OR 506: Final Project

Created in iPython Notebook by Guduguntla Vamshi gudugu@ncsu.edu (mailto:gudugu@ncsu.edu)

1. Portfolio Selection

Objective:

The goal of this problem is to find the optimal portfolio of securities. An optimal portfolio is one that maximizes return on investment, while it mitigates the variance, thereby the risk. The two competing objectives - in **Risk** and **Return**, will formulate the problem as a multi-objective problem.

The objective is defined as:

$$min\, f_1(x), f_2(x)$$

$$such\ that\ x_1 + x_2 + x_3 + x_4 = 1$$

$$subject\ to\ x_1, x_2, x_3, x_4 >= 0$$

$$f_1(x) = 0.5x^T \sigma_c x_i$$

$$f_2(x) = -\sum_{1}^{n} \mu_i x_i$$

- f1 the variance in the returns from the assets
- f2 the return from the assets
- μ_i is the Expected value of the security i and variance σ_i
- σ_c is the covariance between assets i and j
- x_1, x_2, x_3, x_4 are the proportion of the investment in the portfolios - A,B,C,D
- We are going to maximize the returns while minimize the variance

Returns:

- Expected Return of Assets - A,B,C,D
- rit = (r1t, r2t, r3t, r4t)
- The quantities μ and σ are determined from stock market data.

In [44]:
```
Return_table
```

Out[44]:

	A	B	C	D
0	0.00630	0.00660	0.00107	0.02340
1	0.00150	0.00762	-0.06840	-0.07530
2	0.01861	-0.02480	0.02876	0.08761
3	0.03560	-0.05510	0.03200	0.03150
4	0.10110	0.01120	0.11810	0.10390
5	0.09110	0.00190	-0.01123	-0.10090
6	0.09810	0.78910	-0.00121	-0.01210
7	0.10090	0.09120	0.01231	0.09780
8	0.06700	0.07810	0.07910	0.07820
9	0.18190	0.09110	0.08120	0.10120

Further calculations:

- The geometric mean for asset i is calculated as: μ_i is the geometric mean of all the assets i

$$\pi_{i=1}^{10}(1 + r_ij)^{T^{-1}} - 1$$

- The covariance between the assets i and j is calculated as:

$$\sigma_c = (1/T)\sum_{1}^{n}(r_it - r_i)(r_jt - r_j)$$

- where rˆi and rˆj are the arithmetic mean of assets i and j.

The covariance matrix(σ_c) results in:

In [16]:
```
sigma_i_j
```

Out[16]:
```
array([[ 0.00314014,  0.0041508 ,  0.00162361,  0.00143165],
       [ 0.0041508 ,  0.06109909, -0.00167029, -0.00245394],
       [ 0.00162361, -0.00167029,  0.00291397,  0.0031798 ],
       [ 0.00143165, -0.00245394,  0.0031798 ,  0.00563179]])
```

The geometric mean results in:

In [14]:
```
geom_mu
```

Out[14]:
```
[0.06890833538982810l,
 0.08077608424815685 4,
 0.025892621885867273,
 0.03098611592954259]
```

By introducing α and β

$$max\, f(x, y) = \alpha f_1(x, y) + \beta f_2(x, y)$$

- The Pareto frontier is developed as a solution space by weighting the objectives
- The weights are assigned arbitrarily complimenting each other which sum up to 1
- Initially we proceed by equal weighting α = 0.5 and β = 0.5

Non-linear optimization: Using Exterior Penalty method

Method Used:

$$\psi(x, \rho_p) = J(X) + \rho_p * P(X)$$
$$P(X) = \sum_{j=1}^{n} max[0, g_j(X)]^2 + \sum_{k=1}^{m_2} h_k(x)^2$$

- if all constraints are satisfied, then **P(x)=0**
- ρ_p = penalty parameter; starts as a small number and increases
- if ρ_p is small, $\psi(x, \rho_p)$ is easy to minimize but yields large constraint violations
- if ρ_p is large, constraints are all nearly satisfied but optimization problem is numerically ill-conditioned
- if optimization stops before convergence is reached, the design will be infeasible

Penalty Function Used:

$$\psi(x, \rho_p) = J(X) + \rho_p * (\sum_{j=1}^{n} [g_j(X)]^2 + \sum_{k=1}^{m_2} h_k(x)^2)$$

- $g_j(X)$ contains those inequality constraints that are violated at x
- It can be shown that as $\rho_p -> large\ lim(x * (\rho_p)) = x *$
- $\psi(x, \rho_p)$ is defined everywhere

Algorithm Used:

- choose ρ_0 , set $k = 10^0$
- find min $\psi(x, \rho_k)$ => x_k *
- if not converged, set 10^{k+1} > 10^k , k <- k+1 and repeat

Implementation:

In [64]:
```
import numpy as np
from sympy import *
import math
import scipy
import pandas as pd
import warnings
from ggplot import *
%matplotlib inline
warnings.filterwarnings("ignore")
```

Pre-process to compute geometric mean and covariance

In [65]:
```
Return_table  = pd.DataFrame()
Return_table['A'] = np.array([.0063,.0015,.01861,.0356,.1011,.091
1,.0981,.1009,.0670,.1819])
Return_table['B'] = np.array([.0066,.00762,-.0248,-.0551,.0112,.001
9,.7891,.0912,.0781,.0911])
Return_table['C'] = np.array([.00107,-.0684,.02876,.0320,.1181,-.01
123,-.00121,.01231,.0791,.0812])
Return_table['D'] = np.array([.0234,-.0753,.08761,.0315,.1039,-.100
9,-.0121,.0978,.0782,.1012])

def geom_mean():
    mu_A,mu_B,mu_C,mu_D = 1,1,1,1
    for i in range(len(Return_table)):
        mu_A *= (1+Return_table.loc[i,'A'])
        mu_B *= (1+Return_table.loc[i,'B'])
        mu_C *= (1+Return_table.loc[i,'C'])
        mu_D *= (1+Return_table.loc[i,'D'])
    return [mu_A**(.1) - 1,mu_B**.10 - 1,mu_C**.10 - 1,mu_D**.10 -
1]

geom_mu = geom_mean()
sigma_i_j = np.cov(np.array([Return_table['A'], Return_table['B'],
Return_table['C'],Return_table['D']]))
```

Exterior Penalty method implementation

In [78]:
```
def function(w1,w2):
    x, y, z, w = symbols('x y z w',real=True)
    arr = np.matrix([[x],[y],[z],[w]])
    return (((arr.T) * sigma_i_j * arr)*w1 - (geom_mu * arr)*w2).it
em(0)

def constratints():
    x, y, z, w = symbols('x y z w',real=True)
    g1 = x + y + z + w - 1
    g2,g3,g4,g5 = x,y,z,w
    return g1,g2,g3,g4,g5

def penalized_function(mu,g1,g2,g3,g4,g5,w1,w2):
    return function(w1,w2) + mu * (g1**2) + mu*((g2**2) + (g3**2) +
(g4**2) + (g5**2))

def non_linear_solve(mu,g1,g2,g3,g4,g5,starting_vector,w1,w2):
    return np.array(nsolve([diff(penalized_function(mu,g1,g2,g3,g
4,g5,w1,w2),x),diff(penalized_function(mu,g1,g2,g3,g4,g5,w1,w
2),y),diff(penalized_function(mu,g1,g2,g3,g4,g5,w1,w2),z),diff(pena
lized_function(mu,g1,g2,g3,g4,g5,w1,w2),w)],[x,y,z,w],starting_vect
or))

def check_for_constraints(a):
    idx = np.array([])
    for i in range(len(a)):
        if a[i] < 0.:
            idx = np.append(idx,1)
        else:
            idx = np.append(idx,0)
    return idx
```

In [88]:
```
def exterior_penalty(w1,w2):
    x, y, z, w = symbols('x y z w',real=True)
    starting_vector = [1 for _ in range(4)]
    g1,g2,g3,g4,g5 = constratints()

    # penalize everything to get the initial vector
    soln0 = non_linear_solve(1,g1,g2,g3,g4,g5,starting_vector,w1,w
2)
    function_value = function(w1,w2).subs({x:soln0[0],y:soln
0[1],z:soln0[2],w:soln0[3]})

    itr = 13
    for k in range(itr):
        idx = check_for_constraints(soln0)
        soln = non_linear_solve(10**k,g1,g2*idx[0],g3*idx[1],g4*id
x[2],g5*idx[3],starting_vector,w1,w2)
        function_value = function(w1,w2).subs({x:soln[0],y:sol
n[1],z:soln[2],w:soln[3]})
        print 'iteration',k,'Mu: ',10**k,'x1 x2 x3 x4 = ',round(sol
n[0],3),round(soln[1],3),round(soln[2],3),round(soln[3],3),'f(x1,x
2): ',round(function_value,9)
        soln0 = soln
        if abs((g1.subs({x:soln[0],y:soln[1],z:soln[2],w:sol
n[3]}))) < 10**(-17):
            break
    print "Converged Successfully in iterations: ",k
```

```
In [86]: exterior_penalty(0.5,0.5)
```

```
iteration 0 Mu:  1 x1 x2 x3 x4 =  9.932 -0.528 -11.263 2.868 f(x
1,x2):  -0.119667831
iteration 1 Mu:  10 x1 x2 x3 x4 =  3.922 -0.0 -0.001 -2.92 f(x1,x
2):  -0.058116574
iteration 2 Mu:  100 x1 x2 x3 x4 =  1.0 0.0 -0.0 -0.0 f(x1,x2):
-0.032892609
iteration 3 Mu:  1000 x1 x2 x3 x4 =  0.912 0.088 -0.0 -0.0 f(x1,x
2):  -0.033101573
iteration 4 Mu:  10000 x1 x2 x3 x4 =  0.912 0.088 -0.0 -0.0 f(x1,x
2):  -0.033100834
iteration 5 Mu:  100000 x1 x2 x3 x4 =  0.912 0.088 -0.0 -0.0 f(x
1,x2):  -0.03310076
iteration 6 Mu:  1000000 x1 x2 x3 x4 =  0.912 0.088 -0.0 -0.0 f(x
1,x2):  -0.033100752
iteration 7 Mu:  10000000 x1 x2 x3 x4 =  0.912 0.088 -0.0 -0.0 f(x
1,x2):  -0.033100752
iteration 8 Mu:  100000000 x1 x2 x3 x4 =  0.912 0.088 -0.0 -0.0
f(x1,x2):  -0.033100751
iteration 9 Mu:  1000000000 x1 x2 x3 x4 =  0.912 0.088 -0.0 -0.0
f(x1,x2):  -0.033100751
iteration 10 Mu:  10000000000 x1 x2 x3 x4 =  0.914 0.086 -0.0 -0.0
f(x1,x2):  -0.033100646
iteration 11 Mu:  100000000000 x1 x2 x3 x4 =  0.911 0.089 -0.0
-0.0 f(x1,x2):  -0.0331007
iteration 12 Mu:  1000000000000 x1 x2 x3 x4 =  0.833 0.167 -0.0
-0.0 f(x1,x2):  -0.032927723
Converged Successfully in iterations:  12
```

Results:

- The converging function value is **-0.032927723**
- Using the initial value as Mu = 10^0 = 1 final solution of x is **[0.833 0.167 -0.0 -0.0]**
- The algorithm converged in **12 iterations** using the above mu values and tolerance of **10^-8** in constraints

The investment strategy would be:

For equal weights of risk and return α = 0.5 and β = 0.5

- X1 = 83.33%
- X2 = 16.7%
- X3 = 00.00%
- X4 = 00.00%

The **Risk** & **Return**:

In [125]:
```
function1(1.,0.).subs({x:0.833,y:0.167,z:0.,w:0.})
```

Out[125]: 0.00503774215370480

The **Risk** = 0.0050377

In [61]:
```
function1(0.,1.).subs({x:0.833,y:0.167,z:0.,w:0.})
```

Out[61]: -0.0708902494491690

The **Return** = -0.070890

Additional Comments:

- Convergence criteria is: 20 maximum iterations or **constraints satisfied**
- Allowing for tolerance in the constraints around **10^-8**, the iterations are further increased
- Mu values in **12 iterations** increase to the powers of 10.
- The computed values are cross-validated by checking plugging in these values given by the criteria defined above and checking for constraints.

Pareto Frontier: using random uniform weights

Approach:

- We are going to vary α and β by generating **random uniform** numbers between 0 and 1 such that $\alpha + \beta = 1$
- For each combination of α and β, we shall compute the **Return** and **Risk**, by generating the **solution space** for the investment strategy

Implementation:

In [ ]:
```
import random
x, y=symbols('x y',real=True)

def exterior_penalty_test(w1,w2):
    x, y, z, w = symbols('x y z w',real=True)
    starting_vector = [1 for _ in range(4)]
    g1,g2,g3,g4,g5 = constratints()
    soln0 = non_linear_solve(1,g1,g2,g3,g4,g5,starting_vector,w1,w
2)
    function_value = function(w1,w2).subs({x:soln0[0],y:soln
0[1],z:soln0[2],w:soln0[3]})
    itr = 20
    for k in range(itr):
        idx = check_for_constraints(soln0)
        soln = non_linear_solve(10**k,g1,g2*idx[0],g3*idx[1],g4*id
x[2],g5*idx[3],starting_vector,w1,w2)
        function_value = function(w1,w2).subs({x:soln[0],y:sol
n[1],z:soln[2],w:soln[3]})
        soln0 = soln
        if abs((g1.subs({x:soln[0],y:soln[1],z:soln[2],w:sol
n[3]}))) < 10**(-8):
            break
    # x1,x2,x3,x4,risk,return,objective
    return round(soln[0],2),round(soln[1],2),round(soln[2],2),roun
d(soln[3],2),round(function(1.,0.).subs({x:soln[0],y:soln[1],z:sol
n[2],w:soln[3]}),4),round(function(0.,1.).subs({x:soln[0],y:sol
n[1],z:soln[2],w:soln[3]}),4),round(function(w1,w2).subs({x:sol
n[0],y:soln[1],z:soln[2],w:soln[3]}),4)
```

Simulaiton - for 10 trials of random uniform weights:

In [121]:
```
def simulation(trials):
    result = pd.DataFrame()
    for i in range(trials):
    # compute the random uniform weights
        w1 = random.uniform(0,1)
        w2 = 1.0 - w1
        x1,x2,x3,x4,risk,retu,objective =  exterior_penalty_test(w
1,w2)
        result.loc[i,'Risk weight(alpha)'],result.loc[i,'Return wei
ght(beta)'] = round(w1*100,0),round(w2*100,0)
        result.loc[i,'x1'],result.loc[i,'x2'],result.loc[i,'x3'] =
x1,x2,x3
        result.loc[i,'x4'] = x4
        result.loc[i,'Risk'] = risk
        result.loc[i,'Return'] = retu
        result.loc[i,'Objective'] = objective
    return result
```

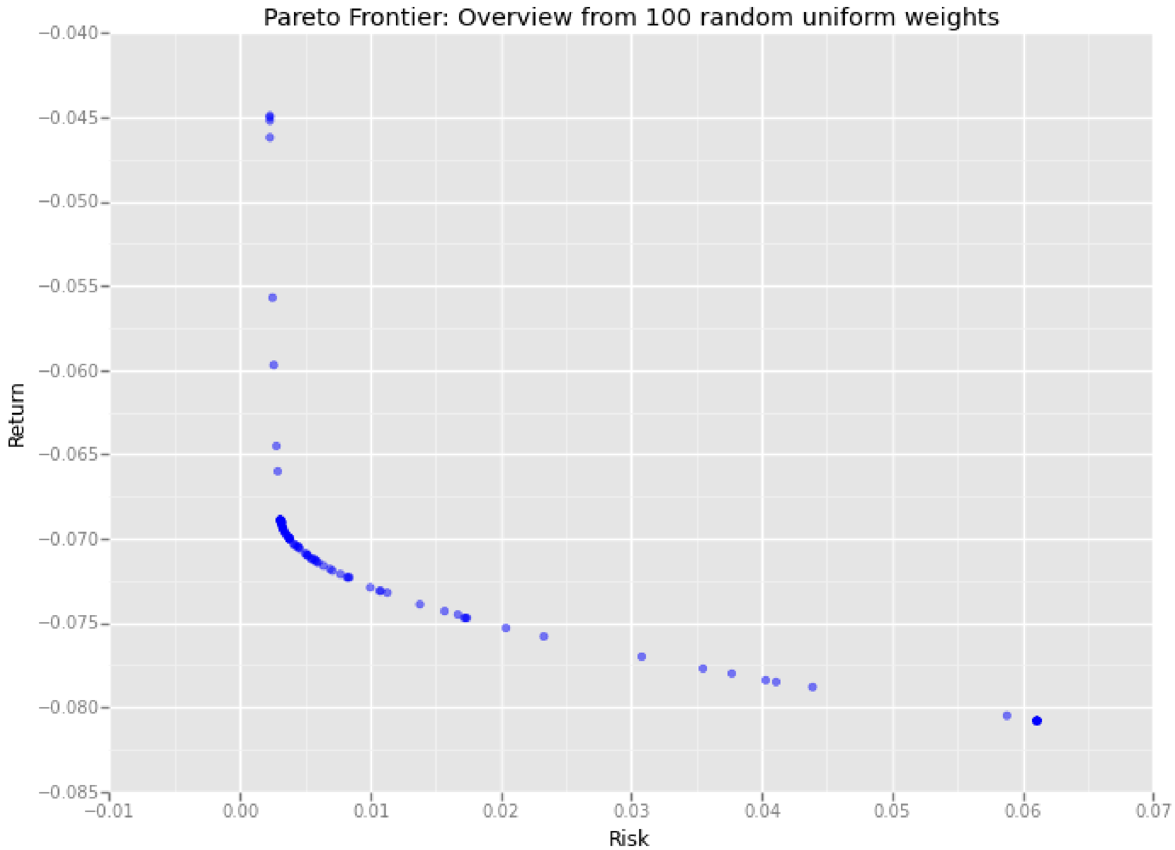
In [122]:
```
result = simulation(10)
result
```

Out[122]:

	Risk weight(alpha)	Return weight(beta)	x1	x2	x3	x4	Risk	Return	Objective
0	69	31	0.97	0.03	-0	-0.00	0.0033	-0.0693	-0.0195
1	58	42	0.94	0.06	-0	-0.00	0.0035	-0.0696	-0.0275
2	35	65	0.82	0.18	-0	-0.00	0.0054	-0.0711	-0.0447
3	32	68	0.79	0.21	-0	-0.00	0.0061	-0.0714	-0.0470
4	44	56	0.88	0.12	-0	-0.00	0.0042	-0.0703	-0.0378
5	94	6	0.93	-0.00	-0	0.07	0.0029	-0.0662	-0.0014
6	66	34	0.96	0.04	-0	-0.00	0.0033	-0.0693	-0.0215
7	39	61	0.85	0.15	-0	-0.00	0.0047	-0.0707	-0.0414
8	42	58	0.87	0.13	-0	-0.00	0.0043	-0.0704	-0.0390
9	24	76	0.68	0.32	-0	-0.00	0.0096	-0.0727	-0.0531

In [133]:
```
df = simulation(100)
```

In [140]:
```
ggplot(aes(x='Risk', y='Return'), data=df) +\
    geom_point(color = 'blue',alpha = 0.5) +\
    ggtitle("Pareto Frontier: Overview from 100 random uniform weig
hts") + \
    stat_smooth(colour='blue',span = 0.0)
```



Out[140]: <ggplot: (290540997)>

Results:

- As expected, we could see the curve for Risk vs Return curve steepest at high returns during minimum risk (inverse realtion)
- The returns decrease slowly, while the risk moves increases

2. Multi-Objective Programming

$$max\ (10x + y, -8x + 2y)$$

$$constraints\ x + y <= 15$$

$$x - y <= 5$$

$$x, y >= 0$$

Objective:

To determine the Pareto frontier for the above multi-goal problem

Method:

- The Pareto frontier is developed as a solution space by weighting the objectives
- The weights are assigned arbitrarily complimenting each other which sum up to 1
- Simple linear programming is used to solve the constrained linear optimization problem

The objective is defined as:

$$f_1(x, y) = 10x + y$$

$$f_2(x, y) = -8x + 2y$$

$$max\ f(x, y) = \alpha f_1(x, y) + \beta f_2(x, y)$$

$$subject\ to\ x + y <= 15$$

$$x - y <= 5$$

$$x, y >= 0$$

$$\alpha + \beta = 1$$

Implementation:

To determine the Pareto frontier:

In [ ]:

```
import numpy as np
from sympy import *
import math
import random
from scipy.optimize import linprog
import pandas as pd

def f1(x,w1):
    return (10*x[0] + y[1])*w1

def f2(x,w2):
    return (-8*x[0] + 2*x[1])*w2


def function(w1,w2):
    x, y=symbols('x y',real=True)
    f1 =  10*x + y
    f2 = -8*x + 2*y
    return -((w1*f1)+(w2*f2))


A = [[1, 1], [1, -1]] # constraint coefficients
b = [15, 5]           # constraint coefficients

c = [-1., -1.5] # objective function coefficients

def linear_optimization(A,b,c):
    x0_bounds = (0., None)
    x1_bounds = (0., None)
    res = linprog(c, A_ub=A, b_ub=b, bounds=(x0_bounds, x1_bound
s),options={"disp": True})
    return res.x
```

In [ ]:

```
result = pd.DataFrame()
for i in range(10):
    x, y=symbols('x y',real=True)
    w1 = random.uniform(0,1)
    w2 = 1.0 - w1
    c = []
    c.append(function(w1,w2).subs({x:1.,y:0.}))
    c.append(function(w1,w2).subs({x:0.,y:1.}))
    result.loc[i,'alpha'] = w1
    result.loc[i,'beta'] = w2

    print linear_optimization(A,b,c)

    result.loc[i,'x'] = linear_optimization(A,b,c)[0]
    result.loc[i,'y'] = linear_optimization(A,b,c)[1]
```

```
In [ ]: result['f1'] = result['alpha'] * (10 * result['x'] + result['y'])
        result['f2'] = result['beta'] * (-8 * result['x'] + 2*result['y'])
        result['f(x,y)'] = result['alpha'] * result['x']  + result['beta']
        * result['y']
```

Results:

```
In [84]: result
```

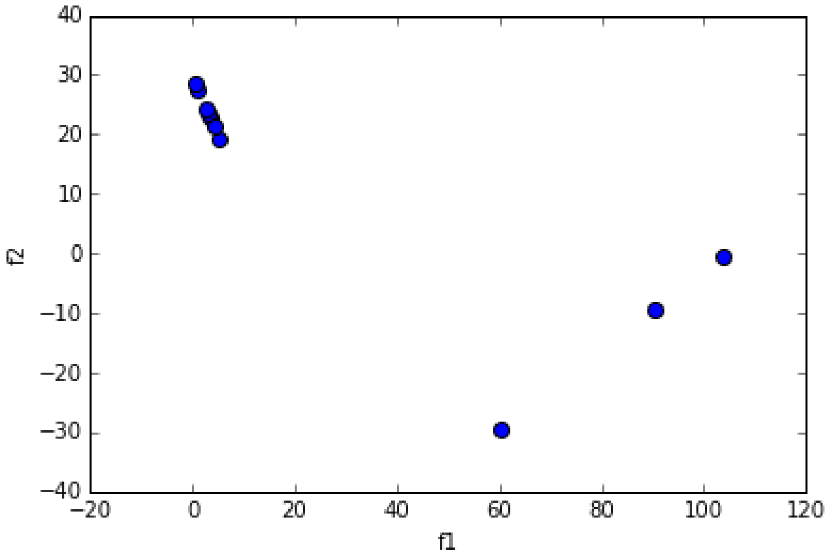
Out[84]:

	alpha	beta	x	y	f1	f2	f(x,y)
0	0.249748	0.750252	0	15	3.746223	22.507553	11.253777
1	0.218162	0.781838	0	15	3.272430	23.455140	11.727570
2	0.200809	0.799191	0	15	3.012133	23.975734	11.987867
3	0.574845	0.425155	10	5	60.358692	-29.760872	7.874223
4	0.863910	0.136090	10	5	90.710556	-9.526296	9.319550
5	0.366507	0.633493	0	15	5.497612	19.004777	9.502388
6	0.085135	0.914865	0	15	1.277023	27.445954	13.722977
7	0.289157	0.710843	0	15	4.337359	21.325283	10.662641
8	0.051592	0.948408	0	15	0.773884	28.452232	14.226116
9	0.991290	0.008710	10	5	104.085499	-0.609668	9.956452

Plot - f1 and f2:

In [107]:
```
import matplotlib.pyplot as plt
import matplotlib.pyplot
import pylab
%matplotlib inline
result.plot(kind='scatter', x='f1', y='f2',s=50)
```

Out[107]: `<matplotlib.axes._subplots.AxesSubplot at 0x7f1bce28aa10>`



Concluding remarks:

- The frontier has two optimal solutions **[0,15]** and **[10,5]**
- The frontier has not many listed points which could be called as optimal.
- The competeting objectives vary with the weights as shown above in the table
- Lower alphas tend to gravitate towards **[0,15]** while lower betas tend to **[10,5]**