Outline

1. Introduction
2. The Bayes Filter
3. Gaussian filters
4. The Kalman filter

Labs:

SLAM Toolbox with Matlab

Assessment:

Labs + Exam (100%)

2.1 The Bayes Filter

- **The Bayes Filter plays principal role in probabilistic Robotics.**

1: **Algorithm Bayes_filter(**$bel(x_{t-1}), u_t, z_t$**):**
2: for all x_t do
3: $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$
4: $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$
5: endfor
6: return $bel(x_t)$

Total Probability Theorem

Bayes Rule

$bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$. **Robot belief of being at state x_t**

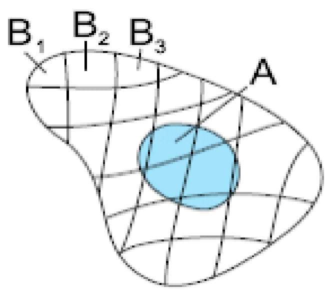
$\overline{bel}(x_t) = p(x_t \mid z_{1:t-1}, u_{1:t})$ **Prior Belief. Prediction of state x_t.**

$p(x_t \mid u_t, x_{t-1})$ **State Transition probability**

$p(z_t \mid x_t)$ **Measurement probability**

Recall …

- **Total Probability Theorem:** Let B be a partition of the sample space S



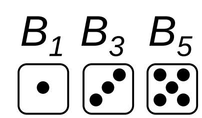
$$B = S \text{ and } B_i \cap B_j = \varnothing \ \forall \mathrm{i} \neq \mathrm{j}$$

$$A = \bigcup_{i}^{n} \left(A \cap B_i \right)$$

$$P(A) = \sum_{k=1}^{n} P(A / B_k) P(B_k)$$

- **Example:** "Imagine we Toss a die"

- What is the probability of getting odd?



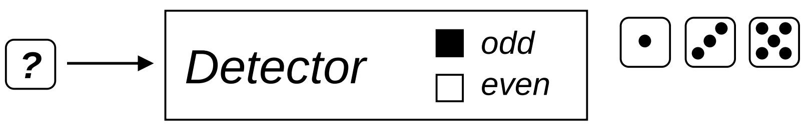
$$P(A_{odd}) = \sum_{k=1}^{6} P(A_{odd} / B_k) P(B_k) = \underbrace{1 \cdot \frac{1}{6}}_{B_1} + \underbrace{0 \cdot \frac{1}{6}}_{B_2} + \underbrace{1 \cdot \frac{1}{6}}_{B_3} + \underbrace{0 \cdot \frac{1}{6}}_{B_4} + \underbrace{1 \cdot \frac{1}{6}}_{B_5} + \underbrace{0 \cdot \frac{1}{6}}_{B_6} = \frac{3}{6}$$

Recall …

- **Bayes Rule:**

$$P(B_i / A) = \frac{P(A / B_i)P(B_i)}{P(A)}$$

- **Example:** Imagine we Toss a die"



- What is the probability of getting a 5 knowing we already got an odd number?

a posteri knowledge

$$P(B_5 / A_{odd}) = \frac{P(A_{odd} / B_5)P(B_5)}{P(A_{odd})} = \frac{1\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

Prior Knowledge

Measurement

Recall …

- **The Bayes Rule can also be written as:**

$$P(B_i / A) = \eta P(A / B_i) P(B_i) \quad , \quad \eta = \frac{1}{P(A)}$$

- **Then, η can be seen as a normalizer** easily computed if $P(A|B_i)$ and $P(B_i)$ are known

$$\left.\begin{aligned} \eta &= \frac{1}{P(A)} \\ P(A) &= \sum_i P(A / B_i) P(B_i) \end{aligned}\right\} \Rightarrow \eta = \frac{1}{\sum_i P(A / B_i) P(B_i)}$$

- **In our example**

$$\eta = \frac{1}{\sum_{i=1..6} P(A_{odd} / B_i) P(B_i)} = \frac{1}{1\frac{1}{6} + 0\frac{1}{6} + 1\frac{1}{6} + 0\frac{1}{6} + 1\frac{1}{6} + 0\frac{1}{6}} = \frac{1}{\frac{3}{6}} = 2$$

$$P(B_5 / A_{odd}) = \eta P(A_{odd} / B_5) P(B_5) = 2 \cdot \left(1 \cdot \frac{1}{6}\right) = \frac{1}{3}$$

2.1 The Bayes Filter

Algorithm Bayes_filter($\boxed{bel(x_{t-1}), u_t, z_t}$ **):**

for all x_t do

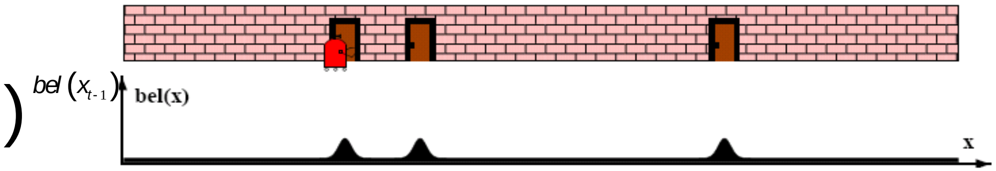
$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx$ *Total Probability Theorem*

$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$ *Bayes Rule*

endfor

return $bel(x_t)$

$$bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$$

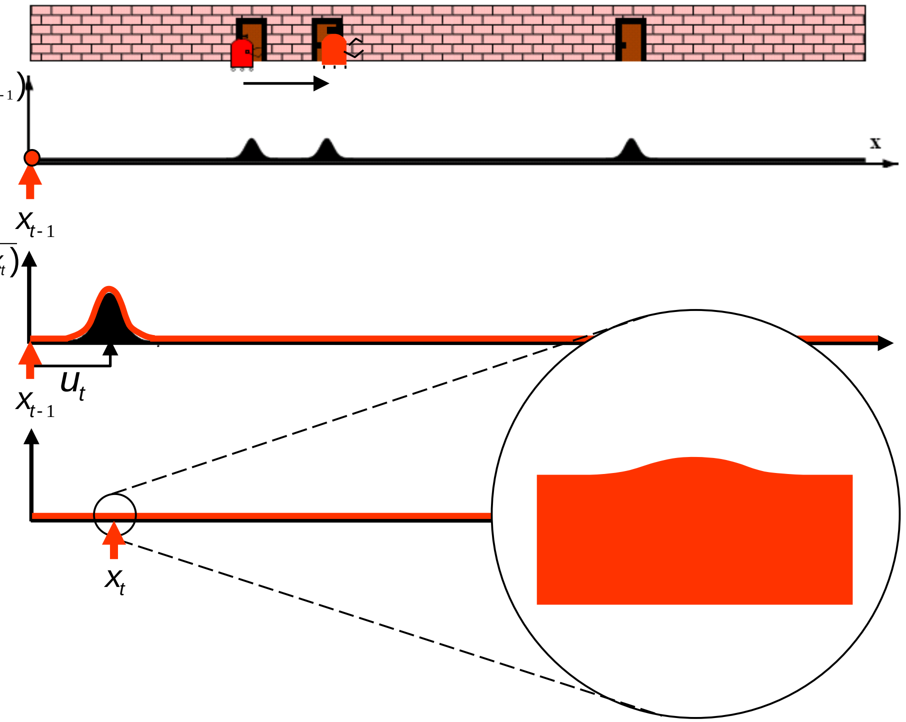


2.1 The Bayes Filter

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$):
for all x_t do
$\boxed{\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx}$ *Total Probability Theorem*
$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$ *Bayes Rule*
endfor
return $bel(x_t)$

$$bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$$

$$p(x_t \mid u_t, x_{t-1})$$



2.1 The Bayes Filter

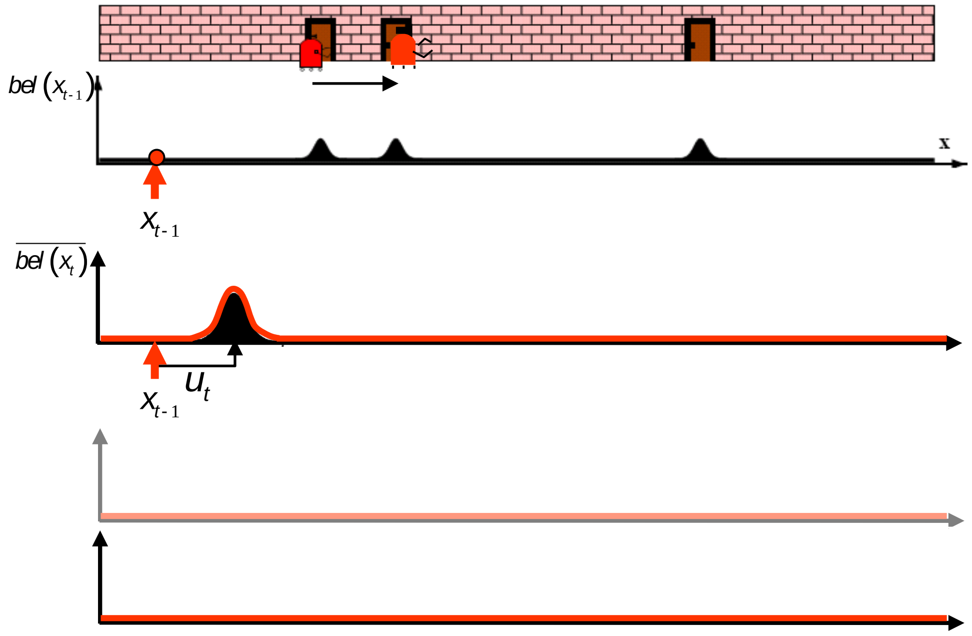
Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$):
for all x_t do
$\boxed{\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\; bel(x_{t-1})\; dx}$
$bel(x_t) = \eta\; p(z_t \mid x_t)\; \overline{bel}(x_t)$
endfor
return $bel(x_t)$

Total Probability Theorem

Bayes Rule

$$bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$$

$$p(x_t \mid u_t, x_{t-1})$$



2.1 The Bayes Filter

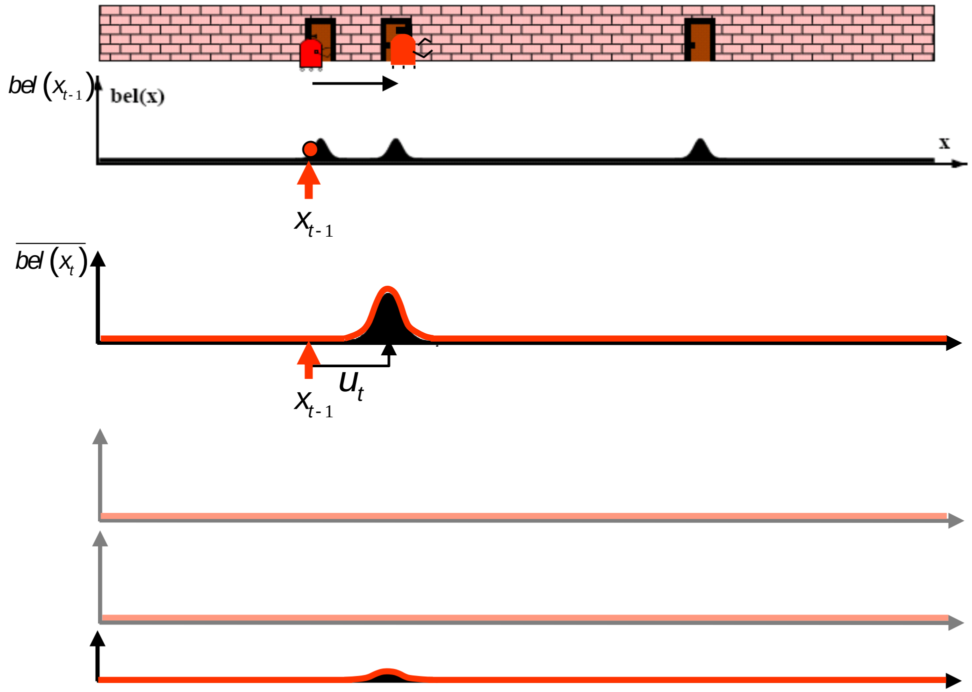
Algorithm Bayes_filter$(bel(x_{t-1}), u_t, z_t)$**:**
for all x_t do
$\boxed{\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx}$
$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$
endfor
return $bel(x_t)$

Total Probability Theorem

Bayes Rule

$$bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$$

$$p(x_t \mid u_t, x_{t-1})$$



2.1 The Bayes Filter

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$):

for all x_t do

$\boxed{\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx}$ *Total Probability Theorem*

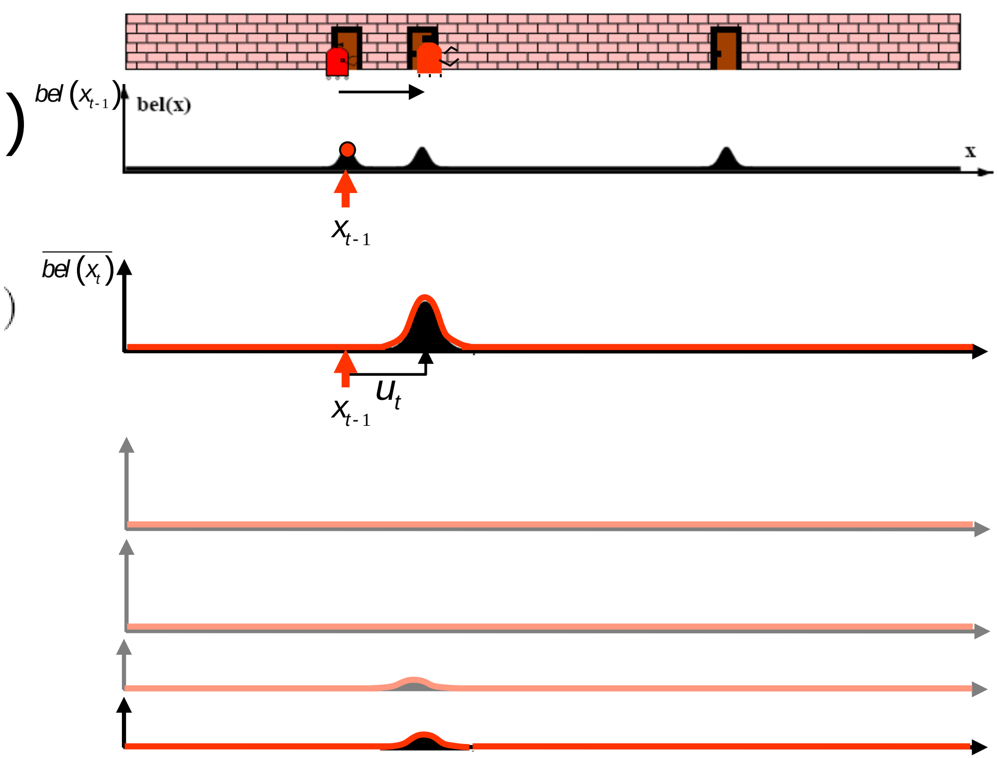
$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$ *Bayes Rule*

endfor

return $bel(x_t)$

$$bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$$

$$p(x_t \mid u_t, x_{t-1})$$

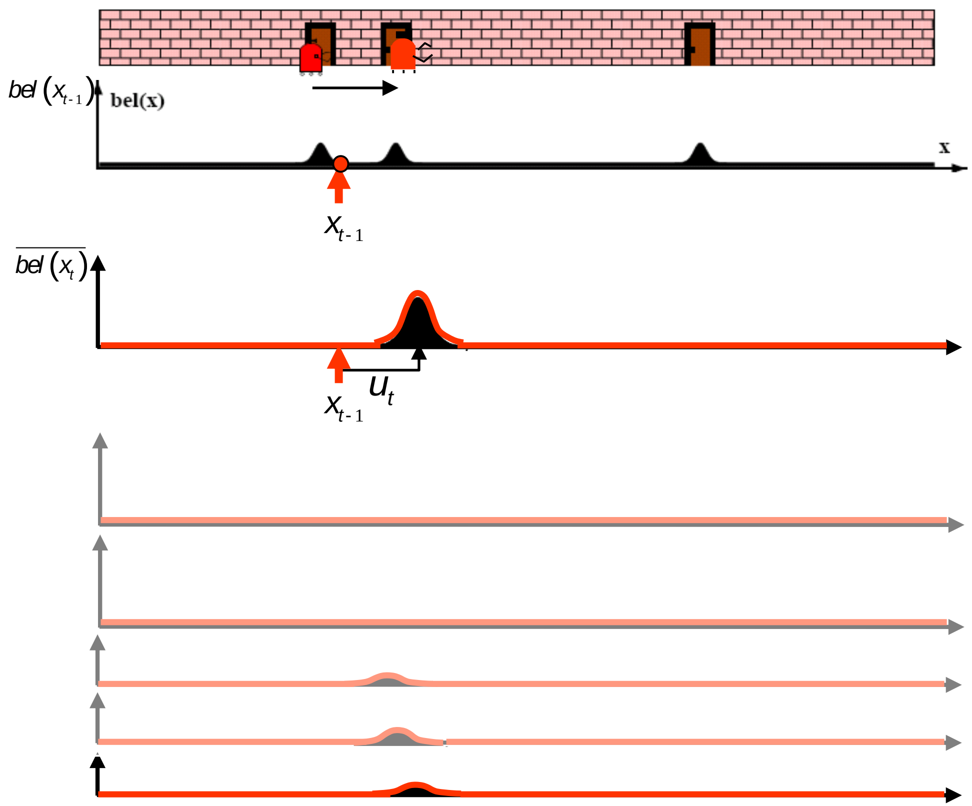


2.1 The Bayes Filter

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$):
  for all x_t do
    $\boxed{\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx}$ *Total Probability Theorem*
    $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$ *Bayes Rule*
  endfor
  return $bel(x_t)$

$$bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$$

$$p(x_t \mid u_t, x_{t-1})$$



2.1 The Bayes Filter

Algorithm Bayes_filter$(bel(x_{t-1}), u_t, z_t)$**:**

for all x_t do

$\boxed{\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx}$ — *Total Probability Theorem*

$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$ — *Bayes Rule*

endfor

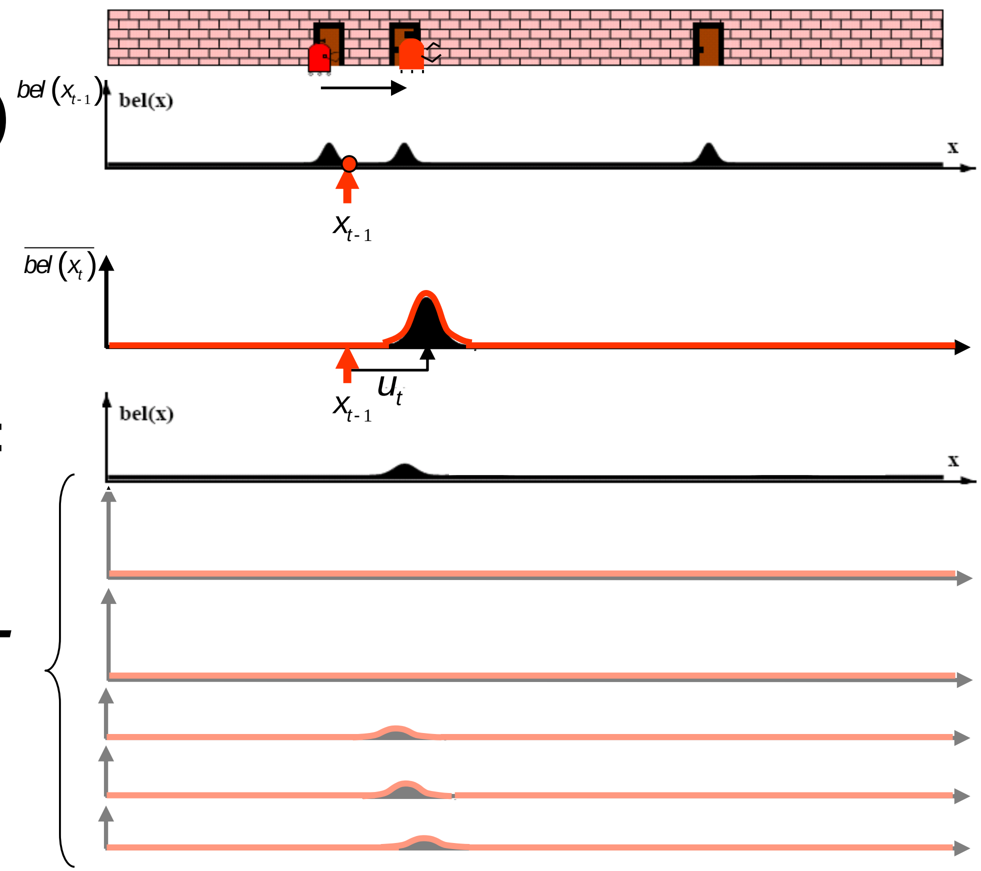
return $bel(x_t)$

$$bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$$

$$p(x_t \mid u_t, x_{t-1})$$

$=$

Σ



2.1 The Bayes Filter

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$):
  for all x_t do
    $\boxed{\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx}$ *Total Probability Theorem*
    $\boxed{bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)}$ *Bayes Rule*
  endfor
  return $\boxed{bel(x_t)}$

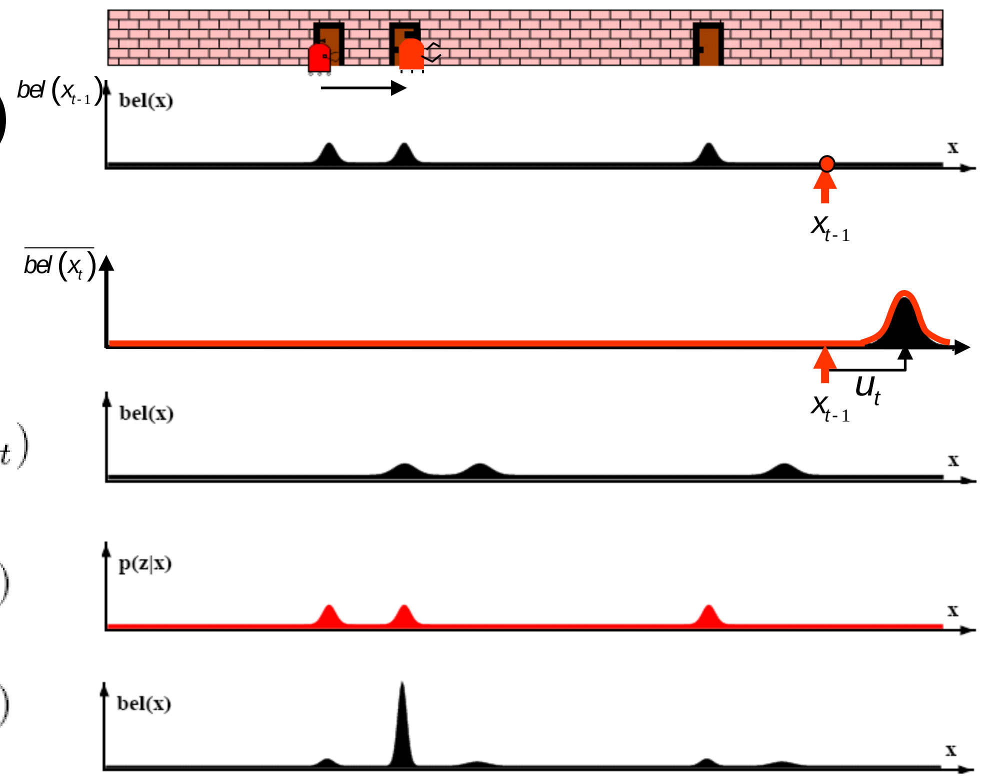
$$bel(x_{t-1}) = p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})$$

$$p(x_t \mid u_t, x_{t-1})$$

$$\overline{bel}(x_t) = p(x_t \mid z_{1:t-1}, u_{1:t})$$

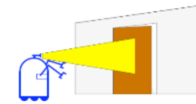
$$p(z_t \mid x_t)$$

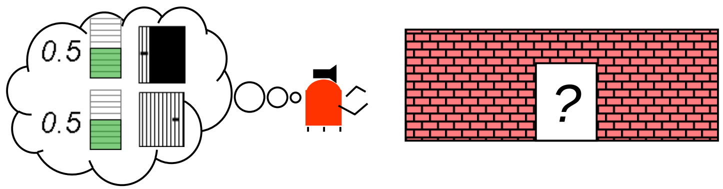
$$bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$$



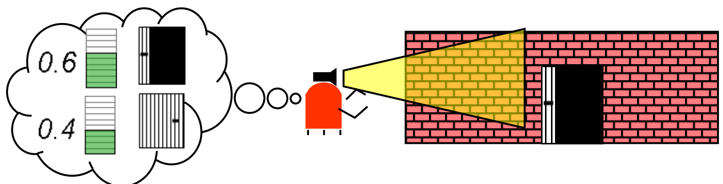
2.2 Ex II: Is the door open or close?

- Example II: A robot wants to discover if a door is open

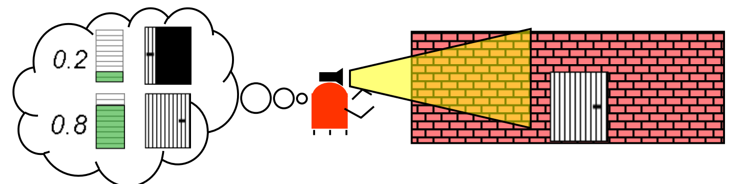




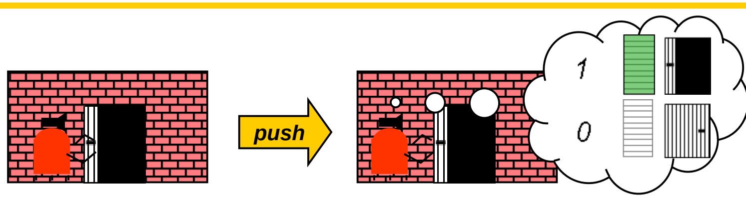
$$bel(X_0 = \text{open}) = 0.5$$
$$bel(X_0 = \text{closed}) = 0.5$$



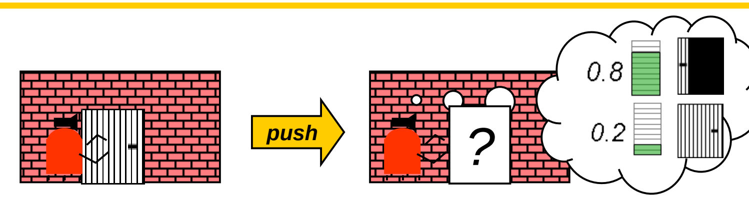
$$p(Z_t = \text{sense\_open} \mid X_t = \text{is\_open}) = 0.6$$
$$p(Z_t = \text{sense\_closed} \mid X_t = \text{is\_open}) = 0.4$$



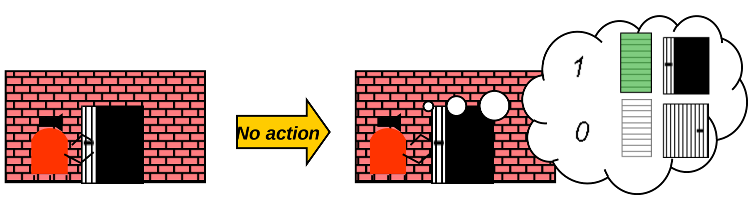
$$p(Z_t = \text{sense\_open} \mid X_t = \text{is\_closed}) = 0.2$$
$$p(Z_t = \text{sense\_closed} \mid X_t = \text{is\_closed}) = 0.8$$



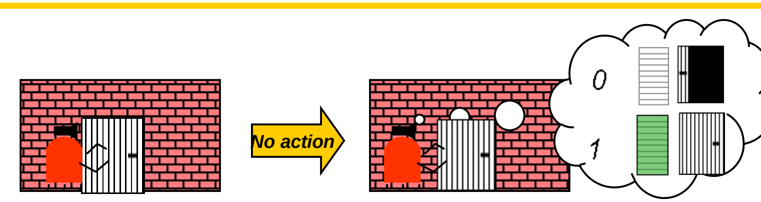
$$p(X_t = \text{is\_open} \mid U_t = \text{push}, X_{t\_1} = \text{is\_open}) = 1$$
$$p(X_t = \text{is\_closed} \mid U_t = \text{push}, X_{t\_1} = \text{is\_open}) = 0$$



$$p(X_t = \text{is\_open} \mid U_t = \text{push}, X_{t\_1} = \text{is\_closed}) = 0.8$$
$$p(X_t = \text{is\_closed} \mid U_t = \text{push}, X_{t\_1} = \text{is\_closed}) = 0.2$$



$$p(X_t = \text{is\_open} \mid U_t = \text{do\_nothing}, X_{t\_1} = \text{is\_open}) = 1$$
$$p(X_t = \text{is\_closed} \mid U_t = \text{do\_nothing}, X_{t\_1} = \text{is\_open}) = 0$$



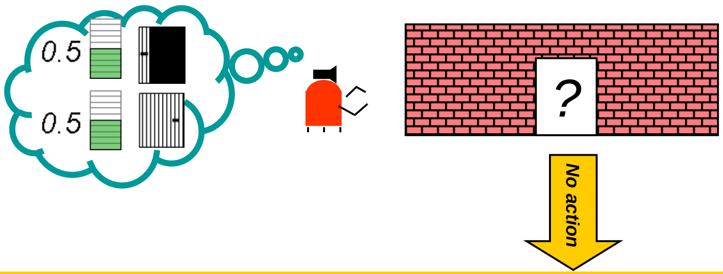
$$p(X_t = \text{is\_open} \mid U_t = \text{do\_nothing}, X_{t\_1} = \text{is\_closed}) = 0$$
$$p(X_t = \text{is\_closed} \mid U_t = \text{do\_nothing}, X_{t\_1} = \text{is\_closed}) = 1$$

2.2 Ex II: Is the door open or close?

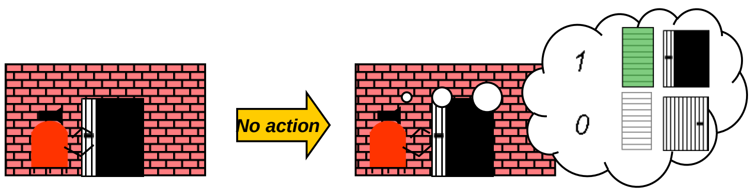
- Example

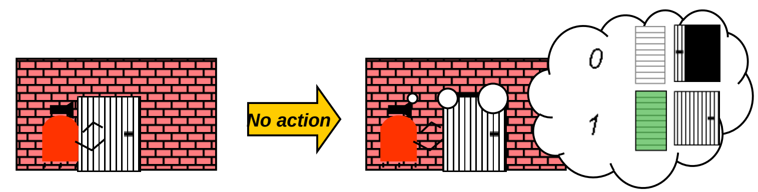
Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$)**:**
  for all x_t do
    $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx$
    $bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$
  endfor
  return $bel(x_t)$

t=t_0



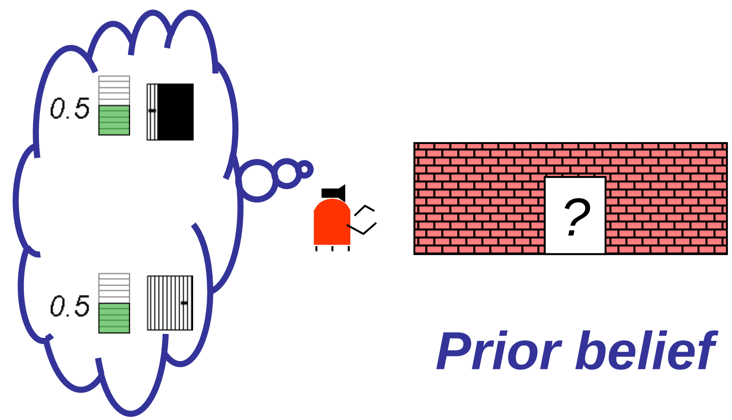
u_1=do_nothing





$$\begin{aligned} &\overline{bel}(X_1 = \text{is\_open}) \\ &= p(X_1 = \text{is\_open} \mid U_1 = \text{do\_nothing}, X_0 = \text{is\_open})\ bel(X_0 = \text{is\_open}) \\ &\quad + p(X_1 = \text{is\_open} \mid U_1 = \text{do\_nothing}, X_0 = \text{is\_closed})\ bel(X_0 = \text{is\_closed}) \\ &= 1 \cdot 0.5 + 0 \cdot 0.5 = 0.5 \end{aligned} \tag{2.46}$$

$$\begin{aligned} &\overline{bel}(X_1 = \text{is\_closed}) \\ &= p(X_1 = \text{is\_closed} \mid U_1 = \text{do\_nothing}, X_0 = \text{is\_open})\ bel(X_0 = \text{is\_open}) \\ &\quad + p(X_1 = \text{is\_closed} \mid U_1 = \text{do\_nothing}, X_0 = \text{is\_closed})\ bel(X_0 = \text{is\_closed}) \\ &= 0 \cdot 0.5 + 1 \cdot 0.5 = 0.5 \end{aligned} \tag{2.47}$$

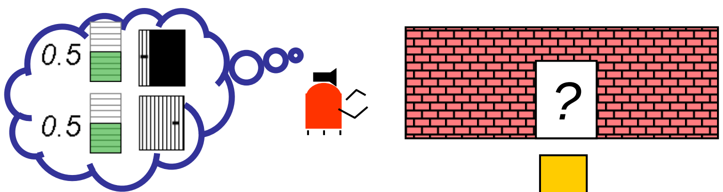


Prior belief

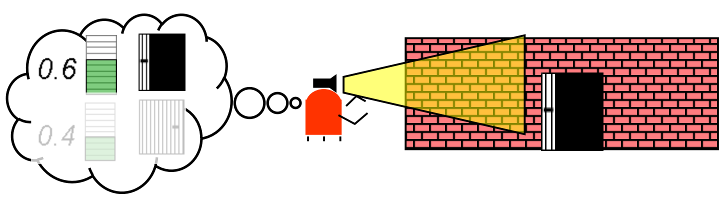
2.2 Ex II: Is the door open or close?

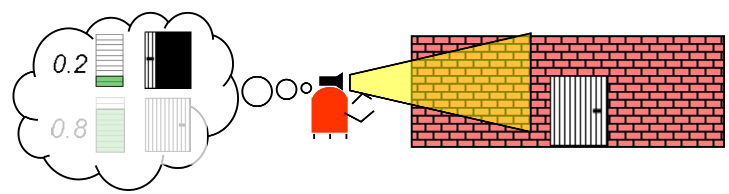
Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$):
for all x_t do
$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})\ dx$
$bel(x_t) = \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t)$
endfor
return $bel(x_t)$

Prior belief



z_1=*sense_open*



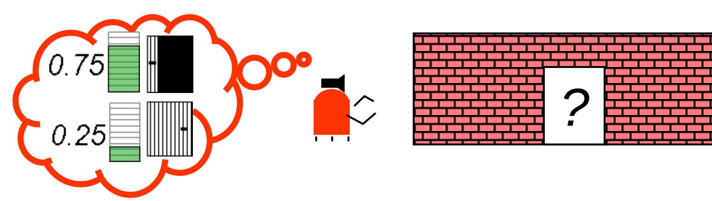


- $bel(X_1 = \text{is\_open})$
  $= \eta\ p(Z_1 = \text{sense\_open} \mid X_1 = \text{is\_open})\ \overline{bel}(X_1 = \text{is\_open})$
  $= \eta\ 0.6 \cdot 0.5 = \eta\ 0.3$ $= 2.5 \cdot 0.3 = 0.75$

- $bel(X_1 = \text{is\_closed})$
  $= \eta\ p(Z_1 = \text{sense\_open} \mid X_1 = \text{is\_closed})\ \overline{bel}(X_1 = \text{is\_closed})$
  $= \eta\ 0.2 \cdot 0.5 = \eta\ 0.1$ $= 2.5 \cdot 0.1 = 0.25$

Now, the normalizer can be computed

$$\eta = (0.3 + 0.1)^{-1} = 2.5$$



Belief at $t = t_1$

2.2 Ex II: Is the door open or close?

Algorithm Bayes_filter($bel(x_{t-1}), u_t, z_t$)**:**

for all x_t do

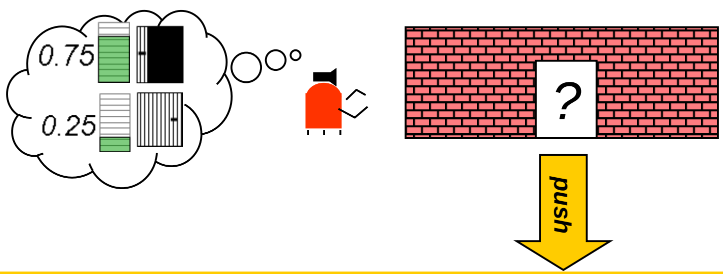
$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\; bel(x_{t-1})\; dx$

$bel(x_t) = \eta\; p(z_t \mid x_t)\; \overline{bel}(x_t)$

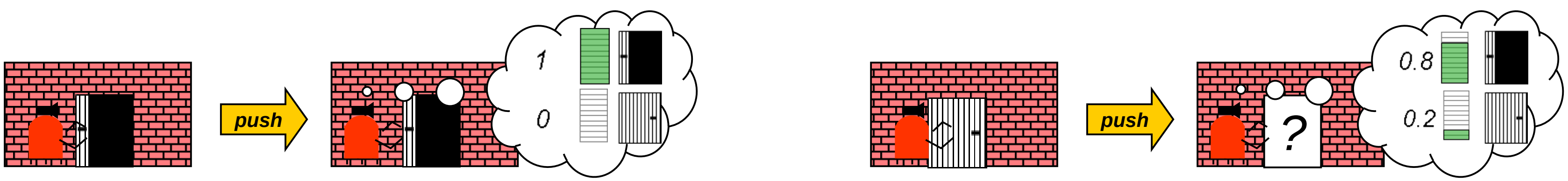
endfor

return $bel(x_t)$

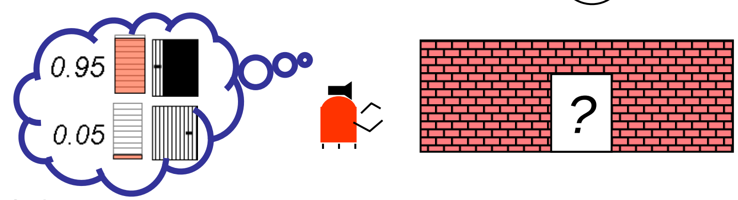
Prior belief

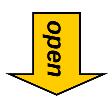


u_2=*push*

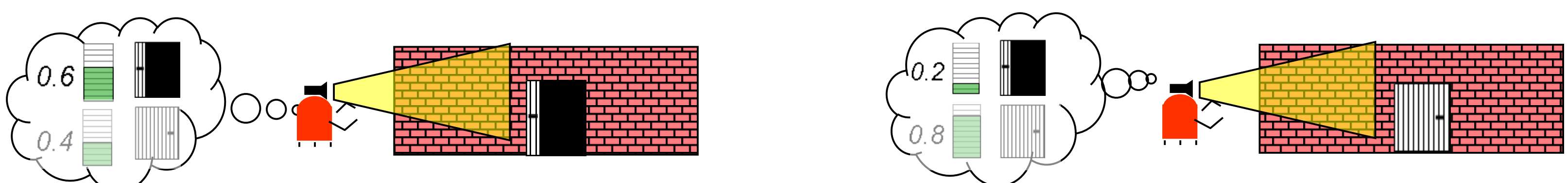


$$\overline{bel}(X_2 = \text{is\_open}) = 1 \cdot 0.75 + 0.8 \cdot 0.25 = 0.95$$
$$\overline{bel}(X_2 = \text{is\_closed}) = 0 \cdot 0.75 + 0.2 \cdot 0.25 = 0.05\,,$$

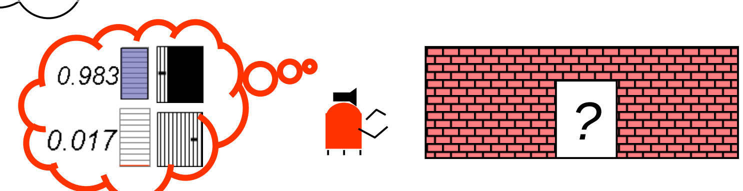




z_2=*sense_open*



$$bel(X_2 = \text{is\_open}) = \eta\, 0.6 \cdot 0.95 \approx 0.983$$
$$bel(X_2 = \text{is\_closed}) = \eta\, 0.2 \cdot 0.05 \approx 0.017\,.$$



2.2 Ex II: Is the door open or close?

- **The Bayes Filter**

  – **Belief of a robot** is the posterior distribution over the state given all past measurements (z) and all past controls (u).

  – **Bayes Filter** is the principal **algorithm for calculating the belief** in robotics.

  – The Bayes Filter makes the ***markov assumption*** according to which the state is a complete summary of the past

$\Rightarrow$ **belief is sufficient to represent the past history of the robot**

2.2 Ex II: Is the door open or close?

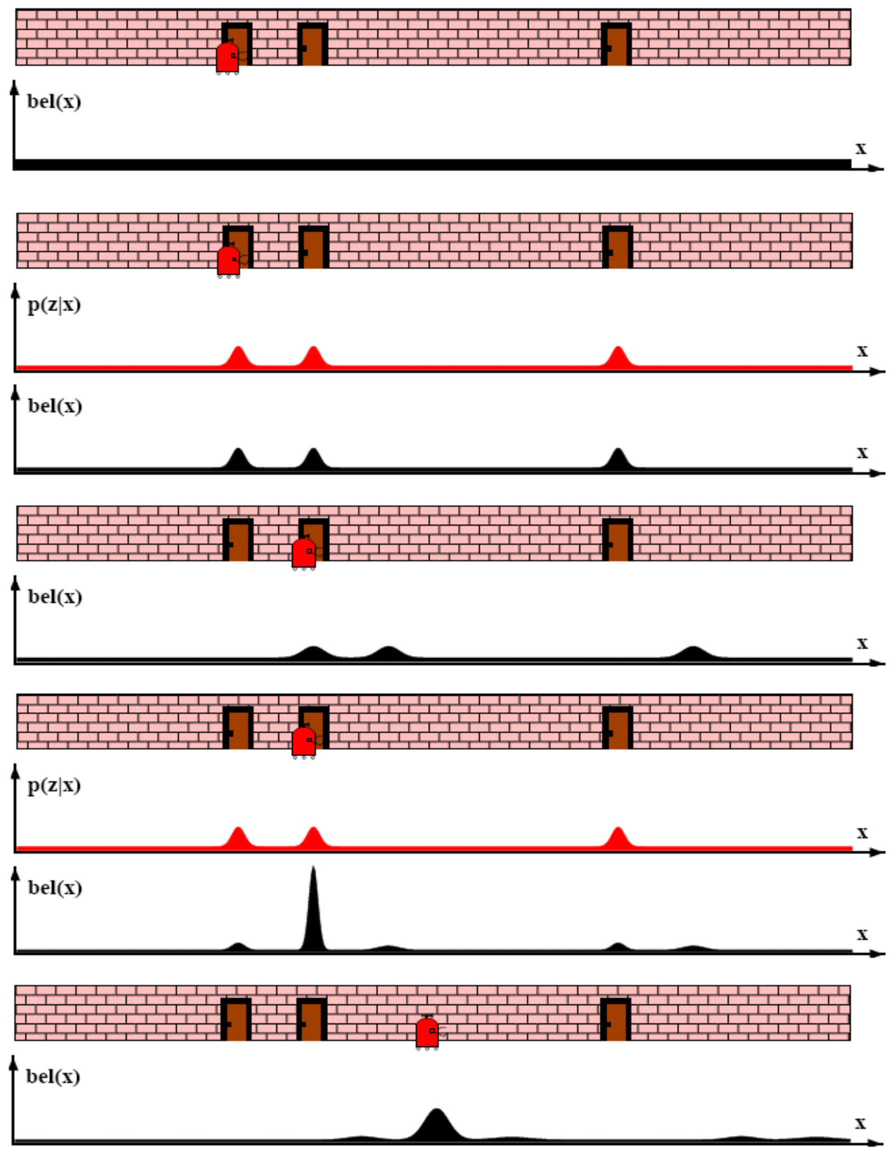
- **The Bayes Filter as shown is not efficient.** There exists probabilistic algorithms that use tractable approximations to the Bayes Filter:

  – **The Gaussian Filters**:
    - **KF**: Kalman Filter
    - **EKF**: Extended Kalman Filter

  – **The Non parametric:**
    - **HF**: Histogram Filter
    - **PF**: Particle Filter

2.3 Map Based Localization

- **Markov Localization:**
  - **Bayes filter**
  - **The real pdf (continuous) is used**
  - **Initial position is unknown**



Algorithm Markov_localization$(bel(x_{t-1}), u_t, z_t, m)$:

for all x_t *do*

$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}, m)\ bel(x_{t-1})\ dx$

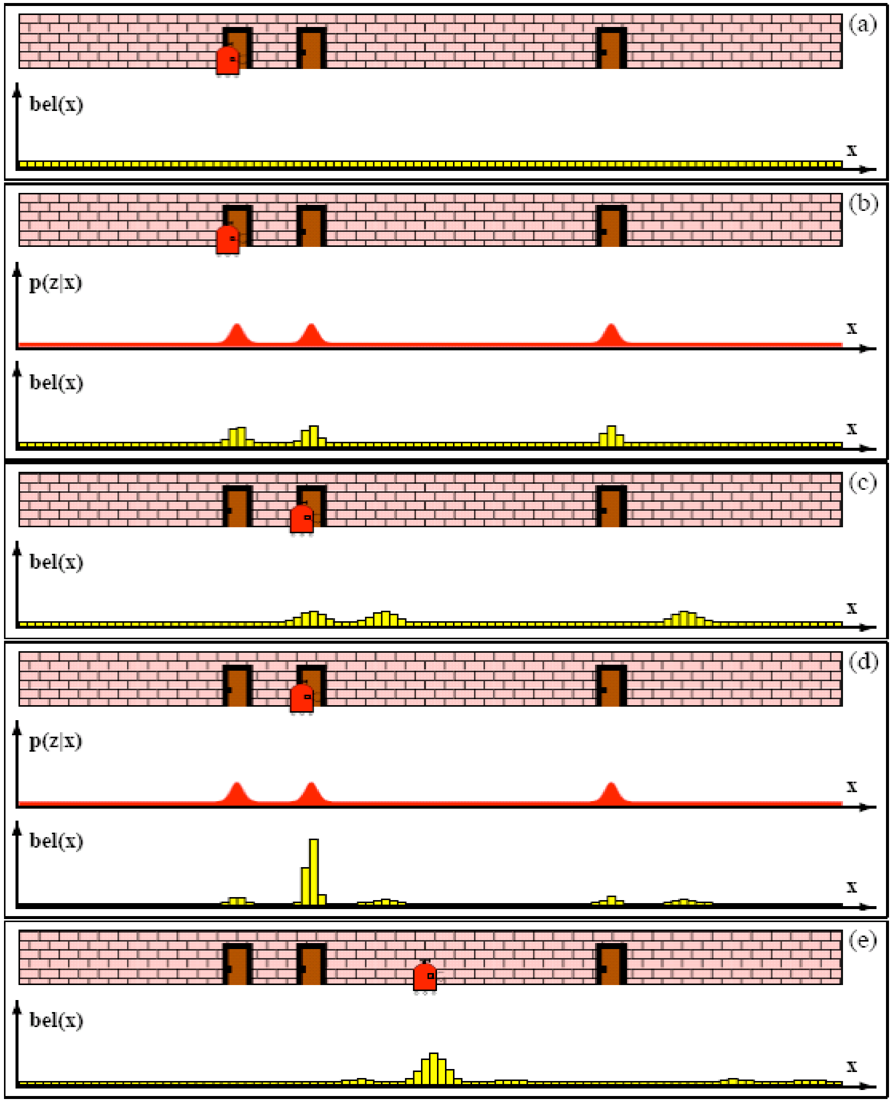
$bel(x_t) = \eta\ p(z_t \mid x_t, m)\ \overline{bel}(x_t)$

endfor

return $bel(x_t)$

2.3 Map Based Localization

- **Grid Localization:**
  - **Histogram filter**
  - **pdf are represented by a probability histogram**
  - **Initial position is unknown**

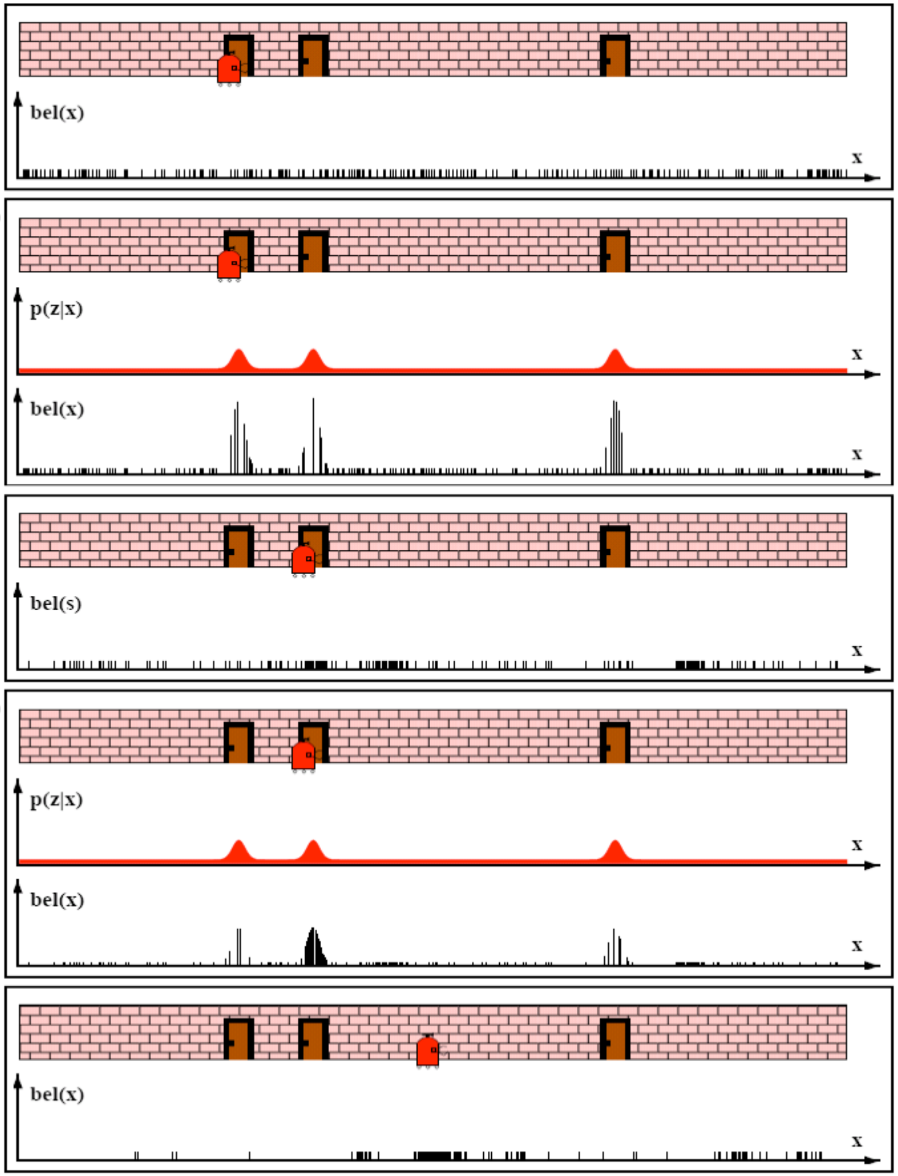


Algorithm Grid_localization($\{p_{k,t-1}\}, u_t, z_t, m$):
  for all k do
    $\bar{p}_{k,t} = \sum_i p_{i,t-1} \text{ motion\_model}(\text{mean}(\mathbf{x}_k), u_t, \text{mean}(\mathbf{x}_i))$
    $p_{k,t} = \eta \text{ measurement\_model}(z_t, \text{mean}(\mathbf{x}_k), m)$
  endfor
  return $\{p_{k,t}\}$

2.3 Map Based Localization

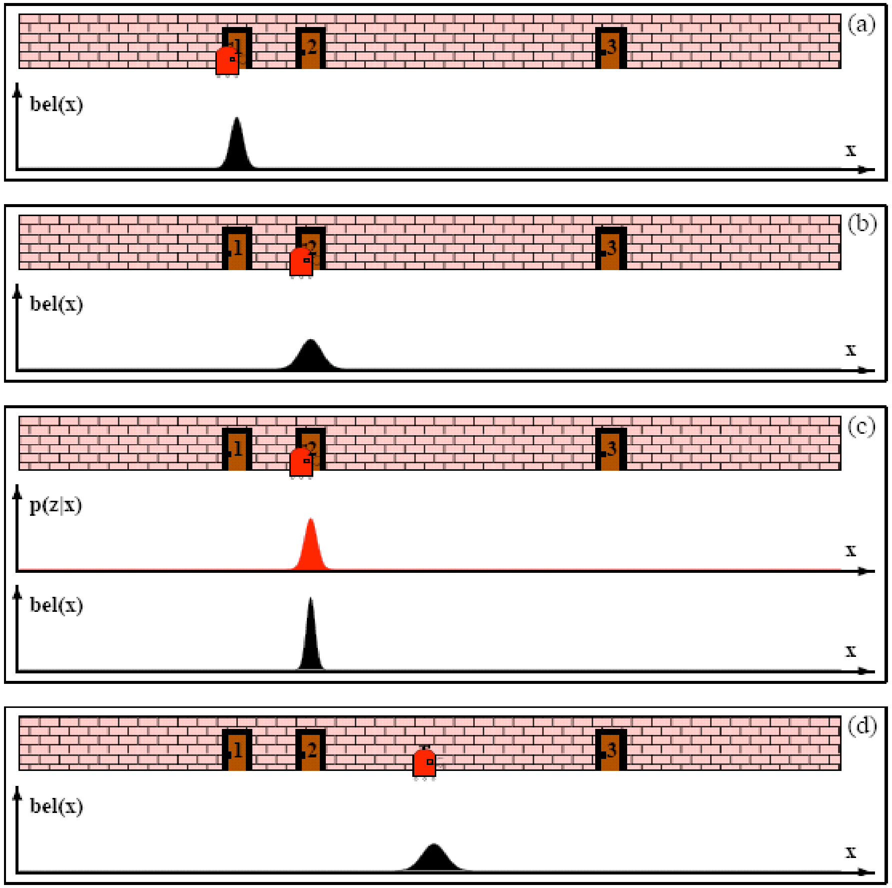
- **Montecarlo Localization:**
  - **Particle filter**
  - **pdf is represented by a set of samples**
  - **The higher the density means higher probability**
  - **Initial position is unknown**

Algorithm MCL$(\mathcal{X}_{t-1}, u_t, z_t, m)$:
  $\bar{\mathcal{X}}_t = \mathcal{X}_t = \emptyset$
  for $m = 1$ to M do
    $x_t^{[m]} = \text{sample\_motion\_model}(u_t, x_{t-1}^{[m]})$
    $w_t^{[m]} = \text{measurement\_model}(z_t, x_t^{[m]}, m)$
    $\bar{\mathcal{X}}_t = \bar{\mathcal{X}}_t + \langle x_t^{[m]}, w_t^{[m]} \rangle$
  endfor
  for $m = 1$ to M do
    draw i with probability $\propto w_t^{[i]}$
    add $x_t^{[i]}$ to $\mathcal{X}_t$
  endfor
  return $\mathcal{X}_t$



2.3 Map Based Localization

- **EKF Localization:**
  - **Extended Kalman Filter**
  - **pdf are represented by uni-modal Gaussians.**
  - **Initial position is known**
  - **Features are distinguishable**



Algorithm: EKF Localization

{- Pose initialization}

$x_0^B = 0;\ P_0^B = 0;$

for k=1 to steps **do**

$\left[x_{R_k}^{R_{k-1}}, Q_k\right] = get\_odometry$

{-EKF prediction}

$\left[x_{k|k-1}^B, P_{k|k-1}^B\right] = move\_vehicle\left(x_{k-1}^B, P_{k-1}^B, x_{R_k}^{R_{k-1}}, Q_k\right)$

$\left[z_k, R_k\right] = get\_measurements$

$\mathcal{H}_k = data\_association\left(x_{k|k-1}^B, P_{k|k-1}^B, z_k, R_k\right)$

{-EKF update}

$\left[x_k^B, P_k^B\right] = update\_position\left(x_{k|k-1}^B, P_{k|k-1}^B, z_k, R_k\right)$

end for