



NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

**OLLSCOIL NA hÉIREANN MÁ NUAD**  
**THE NATIONAL UNIVERSITY OF IRELAND**  
**MAYNOOTH**

**JANUARY 2013 EXAMINATION**

**CS605**

**The Mathematics and Theory  
of Computer Science**

Dr. L. Rapanotti, Dr. A. Winstanley, Mr. T. Naughton

Time allowed: 3 hours

Answer **three** questions

**All questions** carry equal marks

**Additional material allowed:**

One copy of M. Sipser, *Introduction to the Theory of Computation* (PWS, Boston, 1997), containing no annotations or extra pages AND/OR one copy of M. Sipser, *Introduction to the Theory of Computation – Second Edition* (Thomson, Boston, 2005), containing no annotations or extra pages. Please see declaration page at back.

- [25 marks]**
- 1** (a) The language  $\{a^p b^q : q \geq p > 0\}$  is Turing-recognisable.
- [3 marks]
- i. Outline a pseudocode algorithm (using English language, not a Turing machine table of behaviour) for a single-tape Turing machine to recognise this language.
- [2 marks]
- ii. In terms of  $p$  and/or  $q$ , calculate the exact number of timesteps your Turing machine would require to accept a word (some flexibility will be allowed in calculating the exact number of timesteps due to the inherent imprecise nature of your pseudocode algorithm).
- [2 marks]
- iii. Give your answer to part ii for your Turing machine in asymptotic notation.
- (b) Prove that the set of all words over a finite alphabet is countable. [4 marks]
- (c) Prove that the set of all languages over a finite alphabet is uncountable. [10 marks]
- (d) What does a reduction  $A \leq B$  between two problems A and B establish about the relative computability of A and B? What does a polynomial reduction establish about the relative computational complexity of A and B? [4 marks]
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- [25 marks]**
- 2** (a) A student claims that the intersection of two regular languages is a regular language, and attempts to prove it by picking two specific regular languages and showing that their intersection is regular. [10 marks]
- i. What is insufficient with this student's proof?
- ii. Prove that this claim (closure under intersection) is true for all regular languages.
- (b) Let  $L = \{\langle M \rangle : M \text{ is a Turing machine with an input alphabet of } \{a,b\} \text{ and } M \text{ accepts at most one word, i.e. } M \text{ either accepts no words or accepts exactly one word}\}$ . Prove that the complement of L is Turing-recognisable. [12 marks]
- (c) How can we use a reduction to prove nonmembership of a class? [3 marks]

- [25 marks]**
- 3 (a) What are the steps required to prove that a language is NP-complete? [4 marks]
- (b) Let  $\alpha$  be an arbitrary number that is the product of two prime numbers. Prove that the problem of factoring  $\alpha$  is in NP. [4 marks]
- (c) For each of the following languages, prove that it is regular or prove that it is not regular. [10 marks]
- i.  $\{w : w \in \{a, b\}^*, w \text{ contains the same number of substrings } ba \text{ as } ab\}$  (e.g. the string  $aba$  is in this language)
- ii.  $\{w\#ww : w \in \{a, b\}^*\}$
- iii.  $\{uv : u, v \in \{a, b\}^*, |u| = 2|v|\}$
- (d) Outline in detail how one could prove that the set of regular languages is a proper subset of the set of the context-free languages. You must explain each sub-proof required and give a detailed plan for how you would prove each sub-proof. The only theorems you may use (if you wish to) are those you have proved from part (c) of this question and the following [7 marks]
- a language is regular iff it is accepted by a finite automaton
  - a language is context-free iff it is accepted by a pushdown automaton.
- [25 marks]**
- 4 (a) What does a reduction  $A \leq B$  between two problems A and B establish about the relative computability of A and B? What does a polynomial reduction establish about the relative computational complexity of A and B? [4 marks]
- (b) Let the language  $\text{NTWOTIMES}_{C++}$  be defined as  $\text{NTWOTIMES}_{C++} = \{ \langle A, b \rangle : A \text{ is a C++ program, and } b \text{ is an integer variable declared in } A, \text{ and when } A \text{ is run } b \text{ is decremented at most once during the execution of } A \}$ . You are given that  $\text{HALT}_{C++}$  is undecidable.  $\text{HALT}_{C++}$  is defined as  $\text{HALT}_{C++} = \{ \langle B, w \rangle : B \text{ is a C++ function, and } B \text{ halts on its string input } w \}$ . Prove that  $\text{NTWOTIMES}_{C++}$  is undecidable. You may answer this question by assigning a name, mathematical construct, or piece of pseudocode to each of the numbered blanks in the proof template in Figure 1 on page 3. Where blanks have the same number, this denotes that their contents will be the same. Alternatively, you can choose to ignore the template and construct your own proof from scratch. [13 marks]
- (c) Prove that  $\text{NTWOTIMES}_{C++}$  is Turing-recognisable or prove that it is not Turing-recognisable. [4 marks]
- (d) Give a definition of the language  $\text{TWOTIMES}_{C++}$  (the complement of  $\text{NTWOTIMES}_{C++}$ ). Prove that  $\text{TWOTIMES}_{C++}$  is Turing-recognisable or prove that it is not Turing-recognisable. [4 marks]

**Proof.** We will use a mapping reduction to prove the reduction 1. Assume that 2 is decidable. The function  $f$  that maps instances of 3 to instances of 4 is performed by TM  $F$  given by the following pseudocode.

$F =$  “On input  $\langle \underline{5} \rangle$  :

1. Construct the following  $M'$  given by the following pseudocode.

$M' =$  “ 6 ”

2. Output  $\langle \underline{7} \rangle$ ”

Now,  $\langle \underline{7} \rangle$  is an element of 8 iff  $\langle \underline{5} \rangle$  is an element of 9. So using  $f$  and the assumption that 2 is decidable, we can decide 10. A contradiction. Therefore, 2 is undecidable. (This also means that the complement of 2 is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1. Proof template that can be used if you wish.



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**Declaration**

To be signed by the student and collected by an  
invigilator at the beginning of the examination

1. I have searched through my copies of M. Sipser, *Introduction to the Theory of Computation*, first and second editions, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).
2. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.
3. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked may result in a mark of zero.

Print name \_\_\_\_\_ Student number \_\_\_\_\_

Signed \_\_\_\_\_ Date \_\_\_\_\_