Ollscoil Mhá Nuad
Maynooth University

SEMESTER 2
2023-2024

CS605
The Mathematics and Theory of Computer Science

Prof. O. Conlan, Prof. R.J. Farrell, Prof. T. Naughton

Time allowed: 3 hours

Answer at least ***seven*** questions

Your mark will be based on your best ***seven*** answers

All questions carry equal marks

Instructions

	Yes	No	N/A
Formulae and Tables book allowed *(i.e. available on request)*		X	
Formulae and Tables book required *(i.e. distributed prior to exam commencing)*		X	
Statistics Tables and Formulae allowed *(i.e. available on request)*		X	
Statistics Tables and Formulae required *(i.e. distributed prior to exam commencing)*		X	
Dictionary allowed *(supplied by the student)*		X	
Non-programmable calculator allowed		X	
Students required to write in and return the exam question paper		X	
One physical (paper) copy of textbook Michael Sipser, Introduction to the Theory of Computation, without annotations or extra pages *(supplied by the student)*	X		
Students required to sign the declaration page at the end of this document and return it in their answer booklet	X		

Dear student, by starting the examination, you confirm that you have read the instructions in the box below.

<u>Additional material allowed:</u> For this exam you are permitted to bring one physical (paper) copy of the textbook Michael Sipser, *Introduction to the Theory of Computation* (any edition), without annotations or extra pages. Please sign the declaration page at the end of this document and return it in your answer booklet.

[10 marks]

1 For each of the following languages, prove that it is regular or prove that it is not regular.

(a) $\{uv : u,v \in \{0, 1\}^*$, u and v begin with the same symbol$\}$

(b) $\{uvw : u,v,w \in \{0, 1\}^*$, u, v, and w begin with the same symbol$\}$

(c) $\{uv : u,v \in \{0, 1\}^*, |u|=|v|$, u and v begin with the same symbol$\}$

[10 marks]

2 For each of the following languages, prove that it is context-free or prove that it is not context-free.

(a) $\{u\#v : u,v \in \{a, b\}^*, u \neq v\}$

(b) $\{u\#v : u,v \in \{a, b\}^*, |u|=|v|$ and u differs from v by exactly one symbol$\}$

[10 marks]

3 Let the language $\text{EXCEPTION21}_{\text{Java}}$ = {<*M*>: *M* is Java program that takes no input and when *M* runs it does not throw (raise) exception number 21}.

Prove that $\text{EXCEPTION21}_{\text{Java}}$ is undecidable. You are given that HALTS is undecidable. HALTS is defined as HALTS = {<*M, w*> : *M* is a Turing machine that halts on input *w*}. You can use the template in Figure 1 if you wish.

[10 marks]

4 Let the language $\text{LINENO}_{\text{Java}}$ = {<*M*, *n*>: *M* is Java program that takes no input and *n* is an integer, and when *M* is run, line number *n* is never executed}.

Prove that $\text{LINENO}_{\text{Java}}$ is Turing-recognisable, or prove that it is not Turing-recognisable. You are given that HALTS is undecidable. HALTS is defined as HALTS = {<*M*, *w*> : *M* is a Turing machine that halts on input *w*}. You can use the template in Figure 1 if you wish.

[10 marks]

5 Consider the following sextuple of language classes: (regular, context-free, Turing-recognisable, decidable, $\mathcal{P}$, $\mathcal{NP}$).

Each language can be associated with a binary sextuple where symbol 1 denotes membership and 0 denotes nonmembership of the class in question.

For example, if a language was in the first two classes and not in any of the others, it would be associated with the binary sextuple (1, 1, 0, 0, 0, 0).

State the binary sextuple associated with each of the following languages.

(a) The language $L = \{a, aa, ab\}$.

(b) The language $L = \{w : w \in \{a, b\}^*, |w|$ is odd, w has exactly one b positioned exactly in the centre of the word$\}$.

(c) The language $L = \{w : w \in \{a, b\}^*, |w|$ is odd, w has exactly one $b\}$.

(d) The language of chess board configurations for which white can win.

[10 marks]

6 Using the same definition given in Q5 above, state the binary sextuple associated with each of the following languages.

(a) The language of Java programs that contain a `print()` instruction.

(b) The language of Turing machines that go into a state 01.

(c) The language of set systems that can be hit by a set of cardinality 12 (where 'hit' is a technical term defined in the well-known Hitting Set problem).

(d) The language of graphs that have a tour that visits each vertex at least once.

[10 marks]

7 QUARTER = $\{\langle A \rangle : A = \{x_1, ..., x_m\}$ is a set of natural numbers and there exists a subset $A' \subseteq A$ where the sum of the elements in A' is exactly a quarter of the sum of the elements in $A\}$. Prove that this language is in $\mathcal{NP}$.

[10 marks]

8 A coach wishes to assemble a subset of her players to discuss how the squad performed in each match during the previous season. She has lists showing which players played in each of the *M* matches. From her full squad of *P* players, she wishes to pick a subset of *k* players such that she has at least one player from each match. Given that 3-SAT is $\mathcal{NP}$-complete, prove that this problem, called COACH, is $\mathcal{NP}$-complete.

Proof. We will use a mapping reduction to prove the reduction ___1___. Assume that ___2___ is decidable. The function f that maps instances of ___3___ to instances of ___4___ is performed by TM F given by the following pseudocode.

F = "On input $\langle$___5___$\rangle$:

1. Construct the following M' given by the following pseudocode.
   M' = "___6___"
2. Output $\langle$___7___$\rangle$"

Now, $\langle$___7___$\rangle$ is an element of ___8___ iff $\langle$___5___$\rangle$ is an element of ___9___. So using f and the assumption that ___2___ is decidable, we can decide ___10___. A contradiction. Therefore, ___2___ is undecidable. (This also means that the complement of ___2___ is undecidable; the complement of any undecidable language is itself undecidable.)

Figure 1. Proof template that can be used if you wish. Where blanks have the same number, this denotes that their contents will be the same.

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SUMMER 2024 EXAMINATION

CS605

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Declaration

To be signed by each student and returned in their answer booklet at the end of the examination

i. I have searched through my copy of M. Sipser, *Introduction to the Theory of Computation*, any edition, (the Sipser book) and it does not contain any extra pages or annotations (except for annotations that correct minor typographical errors).

ii. I understand that by failing to notify an invigilator of any annotations or extra pages in my copies of the Sipser book, I will receive a mark of zero in this examination. This does not affect any further disciplinary actions that the University may wish to take.

iii. I understand also that directly copying large amounts of material from the Sipser books without substantially tailoring it to the question asked may result in a mark of zero.

Print name ____________________ Student number _________________

Signed ___________________________ Date ______________________