COMBINATORICS AND GRAPH THEORY

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Abstract

In this document, we derive the formula for the outer-independent Roman domination number of the lexicographic product $P_3 \circ P_n$. A detailed proof is presented, along with examples and step-by-step calculations for specific values of n. Graphs are drawn for visualization and correctness verification.

1. Introduction

The concept of the outer-independent Roman domination function (OIRDF) is significant in ensuring efficient domination with specific independence constraints. In this work,

we focus on the lexicographic product $P_3 \circ P_n$ and derive its outer-independent Roman domination number $\gamma_{oiR}(P_3 \circ P_n)$.

2. Definitions and Notations

Definition 1 (Lexicographic Product of Graphs)**.** *The lexicographic product $G \circ H$ of graphs G and H is a graph such that:*

- *The vertex set is $V(G) \times V(H)$.*
- *Two vertices (g_1, h_1) and (g_2, h_2) are adjacent if:*

$$g_1 = g_2 \text{ and } h_1 \text{ is adjacent to } h_2 \text{ in } H, \text{ or } g_1 \text{ is adjacent to } g_2 \text{ in } G.$$

Definition 2 (Outer-Independent Roman Domination)**.** *An outer-independent Roman dominating function (OIRDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that:*

1. *The set of vertices V_0 (where $f(v) = 0$) is independent.*
2. *Every vertex $v \in V_0$ is adjacent to at least one vertex $u \in V_2$ (where $f(u) = 2$). The weight of f is:*

$$w(f) = \sum_{v \in V} f(v),$$

and the outer-independent Roman domination number $\gamma_{oiR}(G)$ is the minimum weight of f.

3. Problem Statement

We aim to find $\gamma_{oiR}(P_3 \circ P_n)$, where P_3 is a path graph with vertices $\{a, b, c\}$ and edges $a - b$ and $b - c$, and P_n is a path graph with n vertices.

4. Detailed Proof

4.1. Graph Structure of $P_3 \circ P_n$

The lexicographic product $P_3 \circ P_n$ has three disjoint P_n-copies corresponding to the ver- tices a, b, c of P_3:

- $P_n(a)$: Copy corresponding to a.
- $P_n(b)$: Copy corresponding to b.
- $P_n(c)$: Copy corresponding to c.

Every vertex in $P_n(b)$ (central P_n) connects to all vertices in $P_n(a)$ and $P_n(c)$, ensuring domination.

4.2. Construction of OIRDF

1. Assign $f(v) = 2$ to $\lceil n/3 \rceil$ vertices in $P_n(b)$ (central P_n) to dominate both $P_n(a)$ and $P_n(c)$.
2. Assign $f(v) = 1$ to all vertices in $P_n(a)$ and $P_n(c)$, ensuring the independence of V_0.
3. Assign $f(v) = 0$ to remaining vertices in $P_n(b)$ such that V_0 forms an independent set.

4.3. Formula

The minimum weight of the OIRDF is:

$$\gamma_{oiR}(P_3 \circ P_n) = \lceil n/3 \rceil + n + 2 \cdot \lfloor n/2 \rfloor.$$

4.4. Proof by Induction

We prove the formula for the outer-independent Roman domination number of the lexicographic product $P_3 \circ P_n$:

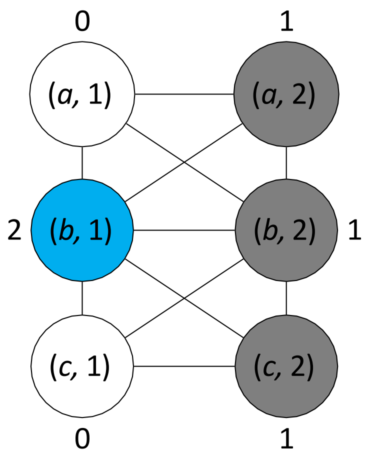
$$\gamma_{oiR}(P_3 \circ P_n) = \lceil n/3 \rceil + n + 2 \cdot \lfloor n/2 \rfloor$$

using induction.

Base Cases

Case $n = 2$**:**

- Assign $f(v) = 2$ to the vertex $(b, 1)$ in $P_n(v_2)$, ensuring it dominates all vertices in $P_n(v_1)$ and $P_n(v_3)$.
- Assign $f(v) = 1$ to the vertices $(a, 2)$, $(b, 2)$, $(c, 2)$, forming an independent set V_0.



The total weight is:

$$w(f) = 2 + 1 + 1 + 1 = 5.$$

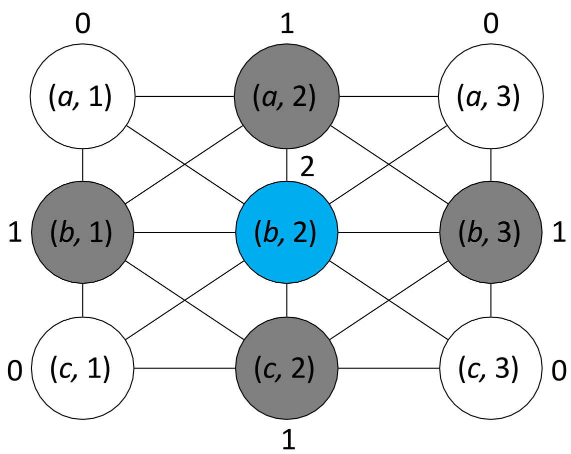
Substituting $n = 2$ in the formula:

$$\gamma_{oiR}(P_3 \circ P_2) = \lceil 2/3 \rceil + 2 + 2 \cdot \lfloor 2/2 \rfloor = 1 + 2 + 2 = 5.$$

The formula holds for $n = 2$.

Case $n = 3$**:**

- Assign $f(v) = 2$ to the vertex $(b, 2)$, ensuring it dominates all vertices in $P_n(v_1)$ and $P_n(v_3)$.
- Assign $f(v) = 1$ to $(a, 2)$, $(b, 3)$, $(c, 2)$, $(b, 1)$ and all other vertices in $P_n(v_2)$ to form an independent set V_0.



The total weight is:

$$w(f) = 2 + 1 + 1 + 1 + 1 + 1 = 8.$$

Substituting $n = 3$ in the formula:

$$\gamma_{oiR}(P_3 \circ P_3) = \lceil 3/3 \rceil + 3 + 2 \cdot \lfloor 3/2 \rfloor = 1 + 3 + 4 = 8.$$

The formula holds for $n = 3$.

Inductive Hypothesis

Assume the formula holds for $n = k$:

$$\gamma_{oiR}(P_3 \circ P_k) = \lceil k/3 \rceil + k + 2 \cdot \lfloor k/2 \rfloor.$$

Inductive Step

We prove the formula holds for $n = k + 1$.

Graph Structure: When transitioning from P_k to P_{k+1}:

- A new vertex group $P_n(v_{k+1})$ is added.
- Additional vertices require domination, and independence of V_0 must be maintained.

Assignment for $n = k + 1$:

- Assign $f(v) = 2$ to $\lceil (k + 1)/3 \rceil$ vertices in $P_n(v_2)$, dominating both:
  - New vertices in $P_n(v_1)$ and $P_n(v_3)$.
  - All internal vertices of $P_n(v_2)$.
- Assign $f(v) = 1$ to all vertices in $P_n(v_1)$ and $P_n(v_3)$, ensuring:
  - Domination is maintained.
  - V_0 remains independent.

Weight Analysis: The weight for P_{k+1} is updated as follows:

$$\gamma_{oiR}(P_3 \circ P_{k+1}) = \lceil (k + 1)/3 \rceil + (k + 1) + 2 \cdot \lfloor (k + 1)/2 \rfloor.$$

Verification:

The derived formula matches the construction for $n = k + 1$. Thus, the formula holds for $n = k + 1$.

Conclusion

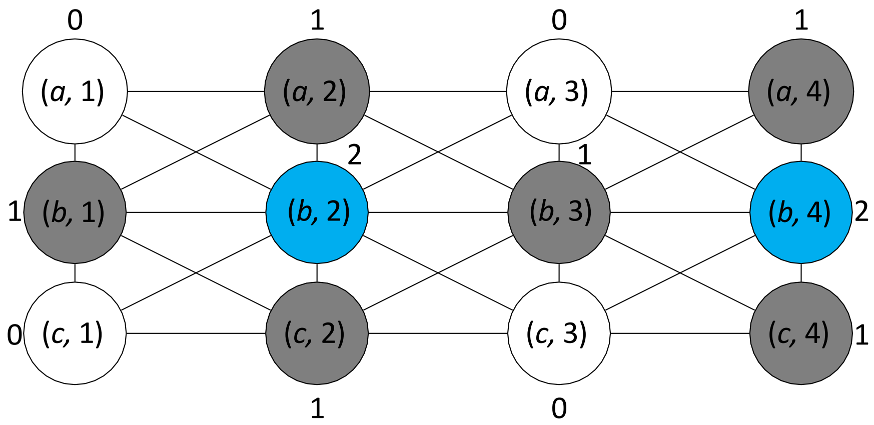
By the principle of mathematical induction, the formula:

$$\gamma_{oiR}(P_3 \circ P_n) = \lceil n/3 \rceil + n + 2 \cdot \lfloor n/2 \rfloor$$

is valid for all $n \geq 2$.

5. Examples

5.1. P_4

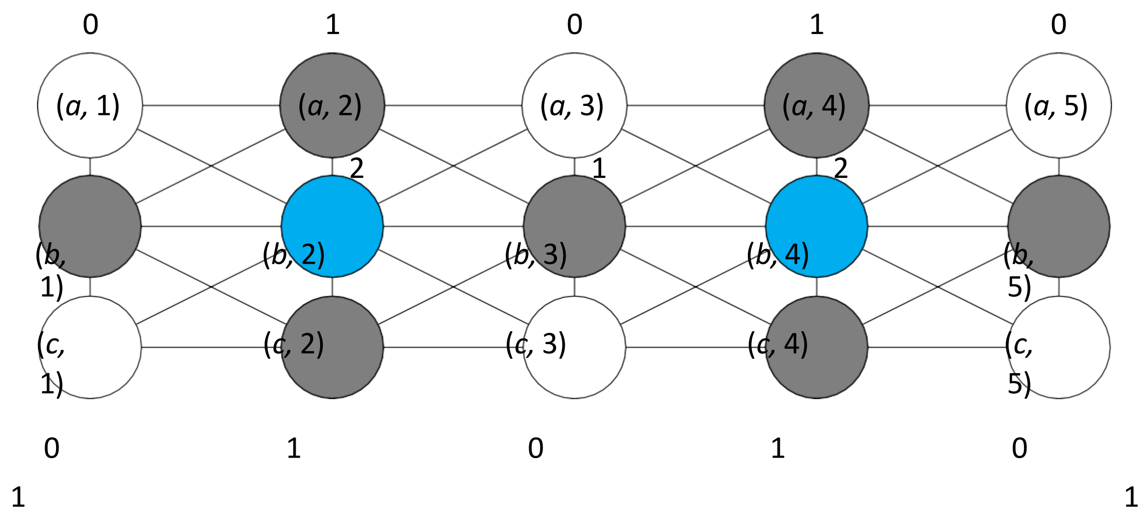


For $P_3 \circ P_4$:

$$\gamma_{oiR}(P_3 \circ P_4) = \lceil 4/3 \rceil + 4 + 2 \cdot \lfloor 4/2 \rfloor = 10.$$

by observing at the graph also we can say that weight of the graph is 10 hence the given formula satisfies.

5.2. p_5



For $P_3 \circ P_5$:

from the graph we can observe that minimum weight of the OIDF is 11

$$\gamma_{oiR}(P_3 \circ P_5) = \lceil 5/3 \rceil + 5 + 2 \cdot \lfloor 5/2 \rfloor = 11.$$

we can observe that we got same answer both by using formula and observation through graph.

5.3. p_6

For $P_3 \circ P_6$:

$$\gamma_{oiR}(P_3 \circ P_6) = \lceil 6/3 \rceil + 6 + 2 \cdot \lfloor 6/2 \rfloor = 14.$$

5.4. p_{10}

For $P_3 \circ P_{10}$:

$$\gamma_{oiR}(P_3 \circ P_{10}) = \lceil 10/3 \rceil + 10 + 2 \cdot \lfloor 10/2 \rfloor = 24.$$

5.5. p_{15}

For $P_3 \circ P_{15}$:

$$\gamma_{oiR}(P_3 \circ P_{15}) = \lceil 15/3 \rceil + 15 + 2 \cdot \lfloor 15/2 \rfloor = 34.$$

6. Conclusion

We have successfully derived the formula for $\gamma_{oiR}(P_3 \circ P_n)$. The correctness is verified using examples and step-by-step calculations for specific values of n.

7. BONUS:CORONA PRODUCT

7.1. definition

Corona Product of P_3 and P_n

The *Corona product* of two graphs G and H, denoted by $G \circ H$, is obtained by replacing each vertex of G with a copy of H, and then adding edges from the neighbors of v in G to every vertex in the copy of H corresponding to v.

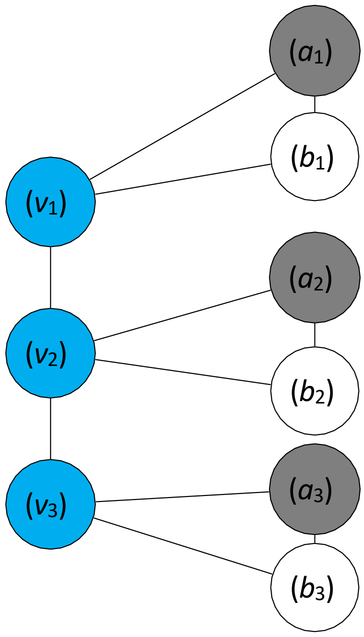
For the Corona product of P_3 (a path on 3 vertices) and P_n (a path on n vertices), the construction is as follows:

1. Take P_3, a path with 3 vertices:

$$P_3 = v_1 - v_2 - v_3$$

2. For each vertex v_i in P_3, replace it with a copy of P_n. So, we get a graph where each v_i is replaced by P_n.

3. For each edge $v_i - v_j$ in P_3, add edges from the neighbors of v_i to all vertices in the copy of P_n corresponding to v_i.



now we assign v_1=v_2=v_3=2 (because it is connected to every other vertex of p_n)and we assign a_1=a_2=a_3=1 to satisfy the OIDF condition

$$\gamma_{oiR}(P_3 \circ P_n) = 3 * 2 + 3 * \lfloor n/2 \rfloor.$$

substituting in formula we get

$$\gamma_{oiR}(P_3 \circ P_n) = 9$$

7.2. Proof by Induction

We considered the base case as $n = 2$, and now we assume this formula satisfies for every k.

Now, we check for $n = k + 1$.

For $n = k + 1$, we again assign 2 for v_n as it is connected to every other vertex of the P_n copy of n. The next step is assigning 1. Here, we assign 1 in such a way that it satisfies OIDF. Hence, we assign 1 to $\lfloor (k+1)/2 \rfloor$ vertices. The minimum weight is $3 \cdot 2 + \lfloor (k+1)/2 \rfloor$. Hence, it satisfies the formula, and the provided formula is proven correct.

8. FINAL RESULT

For lexicographic product :

$$\gamma_{oiR}(P_3 \circ P_n) = \lceil n/3 \rceil + n + 2 \cdot \lfloor n/2 \rfloor$$

For corona product :

$$\gamma_{oiR}(P_3 \circ P_n) = 3 \cdot 2 + 3 \cdot \lfloor n/2 \rfloor.$$