ME7223 :Optimization Methods for Mechanical Design
Assignment 1

INDIAN INSTITUTE OF TECHNOLOGY
MADRAS

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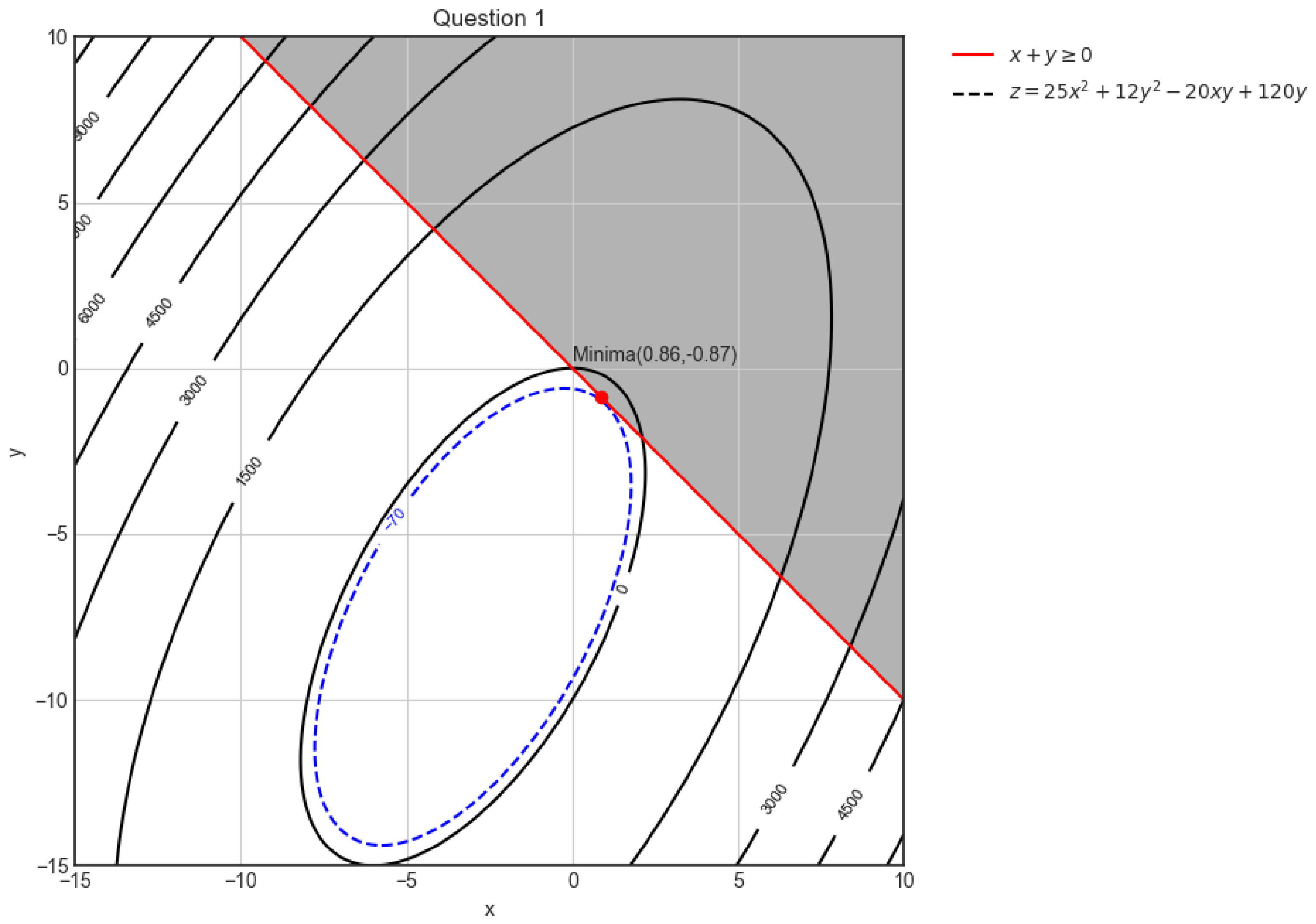
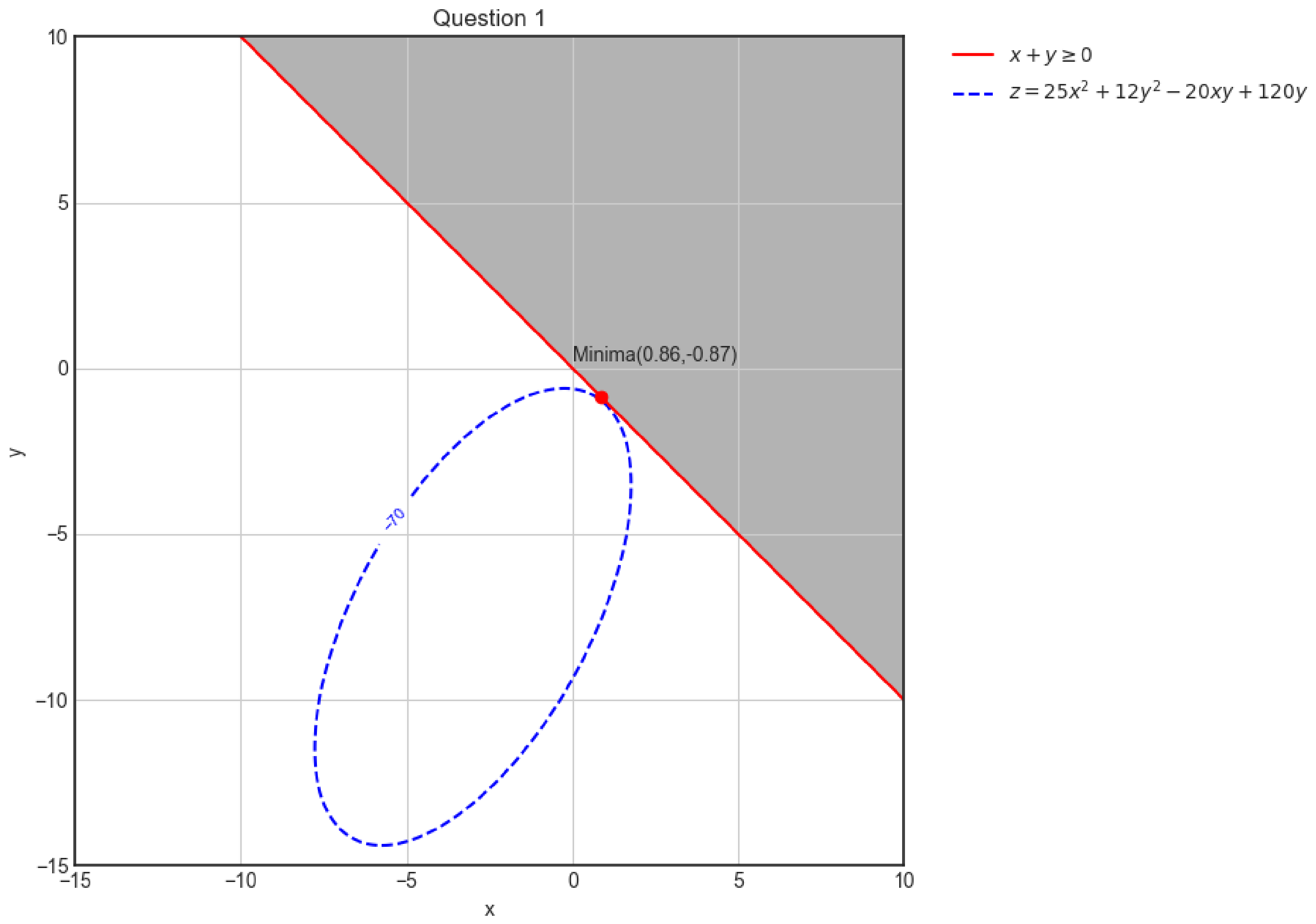
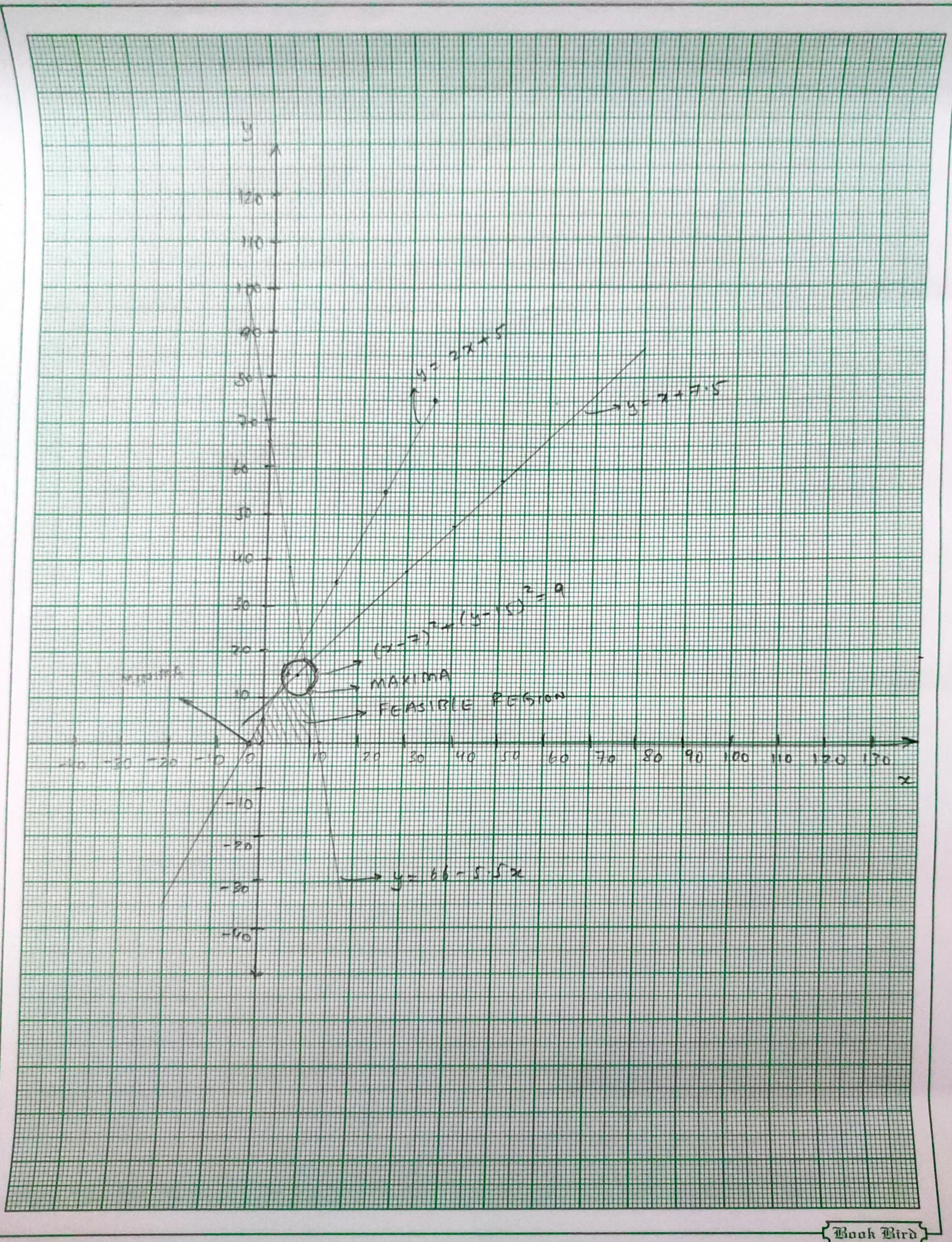


Question 1
Minima(0.86,-0.87)
$x + y \geq 0$
$z = 25x^2 + 12y^2 - 20xy + 120y$
x
y
6000
4500
3000
1500
-70
0
3000
4500

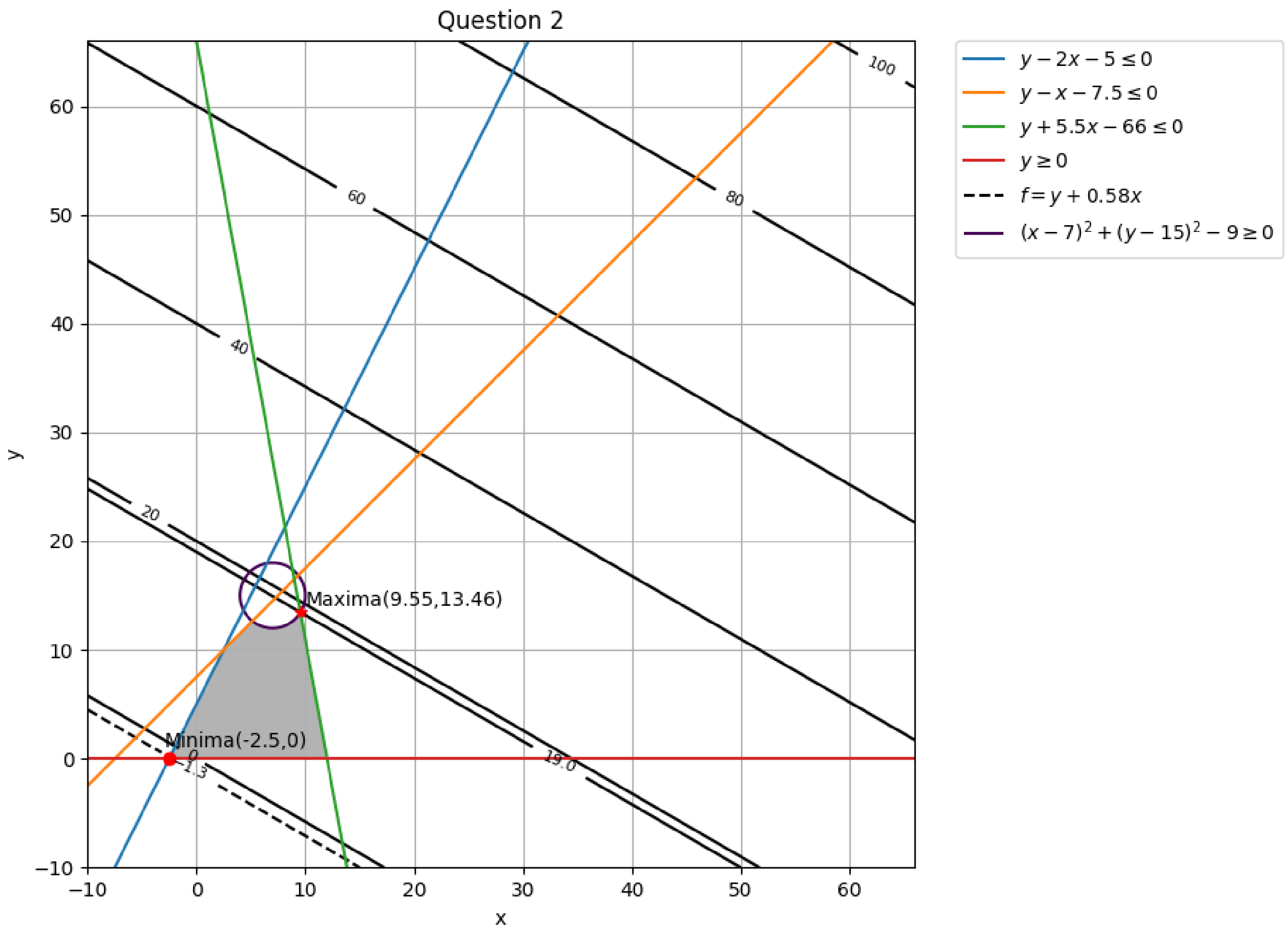


Question 1
Minima(0.86,-0.87)
-70
$x + y \geq 0$
$z = 25x^2 + 12y^2 - 20xy + 120y$
x
y
10
5
0
−5
−10
−15
−15
−10
−5
0
5
10

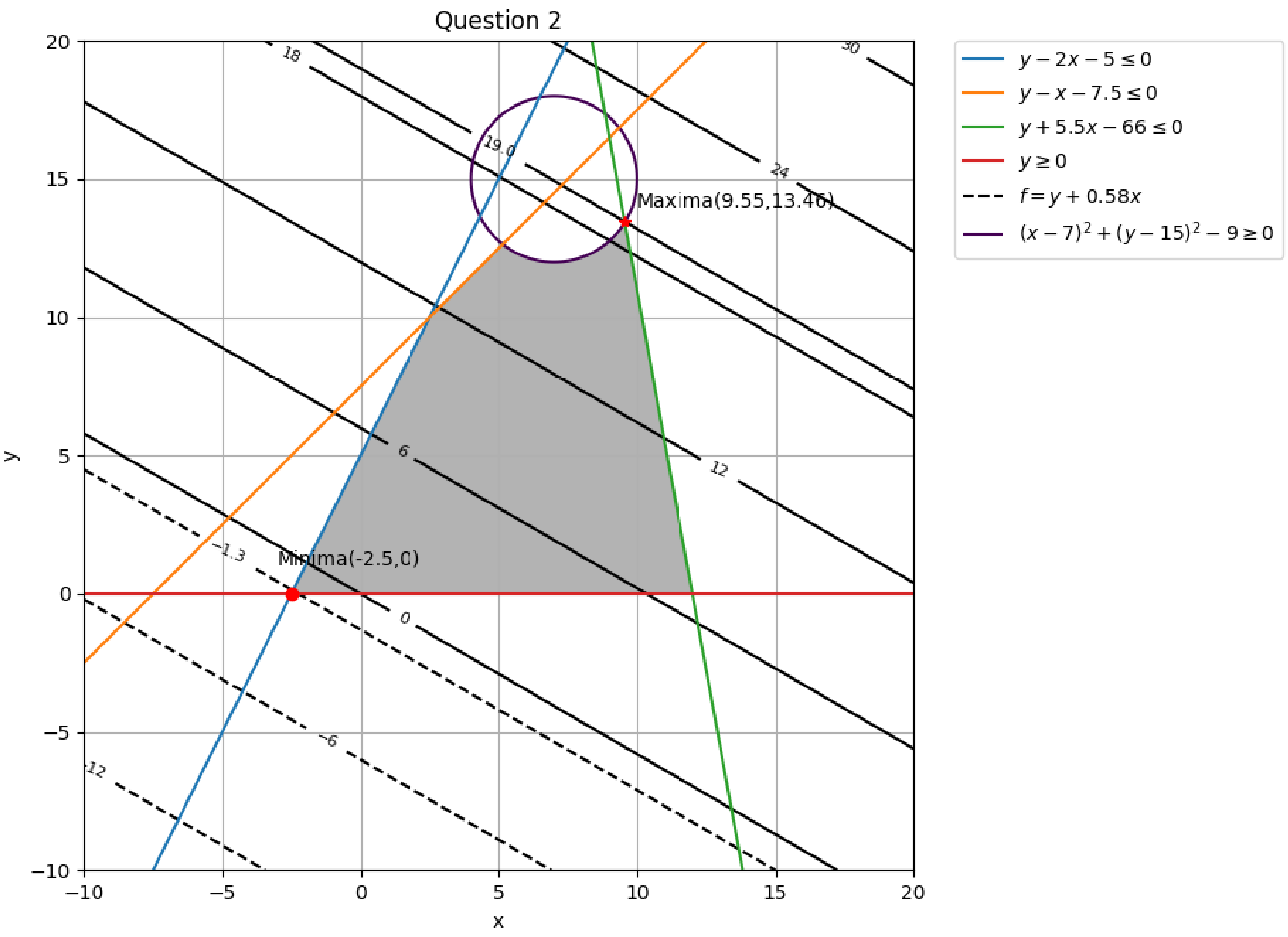
QUESTION - 2



y
x
y = 2x + 5
y = x + 7.5
(x-7)^2 + (y-15)^2 = 9
MAXIMA
MINIMA
FEASIBLE REGION
y = 66 - 5.5x



Question 2
Maxima(9.55,13.46)
Minima(-2.5,0)
$y-2x-5 \leq 0$
$y-x-7.5 \leq 0$
$y+5.5x-66 \leq 0$
$y \geq 0$
$f = y + 0.58x$
$(x-7)^2+(y-15)^2-9 \geq 0$
x
y



Question 2
Maxima(9.55,13.46)
Minima(-2.5,0)
$y - 2x - 5 \leq 0$
$y - x - 7.5 \leq 0$
$y + 5.5x - 66 \leq 0$
$y \geq 0$
$f = y + 0.58x$
$(x-7)^2 + (y-15)^2 - 9 \geq 0$
x
y

(3) $f(x_1 x_2 x_3) = -x_1^2 - x_2^2 + 2x_1x_2 - x_3^2 + 6x_1x_3 + 4x_1 - 5x_3 + 2$ —(1)

Expand in matrix form,

$$f(x) = \frac{1}{2} x^T [A] x + B^T x + c$$

$$= \frac{1}{2} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + B^T x + c$$

$$= \frac{1}{2} \{ a x_1^2 + 2b x_1 x_2 + 2c x_1 x_3 + d x_2^2 + 2e x_2 x_3 + f x_3^2 \} + B^T x + c$$

comparing the above equation with eqn (1)

$$\frac{a}{2} = -1 \quad \frac{d}{2} = -1 \quad \frac{f}{2} = -1$$

$$\frac{1}{2} 2b = 2 \quad \frac{1}{2} 2e = 0 \quad \frac{1}{2} \times 2c = 6$$

$$\therefore \quad a = -2 \quad b = 2 \quad c = 6$$

$$d = -2 \quad e = 0 \quad f = 0$$

We know that A is symmetric for quadratic eqn.

$$\therefore \quad A = \begin{bmatrix} -2 & 2 & 6 \\ 2 & -2 & 0 \\ 6 & 0 & 0 \end{bmatrix}$$

We can see that $a_{11} = -2$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (4-4) = 0$$

$$4 \quad |A| = 72$$

Hence A is indefinite

(4) Potential Energy of particle:-

$$u(x) = 3x^2 - x^3$$

We know that force (F) can be defined as

$$F(x) = -\frac{\partial u}{\delta x}$$

Identifying the equilibrium points:-

$$F(x) = -\frac{\partial u}{\delta x} = -6x + 3x^2$$

The system is said to be in equilibrium if $F = 0$ i.e. force vanishes

$$3x^2 - 6x = 0$$

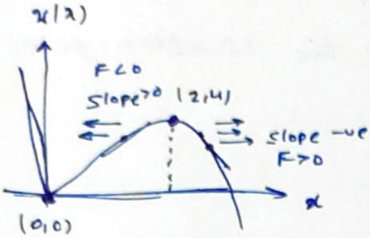
$$3x(x-2) = 0$$

$$x^* = 0 \; ; \; x^* = 2$$

If we take the equilibrium point (2,4) on graph

case (1) and move the particle towards "Right"

then the slope of graph being negative



$\frac{du}{dx} = -ve.$

This implies $F = +ve$

case (2) If we move the particle towards "Left")

then:-

$\frac{du}{dx} = +ve$

This implies $F = -ve$

From above 2 cases we can say that the particle doesnot return to equilibrium position, hence the point is an unstable equilibrium point

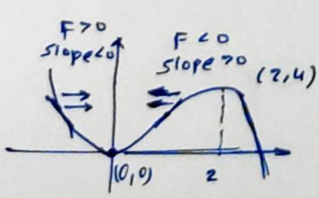
If we take point (0,0) on graph

case (1) If we move the particle towards right

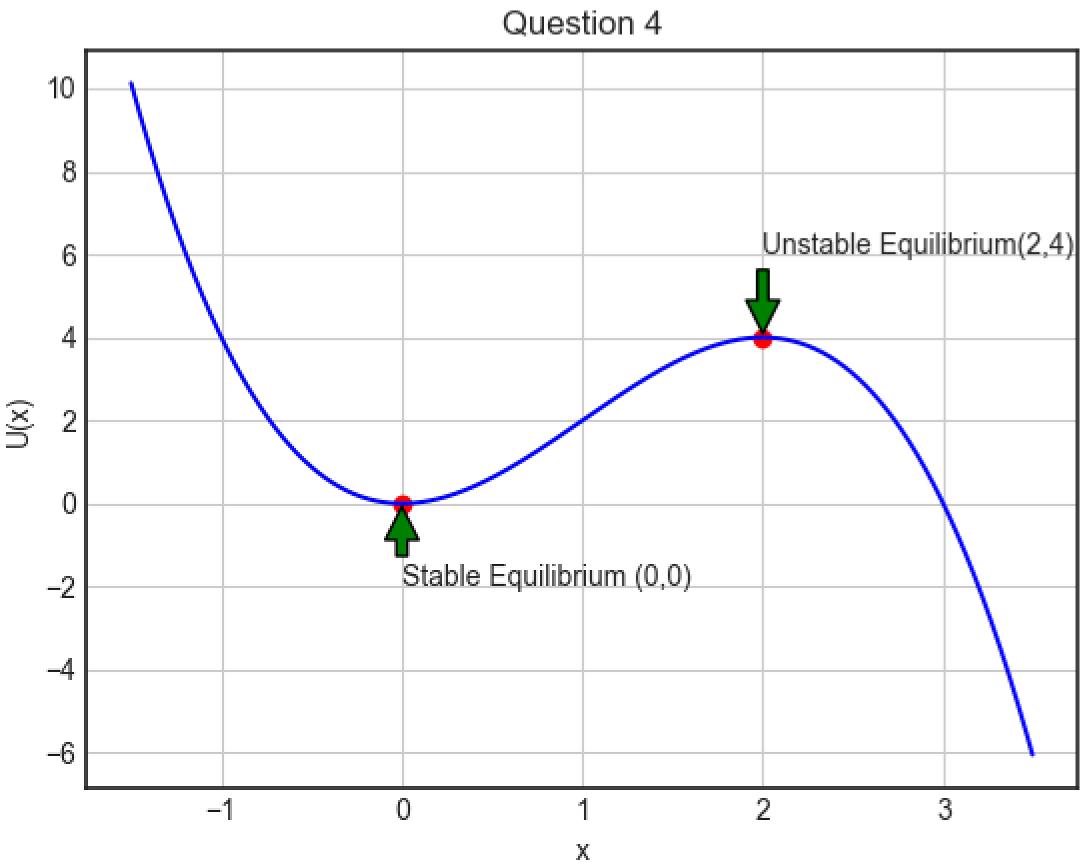
$\frac{du}{dx} = +ve \implies F = -ve$

case (2) If we move the particle towards left

$\frac{du}{dx} = -ve \implies F = +ve$



Here the particle returns to equilibrium position, hence, the point is a stable equilibrium point



Question 4
Unstable Equilibrium(2,4)
Stable Equilibrium (0,0)
U(x)
x
10
8
6
4
2
0
−2
−4
−6
−1
0
1
2
3

(5) Second order Taylor's series :-

$$f(x_1 x_2 x_3) = x_2^2 x_3 + x_1 e^{x_3} \qquad \text{point:- } (1, 0, -2)$$

$$f(x) = f(x^*) + df(x^*) + \frac{1}{2!} d^2 f(x^*)$$

$$df(x) = \sum_{i=1}^{3} h_i \frac{df}{dx_i}$$

$$= h_1 \frac{\partial f}{\partial x_1} + h_2 \frac{\partial f}{\partial x_2} + h_3 \frac{\partial f}{\partial x_3}$$

$$= h_1 \{e^{x_3}\} + h_2 \{2 x_2 x_3\} + h_3 \{x_2^2 + x_1 e^{x_3}\}$$

@ point $(1, 0, -2)$

$$= h_1 \{e^{-2}\} + h_2 \{2 \times 0 \times (-2)\} + h_3 \{(0)^2 + 1 \times e^{-2}\}$$

$$\boxed{df(x^*) = \{h_1 + h_3\} e^{-2}}$$

$$d^2 f(x) = \sum_{i=1}^{3} \sum_{j=1}^{3} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$= h_1^2 \frac{\partial^2 f}{\partial x_1^2} + h_1 h_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + h_1 h_3 \frac{\partial^2 f}{\partial x_1 \partial x_3} + h_2 h_1 \frac{\partial^2 f}{\partial x_2 \partial x_1} + h_2^2 \frac{\partial^2 f}{\partial x_2^2}$$

$$+ h_2 h_3 \frac{\partial^2 f}{\partial x_2 \partial x_3} + h_3 h_1 \frac{\partial^2 f}{\partial x_3 \partial x_1} + h_3 h_2 \frac{\partial^2 f}{\partial x_3 \partial x_2} + h_3 \frac{\partial^2 f}{\partial x_3^2}$$

$$= h_1^2 \{0\} + h_1 h_2 \{0\} + h_1 h_3 \{e^{x_3}\} + h_2 h_1 \{0\} + h_2^2 \{2 x_3\}$$

$$+ h_2 h_3 \{2 x_2\} + h_3 h_1 \{e^{x_3}\} + h_3 h_2 \{2 x_2\} + h_3 \{x_1 e^{x_3}\}$$

$$= 2 h_1 h_3 \{e^{-2}\} + (-4) h_2^2 + h_3 e^{-2}$$

$$\boxed{d^2 f(x^*) = 2 h_1 h_3 e^{-2} - 4 h_2^2 + h_3 e^{-2}}$$

$$f(x^*) = (0)^2 (-2) + 1 \times e^{-2}$$

$$\boxed{f(x^*) = e^{-2}}$$

$$f(x) = e^{-2} + \{h_1 + h_3\} e^{-2} + \frac{1}{2}\{2h_1h_3 e^{-2} - 4h_2^2 + h_3 e^{-2}\} \quad -(1)$$

we know $h = x - x^*$

$$h_1 = x_1 - x_1^*$$
$$= x_1 - 1$$

$$h_2 = x_2 - 0$$

$$h_3 = x_3 - (-2) = x_3 + 2$$

Substituting in eq^n (1)

$$\boxed{f(x) = e^{-2} + \{x_1 - 1 + x_3 + 2\} e^{-2} + \frac{1}{2}\{2(x_1 - 1)(x_3 + 2)e^{-2} - 4 \times (x_2)^2 + (x_3 + 2) e^{-2}\}}$$

Second order Taylor series expansion for

$$f(x) = x_2^2 x_3 + x_1 e^{x_3}$$

(6) For a $\triangle$le ABC find max value of $\sin A + \sin B + \sin C$

Constraint :- $A + B + C = \pi \Rightarrow g = A + B + C - \pi$

$A, B, C > 0$

using Lagrange multiplier method

$$\boxed{L = \sin A + \sin B + \sin C + \lambda \{A + B + C - \pi\}}$$

$\frac{\partial L}{\partial A} = 0 \Rightarrow \cos A + \lambda = 0$

$\frac{\partial L}{\partial B} = 0 \Rightarrow \cos B + \lambda = 0$

$\frac{\partial L}{\partial C} = 0 \Rightarrow \cos C + \lambda = 0$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow A + B + C = \pi$

using the equations we can say

$$\boxed{A = B = C = \frac{\pi}{3}}$$

$$\boxed{\lambda = -\cos\frac{\pi}{3} = -\frac{1}{2}}$$

using the sufficiency condition:-

$L_{11} = \frac{\partial^2 L}{\partial A^2} = -\sin\frac{\pi}{3}$

$L_{22} = \frac{\partial^2 L}{\partial B^2} = -\sin\frac{\pi}{3}$

$L_{33} = \frac{\partial^2 L}{\partial C^2} = -\sin\frac{\pi}{3}$

$L_{12} = L_{21} = L_{32} = L_{23} = L_{13} = L_{31} = 0$

$g_{11} = g_{12} = g_{13} = 1$

$$\begin{vmatrix} -\sqrt{3}/2 - z & 0 & 0 & 1 \\ 0 & -\sqrt{3}/2 - z & 0 & 1 \\ 0 & 0 & -\frac{\sqrt{3}}{2} - z & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

$-3z^2 - 5.196z - 2.249 = 0$

$\boxed{z = -0.866} \Longrightarrow$ Since z is negative we obtain a maximum.

(7) Minimize:

$$f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

subject to:-

$$g_1(x) = x_1 - x_2 = 0$$

$$g_2(x) = x_1 + x_2 + x_3 - 1 = 0$$

(a) Direct substitution:-

$x_1 = x_2$

$x_3 = 1 - 2x_2$

$$\boxed{f(x) = \frac{1}{2}(2x_2^2 + (1-2x_2)^2)}$$

$$= \frac{1}{2}(2x_2^2 + 4x_2^2 - 4x_2 + 1)$$

$$f(x) = 3x_2^2 - 2x_2 + 1/2$$

$$f'(x) = 0$$

$$6x_2 - 2 = 0$$

$\boxed{x_2 = 1/3}$ $\boxed{x_1 = 1/3}$ $\boxed{x_3 = 1/3}$

Hessian matrix:-

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

J is a positive definite matrix

$\therefore \overset{*}{x} = [1/3 \ 1/3 \ 1/3]^T$ is a minima

(b) Constrained variation method

$$J\left(\frac{g_1, g_2}{x_1, x_2}\right) \neq 0 \quad \& \quad J\left(\frac{f, g_1, g_2}{x_k\ x_1\ x_2}\right) = 0$$

$\downarrow$

$$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1 = 2 \neq 0 \rightarrow \text{satisfied}$$

we have 2 constraints & 3 design variables

$m = 2, \ n = 3$

$k = m+1 \Rightarrow 2+1 = 3$

$$J\left(\frac{f, g_1, g_2}{x_k\ x_1\ x_2}\right) = 0$$

$$\begin{vmatrix} x_3 & 0 & 1 \\ x_1 & 01 & 1 \\ x_2 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2x_3 - x_1 - x_2 = 0$

$2x_3 = x_1 + x_2$ from g_1 :- $x_1 = x_2$

So:- $2x_3 + x_3 - 1 = 0$

$$\boxed{x_3 = 1/3}$$

From $g_2 \Rightarrow x_1 = 1/3$ $\quad x_2 = 1/3, \ x_3 = 1/3$

Hessian matrix is positive definite as shown, so

$\overset{*}{x}$ is a minimum.

(c) Lagrangian Multiplies method

$$L = f + \lambda_1 g_1 + \lambda_2 g_2$$

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_1} = 0 \quad \text{—(1)}$$

$$\frac{\partial L}{\partial x_1} = x_1 + \lambda_1 + \lambda_2 = 0 \quad \text{—(2)}$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g_1}{\partial x_2} + \lambda \frac{\partial g_2}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial x_2} = x_2 - \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = \frac{\partial f}{\partial x_3} + \lambda_1 \frac{\partial g_1}{\partial x_3} + \lambda \frac{\partial g_2}{\partial x_3} = 0$$

$$\frac{\partial L}{\partial x_3} = x_3 + \lambda_2 = 0 \quad \text{—(3)}$$

$$\frac{\partial L}{\partial \lambda_1} = g_1 = 0 \quad \text{—(4)} \qquad \frac{\partial L}{\partial \lambda_2} = g_2 = 0 \quad \text{—(5)}$$

From (4) :- $x_1 = x_2$

From (5) :- $x_3 = 1 - 2x_2$

From (3) :- $\lambda_2 = -x_3$

Ean (1) becomes: $x_2 + \lambda_1 + (2x_2 - 1) = 0$

Ean(2) becomes: $x_2 - \lambda_1 + (2x_2 - 1) = 0$

$$2\lambda_1 = 0$$

$$\boxed{\lambda_1 = 0}$$

So $3x_2 = 1$

$$\boxed{x_2 = 1/3} \qquad \boxed{x_1 = x_2 = 1/3}$$

$$\lambda_2 = -1/3$$

Sufficiency condition:-

$$\begin{vmatrix} L_{11}-z & L_{12} & L_{13} & g_{11} & g_{21} \\ L_{21} & L_{22}-z & L_{23} & g_{12} & g_{22} \\ L_{31} & L_{32} & L_{32}-z & g_{13} & g_{23} \\ g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & g_{23} & 0 & 0 \end{vmatrix} = 0$$

For minima z must be positive.

$$L_{11} = \frac{\partial^2 L}{\partial x_1^2} \quad L_{22} = \frac{\partial^2 L}{\partial x_2^2} \quad L_{33} = \frac{\partial^2 L}{\partial x_3^2} \quad L_{12} = L_{21} = \frac{\partial^2 L}{\partial x_1 \partial x_2}$$

$$L_{31} = L_{13} = \frac{\partial^2 L}{\partial x_1 \partial x_3} \qquad L_{23} = L_{32} = \frac{\partial^2 L}{\partial x_2 \partial x_3}$$

$$L_{11} = L_{22} = L_{33} = 1 \qquad L_{13} = L_{23} = L_{12} = 0$$

$$g_{11} = 1 \quad g_{21} = 1 \quad g_{12} = -1 \quad g_{22} = 1 \quad g_{13} = 0 \quad g_{23} = 1$$

$$\begin{vmatrix} 1-z & 0 & 0 & 1 & 1 \\ 0 & 1-z & 0 & -1 & 1 \\ 0 & 0 & 1-z & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{vmatrix} = 0$$

$$6 - 6z = 0$$

$$\boxed{z = 1}$$

$\because$ z is positive, $\overset{*}{x}$ is a minima

(8) (a) Minimize $f(x,y) = x^2 + y^2$

subject to

$g(x,y) = xy = 1$

using Lagrangian multiplier method

$g = -xy + 1 = 0$

$L = f + \lambda g$

$L = x^2 + y^2 + \lambda(1 - xy)$

$\frac{\partial L}{\partial x} = 2x + \lambda(-y) = 0 \quad — ①$

$\frac{\partial L}{\partial y} = 2y + \lambda(-x) = 0 \quad — ②$

$\frac{\partial L}{\partial \lambda} = 1 - xy = 0 \quad — ③$

From ① & ②

$\lambda = \frac{2x}{y} \quad \lambda = \frac{2y}{x}$

From ③

$x = \frac{1}{y}$

$x^2 = y^2$

$\frac{1}{y^2} = y^2 \quad \Rightarrow \quad y^4 = 1$

$y = +1, -1$

$x = +1, -1$

$\lambda = \frac{2(1)}{1} = +2$

∴ The optimum points are :- $(1,1)$ & $(-1,-1)$

(b) Find ∇f and ∇g at the solution point

$\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \quad \nabla g = \begin{bmatrix} -y \\ -x \end{bmatrix}$

@ $x,y = (1,1)$

$\nabla f = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\nabla g = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

@ $x,y = (-1,-1)$

$\nabla f = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

$\nabla g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

At the solution points: ∇f & ∇g are related as

$$\boxed{\nabla f = -\lambda \nabla g}$$

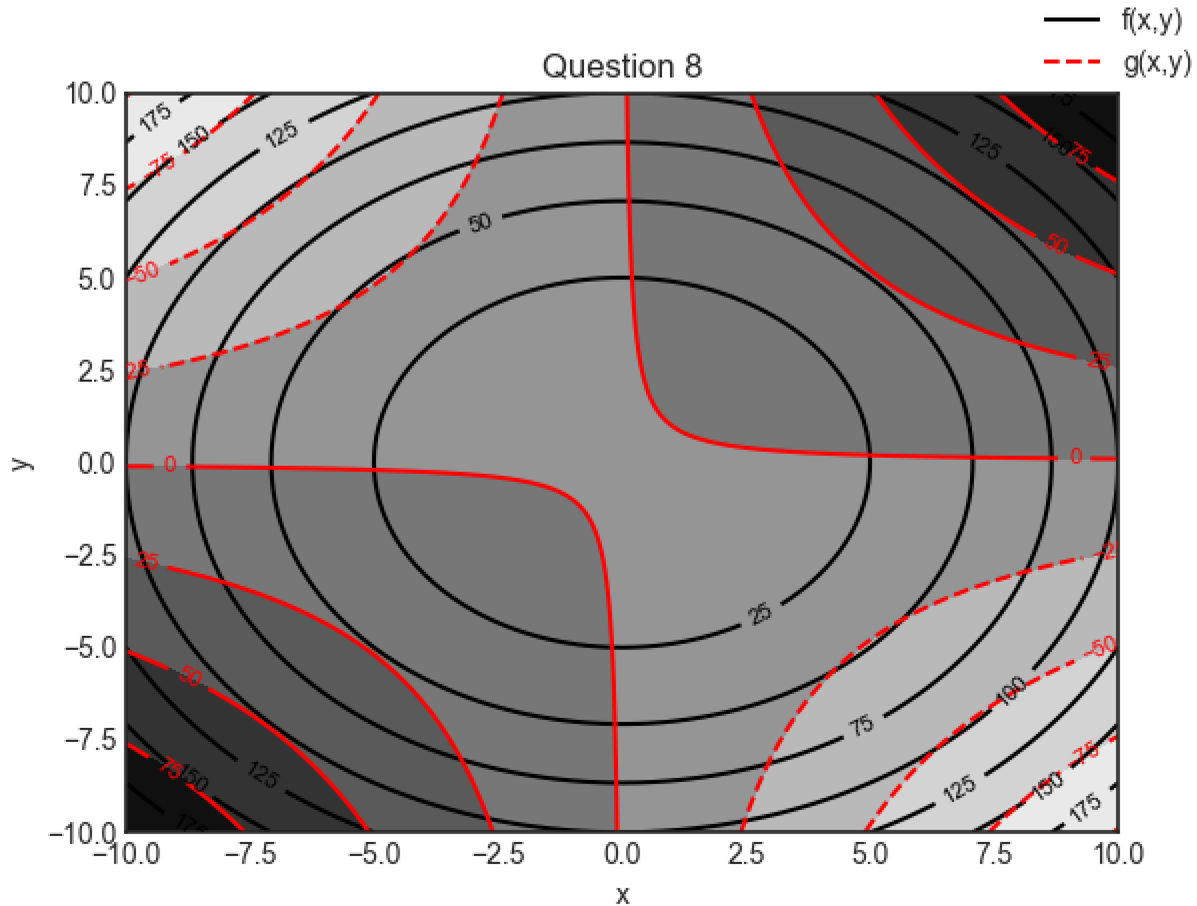
This implies that direction gradient of objective function 'f' is opposite to direction of gradient of constraint 'g' at the solution point

c) The relationship between ∇f & ∇g can be expressed as:-

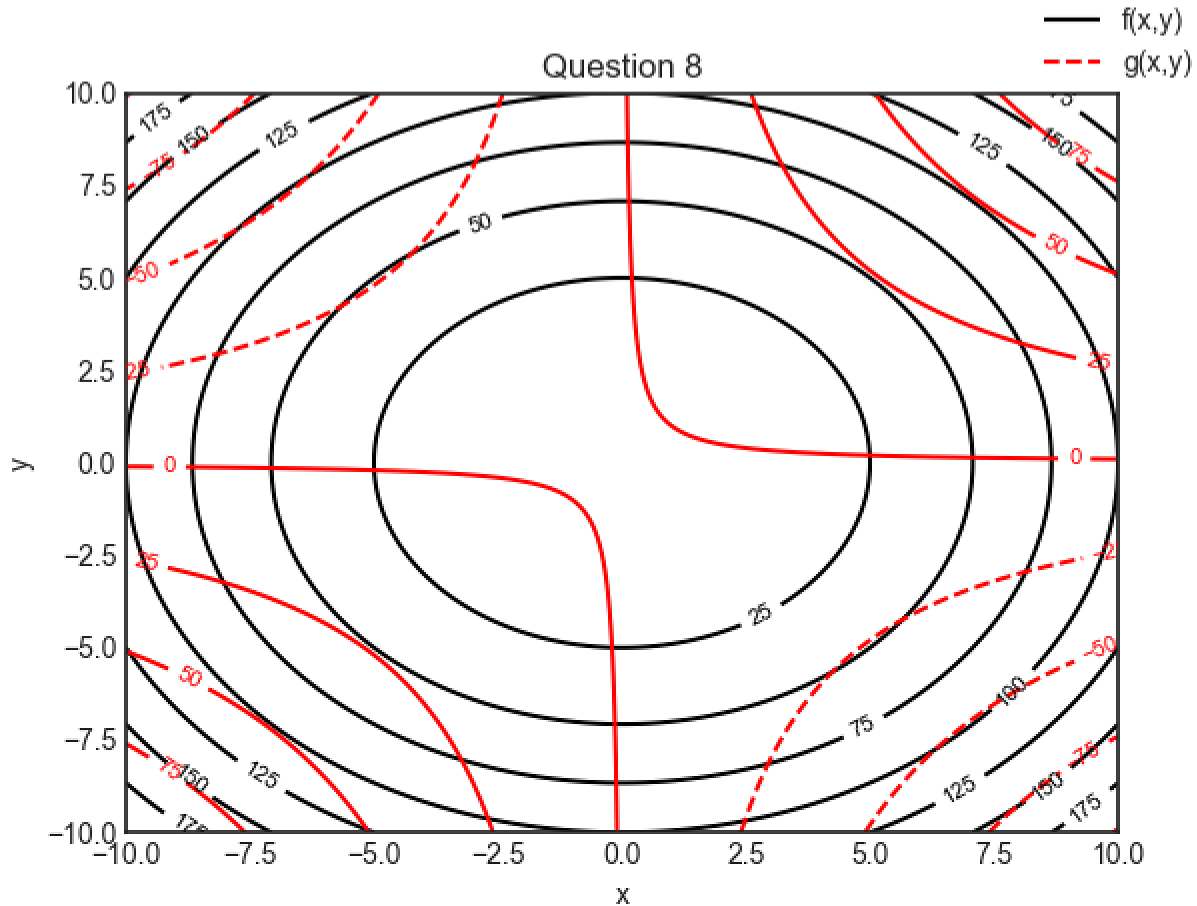
$$-\nabla f = \lambda \nabla g$$

$\Rightarrow \boxed{-\nabla f = \lambda \nabla g}$ at the solution point

It can be seen that ∇f, ∇g & λ are <u>linearly</u> dependent.



f(x,y)
g(x,y)
Question 8
x
y
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
175
150
125
100
75
50
25


f(x,y)
g(x,y)
Question 8
x
y
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
175
150
125
100
75
50
25
0
-25
-50
-75