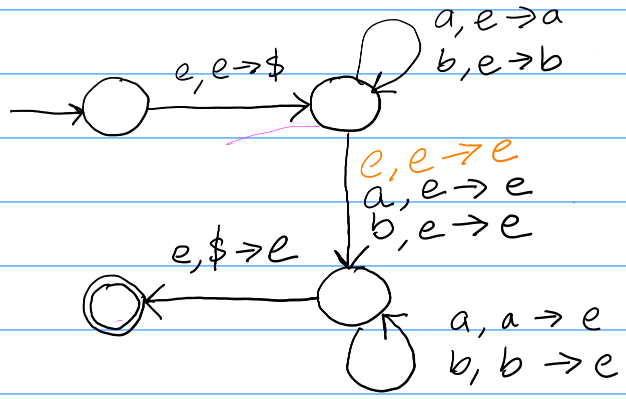
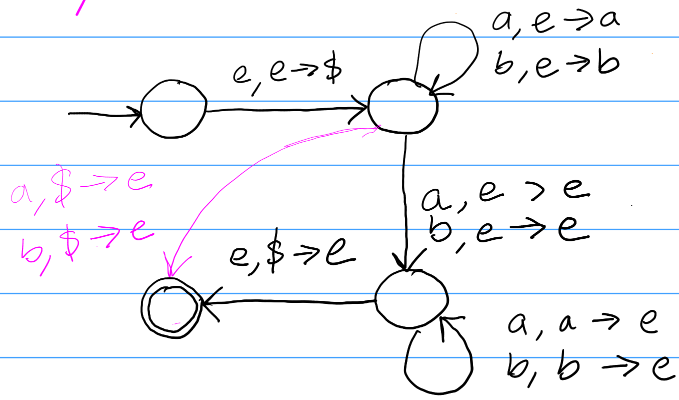
CS605
19 February 2025 (first, a little revision of last week's topics)

$L = \{ w \in \{a, b\}^*, \ w = w^R, \ |w| \text{ is odd} \}$



$L = \{ w \in \{a, b\}^*, \ w = w^R \}$

Student question: does this accept "a" or "b"?



$5 + 3 \times 2 = 7 / 3 \times 9$

$(3+2)^2$

$(3+2) \times (3+2)$

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   1 3 5
 ×   2 4
 ———————
   5 4 0
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Note to Tom: multiply 1 by 4 next.

CS605
19 February 2025 (official start of today's lecture)

Countable sets

A set is countable if it forms a bijection with $\mathbb{N} = \{0, 1, 2, \ldots\ldots\}$

The set $\mathbb{Z}$ is countable.

Proof idea

$\mathbb{Z}$	$\mathbb{N}$
0	0
-1	1
1	2
-2	3
2	4
-3	5
3	6
$\vdots$	$\vdots$

Proof

We will prove this by specifying a bijection f between $\mathbb{N}$ and $\mathbb{Z}$.

$$f : \mathbb{N} \to \mathbb{Z} \qquad f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ \dfrac{x+1}{-2} & \text{if } x \text{ is odd} \end{cases}$$

$$f^{-1} : \mathbb{Z} \to \mathbb{N} \qquad f^{-1}(x) \begin{cases} -2x - 1 & \text{if } x < 0 \\ 2x & \text{otherwise} \end{cases}$$

Bijection $f : \mathbb{N} \to \mathbb{Z}$ proves $\mathbb{Z}$ is countable.

$\text{PRIMES} = \{x \in \mathbb{N},\ x \text{ is prime}\}$ is countable.

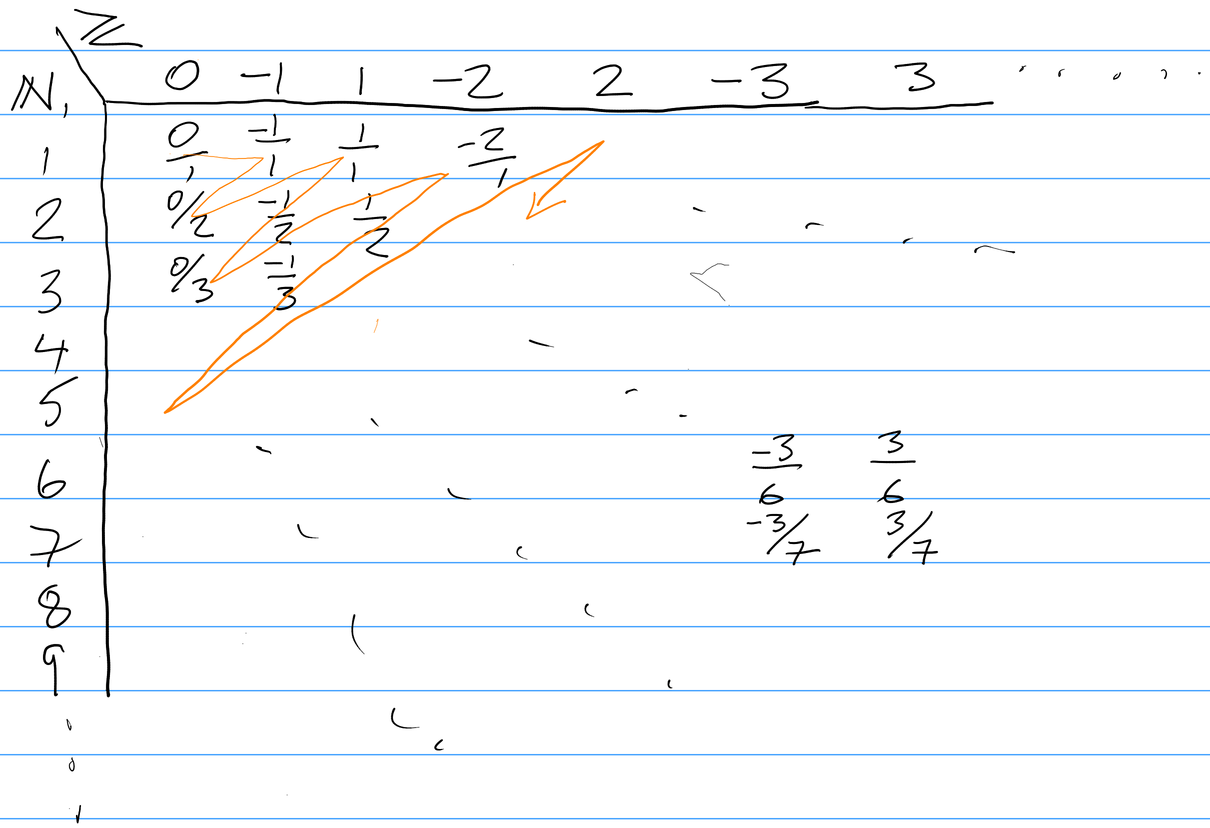
$\text{FIB} = \{x \in \mathbb{N},\ x \text{ is a number is the Fibbonaci sequence}\}$ is countable.

$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z},\ b \in \mathbb{N}_1 \right\}$ is countable.

$\mathbb{N}_1 = \{1, 2, 3, \ldots\}$

Proof idea

We will use a breadth-first search to enumerate $\mathbb{Q}$.



Since we are able to enumerate $\mathbb{Q}$, this proves it is countable.

Aside 2^{Σ^*} is not countable

$\Sigma^* = \{0,1\}^*$ is countable.

Since we can enumerate Σ^* (in lexicographical order, for example) this proves it is countable.

$\{w \in \{0,1\}^*, \; w = w^R\}$ is countable.

$\{a^n b^n : n \geq 0\}$ is countable.

WordDoc = { d : d is a Microsoft™ Word document } is countable.

Proof idea

Save to disk to get a unique bit sequence.

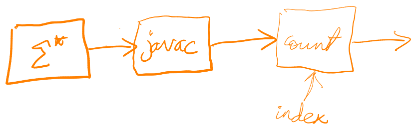
Order sequences lexicographically.

J = { j : j is a Java program contained in a single file } is countable

Proof idea

$\Sigma = \{0, \ldots 9, a, \ldots z, A \ldots Z, +, -, /, *, =, \{, \}, [,], (,), :, ;, ,, ?, |, ', ", \&, \_, <, >, !, \sqcup, \text{TAB}, \hookleftarrow, \sim, \hat{}, ., @, \#, \%,$

An enumerator for J.



The most convincing proof that a set can be enumerated is to define a computer program that lists (outputs) the elements of the set in some order (works every time).

$T = \{ M : M \text{ is a Turing machine} \}$ is countable.

$F = \{ M : M \text{ is a FA} \}$ is countable.

Uncountable sets.

A set is uncountable if it is not countable.

The set $\mathbb{R}_0^1 = \{x : x \in \mathbb{R}, 0 \leq x \leq 1\}$ is uncountable.

Proof We will prove this by contradiction using diagonalisation.

Assume $\mathbb{R}_0^1$ is countable. Therefore there must exist a bijection between $\mathbb{N}$ and $\mathbb{R}_0^1$, and it must be possible to order $\mathbb{R}_0^1$.

Let the following be an arbitrary ordering for $\mathbb{R}_0^1$ with corresponding decimal expansions.

Decimal expansion after the decimal point

$\mathbb{N}$	$\mathbb{R}_0^1$	1	2	3	4	5	6	7	...
0	0	0	0	0	0	0	0	0	...
1	$\pi/10$	3	1	4	1	5	9	2	...
2	$\sqrt{2}/10$	1	4	1	4	1	4	1	...
3	$1/2$	5	0	0	0	0	0	0	...
4	$1/3$	3	3	3	3	3	3	3	...
5	$1/7$	1	4	2	8	5	7	1	...
6	1	9	9	9	9	9	9	9	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Consider an arbitrary such bijection. Consider the decimal expansion of each value in $\mathbb{R}_0^1$ in a table as shown. Extract the diagonal.

Change each digit (not using 9).

Diagonal : 0. 0 1 1 0 3 7 9 . . . . .

Change each digit : 0. 2 4 6 8 1 3 0 . . . . . .

This new number is a value in $\mathbb{R}_0'$, but it is not in our table. A contradiction.

Our description of the bijection was completely general. This contradiction will occur for any ordering (any bijection, any table).

This proves the set $\mathbb{R}_0'$ is uncountable.