Comparison of Block diagram and Signal Flow graph Methods.

Ec A

Block diagram

1. Applicable only to LTI s/m
2. Each element is represented by block
3. Summing point and takeoff point are Separate.
4. Self loop do not exist.
5. Time Consuming
6. Feedback path is present
7. Block diagram reduction rules are used

Signal Flow graph

1. Applicable only to LTI s/m
2. Each Variable is respresented by node.
3. Summing point & takeoff point are not present. They are represented by nodes.
4. Self loop can exist.
5. Requires less time.
6. Feedback loop is present.
7. Mason's ga

Mason's Gain Rule:

- In order to calculate the overall transfer function of a system, an American electronics engineer S.J Mason introduced a rule which is called as the Mason's Gain Rule.

- The Mason's Gain Formula is given by:

$$\boxed{\frac{C(S)}{R(S)} = \sum_{k=1}^{n} \frac{P_k . \Delta_k}{\Delta}}$$

where, $n =$ Number of Forward Paths.

$P_k =$ Forward Path gain of k^{th} forward path.

$\Delta_k =$ Associated Path Factor (cofactor).

$\Delta =$ Determinant of S.F.G.

Δ = Determinant of S.F.G.

= 1 − [Sum of gains of all the Individual loops]
+ [Sum of Products of gains of all possible combinations of two Non-touching Loops]
− [Sum of products of gains of all possible combinations of three Non-touching Loops] + ...

Δ_k = Associated Path Factor.

= The Δ part of S.F.G that is non-touching with the k^{th} forward path.

= 1 − [Sum of gains of all the individual isolated loops]
+ [Sum of Products of gains of all the combinations of two non-touching isolated loops]
− [Sum of products of gains of all the combinations of three non-touching isolated loops] + ...

Steady state error with unit step input

→ $e_{ss} = \lim_{s \to 0} s\,E(s)$, where $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$

$= \lim_{s \to 0} s\left[\dfrac{R(s)}{1+G(s)H(s)}\right]$

→ for unit Step Input

$R(s) = 1/s$

→ steady state error

$e_{ss} = \lim_{s \to 0} s\left[\dfrac{1/s}{1+G(s)H(s)}\right]$

$= \lim_{s \to 0} \dfrac{1}{1+G(s)H(s)}$

$= \dfrac{1}{1+\lim_{s \to 0} G(s)H(s)}$ ⇒ $\boxed{e_{ss} = \dfrac{1}{1+K_p}}$

$$e_{ss} = \frac{1}{1+K_p}, \quad \text{where } K_p = \lim_{s \to 0} G(s)H(s)$$

Type 0 system

→ for type 0 system, number poles at origin is zero.

→ $G(s)H(s) = \dfrac{K(s+z_1)(s+z_2)\ldots}{(s+p_1)(s+p_2)\ldots}$

→ $K_p = \lim_{s \to 0} \dfrac{K(s+z_1)(s+z_2)\ldots}{(s+p_1)(s+p_2)\ldots}$

$= \dfrac{K(z_1)(z_2)\ldots}{(p_1)(p_2)\ldots}$

$=$ Constan

$e_{ss} = \dfrac{1}{1+K_p}$

$= \dfrac{1}{1+\text{Constant}}$

$=$ constant

Calls and notifications will vibrate

$$e_{ss} = \frac{1}{1+K_p}, \quad \text{where } \underline{K_p} = \lim_{s \to 0} G(s)H(s)$$

Type 1 system

→ For Type 1 system, number poles at origin = 1

$$\underline{G(s)H(s)} = \frac{K(s+z_1)(s+z_2)\ldots}{\underline{s}(s+P_1)(s+P_2)\ldots}$$

$$\to K_p = \lim_{s \to 0} \frac{K(s+z_1)(s+z_2)\ldots}{\underline{s}(s+P_1)(s+P_2)\ldots}$$

$$= \frac{1}{0}$$

$$\boxed{K_p = \infty}$$

$$e_{ss} = \frac{1}{1+K_p}$$

$$= \frac{1}{1+\infty}$$

$$\boxed{e_{ss} = 0}$$

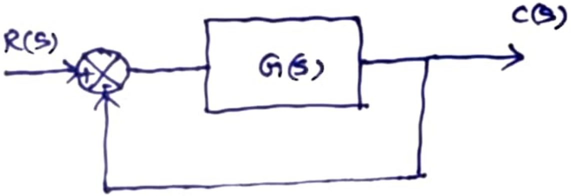
→ For type 2

No of poles at origin = 2

$K_p = \infty$

$e_{ss} = 0$

A unity feedback control system has an open loop transfer function $G(s) = 10/s(s+2)$. Find the rise time, Percentage overshoot, Peak time and Settling time for step input 12 units.



The closed loop transfer function,

$$\frac{C(S)}{R(S)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$\frac{C(S)}{R(S)} = \frac{10/s(s+2)}{1 + (10/s(s+2)) \cdot 1} = \frac{10/s(\cancel{s+2})}{s(s+2) + 10/s(\cancel{s+2})}$$

$$\frac{C(S)}{R(S)} = \frac{10/s(\cancel{s+2})}{s^2 + 2s + 10/s(\cancel{s+2})}$$

$$\frac{C(S)}{R(S)} = \frac{10}{s^2 + 2s + 10}$$

Highest power of S in the denominator is 2 . So it is a second order system

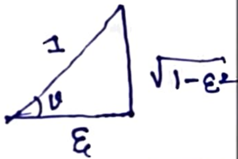
Standard form of second order transfer function $\frac{C(S)}{R(S)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

$$\frac{\omega_n^2}{s^2+2\underbrace{\xi\omega_n}s+\omega_n^2} = \frac{10}{s^2+\underline{2}s+10}$$

$\omega_n^2 = 10$ | $2\xi\omega_n = 2$

$\omega_n = \sqrt{10}$ | $\xi\omega_n = 1$

$\xi = 1/\omega_n = \frac{1}{\sqrt{10}} = 0.316$ $\boxed{\xi = 0.316}$



$\tan\theta = \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}$

$\theta = \tan^{-1}\left(\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}\right)$

$\theta = \tan^{-1}\left(\frac{\sqrt{1-0.316^2}}{0.316}\right)$

$\theta = 1.249$ rad

$T = \frac{1}{\xi\omega_n}$

$= \frac{1}{0.316\sqrt{10}}$

$\boxed{T = 1}$

1 Rise time $(t_r) = \frac{\pi-\theta}{\omega_d} = \frac{3.14-\theta}{\omega_n\sqrt{1-\varepsilon^2}} = \frac{3.14-1.249}{\sqrt{10}\sqrt{1-0.316^2}} = 0.63$ sec

2. $\%M_p = e^{-\xi\pi/\sqrt{1-\varepsilon^2}} \times 100\% = e^{-0.316\times 3.14/\sqrt{1-0.316^2}}$

$= 35.12\%$

3. Peak time $(t_p) = \frac{\pi}{\omega_d} = \frac{3.14}{\omega_n\sqrt{1-\varepsilon^2}} = \frac{3.14}{\sqrt{10}\sqrt{1-0.316^2}} = 1.047$ sec

4. Settling time for 5% Error $= 3T = 3$

For 2% Error $= 4T = 4$

Result: $t_r = 0.6$

$\%M_p =$

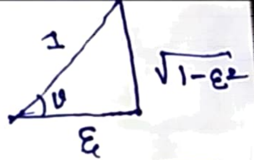
t

$t_s =$ (5% Error)

(2% Error)

$p =$ (for 12 unit)

$$\frac{\omega_n^2}{s^2+\underbrace{2\varepsilon\omega_n}s+\omega_n^2} = \frac{10}{s^2+\underline{2}s+10}$$



$\omega_n^2 = 10$	$2\varepsilon\omega_n = 2$
$\omega_n = \sqrt{10}$	$\varepsilon\omega_n = 1$

$$\varepsilon = 1/\omega_n = \frac{1}{\sqrt{10}} = 0.316 \quad \boxed{\varepsilon = 0.316}$$

$$\tan\theta = \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}\right)$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-0.316^2}}{0.316}\right)$$

$$\theta = 1.249 \text{ rad}$$

1. Rise time $(t_r) = \dfrac{\pi-\theta}{\omega_d} = \dfrac{3.14-\theta}{\omega_n\sqrt{1-\varepsilon^2}} = \dfrac{3.14-1.249}{\sqrt{10}\sqrt{1-0.316^2}} = 0.63$ sec

2. $\% M_p = e^{-\varepsilon\pi/\sqrt{1-\varepsilon^2}} \times 100\% = e^{-0.316\times 3.14/\sqrt{1-0.316^2}}$
$= 35.12\%$

3. Peak time $(t_p) = \dfrac{\pi}{\omega_d} = \dfrac{3.14}{\omega_n\sqrt{1-\varepsilon^2}} = \dfrac{3.14}{\sqrt{10}\sqrt{1-0.316^2}} = 1.047$ sec

4. Settling time for 5% Error $= 3T = 3$
For 2% Error $= 4T = 4$

$$T = \frac{1}{\varepsilon\omega_n} = \frac{1}{0.316\sqrt{10}}$$

$$\boxed{T = 1}$$

Result:

$t_r = 0.63$ sec	$t_s = 3$ (5% Error)
$\% M_p = 35.12\%$	4 (2% Error)
$t_p = 1.047$ sec	$M_p = 4.214$ (for 12 Unit)

$\% M_p = 35.12\%$

for 12 Units $= 35.12 \times 12$

or not using RH criterion

CHATGPT

To apply the Routh-Hurwitz criterion, we need to form the Routh array based on the coefficients of the characteristic equation. The characteristic equation for the given cubic equation is (S^3 + 5S^2 + 25S + 30 = 0).

The Routh array is:

Copy code

```
|1  25
---|---------
S^3| 1  25
S^2| 5  30
S^1| -25
S^0| 30
```

The criterion states that all the elements in the first column of the Routh array must be positive for all the roots to have real parts. In this case, since the element in the second row is (5), which is positive, and the element in the fourth row is (-25), which is negative, not all elements in the first column are positive.

Therefore, based on the Routh-Hurwitz criterion, not all roots have real parts greater than 1.

1. It is valid only for real coefficients of the characteristic equation.
2. It does not provide exact locations of the closed loop poles in left or right half of s-plane.
3. It does not suggest methods of stabilizing an unstable system.
4. Applicable only to linear system.

Ex.1. $s^6 + 4s^5 + 3s^4 - 16s^2 - 64s - 48 = 0$ Find the number of roots of this equation with positive real part, zero real part & negative real part

Sol:

S^6	1	3	-16	-48
S^5	4	0	-64	0
S^4	3	0	-48	0
S^3	0	0	0	

$A(s) = 3S^4 - 48 = 0 \qquad \frac{dA}{ds} = 12S^3$

S^6	1	3	-16	-48
S^5	4	0	-64	0
S^4	3	0	-48	0
S^3	12	0	0	0
S^2	(ε) 0	-48	0	0
S^1	$\frac{576}{\varepsilon}$	0	0	0
S^0	-48			

$$\lim_{\varepsilon \to 0} \frac{576}{\varepsilon} = +\infty$$

Therefore one sign change & system is unstable. Thus there is one root in R.H.S of the s – plane i.e. with positive real part. Now Solve A(s) = 0 for the dominant roots

$A(s) = 3s^4 - 48 = 0$

Put $S^2 = Y$

$\therefore \; 3Y^2 = 48 \qquad \therefore \; Y^2 = 16, \qquad \therefore \; Y = \pm\sqrt{16} = \pm 4$

$S^2 = +4 \qquad S^2 = -4$

$S = \pm 2 \qquad S = \pm 2j$

So, $S = \pm 2j$ are the two parts on imaginary axis i.e. with zero real part. Root in R.H.S. indicated by a sign change is $S = \pm 2$ as obtained by solving A(s) = 0. Total there are 6 roots as n = 6.

Roots with Positive real part = 1
Roots with zero real part = 2
Roots with negative real part = 6 – 2 – 1 = 3

root locus examples step by step | higher order systems |

Education 4u

Sketch the Root Loci of following unity feedback system

$$G(s)H(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}$$

$S_x = 0, -1, -2, -4$

$S_o = -3$

$n = 4 \quad ; \quad m = 1$

A: $n - m = 3$

Angle of Asymptotes $\frac{(2q+1)\pi}{n-m}$ $\quad q = 0, 1, 2$

$q = 0 \quad \frac{\pi}{3} = 60^\circ$

$q = 1 \quad \frac{3\pi}{3} = 180^\circ$

$q = 2 \quad \frac{5(\pi)}{3} = 300^\circ$

Centroid: $\frac{(0-1-2-4) - (-3)}{3} = -\frac{4}{3}$

2:35 / 9:42

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Control System

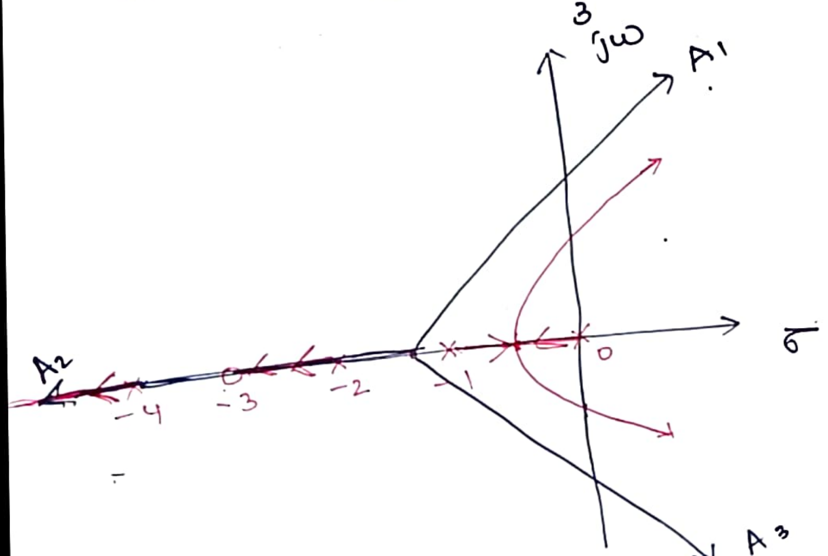
Root locus for Complex Poles

$q=0 \quad \frac{\pi}{3} = 60^\circ$

$q=1 \quad \frac{3\pi}{3} = 180^\circ$

$q=2 \quad \frac{5(\pi)}{3} = 300^\circ$

Centroid: $\frac{(0-1-2-4)-(-3)}{3} = -\frac{4}{3}$



$s(s^2+3s+2)(s+4) + K(s+3) = 0$

$s(s^3+4s^2+3s^2+12s+2s+8) + K(s+3) = 0$

$s(s^3+7s^2+14s+8) + K(s+3) = 0$

$s^4+7s^3+14s^2+8s+Ks+3K=0$

$s^4+7s^3+14s^2+(8+K)s+3K=0$

$s^4+7s^3+14s^2+($

s^4	1	14	$3K$
s^3	7	$8+K$	$-$
s^2	$\frac{7\times 14-8-K}{7} = \frac{90-K}{7}$	$3K$	$-$
s^1	$\frac{\left(\frac{90-K}{7}\right)(8+K)-7\times 3K}{\left(\frac{90-K}{7}\right)}$	$-$	
s^0	$3K$		

$s^4 + 7s^3 + 14s^2 + \dots$

$s^4 + 7s^3 + 14s^2 + (8+k)s + 3k = 0$

s^4	1	14	$3k$
s^3	7	$8+k$	$-$
s^2	$\frac{7\times 14 - 8 - k}{7} = \frac{90-k}{7}$	$3k$	
s^1	$\frac{\frac{90-k}{7}(8+k) - 7\times 3k}{\left(\frac{90-k}{7}\right)}$	$-$	$-$
s^0	$3k$		

$$8 + k - \frac{(21k)}{90-k/7} = 0$$

$$(8+k)(90 - k/7) - 21k = 0$$

$$k = 2$$

$$\frac{k(s+9)}{s(s^2+4s+11)} = 0$$

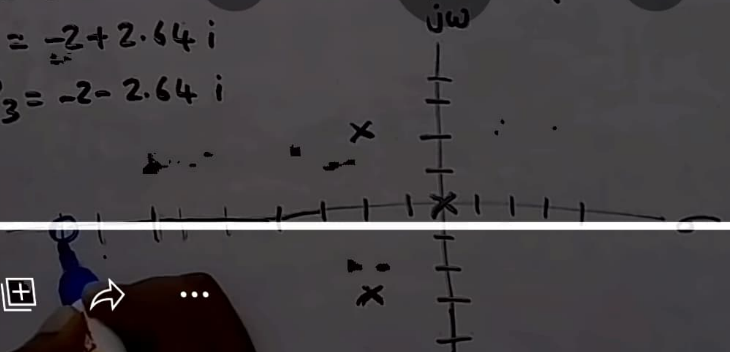
Step-1 – Locate poles and zeros

Poles

$P_1 = 0$
$P_2 = -2+2.64\,i$
$P_3 = -2-2.64\,i$

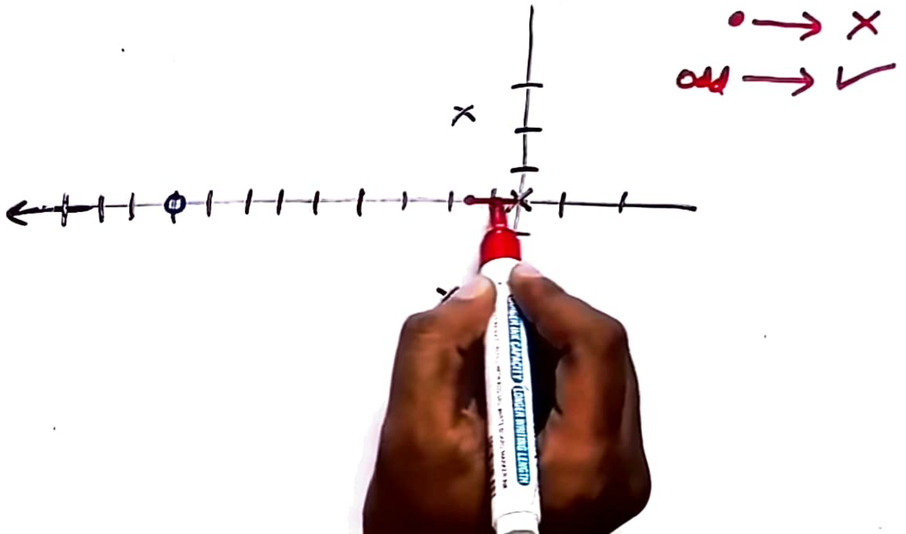
Zeros

$Z_1 = -9$



$$G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$$

step-2 – Find root locus on real axis



$$G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$$

$P_1 = 0$

$P_2 = -2 + 2.64j$

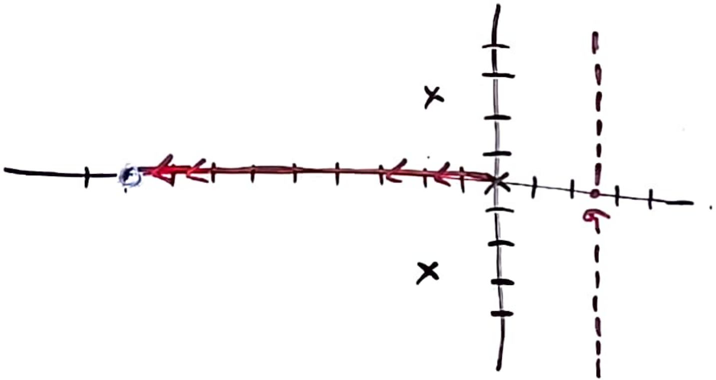
$P_3 = -2 - 2.64j$

$Z_1 = -9$

step-3 Find angle of asymptutes and centeroid

Angle of asymptotes $= \pm 90^\circ, \pm 270^\circ$

$\sigma = 2.5$



$X \rightarrow O$

∞

$$G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$$

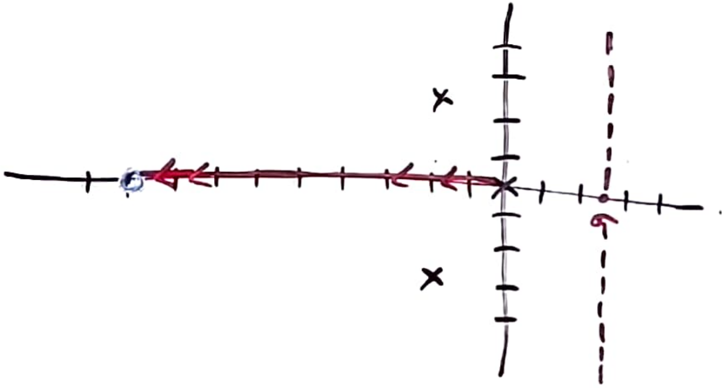
$P_1 = 0$

$P_2 = -2 + 2.64j$

$P_3 = -2 - 2.64j$

$Z_1 = -9$

step-4 find break away and break in point



$\times \to 0$

∞

$$G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$$

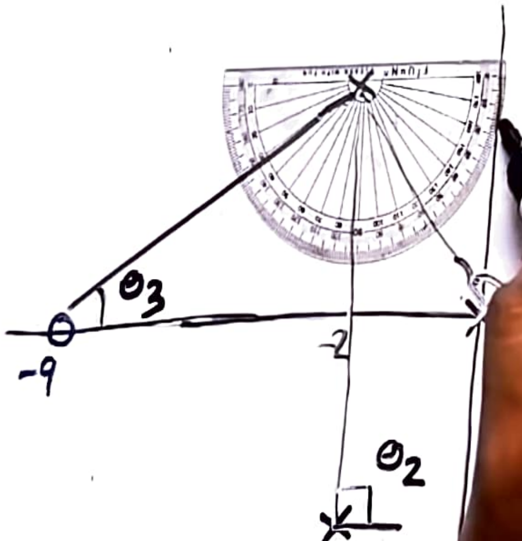
$P_1 = 0$

$P_2 = -2 + 2.64j$

$P_3 = -2 - 2.64j$

$Z_1 = -9$

Step-5 Find angle of departure



$$= 180 - (\theta_1 + \theta_2) + (\theta_3)$$

$$= 180 - (127 + 90) + 21 = -17^\circ$$

$$s(s^2+4s+11)$$

$P_2 = -2 + 2.64j$

$P_3 = -2 - 2.64j$

Find crossing point on imaginary axis

$$C.E = s(s^2+4s+11)+k(s+9) = 0$$

$$s^3+4s^2+11s+ks+9k = 0$$

$$s^3+4s^2+(11+k)s+9k = 0$$

RT

s^3	1	$11+k$
s^2	4	$9k$
s^1	$\frac{44-5k}{4}$	
s^0	$9k$	

$$\frac{44-5k}{4} = 0$$

$$44-5k=0$$

$$44 = 5k$$

$$k = \frac{44}{5} = 8.8$$

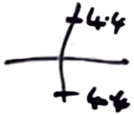
$$A \Rightarrow 4s^2+9k=0$$

$$4s^2 = -9k$$

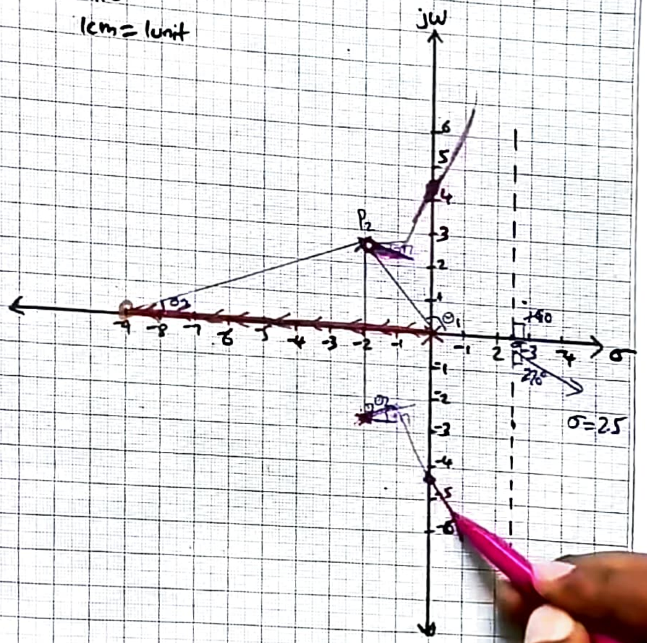
$$4s^2 = -79.2$$

$$s^2 = -79.2$$

$$\omega = \pm 4.4$$



scale
1cm = 1unit



Bandwidth of second order control system | Bandwidth formula | Ban

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Bandwidth (ω_b) :-

$$u = \frac{\omega}{\omega_n}$$

When $u = u_b$, the magnitude M of the closed loop system is $\frac{1}{\sqrt{2}}$ (or -3db)

$$M = \frac{1}{\left[(1-u_b^2)^2 + 4\xi^2 u_b^2\right]^{1/2}} = \frac{1}{\sqrt{2}}$$

$$\left[(1-u_b^2)^2 + (4\xi^2 u_b^2)\right]^{\cancel{1/2} \times \cancel{2}} = \sqrt{2}^{\times 2} \qquad (2)^{\cancel{1/2} \times \cancel{2}}$$

$$(1-u_b^2)^2 + 4\xi^2 u_b^2 = 2$$

$$1 + u_b^4 - 2u_b^2 + 4\xi^2 u_b^2 = 2$$

$$1 + u_b^4 - 2u_b^2(1 - 2\xi^2) - 2 = 0$$

$$u_b^4 - 2u_b^2(1-2\xi^2) - 1 = 0$$

Let $x = u_b^2$ $\quad x^2 - 2(1-2\xi^2)x - 1 = 0$

7:33 / 11:29

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Phase Margin formula(Second order System)

Bandwidth (ω_b) :-

$$u = \frac{\omega}{\omega_n}$$

When $u = u_b$, the magnitude M of the closed loop system is $\frac{1}{\sqrt{2}}$ (or -3db)

$$M = \frac{1}{\left[(1-u_b^2)^2 + 4\xi^2 u_b^2\right]^{1/2}} = \frac{1}{\sqrt{2}}$$

$$\left[(1-u_b^2)^2 + (4\xi^2 u_b^2)\right]^{\cancel{1/2} \times \cancel{2}} = \sqrt{2}^{\times 2} \qquad (2)^{\cancel{1/2} \times \cancel{2}}$$

$$(1-u_b^2)^2 + 4\xi^2 u_b^2 = 2$$

$$1 + u_b^4 - 2u_b^2 + 4\xi^2 u_b^2 = 2$$

$$1 + u_b^4 - 2u_b^2(1 - 2\xi^2) - 2 = 0$$

$$u_b^4 - 2u_b^2(1-2\xi^2) - 1 = 0$$

Let $x = u_b^2$ $\qquad x^2 - 2(1-2\xi^2)x - 1 = 0$

$$x^2 - 2(1-2\varepsilon^2)x - 1 = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\therefore\ x = \frac{-[-2(1-2\varepsilon^2)] \pm \sqrt{(-2(1-2\varepsilon^2))^2 - 4\times 1\times -1}}{2\times 1}$$

$a = 1$

$b = 2\ 1-2\varepsilon$

$$x = \frac{2(1-2\varepsilon^2) \pm \sqrt{4(1-2\varepsilon^2)^2+4}}{2}$$

$$x = \frac{2(1-2\varepsilon^2) \pm 2\sqrt{(1-2\varepsilon^2)^2+1}}{2}$$

$$x = (1-2\varepsilon^2) \pm \sqrt{1+4\varepsilon^4-4\varepsilon^2+1}$$

$$x = (1-2\varepsilon^2) \pm \sqrt{2-4\varepsilon^2+4\varepsilon^4}$$

Let us take only the +ve sign

$$\therefore\ x = (1-2\varepsilon^2) + \sqrt{2-4\varepsilon^2+4\varepsilon^4}$$

$\underline{u_b = \sqrt{x}}$ $\therefore$ $u_b = \sqrt{1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}}$

$u_b^2 = x$

$u_b = \frac{\omega_b}{\omega_n}$ $\frac{\omega_b}{\omega_n} = \sqrt{1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}}$

$$\boxed{\omega_b = \omega_n \left[1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4} \right]^{1/2}}$$

Phase Margin(γ) :-

The open loop transfer function of second order system,

$$G(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)}$$

Let $s = j\omega$

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)(j\omega + 2\xi\omega_n)}$$

The plots of $\left[1+j2\xi\left(\frac{\omega}{\omega_n}\right)-\left(\frac{\omega}{\omega_n}\right)^2\right]$, ... similar to the plots of complex conjugate poles, i.e.

$$\frac{1}{\left[1+j2\xi\left(\frac{\omega}{\omega_n}\right)-\left(\frac{\omega}{\omega_n}\right)^2\right]}$$

7.4.2 General Procedure for Constructing Bode Plots

The following steps are generally involved in constructing the Bode plot for a given $G(j\omega)H(j\omega)$.

1. Rewrite the sinusoidal transfer function as a product of basic factors in time constant form as discussed above.
2. Identify the corner frequencies associated with each one of these basic factors.
3. Knowing the corner frequencies, draw the asymptotic magnitude plot. This plot consists of straight line segments with line slope changing at each corner frequency by +20 dB/decade for a zero and –20 dB/decade for a pole ($\pm 20m$ dB/decade for a zero or pole of multiplicity m). For a complex conjugate zero or pole the slope changes by ± 40 dB/decade.
4. From the error graphs of Figure 7.25, determine the corrections to be applied to the asymptotic plot.
5. Draw a smooth curve through the corrected points such that it is asymptotic to the line segments. This gives the actual log magnitude plot.
6. Calculate the total phase angle of $G(j\omega)H(j\omega)$ at different frequencies and plot the resultant phase plot.

drawn from the point $(-1 + j0)$. Thus, the encirclement of the origin by the contour Γ_q is equivalent to the encirclement of the point $(-1 + j0)$ by the contour Γ_{GH} as shown in Figure 8.7.

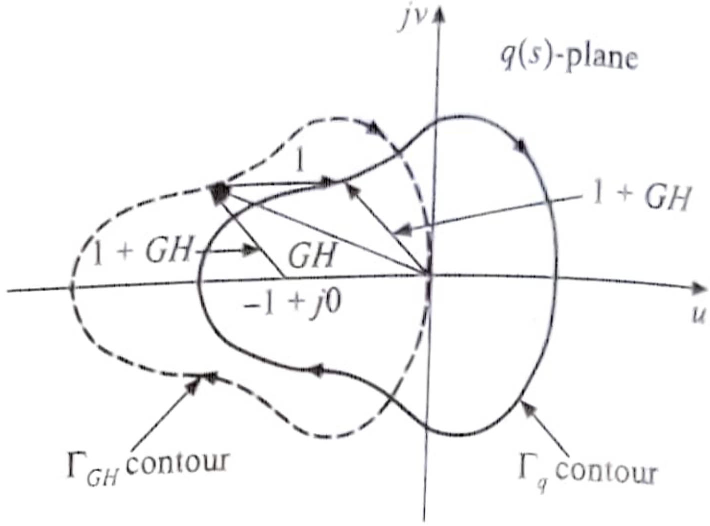

Figure 8.7 Γ_{GH} and Γ_q contours.

The Nyquist stability criterion may now be stated as follows:

If the contour Γ_{GH} of the open-loop transfer function $G(s)H(s)$ corresponding to the Nyquist contour in the s-plane encircles the point $(-1 + j0)$ in the counterclockwise direction as many times as the number of right-half s-plane poles of $G(s)H(s)$, the closed-loop system is stable.

In the commonly occurring case of the open-loop stable systems, the closed-loop system is stable if the contour Γ_{GH} of $G(s)H(s)$ does not encircle the $(-1 + j0)$ point, i.e. the net encirclement is zero.

The mapping of the Nyquist contour into the contour Γ_{GH} is carried out as follows:

1. The mapping of the imaginary axis is carried out by substitution of $s = j\omega$ in $G(s)H(s)$. This converts the mapping function into a frequency function of $G(j\omega)H(j\omega)$.
2. In physical systems $(m \leq n)$,

$$\underset{\substack{s \to Re^{j\phi} \\ R \to \infty}}{\mathrm{Lt}} G(s)H(s) = \text{real constant (it is zero if } m < n)$$

Thus, the infinite arc of the Nyquist contour maps into a point on the real axis.

The complete contour Γ_{GH} is thus the polar plot of $G(j\omega)H(j\omega)$ with ω varying from $-\infty$ to ∞. This is usually called the Nyquist plot or locus of $G(s)H(s)$. The Nyquist plot is symmetrical about the real axis since $G^*(j\omega)H^*(j\omega) = G(-j\omega)H(-j\omega)$.

Example 8.1 Draw the Nyquist plot and comment on the stability of the system represented by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\left[\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau\right]$$

In terms of STM,

$$\mathbf{x}(t) = \phi(t)\mathbf{x}(0) + \int_0^t \phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

If the initial time is t_0 instead of $t = 0$, the solution of the non-homogeneous state equation, also called the state transition equation becomes

$$\mathbf{x}(t) = \phi(t-t_0)\mathbf{x}(t_0) + \int_0^t \phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Significance of the STM: Since the STM satisfies the homogeneous state equations, it represents the free response of the system. In other words, it governs the response that is excited by the initial conditions only. The STM is dependent only on the system matrix **A**, and therefore it is sometimes referred to as the STM of **A**. As the name implies, the STM describes the change of state from the initial time $t = 0$, to any time t, when the inputs are zero.

Properties of STM

1. $\phi(0) = \mathbf{I}$, *Proof:* $\phi(0) = e^{\mathbf{A}\times 0} = \mathbf{I}$

2. $\phi^{-1}(t) = \phi(-t)$ *Proof:* $\phi^{-1}(t) = \dfrac{1}{\phi(t)} = \dfrac{1}{e^{\mathbf{A}t}} = e^{-\mathbf{A}t} = \phi(-t)$

3. $\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0)$ for any t_2, t_1, t_0

   Proof: $[\phi(t_2 - t_1)\phi(t_1 - t_0) = e^{\mathbf{A}(t_2-t_1)} \cdot e^{\mathbf{A}(t_1-t_0)} = e^{\mathbf{A}(t_2-t_1+t_1-t_0)} = e^{\mathbf{A}(t_2-t_0)} = \phi(t_2 - t_0)]$

4. $[\phi(t)]^k = \phi(kt)$ *Proof:* $[\phi(t)]^k = \phi(t)\cdot\phi(t)\cdots k \text{ times} = e^{\mathbf{A}t}\cdot e^{\mathbf{A}t}\cdots k \text{ times} = e^{\mathbf{A}kt} = \phi(kt)$

5. $\phi(t_1 + t_2) = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

   Proof: $\phi(t_1 + t_2) = e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}\cdot e^{\mathbf{A}t_2} = e^{\mathbf{A}t_2}\cdot e^{\mathbf{A}t_1} = \phi(t_1)\cdot\phi(t_2) = \phi(t_2)\cdot\phi(t_1)$

$$= \phi(s)\ \mathbf{x}(0) + \phi(s)\ \mathbf{BU}(s)$$

$$= \phi(s)\ [\mathbf{x}(0) + \mathbf{BU}(s)]$$

Taking the inverse Laplace transform on both sides,

$$\mathbf{x}(t) = \mathbf{L}^{-1}[\phi(s)\ [\mathbf{x}(0) + \mathbf{BU}(s)]]$$

$$= \phi(t)\ [\mathbf{x}(0)] + \mathbf{L}^{-1}\ [\phi(s)\ \mathbf{BU}(s)]$$

Applying convolution theorem,

$$\mathbf{x}(t) = \phi(t)\mathbf{x}(0) + \int_0^t \phi(t-\tau)\,\mathbf{Bu}(\tau)\,d\tau$$

$$= \phi(t)\left[\mathbf{x}(0) + \int_0^t \phi(-\tau)\,\mathbf{Bu}(\tau)\,d\tau\right]$$

This solution of linear non-homogeneous state equation is called the state transition equation.

Example 10.23 Obtain the STM for the state model whose **A** matrix is given by

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ (c) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Solution: (a) For the given system matrix **A**

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}$$

$$\therefore \quad \phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\text{adj}\,[\text{sI} - \text{A}]}{|\text{sI} - \text{A}|} = \frac{\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}^T}{(s-1)^2} = \frac{\begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}}{(s-1)^2}$$

$$\therefore \quad \text{STM} = \phi(t) = L^{-1}[\phi(s)] = L^{-1}[s\mathbf{I} - \mathbf{A}]^{-1} = L^{-1}\begin{bmatrix} \dfrac{1}{s-1} & \dfrac{1}{(s-1)^2} \\ 0 & \dfrac{1}{s-1} \end{bmatrix} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

(b) For the given system matrix **A**

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}$$

$$\phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\text{Adj}\,[s\mathbf{I} - \mathbf{A}]}{|s\mathbf{I} - \mathbf{A}|} = \frac{\begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix}^T}{|s\mathbf{I} - \mathbf{A}|} = \frac{\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}}{(s+1)^2}$$

$$\text{STM} = \boldsymbol{\phi}(t) = L^{-1}\boldsymbol{\phi}(s) = L^{-1}[s\mathbf{I} - \mathbf{A}]^{-1} = L^{-1}\begin{bmatrix} \dfrac{s+2}{(s+1)^2} & \dfrac{1}{(s+1)^2} \\ \dfrac{-1}{(s+1)^2} & \dfrac{s}{(s+1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

(c) For the given system matrix **A**

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\boldsymbol{\phi}(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\text{Adj}\,[s\mathbf{I} - \mathbf{A}]}{|s\mathbf{I} - \mathbf{A}|} = \frac{\begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}^T}{|s\mathbf{I} - \mathbf{A}|} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{s^2 + 3s + 2}$$

$$\therefore \quad \text{STM} = \boldsymbol{\phi}(t) = L^{-1}\boldsymbol{\phi}(s) = L^{-1}[s\mathbf{I} - \mathbf{A}]^{-1} = L^{-1}\begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} L^{-1}\left(\dfrac{s+3}{(s+1)(s+2)}\right) & L^{-1}\left(\dfrac{1}{(s+1)(s+2)}\right) \\ L^{-1}\left(\dfrac{-2}{(s+1)(s+2)}\right) & L^{-1}\left(\dfrac{s}{(s+1)(s+2)}\right) \end{bmatrix}$$

$$= \begin{bmatrix} L^{-1}\left(\dfrac{2}{s+1} - \dfrac{1}{s+2}\right) & L^{-1}\left(\dfrac{1}{s+1} - \dfrac{1}{s+2}\right) \\ L^{-1}\left(\dfrac{-2}{s+1} + \dfrac{2}{s+2}\right) & L^{-1}\left(\dfrac{-1}{s+1} + \dfrac{2}{s+2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

10.7.3 Computation of the STM using Cayley–Hamilton Theorem

The STM may be computed using the technique based on the Cayley–Hamilton theorem. For large systems this method is far more convenient computationally compared to other methods.

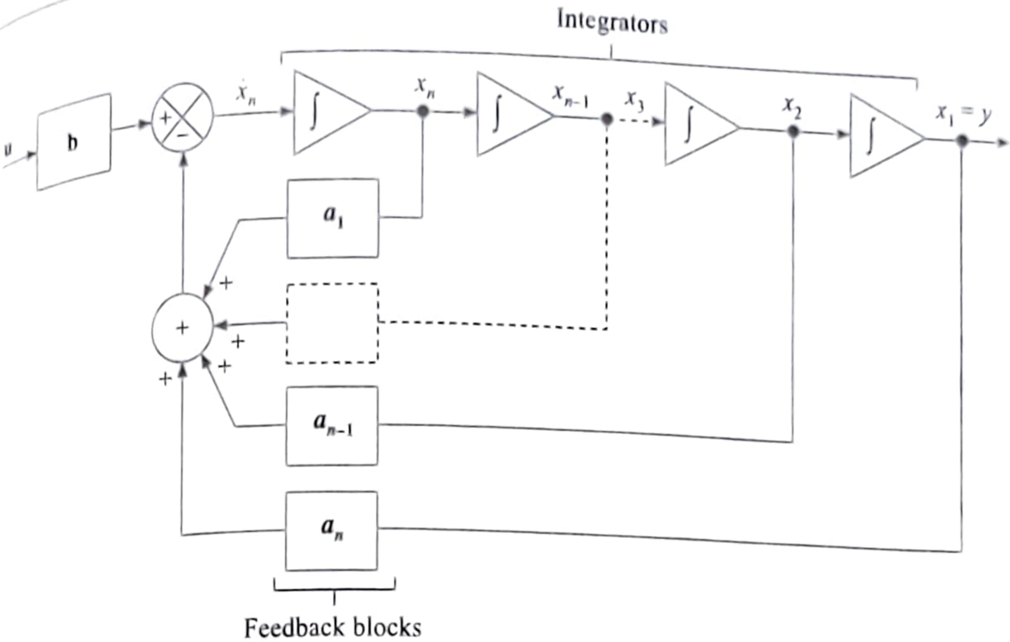

Figure 10.15 Block diagram representation of the state model for $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$ and $y = \mathbf{CX}$.

Example 10.10 Obtain a state model for the system described by

$$T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3 + 6s^2 + 10s + 5}$$

Solution: The differential equation corresponding to the given transfer function is obtained by cross-multiplying and taking the inverse Laplace transform. So, we have

$$\dddot{y} + 6\ddot{y} + 10\dot{y} + 5y = u$$

Since the derivatives of the input are not present in the differential equation, phase variables can be selected as the state variables. Therefore,

$$x_1 = y \qquad \text{i.e.} \qquad y = x_1$$
$$x_2 = \dot{y} = \dot{x}_1 \qquad \dot{x}_1 = x_2$$
$$x_3 = \ddot{y} = \dot{x}_2 \qquad \dot{x}_2 = x_3$$
$$\dddot{y} = -6\ddot{y} - 10\dot{y} - 5y + u \qquad \dot{x}_3 = -5x_1 - 10x_2 - 6x_3 + u$$

$x_3 = \ddot{y}$

$\dot{x}_3 = \dddot{y}$

Therefore, the state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -10 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 10.11 Obtain a state model of the system described by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{5}{s^3 + 6s + 7}$$

Solution: The transfer function does not have any zeros. So the matrix **A** will be in bush (companion) form with the elements in the last row as −7, −6, 0. The matrix **B** has the last element as 5 and all other elements as zeros.

Expressing the system in terms of the differential equation by cross-multiplying the terms of the transfer function and taking the inverse Laplace transform, we have

$$\dddot{y} + 6\dot{y} + 7y = 5u$$

Define the state variables as

$$x_1 = y.$$
$$x_2 = \dot{y} = \dot{x}_1 .$$
$$x_3 = \ddot{y} = \ddot{x}_1 = \dot{x}_2$$

Equating the highest-order term $\dddot{y}$ to all other terms in the differential equation, we have

$$\dddot{y} = -6\dot{y} - 7y + 5u$$

i.e.

$$\dot{x}_3 = -6x_2 - 7x_1 + 5u$$

So, the first-order differential equations constituting the state equations are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -7x_1 - 6x_2 + 5u$$

The output equation is

$$y = x_1$$

The state model based on the above equations is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The state model in vector-matrix form is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -5 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ -4\\ 25\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Example 10.13 The transfer function of a control system is given by

$$\frac{Y(s)}{U(s)}=\frac{s^2+3s+4}{s^3+2s^2+3s+2}$$

Obtain a state model.

Solution: The corresponding differential equation obtained by cross-multiplying and taking the inverse Laplace transform is

$$\dddot{y}+2\ddot{y}+3\dot{y}+2y=\ddot{u}+3\dot{u}+4u$$

Comparing this differential equation with the standard differential equation of a third-order system, we have

$$\dddot{y}+a_1\ddot{y}+a_2\dot{y}+a_3y=b_0\dddot{u}+b_1\ddot{u}+b_2\dot{u}+b_3u$$

i.e. $\quad a_1=2,\ a_2=3,\ a_3=2 \quad$ and $\quad b_0=0,\ b_1=1,\ b_2=3,\ b_3=4$

Therefore,

$$\beta_0=b_0=0$$

$$\beta_1=b_1-a_1\beta_0=1-2\times 0=1$$

$$\beta_2=b_2-a_2\beta_0-a_1\beta_1=3-3\times 0-2\times 1=1$$

$$\beta_3=b_3-a_3\beta_0-a_2\beta_1-a_1\beta_2=4-2\times 0-3\times 1-2\times 1=-1$$

$b_1 - a_1\beta_0$

$b_2 - a_2\beta_0 - a_1\beta_1$

$b_3 - a_3\beta_0 - a_2\beta_1$

The state variables are as follows:

$$x_1=y-\beta_0u$$

$$x_2=\dot{x}_1-\beta_1u$$

$$x_3=\dot{x}_2-\beta_2u$$

Also $\quad \dot{x}_3=-a_3x_1-a_2x_2-a_1x_3+\beta_3u$

2

The state and output equations are as follows:

$$y=x_1+\beta_0u=x_1$$

$$\dot{x}_1=x_2+\beta_1u=x_2+u$$

$$\dot{x}_2=x_3+\beta_2u=x_3+u$$

$$\dot{x}_3=-2x_1-3x_2-2x_3-u$$

Hence the state model is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & -3 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

An alternative method using signal flow graphs is presented as follows:

Alternative way of obtaining the state model using signal flow graph when the transfer functions have poles and zeros: Let us consider an nth-order transfer function

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

Let the state variables be $x_1, x_2, \ldots, x_n$.

The signal flow graph must have n integrators. By dividing the numerator and denominator of $T(s)$ by s^n, the above transfer function may be rearranged as

$$T(s) = \frac{b_0 + b_1 s^{-1} + b_2 s^{-2} + \ldots + b_{n-1} s^{-n+1} + b_n s^{-n}}{1 + a_1 s^{-1} + a_2 s^{-2} + \ldots + a_{n-1} s^{-n+1} + a_n s^{-n}}$$

$$= \frac{b_0 + b_1 s^{-1} + b_2 s^{-2} + \ldots + b_{n-1} s^{-n+1} + b_n s^{-n}}{1 - (-a_1 s^{-1} - a_2 s^{-2} - \ldots - a_{n-1} s^{-n+1} - a_n s^{-n})}$$

Earlier we have seen that the transfer function and signal flow graph are related by Mason's gain formula:

$$T(s) = \sum_k \frac{M_k \Delta_k}{\Delta}$$

where M_k = the path gain of the kth forward path

Δ (Determinant of the signal flow graph)

= 1 – (sum of loop gains of all individual loops)

+ (sum of gain products of all possible combinations of two non touching loops)

– (sum of gain products of all possible combinations of three non touching loops) +...

Δ_k = the value of Δ for that part of the graph not touching the kth forward path.

Comparing the above expressions for $T(s)$, we observe that the signal flow graph for $T(s)$ may consist of

1. n feedback loops (touching each other) with gains $-a_1 s^{-1}, -a_2 s^{-2}, \ldots, -a_{n-1} s^{-n+1}, -a_n s^{-n}$
2. $n + 1$ forward paths which touch the loops and have gains $b_0, b_1 s^{-1}, b_2 s^{-2}, \ldots, b_{n-1} s^{-n+1}, b_n s^{-n}$

The signal flow graph configuration which satisfies the above requirements is shown in Figure 10.16.

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s + a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s + a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s + a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2 + \omega^2}$
$\cos(\omega t + \theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2 + \omega^2}$
$e^{-at} \sin \omega t$	ω