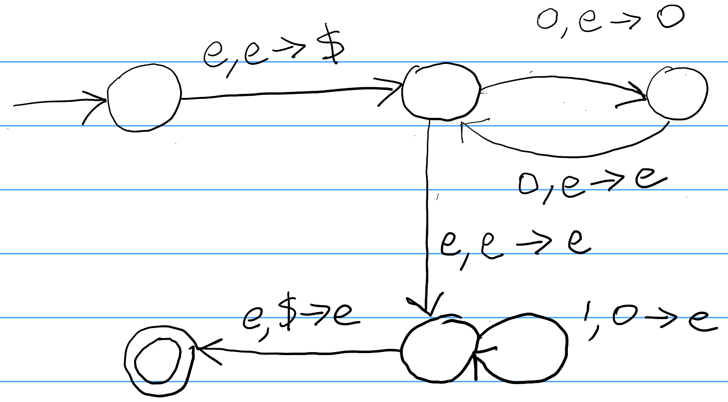
CS605
12 February 2025

$L4b \; \{0^{2n}1^n : n \geq 0\}$ → Two options → Push all zeros (you guys in lab)
→ Push half zeros (you guys in lab)



$L4_c \; \{w : w \in \{0,1\}^*, w$ has an equal number of 0s as 1s$\}$

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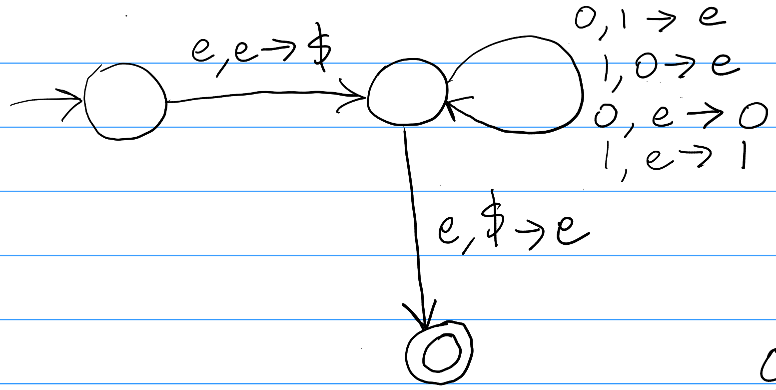

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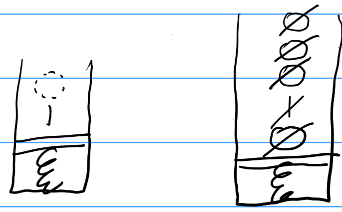

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For today's lab:

$W12A = \{ w \in \{0,1\}^* : w = 0^n1^n \text{ or } w = 1^n0^n, \; n \geq 0\}$

Context-free languages (CFL, c.f. languages)

DEFINITION
~~THEOREM~~ **2.20**

A language is context free if and only if some pushdown automaton recognizes it.

DEFINITION
~~LEMMA~~ **2.27**

If a pushdown automaton recognizes some language, then it is context free.

COROLLARY 2.32

Every regular language is context free.

Proof
We will prove this by constructing an equivalent PDA from an arbitrary FA.

Let L be a reg. lang. Since L is reg., a FA M exists to recognize it. Modify M by adding two e symbols to each row of the transition function as follows:

$$s, r, n \longrightarrow s, r, e, n, e$$

M is now a PDA that recognises the same language L.

According to Defn 2.27, this proves that L is c.f.

This procedure is completely general and works for every regular lang. L, therefore this proves that every reg. lang. is c.f.

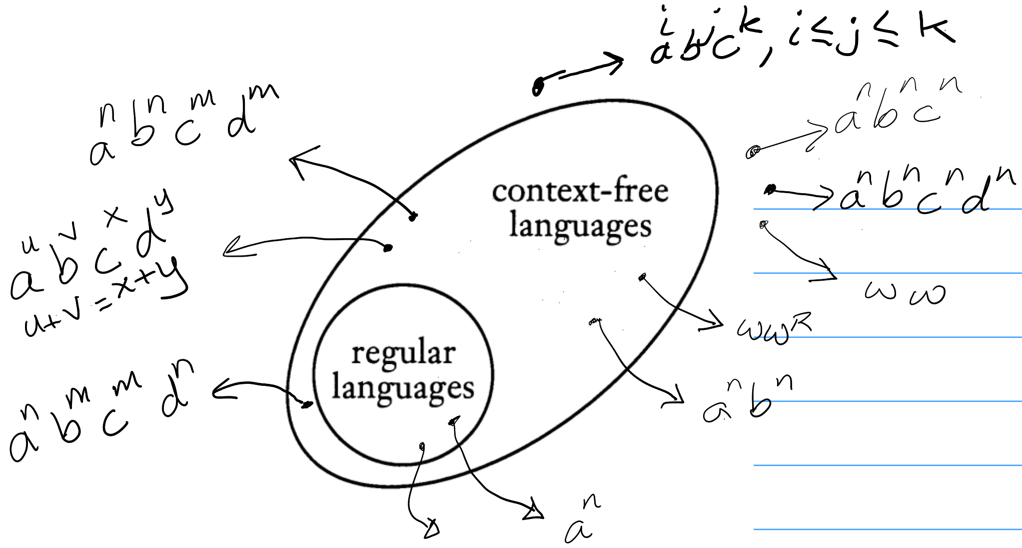


FIGURE 2.33
Relationship of the regular and context-free languages

2.3

NON-CONTEXT-FREE LANGUAGES

In this section we present a technique for proving that certain languages are not context free. Recall that in Section 1.4 we introduced the pumping lemma for showing that certain languages are not regular. Here we present a similar pumping lemma for context-free languages. It states that every context-free language has a special value called the ***pumping length*** such that all longer strings in the language can be "pumped." This time the meaning of *pumped* is a bit more complex. It means that the string can be divided into five parts so that the second and the fourth parts may be repeated together any number of times and the resulting string still remains in the language.

THE PUMPING LEMMA FOR CONTEXT-FREE LANGUAGES

THEOREM 2.34

Pumping lemma for context-free languages If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.

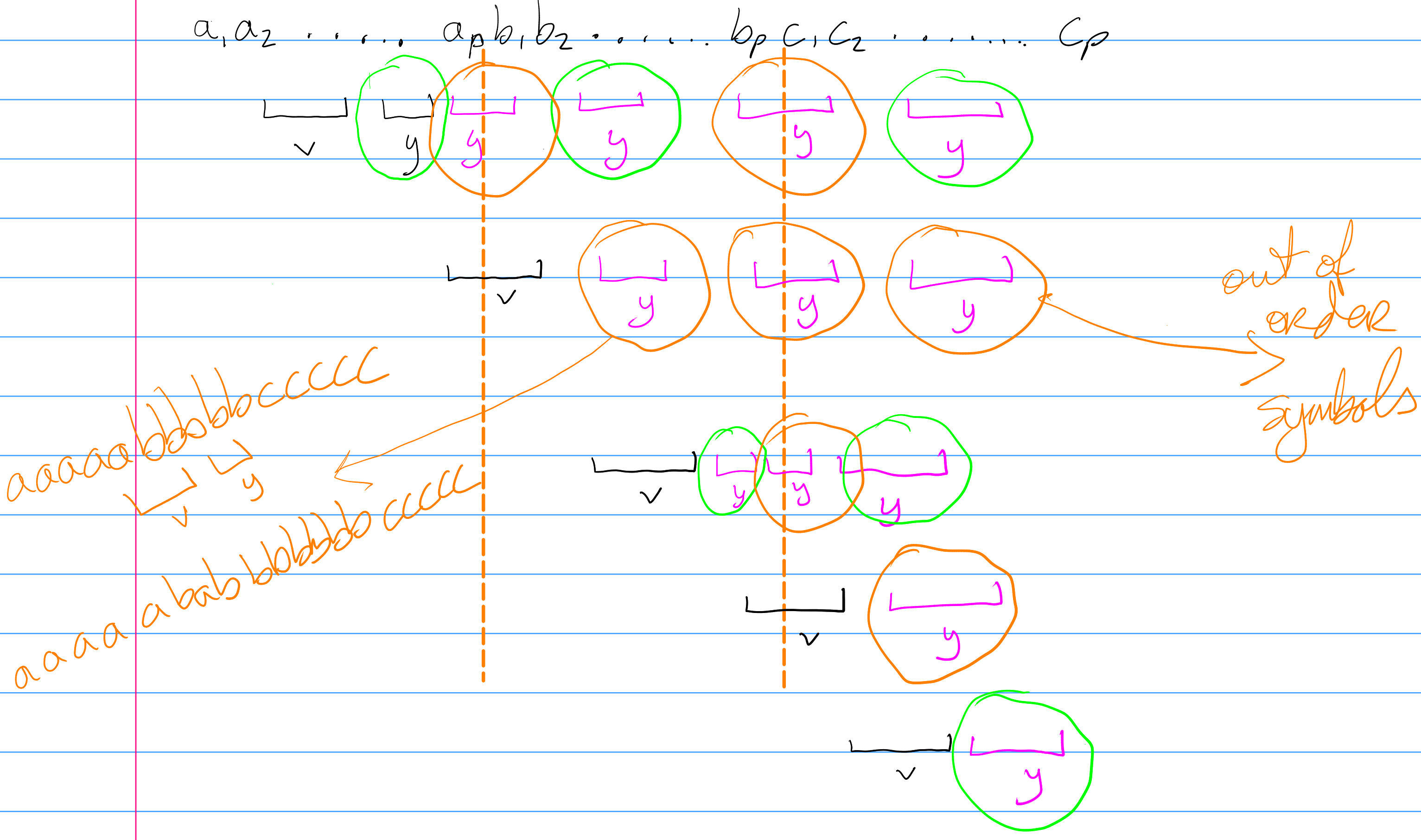
When s is being divided into $uvxyz$, condition 2 says that either v or y is not the empty string. Otherwise the theorem would be trivially true. Condition 3 states that the pieces v, x, and y together have length at most p. This technical condition sometimes is useful in proving that certain languages are not context free.

EXAMPLE 2.36 ....

Use the pumping lemma to show that the language $B = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$ is not context free.

Proof

We assume that B is a CFL and obtain a contradiction. Let p be the pumping length for B that is guaranteed to exist by the pumping lemma. Select the string $s = \mathtt{a}^p\mathtt{b}^p\mathtt{c}^p$. Clearly s is a member of B and of length at least p. The pumping lemma states that s can be pumped, but we show that it cannot. In other words, we show that no matter how we divide s into $uvxyz$, one of the three conditions of the lemma is violated.



All 13 ways of dividing s into uvxyz can be grouped into two categories:

1. v or y has a mixture of symbols

This will not produce words in B when we pump up, because we will have out of order symbols.

2. Both v and y have just one symbol type each (e.g. ($v = aa..aa$ and $y = aa..a$) or ($v = aa..a$ and $y = bb..b$)).

When we pump up, we'll leave out at least one symbol type, so words will not be in the lang. either.

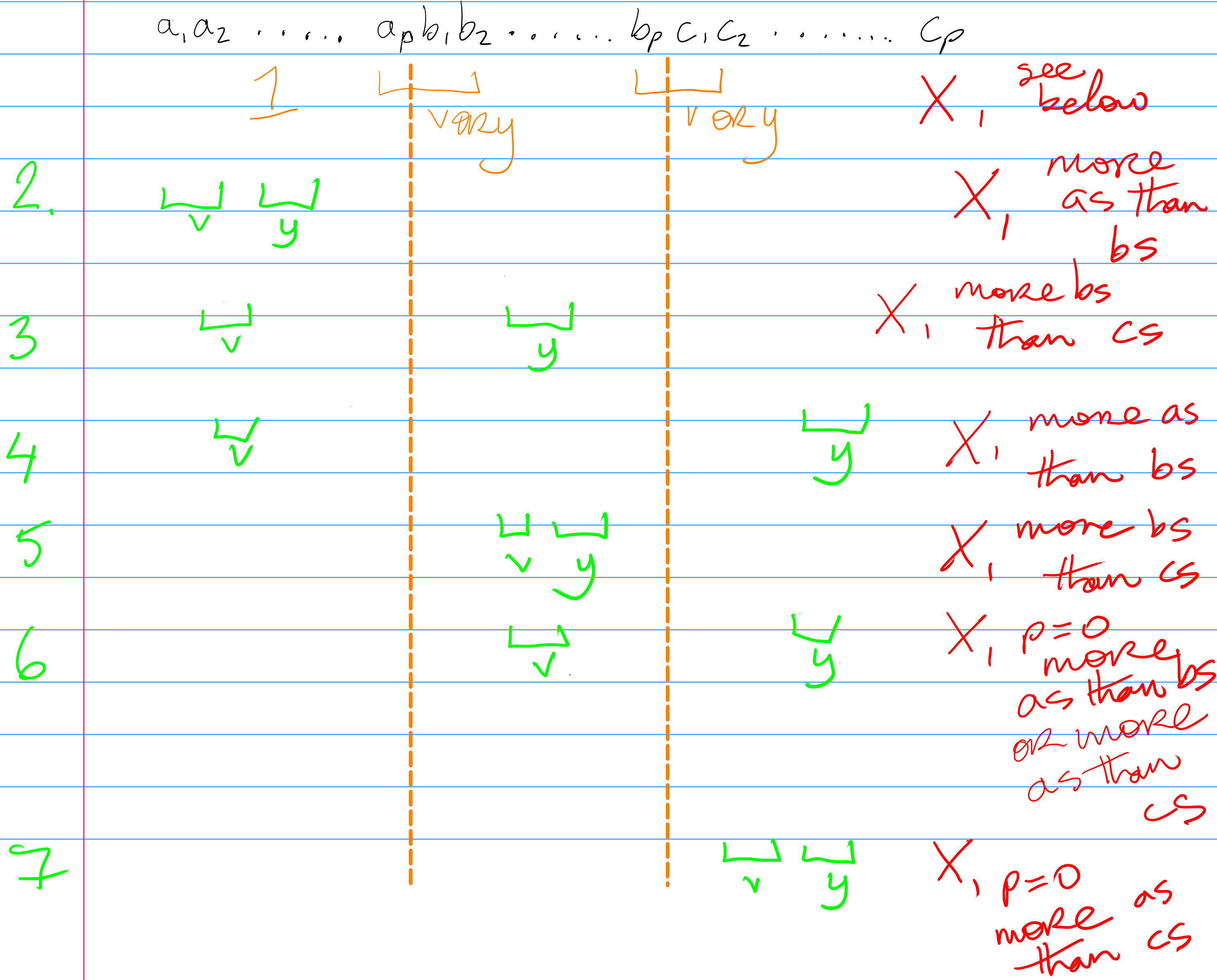
One of these cases must occur. Because both cases result in a contradiction, a contradiction is unavoidable. So the assumption that B is a CFL must be false. Thus we have proved that B is not a CFL. ∎

EXAMPLE 2.37

Let $C = \{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$. We use the pumping lemma to show that C is not a CFL. This language is similar to language B in Example 2.36, but proving that it is not context free is a bit more complicated.

Proof

We will prove this using the pumping lemma for CFLs. Assume C is c.f. in order to generate a contradiction. Let $s = a^p b^p c^p$. $s \in C$ and has a length greater than p, so the pumping lemma states it can be pumped according to three conditions. Let's look at all the ways s can be divided into $uvxyz$



1 v or y have a mixture of symbols: when we pump we'll get out of order symbols.

One of those 7 cases must occur when dividing the word $s = uvxyz$. Because each one breaks one of the three conditions, we have a contradiction.

This proves that C is not c.f.

EXAMPLE 2.38

Let $D = \{ww \mid w \in \{0,1\}^*\}$. Use the pumping lemma to show that D is not a CFL. Assume that D is a CFL and obtain a contradiction. Let p be the pumping length given by the pumping lemma.

Proof

On next page...

Proof

We will prove this using the pumping lemma for CFLs. Assume D is c.f. in order to generate a contradiction. Let $s = 0^p 1 0^p 1$ $s \in D$ and has a length greater than p, so the pumping lemma states it can be pumped according to three conditions. Let's look at all the ways s can be divided into $uvxyz$

$$\underbrace{0_1 0_2 \ldots}_{u}\underbrace{\ldots\ldots}_{v}\underbrace{0_p 1 0_1 0_2}_{x}\underbrace{\ldots\ldots}_{y}\underbrace{0_p 1}_{z}$$

$p=1$ $\quad 000\underbrace{0}_{v}1\,\underbrace{0}_{y}00001$

$p=2$ $\quad 000001000001$ X bad choice of s

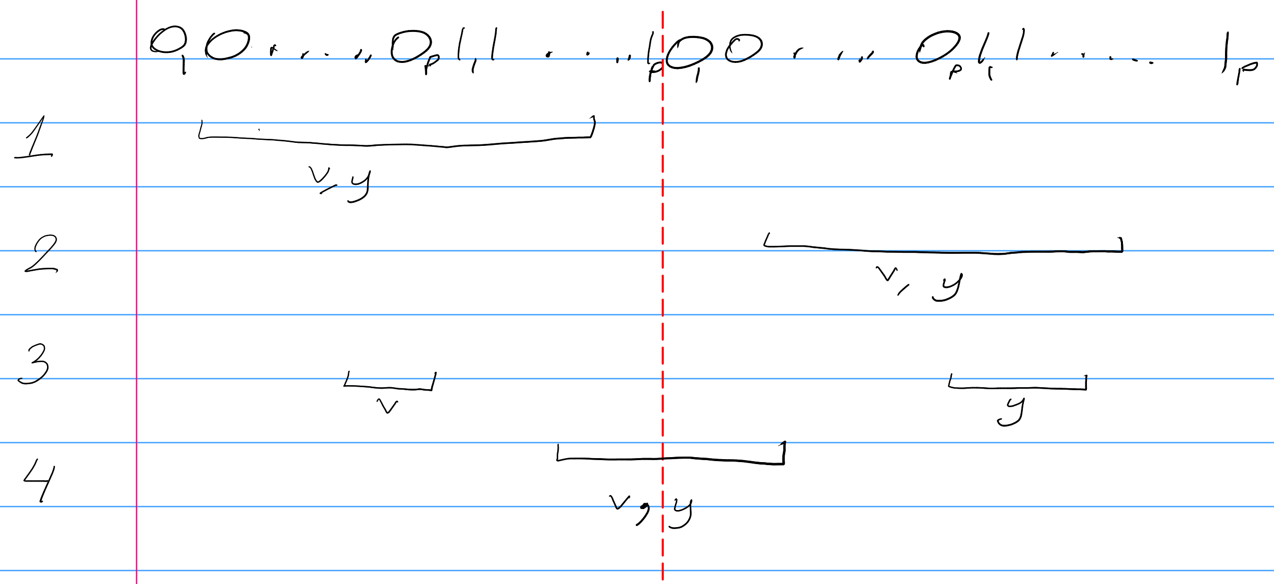
$p=0$ $\quad 00010001$

This word can be pumped by choosing v = "0" and y = "0".
Showing a word in D can be pumped proves nothing about D:
D might still be c.f. or not c.f., we don't know.
We have made no progress yet in our proof -- zero marks.
What we need to find is at least one word s that cannot be pumped, because the pumping lemma says that if D is a c.f. language, then ALL words in the language can be pumped for all values of i and the other two rules.

So, we need to find just one word s in D, such that searching exhaustively for all possible ways to divide it into s=uvxyz proves that there is no way to pump it. When showing it can't be pumped for each possiobility, we just need to find a single i=0,1,... for which it can't be pumped, or show that it breaks one of the other two rules, because the pumping lemma states that all words in D can be pumped without breaking any of the three rules.

Let's see if we can make progress with a different choice of s...

Different choice of s. Let $s = 0^p 1^p 0^p 1^p$.



1 v and y both on l.h.s. of word.

X_1 When we pump up, a 1 will appear at the beginning of the second w, which isn't allowed because the first w begins with 0.

2. v and y both on r.h.s. of word.

X_1 When we pump up, a 0 will appear at the end of the first w, which isn't allowed because the second w ends with a 1.

3. v is on the l.h.s. and v is on the r.h.s. and v has at least one 0 or y has at least one 1.

X_3 v and y cannot be so far apart because there are p 1s and p 0s either side of the midpoint.

4. v is on the l.h.s. and v is on the r.h.s. and v has some 1s or y has some 0s.

X_1 When pumping down, either we will lose 1s from the end of the first w or we will lose some 0s from the beginning of the second word. In either case, the numbers of 0s and 1s won't match so the word will not be in D.

All possible ways of dividing and pumping s falls into one of these four categories, and none of them allows s to be pumped.

Since s is a word in D of length at least p and we have shown that s cannot be pumped, this is a contradiction. Therefore D must not be c.f.