CS 726: Homework 3 (Due Feb 18 , 2019)

Write your answers in the space provided. You are expected to solve each question on your own. Do not try to search the answers from any external sources, like the web. You are allowed to discuss a few questions with your classmates provided you mention their names.

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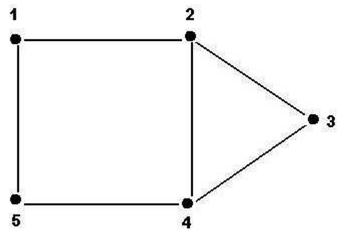
1. A sender wants to send n symbols $x_1, \ldots, x_n$ via a noisy channel to a receiver. Each symbol can take m possible values $0, 1, \ldots, m-1$ and the channel can corrupt any of the symbols to an arbitrary other symbol with probability σ. Assume the corruption of each bit happens independently of the other. The sender sends K additional symbols $w_1, \ldots, w_K$ to help correct some of the errors. Each symbol w_j is set such that $w_j + x_{j1} + \ldots x_{jr}$ is a perfect multiple of m where $\{x_{j1}, \ldots, x_{jr}\}$ is a subset of size r of $x_1, \ldots x_n$. The extra bits can also get corrupted in the same way. Let $x'_1, \ldots x'_n, w'_1, \ldots, w'_K$ denote the bits seen by the receiver.

   (a) Draw a Bayesian network expressing the relationship between the different variables including the values of the various potentials. You can assume $n = 4, K = 2, m = 3, r = 3$, $\{x_{11}, x_{12}, x_{13}\} = \{x_1, x_2, x_3\}$ and $\{x_{21}, x_{22}, x_{23}\} = \{x_4, x_2, x_3\}$ ..2

   (b) A receiver R obtains $\bar{x'_1}, \ldots \bar{x'_n}, \bar{w'_1}, \ldots, \bar{w'_K}$ as values of the $n+K$ symbols. Call this set E, the evidence set whose values are known. Now, R is interested in obtaining the most likely values of the $x_1, \ldots x_n, w_1, \ldots w_K$ variables. Draw the junction tree (JT) for this inference task for the example graph you drew earlier. In the graph you can treat the values of E as constant. Show the different steps of moralizing, triangulating, creating the JT, and assigning potentials to the JT cliques. ..3

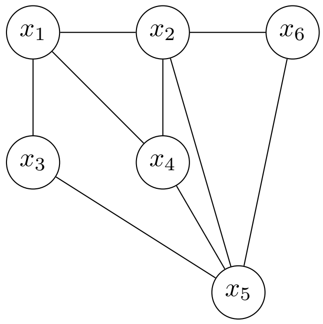
   (c) In general, what is the complexity of the above inference task in terms of quantities n, K, m, r? Justify your answer. ..2

2. The maximum independent set (MIS) of a graph is defined as the largest set of vertices such that there is no edge between them. For example, in this graph, the MIS is two consisting of $\{3, 5\}$ or $\{3, 1\}$ or $\{1, 4\}$ (there are more).



   Encode the problem of finding the MIS of a graph H over n nodes as a problem of MAP inference in a suitably constructed undirected graphical model (UGM) $P(X_1, \ldots X_m)$. Clearly show the graph of the UGM and the potentials such that $\text{argmax}_{X_1,\ldots,X_m} P(X_1, \ldots X_m)$ will give us a solution to the original MIS problem. (Higher marks for a positive distribution.) ..4 Let G be the required UGM. The nodes and edges in G are identical to those in H. Each variable in G is binary and all the node potential (in log form) are defined as follows. $\theta(1) = 1, \theta(0) = 1$. Edge potentials between any two nodes: $\theta(1, 1) = -2N, \theta(0, 1) = \theta(1, 0) = \theta(0, 0) = 0$. The exponent of all these potentials is positive. Thus, we get a positive distribution. The MAP solution gives as the maximum independent set those vertices that have been assigned a label 1. It is easy to see that the MAP will always give rise to an independent set because the $\theta(1, 1) = -2N$ ensures that any solution where adjacent nodes are labeled 1 can always be improved by flipping a node to 0.

3. For the undirected H below, perform the following operations



(a) Triangulate H ..1

(b) Identify the maximal cliques using the method discussed in class. [Just listing the maximal cliques will carry no marks.] ..2

(c) Draw the weighted graph over clique nodes and find its maximum spanning tree to get the junction tree of H. ..2

(d) Let C be set of variables that form a leaf node of a junction tree (JT), and let $S \subset C$ be the separator set that connects C to the rest of the JT. What is the minimum amount of work required to compute $\Pr(C - S|S)$. Let n be the number of variables, m the cardinality of each variable, and k the size of the largest clique in JT. ..3

$\Pr(C - S|S) = \frac{\Pr(C)}{S} = \frac{\sum_{\mathbf{x}_{V-C}} \prod_F \psi_F(\psi_F(\mathbf{x}_F))}{\sum_{\mathbf{x}_{V\S}} \prod_F \psi_F(\psi_F(\mathbf{x}_F))}$ By the running intersection property, all factors $F \neq C$ contain no variable from $C - S$. Thus, we can rewrite the numerator and denominator and obtain that $\Pr(C - S|S) = \frac{\psi_C(\mathbf{x}_C)}{\sum_{\mathbf{x}_{C-S}} \psi_C(\mathbf{x}_C)}$

4. Let G be a chordal graph with a junction tree T. We add an edge between two non-adjacent vertices x_i, x_j in G to get a new graph G'.

(a) Show how you will modify T to get the junction tree T' for G' under the following cases. Justify your answer.

i. x_i, x_j are simplicial and their neighbors in G are the same. ..2
let ci and cj be the cliques which contain xi and xj respectively and let sij be common neighbors. Add xj to ci. Let Pij be the path between ci and cj. Connect all nbrs of cj that are not on Pij, to node ci.

ii. There is no path between x_i and x_j in G. ..2
Easy. Just create a new clique node C with xixj and connect C to a clique that contains xi and another clique that contains xj.

(b) Show a case where T and T' have no cliques in common and T has at least three nodes. ..3
If G is an open chain and we add an edge connecting the two ends. Then triangulation will lead to creating cliques over three variables at a time. The original junction tree is only over variable pairs in a chain.

Total: 26