Ollscoil Mhá Nuad
Maynooth University

SEMESTER 1
2024-2025

CS370
Computation and Complexity

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Time allowed: 2 hours

Answer ***all seven*** questions

All questions carry equal marks

Instructions

	Yes	No	N/A
Formulae and Tables book allowed *(i.e. available on request)*		X	
Formulae and Tables book required *(i.e. distributed prior to exam commencing)*		X	
Statistics Tables and Formulae allowed *(i.e. available on request)*		X	
Statistics Tables and Formulae required *(i.e. distributed prior to exam commencing)*		X	
Dictionary allowed *(supplied by the student)*		X	
Non-programmable calculator allowed		X	
Students required to write in and return the exam question paper	X		
One single-sided A4 sheet of notes allowed (supplied by the student)	**X**		
Students required to return the single-sided A4 sheet (details below)	**X**		

One A4 sheet of notes can be brought into the exam.
Only one side of the A4 sheet can have writing/drawings, the other is for name and signature.
All content must be original hand-written or hand-drawn; no typed, printed, photocopied, or computer-generated content.
The A4 sheet must be returned with this exam question paper.
Only the answers on this exam question paper will be marked – rough work on extra pages will not be marked. If your answer requires more space than the exam question paper allows, refer to specific page number(s) in your answer booklet where your answer continues.
Write your name and student number below.

Name ..... Student number .....

[10 marks]

1 Prove the following set is countable by constructing a TM M_1 to write out the elements of the set in a lexicographical ordering of your choosing:

LibreOfficeWriterDocs = {<*D*>: *D* is a valid LibreOffice Writer document}.

Write out the files with filenames a.odt, aa.odt, aaa.odt, and so on.

You are given the following definition. A LibreOffice Writer document is a binary file containing a sequence of bits such that when the command-line program “writerlo” runs on that file, it decides whether it contains a valid document.

You must use the template below and must write on this exam question sheet.

Proof

We will prove that the set is countable by constructing a TM M_1 to write out the elements of the set in lexicographical order.

M_1 = “Ignore the input:

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .“

Since M_1 writes out the elements of the set in lexicographical order without skipping any, this proves that the set is countable.

[10 marks]

2 Prove that the following language is decidable. You must give your answer on this exam question sheet.

L_2 is the language of finite automata that accept at least one word of length greater than 5. It is defined formally as
$L_2 = \{\langle M \rangle : M \text{ is a finite automaton that accepts at least one word } w \text{ where } |w| > 5\}$.

Proof

[10 marks]

3 Prove that the problem associated with language L_3 defined below is undecidable. You are given that HALT = {<*M*, *w*>: *M* is a Turing machine and *M* halts on *w*} is undecidable. Use the template provided to perform a mapping reduction. You must give your answer on this exam question sheet.

The language L_3 is defined as
L_3 = {<*M*>: *M* is a Turing machine that accepts at least one word of length greater than 5, i.e. *M* accepts at least one word *w* where |*w*| > 5}.

Note, in the template below, some blanks have a small subscript number. Blanks with the same subscript number must have the same value.

Proof

We will use a mapping reduction to prove the reduction

. . . . . ≤ . . . .

Assume that . . . . . $_1$ is decidable.

The transition function *f* that maps instances of . . . . . to instances of

. . . . . is given by TM F given by the following pseudocode.

F = "On input <. . . . . $_2$>:

1. Construct the following M' given by the following pseudocode.

   M' = ". . . . . .
   . . . . . .
   . . . . . .
   . . . . . .
   . . . . . .
   . . . . . ."

2. Output <. . . . . $_3$>."

Now, <. . . . . $_3$> is an element of . . . . . iff <. . . . . $_2$>
is an element of . . . .

So, using *f* and the assumption that . . . . . $_1$ is decidable, we can decide

. . . . A contradiction.

Therefore, ... $_1$ is undecidable. (This also means that the complement of

. . . . . $_1$ is undecidable; the complement of any undecidable language is itself undecidable.)

[10 marks]

4 Prove that the complement of language L_3 defined previously is not Turing-recognisable. Refer to your proof that L_3 is undecidable in your answer. You must give your answer on this exam question sheet.

Proof

[10 marks]

5 Prove that the following language is in $\mathcal{P}$ or prove that it is $\mathcal{NP}$-hard. You must give your answer on this exam question sheet.

L_5 = {<*L*>: *L* is a collection of *M* sets of integers, where at least one integer *a* exists such that each set contains *a*}.

This language has an equivalent definition of L_5 = {<*L*>: *L* is a collection of *M* sets of integers, where all *M* sets have at least one integer in common}.
As examples, <{{1, 2}, {1, 3}, {1, 2, 3}}> $\in$ L_5
and <{{1, 2}, {1, 3}, {2, 3}, {2}, {1, 2, 3}}> $\notin$ L_5.

Proof

[10 marks]

6 Prove that the language L_6 defined below is in $\mathcal{NP}$. You must give your answer on this exam question sheet.

The Students' Union wishes to find out if the noise level was too high during any of the x exams in the January 2025 exam period, but only has time to ask k students. The Union has lists E_1, E_2, ..., E_x of all students who attended each of the x exams. Each E_i is a subset of S (the set of all students in the University). Students can have attended any number of exams from 0 to x.

This problem can be defined as the language $L_6 = \{<J, k> : J = \{E_1, E_2, ..., E_x\}$ is a set containing a list E_i of students attending each exam i in the University. There exists a subset of S of size k such that each exam was attended by at least one of those k students$\}$.

Proof

[10 marks]

7 (a) Use a reduction to prove that the language L_6 defined previously is $\mathcal{NP}$-complete. You are given that 3-SAT is $\mathcal{NP}$-complete. Refer to your proof that L_6 is in $\mathcal{NP}$ in your answer. You must give your answer on this exam question sheet. [7 marks]

(b) Give an example of the output of your reduction for the input $C = (a \vee \overline{b} \vee \overline{d}) \wedge (\overline{b} \vee c \vee d) \wedge (a \vee d \vee \overline{e}) \wedge (\overline{b} \vee c \vee \overline{e}) \in$ 3-SAT. [3 marks]

Proof