tacks spaced so as to represent the supports on the real beam. See Figure 33. A load diagram of the real beam is shown at the bottom. Notice that the thumb tacks used for supports of the wire must be located vertically so as to function in the opposite direction to reactions on the real beam.

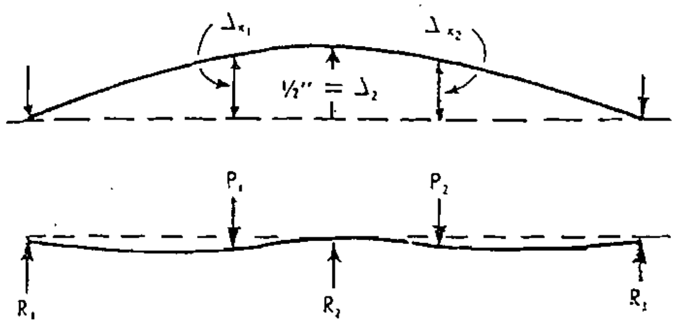
The point of the model beam at the reaction in question (R_1) is raised upward some convenient distance, for example ½" or 1", and the deflection curve of the wire beam is traced in pencil. This is shown immediately below the model.

The final value for the reaction (R_1) is equal to



$$R_1 = +P_1 \frac{\Delta_{x_1}}{\Delta_1} - P_2 \frac{\Delta_{x_2}}{\Delta_1}$$

FIGURE 33



$$R_2 = +P_1\frac{\Delta_{x_1}}{\Delta_2} + P_2\frac{\Delta_{x_2}}{\Delta_2}$$

FIGURE 34

the sum of the actual applied forces multiplied by the ratio of their ordinates of this curve to the original displacement at R_1.

The influence curve for the central reaction (R_2) may also be found in the same manner. See Figure 34. Deflection curve of the wire model is shown first and then the load diagram of the real beam.

Problem 3

A continuous beam has 5 concentrated loads and 4 supports. The problem is to find the reactions at the supports.

The reactions are found by comparing the ordinates of the deflection curve of a wire representing the beam. See Figure 35, where the critical dimensions appear on the (upper) load diagram.

For the ends, reactions R_1 and R_4, displace the end of the wire a given amount as shown. The portion of each applied load (P) to be transferred to the reaction R_1 is proportional to the ordinate of the deflection curve under the load (P) and the given displacement at R_1.

For the interior reactions R_2 and R_3, displace the wire a given amount at R_2. From the ordinates of this

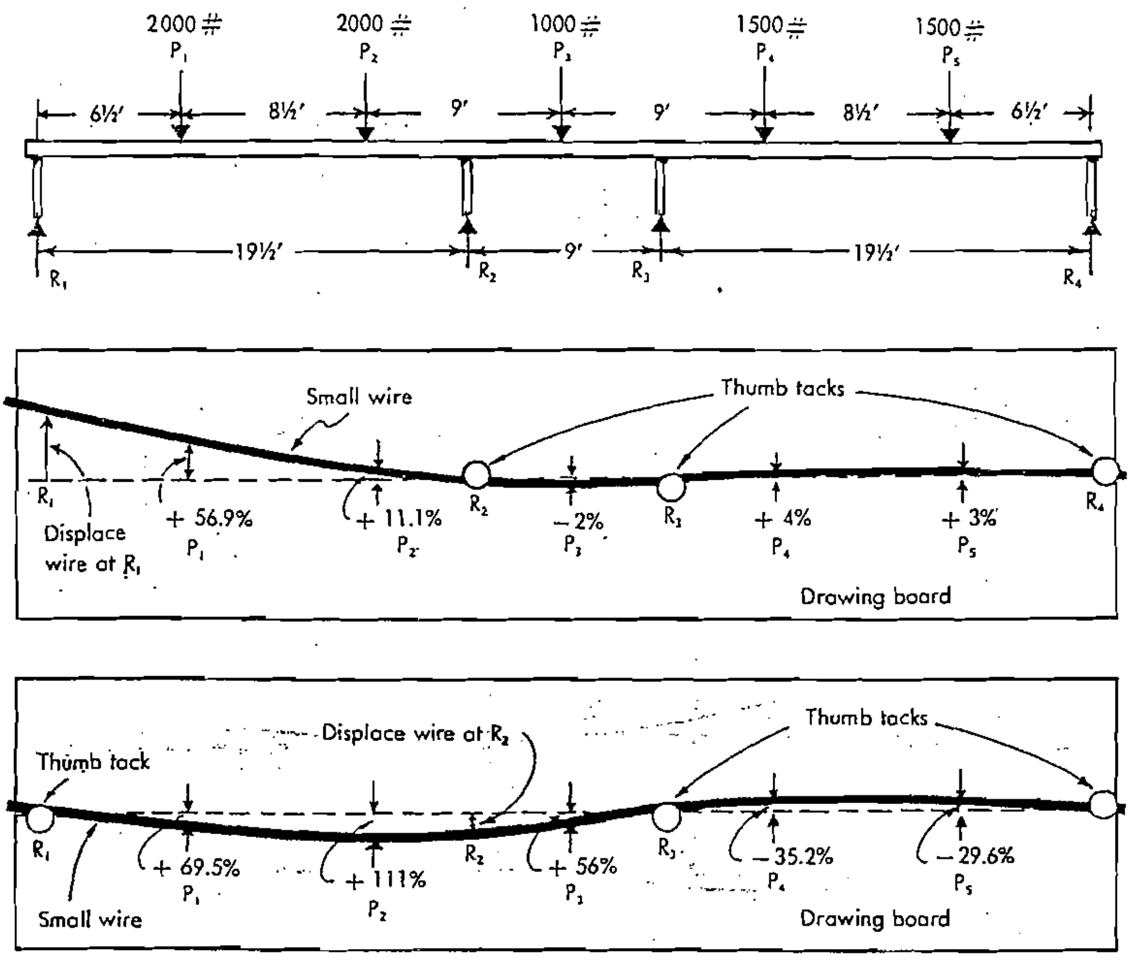


FIGURE 35

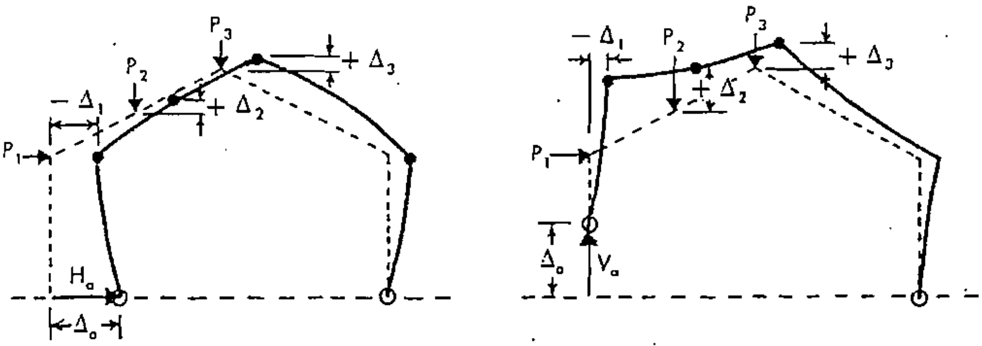


FIGURE 36

deflected wire, determine the ratios of each applied load (P) for the reaction at R_2.

The computation of forces for the reactions R_1 and R_4 is as follows:

$$R_1 = +.569\ P_1 + .111\ P_2 - .02\ P_3 + .04\ P_4 + .03\ P_5$$

$$= .569(2000\#) + .111(2000\#) - .02(1000\#) + .04(1500\#) + .03(1500\#)$$

$$= \underline{+\ 1445 \text{ lbs}}$$

$$R_2 = +.695\ P_1 + 1.11\ P_2 + .56\ P_3 - .352\ P_4 - .296\ P_5$$

$$= .695(2000\#) + 1.11(2000\#) + .56(1000\#) - .352(1500\#) - .296(1500\#)$$

$$= \underline{+\ 3198 \text{ lbs}}$$

Reactions R_3 and R_4 can be found in like manner.

Application to Frames

This same method may be extended to the analysis of frames. If the frame has a constant moment of inertia, a stiff wire may be bent into the shape of the frame. If the frame has a variable moment of inertia, the model may be made of a sheet of plastic or cardboard proportioned to the actual moments of inertia.

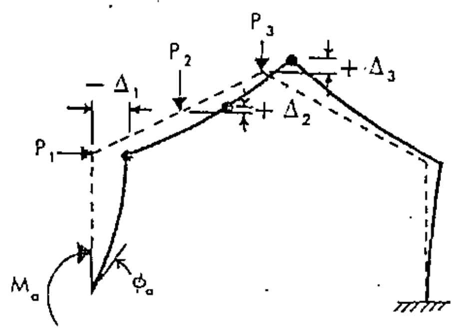


FIGURE 37

Reactions, either horizontal (H) or vertical (V) at the supports, may be found by displacing the frame at the support a given amount in the direction of the desired reaction. See Figure 36. The outline of the displaced model frame is traced in pencil, and this becomes the curve showing the influence of any load (at any point) upon this reaction.

The displacement of each point of the model frame (Δ) where a load is applied is measured in the same direction as the application of the load, and the resulting reaction may be computed from the following:

horizontal reaction

$$H_a = P_1\left(\frac{-\Delta_1}{\Delta_a}\right) + P_a\left(\frac{+\Delta_2}{\Delta_2}\right) + P_3\left(\frac{+\Delta_3}{\Delta_a}\right)$$

vertical reaction

$$V_a = P_1\left(\frac{-\Delta_1}{\Delta_a}\right) + P_2\left(\frac{+\Delta_2}{\Delta_a}\right) + P_3\left(\frac{+\Delta_3}{\Delta_a}\right)$$

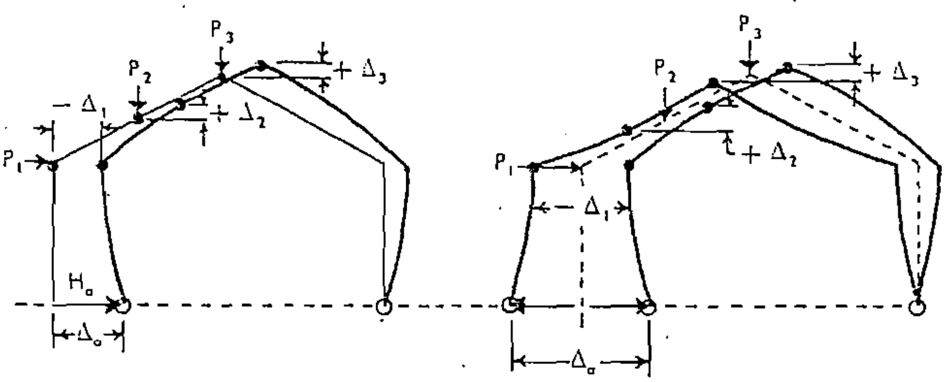
Moments at the ends of the frame (or at any point in the frame) may be found by rotating the point in question a given angle (ϕ_a) and again drawing the resulting displaced model frame. See Figure 37.

The displacement of each point of the model frame (Δ) where a load is applied is measured in the same direction as the application of the load, and the resulting moment may be computed from the following:

moment at left-hand support

$$M_a = \frac{P_1\ (-\Delta_1) + P_2\ (+\Delta_2) + P_3\ (+\Delta_3)}{\phi_a}$$

It is necessary to displace the model a considerable distance in order that some accuracy may be obtained in the readings. Therefore, some error may be introduced because the final shape of the frame may alter the real load conditions. This error can be reduced greatly by measuring the displacements between one



(a) Measuring displacement of model frame from initial condition to displaced condition

(b) Measuring displacement of model frame from one displaced condition to an equal and opposite displaced condition

FIGURE 38

condition and the opposite condition. See Figure 38.

This method of equal to opposite displacement may also be applied to moments in which the frame is rotated an equal amount in both directions, and measurements taken from one extreme to the other.

10. INFLUENCE LINE FOR DEFLECTION

In like manner, the use of a wire model based on Maxwell's Theorem of Reciprocal Deflection is useful in finding the deflections of a beam under various loads or under a moving load.

If a 1-lb load is placed at a particular point on a beam, the resulting deflection curve becomes the plot of the deflection (Δ) at this point as the 1-lb load is moved across the length of the beam. This is called the influence line for deflection at this particular point.

TABLE 4—Incremental Deflections of Real Beam

Point	Load (Lbs)	Ordinate x 10^{-3}	Deflection At Free End (In.)
0	100	0	0
3'	150	— .60	— .030
8'	300	—1.06	— .318
15'	400	—1.60	— .640
21'	750	—1.56	—1.170
23'	750	—1.36	—1.020
28'	375	— .70	— .262
33'	150	+ .70	+ .105
37'	325	+2.00	+ .650
40'	100	+3.25	+ .325
Total	3300 lbs		—2.360"

Problem 4

To determine the deflection of the overhung portion of this trailer, Figure 39, under the various loads. Assume a cross-section moment of inertia (I) of 2 × 11.82 in.^{4}.

Using the standard beam formula for this type of beam, the deflection of the free (right) end is determined for a 1-lb load placed at that point:

$$\Delta_{end} = \frac{P\ a^2}{3\ E\ I}(L + a)$$

$$= \frac{1\#\ (120)^2}{3(30 \times 10^6)(2 \times 11.82)}(360 - 120)$$

$$= 3.25 \times 10^{-3} \text{ inches}$$

A wire model of this beam is held at the two supports (trailer hitch and the wheel assembly) with thumb tacks on a drawing board. The outer end is displaced an amount equal to 3.25 on a suitable scale. The deflection curve is traced in pencil from this displaced wire beam. The ordinates of this resulting deflection curve become the actual deflections at the free end as the 1-lb load is moved across the length of the beam.

Multiplying each of the loads on the real beam by the ordinate at that point gives the deflection at the free end caused by each load on the real beam. See Table 4. Summing these incremental deflections gives the total deflection:

Δ = 2.36" upward

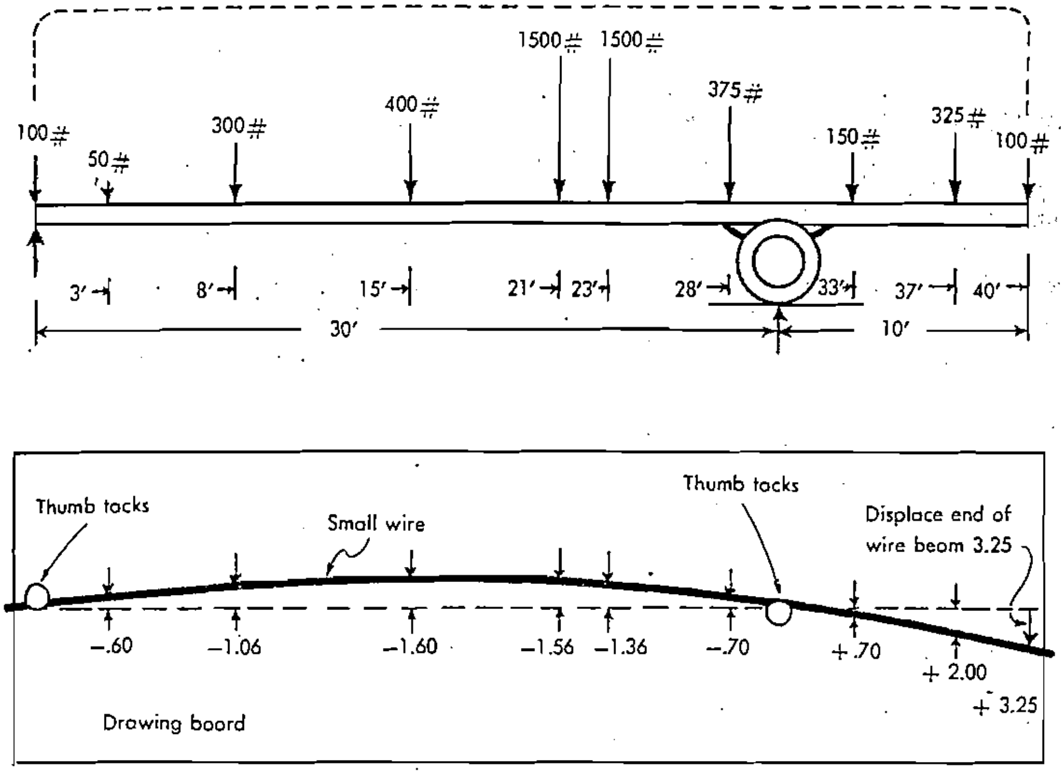
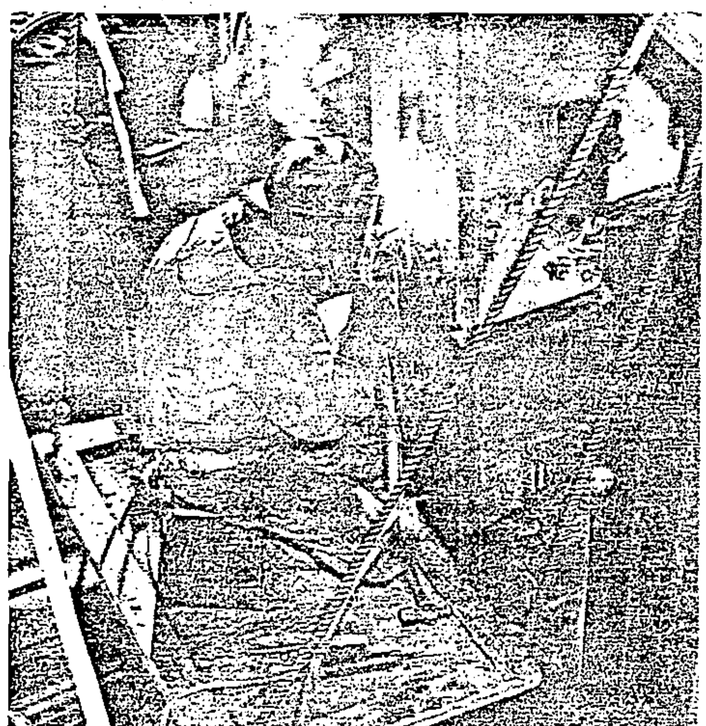
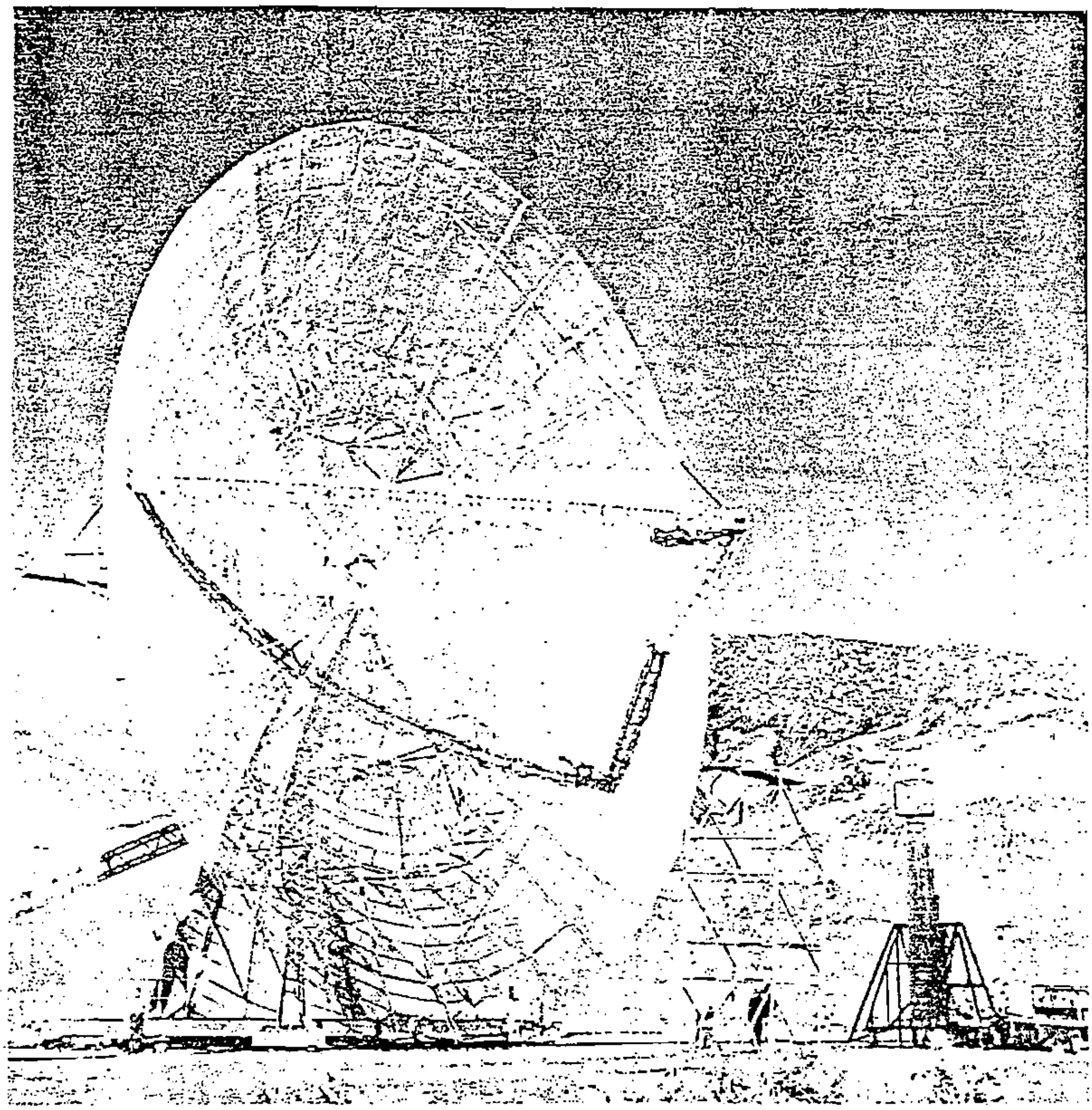


FIGURE 39

Erection of the 32-story Commerce Towers in Kansas City, Missouri was speeded with the aid of modern semi-automatic arc welding. Field use of self-shielding cored electrode quadrupled the rate of weld metal deposition. The weldor shown here is making a field splice of two sections of the heavy building column.





Complex antenna systems needed in age of space communications are sensitive to bending deflections caused by high wind loads. Good engineering, including the specification of high strength steels and rigid welded connections, is essential to the satisfactory performance of such structures. In the parabolic antenna dish shown, 6400 sq ft of expanded metal mesh are welded to a space frame of tubular welded trusses.