MATH 317: SPRING 2016 **ANSWERS** **EXAM 1**

1. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be vectors in $\mathbb{R}^n$.

(3pts) (a) Define the dot product of $\mathbf{x}$ and $\mathbf{y}$.

Answer:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

(3pts) (b) What does it mean to say that $\mathbf{x}$ and $\mathbf{y}$ are *parallel*?

Answer: The vectors $\mathbf{x}$ and $\mathbf{y}$ are *parallel* if they are scalar multiples of each other. That is, there exists a real number c such that $\mathbf{x} = c\,\mathbf{y}$.

(4pts) (c) What does it mean to say that $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal*?

Answer: The vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if their dot product is 0. That is, $\mathbf{x} \cdot \mathbf{y} = 0$.

(4pts) (d) Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are also vectors in $\mathbb{R}^n$. What does it mean to say that $\mathbf{x}$ is a *linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$?

Answer: $\mathbf{x}$ is a *linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if there exist scalars $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n.$$

(10pts) 2. Recall Prop. 2.1 of our text says that the dot product satisfies the following properties: for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

1. $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$;
2. $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2 \geq 0$, with equality if and only if $\mathbf{x} = 0$;
3. $(c\mathbf{x}) \cdot \mathbf{y} = c(\mathbf{x} \cdot \mathbf{y})$;
4. $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.

Prove that if $\mathbf{x}$ is orthogonal to each of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$, then $\mathbf{x}$ is orthogonal to every linear combination of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$. (For full credit, identify places in your proof where properties from the list above are used; to refer to these properties, use the letters given above.)

Answer: Suppose $\mathbf{x}$ is orthogonal to each of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$. Then,

$$\begin{aligned}
\mathbf{x} \cdot (c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) &= \mathbf{x} \cdot (c_1\mathbf{v}_1) + \cdots + \mathbf{x} \cdot (c_k\mathbf{v}_k) && (\because \text{Prop. 2.1 (4)}) \\
&= c_1(\mathbf{x} \cdot \mathbf{v}_1) + \cdots + c_k(\mathbf{x} \cdot \mathbf{v}_k) && (\because \text{Prop. 2.1 (1) and (3)}) \\
&= c_1 0 + \cdots + c_k 0 && (\because \mathbf{x} \cdot \mathbf{v}_i = 0 \text{ for all } 1 \leq i \leq k) \\
&= 0.
\end{aligned}$$

(5pts) 3. State a theorem about existence and uniqueness of solutions to the system $A\mathbf{x} = \mathbf{b}$. You may state more than one theorem if you wish, but quality is better than quantity. Only write what you know is true and **carefully state your assumptions**. If you make a broad statement that, in fact, only applies under a narrow set of conditions, and you leave out those conditions, then you will not receive very much credit. (Use only the space provided below.)

Answer: There is more than one good answer to this. Here's one possibility, which is probably the most helpful when answering the questions on the next page. (Other possible answers are given on the last page below.)

Let $A \in \mathbb{R}^{m \times n}$. Then the equation $A\mathbf{x} = \mathbf{b}$

1. is consistent for all $\mathbf{b} \in \mathbb{R}^m$ when $\operatorname{rank}(A) = m$ (=number of rows of A);
2. has at most one solution when $\operatorname{rank}(A) = n$ (=number of columns of A).

Recall that $\operatorname{rank}(A) = m$ iff the echelon form of A has no rows consisting of all zeros; $\operatorname{rank}(A) = n$ iff the echelon form of A has a pivot in every column.

4. For the given matrix, write down the letters corresponding to true statements in each case. Select from the following statements:

   A. For every $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ is consistent.
   B. For every $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ is inconsistent.
   C. For every $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has exactly one solution.
   D. For every $\mathbf{b} \in \mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
   E. There exists $\mathbf{b} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ is consistent.
   F. There exists $\mathbf{b} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent.
   G. There exists $\mathbf{b} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ has exactly one solution.
   H. There exists $\mathbf{b} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

(4pts) (a) If the matrix A has reduced echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

then which of the statements (a)–(h) above is true? (Select all that apply.)

A. B. ***C.*** D. ***E.*** F. ***G.*** H.

(4pts) (b) If the matrix A has reduced echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

then which of the statements (a)–(h) above is true? (Select all that apply.)

A. B. C. D. ***E.*** ***F.*** ***G.*** H.

(4pts) (c) If the matrix A has reduced echelon form $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$,

then which of the statements (a)–(h) above is true? (Select all that apply.)

A. B. C. ***D.*** ***E.*** F. G. ***H.***

(4pts) (d) If the matrix A has reduced echelon form $\begin{bmatrix} 1 & 0 & -1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

then which of the statements (a)–(h) above is true? (Select all that apply.)

A. B. C. D. ***E.*** ***F.*** G. ***H.***

5. Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{n\times m}$, and $\mathbf{b} \in \mathbb{R}^m$. For each statement below, either prove the claim or write FALSE and give a counter-example.

(5pts) (a) **Claim:** If $AB = I_m$, then a solution to $A\mathbf{x} = \mathbf{b}$, if it exists, is unique.

> **Answer:** FALSE. Let $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and suppose $\mathbf{b} = [0]$. Then there are infinitely many solutions to $A\mathbf{x} = \mathbf{b}$, since every $\mathbf{x} = (x_1, x_2)$ satisfying $x_1 + x_2 = 0$ is a solution.

(5pts) (b) **Claim:** If $CA = I_n$, then a solution to $A\mathbf{x} = \mathbf{b}$, if it exists, is unique.

> **Answer:** Suppose $A\mathbf{x} = \mathbf{b}$ has a solution, call it $\mathbf{x}'$. Suppose $\mathbf{x}''$ is another solution, so that $A\mathbf{x}'' = \mathbf{b}$. Then $A\mathbf{x}' = A\mathbf{x}''$, so $BA\mathbf{x}' = BA\mathbf{x}''$, so $I_n\mathbf{x}' = I_n\mathbf{x}''$, so $\mathbf{x}' = \mathbf{x}''$.

(5pts) (c) **Claim:** If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, then for every $\mathbf{b} \in \mathbb{R}^m$ there is exactly one solution to $A\mathbf{x} = \mathbf{b}$.

> **Answer:** FALSE. Let $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = [0]$. That is,
>
> $$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{if and only if} \quad x_1 = 0.$$
>
> In this case, *if the equation* $A\mathbf{x} = \mathbf{b}$ *has a solution*, then it is unique. However, it is not true that $A\mathbf{x} = \mathbf{b}$ always has a unique solution, since it may be inconsistent. For example, continuing with the matrix $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ above, if $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $A\mathbf{x} = \mathbf{b}$ has no solution.

6. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(8pts) (a) Find A^{-1}.

Answer:

$$A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

(5pts) (b) Use your answer to (a) to solve $A\mathbf{x} = \mathbf{b}$.

Answer:

$$\mathbf{x} = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$$

(2pts) (c) Use your answer to (b) to express $\mathbf{b}$ as a linear combination of the columns of A. (Fill in the blanks with your answers.)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \underline{\ -3\ } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \underline{\ 4\ } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \underline{\ 2\ } \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Alternative answers to Question 3:

- Suppose the system $A\mathbf{x} = \mathbf{b}$ is consistent. Then it has a unique solution if and only if the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens exactly when $\operatorname{rank}(A)$ is the number of columns of A.
- Let $A \in \mathbb{R}^{n\times n}$. The following are equivalent:
    1. A is nonsingular.
    2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
    3. For every $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution (indeed, a unique solution).