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## **Topological Josephson Tunnel junctions**

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## **Abstract**

Topological superconductivity is relatively fresh topic of condensed matter physics. Being a rich platform for intriguing and beautiful problems, it also has a huge and unrevealed potential for technology, especially for quantum computing.

The notion of topological superconductivity is closely related to a possibility of presence of a Majorana state — special topologically protected state, usually localized near some inhomogeneity in a topological superconductor. Despite the numerous theoretical proposals of construction this state in condensed matter, little experimental signatures of them are obtained.

In this work the system of two one-dimensional superconducting wires connected with a tunnel junction is considered. Under special conditions this system can host a Majorana fermion. The properties of this system, such as supgap states, stationary supercurrent and ionization under oscillating external voltage are studied. The results of this work have the potential in developing a new technique of detecting Majorana fermions in such systems.

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# Chapter 1

## Introduction

The system, considered in this work, is a pair of 1D superconductors connected with a Josephson junction. For all the discussion presented it's crucial for one of superconductors to be topological.

Topological superconductivity is relatively fresh topic in physics. On the one hand it's being connected to particle physics through the notion of Majorana fermion – the particle coinciding with it's own antiparticle[1]. It can be looked for not only in Standart models' context[2][3][4], but also as a state in solids[5][6][7][8][9][10][11][12][13]. Despite the difference between these entities, there is a clear analogy between majoranas in condensed matter and majoranas in particle physics.

On the other hand topological superconductivity is of interest to quantum computation community as a platform to build fault tolerant quantum memory. Although significant difficulties has appeared on this way, the intention to realize this program is still strong and gives the motivation to build a superconducting samples, which demonstrates signatures of nontrivial topology and presence of Majorana fermions.

The proposition of using superconducting wires as a carriers of Majorana fermions came from a seminal work of Kitaev [6]. The key ingredient of this system was a p-wave superconductivity assumed to be present in a wire. There was showed, that under certain conditions the Majroana state ca be present at the end of the wire. After some time another propositions[14][15] appeared, based on semiconductor-superconductor heterostructures with s-wave superconductivity, external magnetic field and spin-orbit coupling. It was showed, that the sign of quantity  $g = B = \sqrt{B^2 + \mu^2}$ , where  $B$  is a magnetic field,  $\Delta$  is the absolute value of superconducting order parameter and  $\mu$  is a chemical potential can be used as a topological index, and a Majorana state will appear where  $\text{sgn } g$  is changing.

The model, considered is this work is close to the ones used in [14][15]. It consists of two superconducting wires connected with a tunnel junction. However instead of domain wall of  $\text{sgn } g$ , a tunnel barrier between areas with  $g > 0$  and  $g < 0$  is introduced. This model formulated in detail in chapter 2. The spectrum of this model and stationary supercurrent are studied in chapter 3 and the ionization under small external oscillating voltage is considered in chapter 4. Chapter **LINK to discssion chapter** stands for the discussion of obtained results and their possible experimental realization and use, while chapter **LINK to conclusion chapter** concludes the study.

## Chapter 2

# The model

The model studied here is mostly inspired by the works [14] and [15]. Here, however, some more complications are present to make it better reflecting possible experimental situation and catch some higher energy effects that in [14] and [15].

### 2.1 Problem statement

The system under consideration consists of two 1D s-wave superconducting wires connected with a tunnel junction. There is a strong spin-orbit coupling assumed to be present and external magnetic field is applied in the direction perpendicular to the wire. The Hamiltonian of the bulk of each wire, written in the Bogoliubov-de Gennes formalism, is similar to the ones presented in [14] and [15]:

$$\mathcal{H} = \int dy \Psi^\dagger(y) H \Psi(y) \quad \Psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \\ \psi_\downarrow^\dagger \\ -\psi_\uparrow^\dagger \end{pmatrix} \quad (2.1)$$

$$H = \left( \frac{p^2}{2m} - \mu_0 \right) \tau_z + up\sigma_z\tau_z + B\sigma_x + \Delta\tau_\phi \quad (2.2)$$

Here  $\sigma_i$  and  $\tau_i$  are Pauli matrices in spin and particle-hole subspaces respectively,  $\tau_\phi = \tau_x \cos \phi - \tau_y \sin \phi$ , with  $\phi$  being a superconducting phase,  $\mu_0$  is a chemical potential,  $B$  is an external magnetic field,  $\Delta$  is the absolute value of superconducting order parameter and  $u$  is spin-orbit coupling constant with the dimension of velocity. The wire is being aligned along the y-axis, while the direction of the magnetic field coincides with x-axis. Note, that only one component of spin-orbit is nonzero due to 1D nature of the problem.

The tunnel junction is introduced by applying an external electrical field. Its potential profile  $U(y)$  is presented on figure 2.1(a). Inside each wire the potential is assumed to be homogeneous, though its value can be different to the right and to the left of the junction. The junction itself is modeled by a sharp peak of the potential.

To take this into account one should include an additional term  $U(y)\tau_z$  in (2.2). However this term can be combined with the second term of by (2.2) by introducing an effective chemical

potential  $\mu(y) = \mu_0 - U(y)$  (see figure 2.1(b)). From now on all presence of the external field will be hidden in  $\mu(y)$ .

The superconducting phase  $\phi$  in left and right wires,  $\phi_L$  and  $\phi_R$ , can also be different. The phase inside the barrier is undefined as  $\Delta(y) = 0$  there.

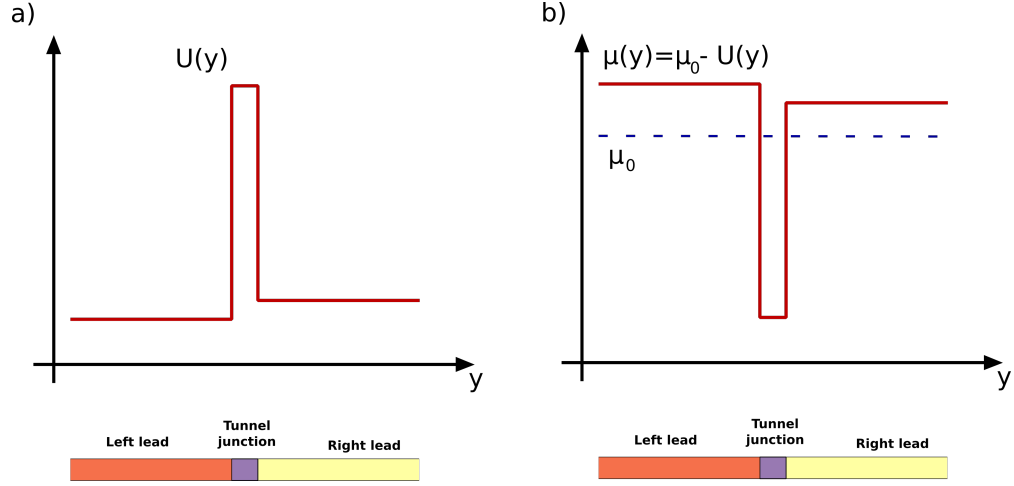


Figure 2.1: (a)  $y$ -profile of external electrical field. (b)  $y$ -profile of effective chemical potential

Finally, the BdG Hamiltonian for the model reads:

$$H = \left( \frac{p^2}{2m} - \mu(y) \right) \tau_z + up\sigma_z\tau_z + B\sigma_x + \Delta(y) \tau_{\phi(y)} \quad (2.3)$$

with

$$\mu(y) = \begin{cases} \mu_L, & -\frac{L}{2} < y \\ \mu_b, & -\frac{L}{2} < y < \frac{L}{2} \\ \mu_R, & \frac{L}{2} < y \end{cases} \quad \Delta(y) = \begin{cases} \Delta, & y > \frac{L}{2}, y < -\frac{L}{2} \\ 0, & -\frac{L}{2} < y < \frac{L}{2} \end{cases} \quad (2.4)$$

$$\phi(y) = \begin{cases} \phi_L, & -\frac{L}{2} < y \\ \phi_R, & \frac{L}{2} < y \end{cases} \quad (2.5)$$

with  $L$  being the size of the junction. Note, that the parameters  $B$ ,  $u$ ,  $\Delta$  and  $m$  are taken to be constant across the system.

This setup is close to one of the models considered by Oreg et al. in [14] (*"Spatially varying  $\mu$ "* section). The difference is in the profile of  $\mu(y)$  – in [14] there is a step in effective chemical potential, while here this function possesses a well.

In [14] it's also discussed that the Majorana fermion appears at the inhomogeneity if the difference  $B - \sqrt{\mu^2 + \Delta^2}$  is greater than zero on one side of the step in  $\mu(y)$  and lesser than zero on another side of it. As will be shown further, this is also relevant to the system presented here.

Note, that if  $B > |\Delta|$  this condition can always be satisfied by choosing appropriate  $\mu_L$  and  $\mu_R$ .

When the two wires of different sign of  $g$  are assumed in this work, the trivial wire will always be on the left, while the topological wire will always be on the right.

The model, described by (2.3) and (2.4) possesses a big number of external parameters. Different areas in this parameter space require different approaches and sometimes lead to completely different physics. Here the certain experimentally reasonable constraints are assumed:

$$\mu_L, \mu_R \ll B \sim \Delta \ll mu^2 \ll |\mu_b| \quad (2.6)$$

The experimental justification of this choice is given in the section **SECTION ABOUT REALIZATION**, while call for it from theoretical point of view will arise further in this chapter.

## 2.2 The dispersion of a homogeneous wire

Before discussing the properties of the junction it's necessary to consider a dispersion of a homogeneous wire modeled with the Hamiltonian (2.2). Although this can be done exactly, it's instructive to obtain this dependence step by step, starting with a simpler model and adding new terms until the Hamiltonian (2.2) is restored.

The starting point is the Hamiltonian consisting only of kinetic energy and chemical potential terms:  $H = \frac{p^2}{2m} - \mu$ . It has simple parabolic dispersion presented at fig. 2.2(a). When the spin is introduced and spin-orbit coupling term  $up\sigma_z$  is added, the parabola splits in two (fig. 2.2(b)), each one corresponding to it's own z-protection of the spin. After introducing a magnetic field with  $B\sigma_x$  term, the gap at the intersection opens (fig. 2.2(c)). The next step is introducing the BdG formalism, by adding the multiplier  $\tau_z$  elsewhere except for magnetic field term:  $H = \left(\frac{p^2}{2m} - \mu_0\right)\tau_z + up\sigma_z\tau_z + B\sigma_x$ . This procedure doubles the spectra in a way that each eigenvector with energy  $E$  obtains a partner eigenvector with energy  $-E$ , so additional two energy branches appear, being a mirror reflection of initial dispersion. This is presented at fig. 2.2(d), with the dashed lines being BdG partners. The last step is adding the superconducting term  $\Delta\tau_\phi$ , which opens the gap where the dashed and the solid lines are intersected (fig. 2.2(f)).

As was mentioned before, the dispersion can be found explicitly. As was pointed in [14], it can be done by squiring the Hamiltonian (2.2) twice and solving a resulting biquadratic equation, leading to:

$$E_{1,2}^2(p) = B^2 + \Delta^2 + \xi_p^2 + (up)^2 \pm 2\sqrt{B^2\Delta^2 + B^2\xi_p^2 + (up)^2\xi_p^2} \quad (2.7)$$

with  $\xi_p = \frac{p^2}{2m} - \mu$ . This dependence, presented at fig. 2.2(f), has two positive and two negative branches, as any BdG dispersion with electron-hole symmetry does. It further discussion only positive branches are considered, if opposite is not mentioned.



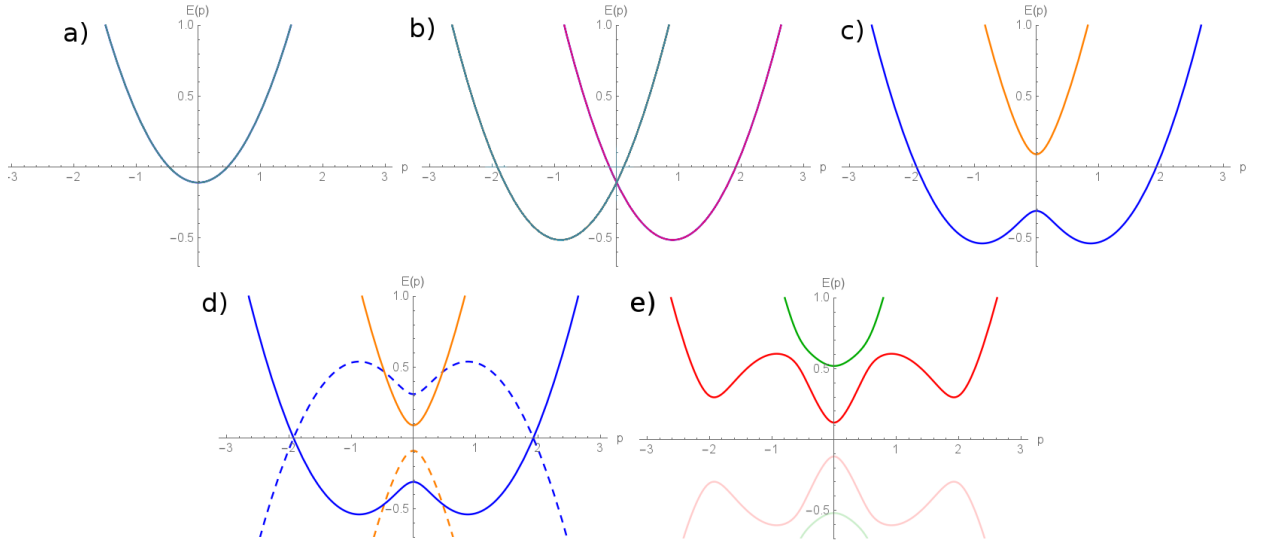


Figure 2.2: The dispersion of different Hamiltonians: a) mere kinetic energy and chemical potential:  $H = \frac{p^2}{2m} - \mu$  b) spin-orbit coupling added:  $H = \frac{p^2}{2m} - \mu_0 + up\sigma_z$  c) magnetic field added:  $H = \frac{p^2}{2m} - \mu_0 + up\sigma_z + B\sigma_x$  d) BdG formalism introduced:  $H = \left(\frac{p^2}{2m} - \mu_0\right) \tau_z + up\sigma_z \tau_z + B\sigma_x$  e) The complete Hamiltonian of homogeneous wire:  $H = \left(\frac{p^2}{2m} - \mu_0\right) \tau_z + up\sigma_z \tau_z + B\sigma_x + \Delta\tau_\phi$ . The parameters of the Hamiltonians for the plotting are:  $B = 0.2$ ,  $\Delta = 0.3$ ,  $u = 0.9$ ,  $m = 1$ ,  $\mu = 0.11$

If the constrains (2.6) are assumed, the lower branch of this spectra has three minima: one of them is at  $p = 0$  exactly, and two another are at  $p = \pm 2mu$  in the leading order. The last two are not very interesting – the energy there is approximately equal to  $\Delta$ , as it should be due to perturbative introduction of superconducting term. On the contrary, the minimum at  $p = 0$ , which is given by[14]:

$$E_2(0) = |g|, \quad g = B - \sqrt{\Delta^2 + \mu^2} \quad (2.8)$$

is the most important peculiarity of the spectrum. First, as  $\mu \ll B \sim \Delta$ , it's the true gap of the spectrum as  $\left|B^2 - \sqrt{\Delta^2 + \mu^2}\right| \approx \left|B - \Delta - \frac{\mu^2}{2\Delta}\right| \ll \Delta$ . Second, the sign of  $g$  defines where the wire can or cannot host the Majorana state near some inhomogeneity. Here it's useful to introduce the terminology: if  $g > 0$  the wire is called "topological", otherwise it's called "trivial". In [14] and [15] it was derived, that the contact of trivial and topological wire hosts a Majorana state. It can also be shown (see section **ENTER SECTION**), that this state is present on the end of a topological wire and isn't there for a trivial one.

Note, that when two wires are considered, there are two gaps,  $g_{L,R} = B - \sqrt{\Delta^2 + \mu_{L,R}^2}$ . When the magnetic field  $B$  is close to  $\Delta$ , one can change the signs of  $g_{L,R}$  by changing  $\mu_{L,R}$  respectively.

It's instructive to clarify the place of  $g_{L,R}$  in the paramter hierarchy of the problem. As

$\mu_L \sim \mu_R \ll B \sim \Delta$  and  $g_L = B - \sqrt{\Delta^2 + \mu_L^2} < 0$ , one can figure out that  $\mu_R \sim \mu_L \gtrsim (B - \Delta)(B + \Delta)$ . Taking that into account and noticing that  $g_{L,R} \approx B - \Delta - \frac{\mu^2}{2\Delta^2}$  and introducing  $\beta = B - \Delta$  one finds  $g_{L,R} \lesssim \frac{\beta^2}{\Delta}$ , while  $\mu_{L,R} \gtrsim \sqrt{\beta\Delta}$ , so  $\sqrt{\beta} \ll \mu_L, \mu_R$ .

## 2.3 High and low modes

Though the wavefunctions of (2.2) can be found explicitly, their form is enough complicated to stall any further analysis. However, as the spin-orbit energy is assumed to be the biggest energy scale for a homogeneous wire, one can reduce the Hamiltonian (2.2) to:

$$H = \left( \frac{p^2}{2m} + up\sigma_z \right) \tau_z \quad (2.9)$$

in this problem it's reasonable to assume the low energy limit, as all the Majorana physics should live at the energies of the order of  $g$ . Thus the energy term must be omitted in Schroedinger equation, which leads to two types of momenta:  $p_{short} \approx \pm 2mu$  and  $p_{long} \approx 0$  and, respectively two types of wavefunctions: shortwave and longwave ones. The fact that  $p_{long}$  is equal to zero means, that the approximation (2.9) is insufficient to describe them. However, to deal with long-wave wavefunctions one can omit the quadratic term in (2.2) and work with linearized Hamiltonian, similar to the ones used in [14] and [15].

## Chapter 3

# Stationary properties

The stationary properties of the system are defined by it's spectrum. In this chapter the boundary condition is introduced, wavefunctions of the homogeneous wires are obtained, undergap states are investigated and the stationary supercurrent is estimated.

### 3.1 Boundary condition

To obtain the spectrum of the system it's necessary to find the boundary conditions. As the barrier chemical potential is the biggest energy parameter of the problem, the wave-functions there are defined by the Hamiltonian:

$$H(y) = \left( \frac{p^2}{2m} + \mu_b \right) \tau_z, \quad -\frac{L}{2} < y < \frac{L}{2} \quad (3.1)$$

as the low energies are the under consideration, in Sroedinger equation the energy term can be omitted, so  $p_b \approx \pm i\sqrt{2m\mu_b}$ . One can solve the problem given by (3.1) and match the values of the wavefunction and it's derivatives on the left and on the right of the barrier to obtain:

$$\begin{cases} \psi_L + b\partial_y\psi_L = t(\psi_R + b\partial_y\psi_R) \\ \psi_R - b\partial_y\psi_R = t(\psi_L - b\partial_y\psi_L) \end{cases} \quad (3.2)$$

here  $\psi_{L,R} = \psi(\mp \frac{L}{2})$ ,  $b = (2m\mu_b)^{-\frac{1}{2}}$  — the penetration depth for the particle inside the barrier and  $t = e^{-\frac{L}{b}}$  — the tunneling constant assumed to be small:  $t \ll 1$ . This condition reads, that the size of the barrier  $L$  should be much bigger than the penetration depth  $b$ .

This condition is invariant under the combined action  $L \leftrightarrow R$ ,  $y \rightarrow -y$ . To simplify the further analysis one can reverse the direction in the left wire and put both ends of the wires from  $y = \frac{L}{2}$  to  $y = 0$ . The boundary condition then becomes:

$$\begin{cases} \psi_L - b\partial_y\psi_L = t(\psi_R + b\partial_y\psi_R) \\ \psi_R - b\partial_y\psi_R = t(\psi_L + b\partial_y\psi_L) \end{cases} \quad (3.3)$$

This transformation is illustrated on the fig 3.1. Note

The boundary condition (3.3) can be rewritten with introducing the spinor  $\Psi = (\psi_L, \psi_R)^T$  and Pauli matrices  $\hat{s}_i$  in LR space:

$$(1 - t\hat{s}_x) \Psi - (1 + t\hat{s}_x) b\partial_y \Psi = 0 \quad (3.4)$$

since for all  $t \neq 1$  (recall, that  $t \ll 1$ ) the matrix is  $1 \pm t\hat{s}_x$  in reversible. Multiplying the last equation by  $(1 - t\hat{s}_x) / (1 + t^2)$  one obtain:

$$\left(1 - 2\tilde{t}\hat{s}_z - \tilde{b}\partial_y\right) \psi = 0 \quad (3.5)$$

where  $\tilde{t} = \frac{t}{1+t^2}$ ,  $\tilde{b} = \frac{1-t^2}{1+t^2}b$ . In the leading order on  $t$ , which corresponds to the tunneling limit,  $\tilde{t} = t$ ,  $\tilde{b} = b$ .

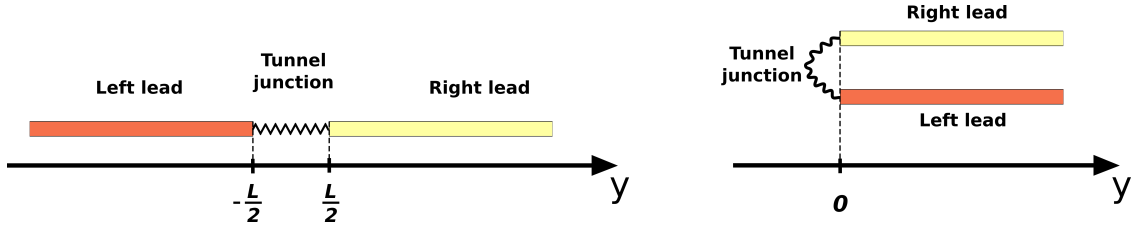


Figure 3.1: Illustration of switching the direction of left wire

One can argue, that in tunneling limit the second and the third term in (3.5) are much smaller than the first one and should not be taken when the leading order is considered. However, if the second terms is omitted, the leads become efficiently disconnected, and no tunnel effects can be found. The same is true for the third term — if it's not present, the boundary condition immediately implies  $\Psi(0) = 0$ , so the wires become disconnected again.

## 3.2 High momentum modes

As was pointed in section 2.3, there are two shortwave and longwave wavefunctions inside the wire, and the first ones can be described with the Hamiltonian (2.9). However, if one is looking for the localized states, even the longwave modes should be taken decaying. To obtain this, one needs to add a restore superconducting term in (2.9), so the spectrum become gapped and the momenta can get an imaginary part. So, for shortwave modes one should consider a Hamiltonian:

$$H = \left( \frac{p^2}{2m} - up\hat{s}_z\sigma_z \right) \tau_z + \Delta\tau_\phi \quad (3.6)$$

here the multiplier  $\hat{s}_z$  is added in the spin-orbit coupling term, as the direction of the left wire is inverted, so to write a correct Hamiltonian for LR space, one needs to change  $p$  to  $-p$  for the left wire — which is exactly adding  $-s_z$  multiplier to each momentum.

Denoting  $\eta = \frac{p^2}{2m} - up\hat{s}_z\sigma_z$ , one can rewrite (3.6) as  $H = \eta\tau_z + \Delta\tau_\phi$ . As  $\hat{s}_z\sigma_z$  commutes with  $H$  one can treat it as a number, so the dispersion is  $E^2 = \eta^2 + \Delta^2$  (the number corresponding to eigenstate of  $\hat{s}_z$  will be denoted as  $s_z$  while the number, corresponding to the eigenstate of  $\sigma_z$  will be denoted as  $\varsigma_z$ ). Thus  $\eta = \pm i\sqrt{\Delta^2 - E^2}$ , as the case  $|E| < \Delta$  is assumed. For momenta one can write the equation:

$$p^2 - 2mus_z\varsigma_z p - 2m\eta = 0 \quad (3.7)$$

which for shortwave momenta gives  $p_{short} \approx 2mus_z\varsigma_z + \frac{\eta}{u}s_z\varsigma_z$ . Choosing the sign of  $\eta$  in a way, that the wavefunction decays at  $x \rightarrow \infty$ , one can obtain:

$$p_{short} \approx 2mus_z\varsigma_z + i\frac{\sqrt{\Delta^2 - E^2}}{u} \quad (3.8)$$

Now the wavefunction can be constructed by putting (3.8) into the Schroedinger equation  $(\eta\tau_z + \Delta\tau_\phi)\Psi = E\Psi$ . The solutions are:

$$\Psi_{s_z, \varsigma_z}(x) = \begin{pmatrix} 1 \\ e^{i(s_z\varsigma_z\gamma + \phi_{s_z})} \end{pmatrix}_{eh} e^{2imus_z\varsigma_z x - \frac{\sqrt{\Delta^2 - E^2}}{u}x} |s_z, \varsigma_z\rangle \quad (3.9)$$

where  $|s_z, \sigma_z\rangle$  are eigenvectors of matrix  $\hat{s}_z\sigma_z$ ,  $\gamma = -\frac{\pi}{2} + \arcsin \frac{E}{\Delta}$  and  $\phi_1 = \phi_L$ ,  $\phi_{-1} = -\phi_R$ . Thus the longwave part of eqigenstate can be written as:

$$\Psi_{long} = \sum_{s_z=\pm 1} \sum_{\varsigma_z=\pm 1} C_{s_z, \varsigma_z} \Psi_{s_z, \varsigma_z}(x) \quad (3.10)$$

### 3.3 Eliminating longwave modes from boundary condition

As the Majorana mode acts at low energies, it's expected to be predominantly longwave. This argument is in accord with [14] and [15], where the majorana state was an eigenstate of a linearized Hamiltonian, which is relevant only for longwave physics. So, it's reasonable to eliminate the shortwave modes from the problem, reformulating the boundary condition (2.9).

The wave function can be decomposed in shortwave and longwave parts:  $\Psi = \Psi_{high} + \Psi_{low}$ . inserting it into the(2.9) and using the fact, that  $p_{low} \ll p_{high} \approx 2mus_z\sigma_z$ , one can obtain at the

boundary:

$$(1 - 2t\hat{s}_x) \Psi_{low} + (1 - 2t\hat{s}_x - 2ibum\hat{s}_z\sigma_z) \Psi_{high} = 0 \quad (3.11)$$

Multiplying by the  $(1 - 2t\hat{s}_x)^{-1}$  and eliminating  $t^2$  terms, obtain:

$$\Psi_{low} = (-1 + i\zeta (1 + 2t\hat{s}_x) s_z \sigma_z) \Psi_{high} \quad (3.12)$$

with  $\zeta = 2bum$ .

Now, using the expansion (3.10) and renormalizing the coefficients:  $C_{s_z \varsigma_z} \rightarrow -(1 - i\zeta s_z \varsigma_z) C_{s_z \varsigma_z}$  one can rewrite the boundary condition for  $\varsigma_z$  spin component of the wavefunction as:

$$\Psi_{low, \varsigma_z} = \left(1 + \frac{2i\zeta t \hat{s}_z \sigma_z}{1 + i\zeta \hat{s}_z \sigma_z}\right) \sum_{s_z = \pm 1} C_{s_z, \varsigma_z} \begin{pmatrix} 1 \\ e^{i(s_z \varsigma_z \gamma + \phi_{s_z})} \end{pmatrix}_{eh} e^{2im u s_z \varsigma_z x - \frac{\sqrt{\Delta^2 - E^2}}{u} x} |s_z, \varsigma_z\rangle \quad (3.13)$$

This can be multiplied by  $\left(1 + \frac{2i\zeta t \hat{s}_z \sigma_z}{1 + i\zeta \hat{s}_z \sigma_z}\right)^{-1}$ , which up to a  $t^2$  correction yields:

$$\left(1 - \frac{2i\zeta t \hat{s}_z \sigma_z}{1 + i\zeta \hat{s}_z \sigma_z}\right) \Psi_{low, \varsigma_z} = \sum_{s_z = \pm 1} C_{s_z, \varsigma_z} \begin{pmatrix} 1 \\ e^{i(s_z \varsigma_z \gamma + \phi_{s_z})} \end{pmatrix}_{eh} e^{2im u s_z \varsigma_z x - \frac{\sqrt{\Delta^2 - E^2}}{u} x} |s_z, \varsigma_z\rangle \quad (3.14)$$

For each  $\varsigma_z$  the above equation can be interpreted as the requirement that the l.h.s. 4-vector (in LR- and eh-spaces) lies in the 2d linear space  $L_2$  spanned by the two vectors in the sum in the r.h.s.. This can be reformulated as the requirement that the l.h.s. be orthogonal to the complementary 2d space  $\bar{L}_2$ . There are two basic vectors  $\bar{\Psi}_{s_z \varsigma_z}$  ( $s_z = \pm 1$ ) spanning  $\bar{L}_2$  for each  $\varsigma_z$ :

$$\bar{\Psi}_{s_z \varsigma_z} = \begin{pmatrix} 1 \\ -e^{i(s_z \varsigma_z \gamma + \phi_{s_z})} \end{pmatrix} |s_z, \varsigma_z\rangle \quad (3.15)$$

Thus one needs to multiply (3.14) by  $(\bar{\Psi}_{+\varsigma_z}, \bar{\Psi}_{-\varsigma_z})$  from the left and, after all evaluating the matrix product, find the boundary condition on longwave modes in the form:

$$\begin{pmatrix} 1 & -e^{-i(\varsigma_z \gamma - \phi_L)} & A & -Ae^{-i(\varsigma_z \gamma - \phi_L)} \\ A^* & -A^* e^{i(\varsigma_z \gamma + \phi_R)} & 1 & -e^{i(\varsigma_z \gamma + \phi_R)} \end{pmatrix} \Psi_{long, \varsigma_z} = 0 \quad (3.16)$$

here  $A = -\frac{2i\zeta t \varsigma_z}{1 + i\zeta \varsigma_z}$  and the elements are ordered as  $(Le, Lh, Re, Rh)$ .

When studying wavefunctions in superconductors, it is more convenient to work with zero phase  $\phi$ . This can be achieved by gauging the phase difference into the boundary condition. In-

deed, suppose  $H_\phi$  describes a wire with phase  $\phi$ . Then,  $H_\phi = U_\phi^\dagger H_0 U_\phi$  with  $U_\phi = \text{diag}(1, e^{i\phi})_{eh}$  and the wave functions are also related via unitary rotation  $\psi_\phi = U_\phi^\dagger \tilde{\psi}$ . So the transform  $U^\dagger = \text{diag}(1, e^{-i\phi_L}, 1, e^{-i\phi_R})_{Le,Lh,Re,Rh}$  will eliminate all the phases from the wires and put them into boundary condition. Substituting  $\Psi_{low,\varsigma z} = U^\dagger \tilde{\Psi}$  into (3.16) one arrives at an even simpler boundary condition on the zero-phase function  $\tilde{\Psi}$ :

$$\begin{pmatrix} 1 & -e^{-i\varsigma z \gamma} & A & -Ae^{-i(\varsigma z \gamma + \varphi)} \\ A^* & -A^* e^{i(\varsigma z \gamma + \varphi)} & 1 & -e^{i\varsigma z \gamma} \end{pmatrix} \tilde{\Psi}_{long,\varsigma z} = 0 \quad (3.17)$$

where  $\varphi = \phi_R - \phi_L$ . Note, that any physical quantity can depend only on phase difference  $\varphi$ , but not on  $\phi_L$  or  $\phi_R$  separately.

It's also convenient to rewrite it in the form acting on the left and right wire wavefunctions:

$$M_L \tilde{\psi}_{low}^L + M_R \tilde{\psi}_{low}^R = 0$$

$$M_L = \begin{pmatrix} 1 & -e^{-i\sigma_z \gamma} \\ A^* & -A^* e^{i(\sigma_z \gamma + \varphi)} \end{pmatrix}_{eh} \quad M_R = \begin{pmatrix} A & -Ae^{i(\sigma_z \gamma + \varphi)} \\ 1 & -e^{i\sigma_z \gamma} \end{pmatrix}_{eh} \quad (3.18)$$

This for is especially useful for finding subgap states localized near the barrier.

### 3.4 Low momenta and linearized Hamiltonian

To utilize boundary condition (3.16) or (3.17), it's necessary to find low momenta wavefunctions in homogenous wire (this functions constitute  $\psi_{low}$  in (3.16) and (3.17)). For this purpose one can use a linearized version of the Hamiltonian (2.2), like in [14] and [15]:

$$H = -\mu\tau_z + up\sigma_z\tau_z + B\sigma_x + \Delta\tau_x \quad (3.19)$$

here the zero phase  $\phi$  is assumed and  $\mu$  can be equal  $\mu_L$  or  $\mu_R$  depending on the wire considered. As was mentioned before, the nonzero phase can be restored by using  $U_\phi$  matrix. This hamiltonian is valid only for the right wire. To obtain the solution in the left wire one needs to reverse the sign of  $p$  in (2.2). Instead of doing so, the unitary transform  $\psi_L = \sigma_x \psi_R$  can be utilized, as for (2.2)  $H(-p) = \sigma_x H(p) \sigma_x$ .

Remembering, that  $\beta = B - \Delta \ll B, \Delta$ , one can treat this Hamiltonian penetratively,

decomposing it as  $H = H_0 + V_0$ :

$$H_0 = up\sigma_z\tau_z + \Delta(\sigma_x + \tau_x) \quad (3.20)$$

$$V = -\mu\tau_z + \beta\sigma_x \quad (3.21)$$

As  $H_0$  commutes with  $\sigma_x\tau_x$ , it's convenient to rewrite it in the basis of common eigenstates of  $\sigma_x$  and  $\tau_x$ . Denoting them as  $|\sigma_x, \tau_x\rangle$  and arranging the order as  $(|+, +\rangle, |-, -\rangle, |+, -\rangle, |-, +\rangle)$  one can rewrite  $H_0 + V$  as:

$$H_0 = \begin{pmatrix} 2\Delta & up & 0 & 0 \\ up & -2\Delta & 0 & 0 \\ 0 & 0 & 0 & up \\ 0 & 0 & up & 0 \end{pmatrix} \quad V = \begin{pmatrix} \beta & 0 & -\mu & 0 \\ 0 & -\beta & 0 & -\mu \\ -\mu & 0 & \beta & 0 \\ 0 & -\mu & 0 & -\beta \end{pmatrix} \quad (3.22)$$

It's easy to see, subspace  $\text{Span}(|+, +\rangle, |-, -\rangle)$  require no perturbation to obtain the eigenstates in the leading order. Indeed, diagonalizing the upper subblock of  $H_0$ , one finds, that  $E = \sqrt{(2\Delta)^2 + (up)^2}$ . When the low energy states are the objects of interest ( $E \sim g_{L,R}$ ), one finds, that  $p = \pm \frac{i\Delta}{2u}$  in the leading order, and the corresponding eigenstates are  $|+, +\rangle \pm i|-, -\rangle$ . If

The another two eigenstates are a little bit more complicated. Diagonalizing the lower subblock of  $H_0$ , one immediately finds, that  $E = \pm up$ . This corresponds to the fact, that  $H_0$  is the version of  $H$  with a closed gap  $g$  on lower branch (see fig. 2.2,(e)), so in the zero order this states cannot form anything localized at all. To find them correctly, one needs to take into account the perturbation  $V$  and solve the secular equation using the following ansatz:

$$\psi = r_1 |+, +\rangle + r_2 |-, -\rangle + q_1 |+, -\rangle + q_2 |-, +\rangle \quad (3.23)$$

with  $r_i \ll q_j$  for all pairs  $(i, j)$ . In the leading order (remember, that both  $E$  and  $up$  are of the order of  $g_{L,R}$  now) this results to a couple of equations:

$$\begin{cases} \left(-E + B - \Delta - \frac{\mu^2}{2\Delta}\right) q_1 + upq_2 = 0 \\ upq_1 + \left(-E - B + \Delta + \frac{\mu^2}{2\Delta}\right) q_2 = 0 \end{cases} \quad (3.24)$$

recall, that at this precision  $g = B - \Delta - \frac{\mu^2}{2\Delta}$  and find  $E^2 = g^2 + u^2p^2$ .

For this states the momenta are of the order of  $g/u$ , so it's reasonable to name them long-wave states. Than the states with momenta  $\pm \frac{i\Delta}{2u}$  will be called mediumwave states. Now it's time to present this wavefunctions in original BdG basis. The expressions here are relevant only for the right lead and for  $E > 0$ . To find the wavefunctions in the left lead, the transform  $\psi_L = \sigma_x\psi_R$



can be used, while for finding the negative energy states one can utilize electron-hole transform:

$\psi_{E<0} = \tau_y \sigma_y K \psi_{E>0}$  with  $K$  being a complex conjugation operator.

For  $E > 2\Delta$  medium wave states are:

$$\psi_{medium}^{out, in} \Big|_{E>2\Delta} = \begin{pmatrix} 1 \\ \frac{E \mp \sqrt{E^2 - 4\Delta^2}}{2\Delta} \\ \frac{E \mp \sqrt{E^2 - 4\Delta^2}}{2\Delta} \\ 1 \end{pmatrix} e^{\frac{\pm ix \sqrt{E^2 - 4\Delta^2}}{u}} \quad (3.25)$$

For  $E > 2\Delta$  medium wave states are:

$$\psi_{medium}^{grow, dec} \Big|_{E<2\Delta} = \begin{pmatrix} 1 \\ \frac{E \pm i\sqrt{4\Delta^2 - E^2}}{2\Delta} \\ \frac{E \pm i\sqrt{4\Delta^2 - E^2}}{2\Delta} \\ 1 \end{pmatrix} e^{\frac{\pm x \sqrt{4\Delta^2 - E^2}}{u}} \quad (3.26)$$

For  $E > |g|$  longwave states are:

$$\psi_{long}^{out, in} \Big|_{E>g} = \begin{pmatrix} 1 \\ \frac{E \mp \sqrt{E^2 - g^2}}{g} \\ -\frac{E \mp \sqrt{E^2 - g^2}}{g} \\ -1 \end{pmatrix} e^{\pm \frac{ix \sqrt{E^2 - g^2}}{u}} \quad (3.27)$$

For  $E < |g|$  longwave states are:

$$\psi_{long}^{grow, dec} \Big|_{E<g} = \begin{pmatrix} 1 \\ \frac{E \pm i\sqrt{g^2 - E^2}}{g} \\ -\frac{E \pm i\sqrt{g^2 - E^2}}{g} \\ -1 \end{pmatrix} e^{\pm \frac{x \sqrt{g^2 - E^2}}{u}} \quad (3.28)$$

### 3.5 Subgap states

To find the bound states one needs to make two linear combinations (each for its own wire) of decaying wave functions from (3.26) and (3.28) at  $x = 0$  and put them into boundary condition (3.3). For the right wire they can be taken directly from (3.26), (3.28), while for the left wire they should be multiplied by  $\sigma_x$  from the left (see the beginning of section 3.4). These linear

combinations can be written as:

$$\tilde{\psi}_L = C_{medium}^L \sigma_x \psi_{medium}^{dec} + C_{long}^L \sigma_x \psi_{L,long}^{dec} \quad \tilde{\psi}_R = C_{medium}^R \psi_{medium}^{dec} + C_{long}^R \psi_{R,long}^{dec} \quad (3.29)$$

where  $C_{medium}^L, C_{long}^L, C_{medium}^R, C_{long}^R$  are the undefined coefficients. Note, that the spinor  $\psi_{medium}^{dec}$  is the same for the left and for the right wires.

Putting these combinations into boundary condition 3.3, one can obtain four equations for these coefficients. If this system has a solution at energy  $E_0$ , than there is a bound state with this energy. The condition of solvability can be written as:

$$\det F = 0 \quad (3.30)$$

where matrix  $F$  is given by:

$$F = \begin{pmatrix} M_L \sigma_x \psi_{medium}^{dec}, & M_L \sigma_x \psi_{L,long}^{dec}, & M_R \psi_{medium}^{dec}, & M_R \psi_{R,long}^{dec} \end{pmatrix} \quad (3.31)$$

As  $E \sim g_{L,R} \ll \Delta$ , the mediumwave spinor can be taken in it's low-energy form: In the most part of this work the low energy version:  $\psi_{medium}^{dec} \approx (1, -i, -i, 1)^T$  will be used.

For dealing with  $\psi_{long}^{dec}$  it's convenient to introduce two quantities  $\chi_{L,R}$ :

$$\cosh \chi_{L,R} = \frac{g_{L,R}}{\sqrt{g_{L,R}^2 - E^2}} \quad \sinh \chi_{L,R} = \frac{E}{\sqrt{g_{L,R}^2 - E^2}} \quad (3.32)$$

If  $g > 0$ , the corresponding parameter  $\chi$  is real and the monotonously growing with  $E$ . For  $g < 0$  the corresponding is complex, but can be parametrized as  $\chi = -\tilde{\chi} + i\pi$  with real and monotonous  $\tilde{\chi}$ .

With this parameters the spinors  $\psi_{L(R),long}^{dec}$  at  $x = 0$  can be written as:

$$\psi_{L(R),long}^{dec} = \begin{pmatrix} -\sinh \chi_{L(R)} - i \\ -\cosh \chi_{L(R)} \\ \cosh \chi_{L(R)} \\ \sinh \chi_{L(R)} + i \end{pmatrix} \quad (3.33)$$

This parametrization completes the toolset used for studying the subgap states.

### 3.5.1 The case of zero tunneling

Consider first the the equation (3.30) with  $t = 0$ . This corresponds to absolutely unpenetrable barrier, or, which is same, to independent wires ended with a vacuum. The computation of the

determinant in (3.30) becomes a rather easy problem and results in:

$$\det F \Big|_{t=0} = -16 (i \sinh(\chi_L) + \cosh(\chi_L) - 1) (i \sinh(\chi_R) + \cosh(\chi_R) - 1) \quad (3.34)$$

If both  $g_L, g_R$  are negative (triv-triv junction), this determinant cannot be equal to zero at all, as  $\cosh \chi_{L,R}$  are also negative and the real part in each braces always non zero. If one of  $g_L, g_R$  (triv-top junction), say  $g_R$  there is only one solution at  $E = 0$ . If there both  $g_L, g_R$  (top-top junction) are positive, there are two solutions at  $E = 0$ .

This result proves, that the presence of Majorana mode in a isolated wire is defined only by the sign of  $g$  and justifies the notion of topology in this system.

### 3.5.2 Small tunneling

To take into account the tunneling effect one may decompose  $\det F$  in  $t$ . Note, that there is no first order in  $t$  due to the structure of boundary condition (3.17). The decomposition can be written as:

$$\det F = d_0 + d_2 t^2 + \dots \quad (3.35)$$

where  $d_0$  is given by (3.34).  $d_2$  can be computed in a same way, but appears to be a rather complex formula. However, in tunneling limit the second correction in (3.35) should be much smaller than the first one, so the only values of  $E$  that should be considered are the ones with  $d_0$  is close to zero.

For triv-triv junction there are no such points, so in the tunneling limit there are no bound states for this case.

For triv-top junction there is a solution for  $E = 0$ , which corresponds to  $\chi_L = i\pi, \chi_R = 0$ . Computing  $d_2$  for this parameters, one finds that it's exactly zero, so there is no correction to Majorana energy — as at should be, as it's protected by a particle-hole symmetry.

For top-top junction the situation is more interesting. In that case there are two solutions at  $E = 0$ , which should split for nonzero  $t$ . Calculating  $d_2$  at  $E = 0$  and decomposing  $d_0$  for small  $E$ , one finds:

$$d_0 = \frac{16E^2}{g_R g_L} \quad d_2 = -\frac{256t^2 \zeta^4}{(1 + \zeta^2)^2} \cos^2 \frac{\phi}{2} \quad (3.36)$$

Using, that  $\zeta = 2bum \ll 1$ , one can find the energy levels:

$$E_{1,2} = \pm 4t \zeta^2 \sqrt{g_R g_L} \cos \frac{\phi}{2} \quad (3.37)$$

this answer is relevant only if these level are well below the gaps:  $t \zeta^2 \sqrt{g_R g_L} \ll \min [g_R, g_L]$ .

### 3.6 Stationary supercurrent

The stationary supercurrent for a Josephson contact is defined by [16]:

$$I = -\frac{2e}{\hbar} \sum_p \tanh\left(\frac{\varepsilon_p}{2k_B T}\right) \frac{d\varepsilon_p}{d\varphi} - \frac{4e}{\hbar} k_B T \int_{cont.} d\varepsilon \log\left[2 \cosh\left(\frac{\varepsilon}{2k_B T}\right)\right] \frac{d\rho}{d\varphi} + \frac{2e}{\hbar} \frac{d}{d\varphi} \int dy \frac{|\Delta|^2}{|c|} \quad (3.38)$$

Here  $\varepsilon_p$  are the energies of the states localized near the barrier,  $\rho$  is the density of states and  $c(\mathbf{r})$  is the interaction constant of the BCS theory,  $\varphi$  is a phase difference,  $k_B$  is a Boltzmann's constant and  $T$  is the temperature. The first term comes from the discrete spectra and the sum is taken over all states in it, the second term is the current from continuous spectra and the third term comes from inhomogeneity of the order parameter. As pointed in [16], despite being generally nonzero, this last term doesn't contribute when step-model functions  $\Delta$  like in (2.4) are used.

After ommiting the third term and taking the low temperature limit one rewrites (3.38) as:

$$I = -\frac{2e}{\hbar} \sum_p \frac{d\varepsilon_p}{d\varphi} - \frac{2e}{\hbar} \int_{cont.} \varepsilon d\varepsilon \frac{d\rho}{d\varphi} \quad (3.39)$$

Here the only unknown quantity is the density of states. As there is a derivative over  $\phi$  taken, one need to find the phase dependent part of  $\rho$  only. It can be done by using the relation between the density of states and the scattering matrix [17]:

$$\rho(\phi) = \frac{1}{2\pi i} \frac{\partial}{\partial \varepsilon} \log \det S + const. \quad (3.40)$$

As was showed in the section 3.5, there are no undergap states in triv-top contact except Majorana state. But this state lays exactly on zero energy regardless of the phase difference, so the derivative in the first term of (3.39) will be zero and no supercurrent from the Majorana state is present.

The scattering matrix can be found with the help of boundary condition (3.17) and the results of section (3.4). It's dimension depends of the number of propagating modes at given energy. In this section only the triv-top. contact will be considered, while the scattering matrices and DOS for other types of contacts will be given in the appendix.

The process of building s-matrix is the following. Firstly, it's necessary to consider an energy and all the propagating wavefunctions at this energy. Among them there are wave functions, localized near the barrier ( $\text{Imp} > 0$ ), propagating towards the barrier ( $p < 0$ ), and propagating from

the barrier ( $p < 0$ ). Let's denote these three sets as  $X_{local}$ ,  $X_{in}$  and  $X_{out}$ . Each wavefunction  $\psi_i$  may contribute with it's own coefficient  $C_i$  — denote the vectors of the coefficients  $\vec{Y}_{local}$ ,  $\vec{Y}_{in}$  and  $\vec{Y}_{out}$ . The s-matrix connects the coefficients from  $\vec{Y}_{in}$  to  $\vec{Y}_{out}$  in a following way:

$$\vec{Y}_{out} = \hat{S}\vec{Y}_{in} \quad (3.41)$$

to find a row of  $\hat{S}$ , one should be taken equal to unity, than take the functions from  $X_{local}$ ,  $X_{out}$  and one chosen function  $\Psi_{in,chosen}$  from  $X_{in}$  and make the linear combination of the form:  $\Upsilon = \psi_{chosen}^{in} + c_1^{out}\Psi_1^{out} + c_2^{out}\Psi_2^{out} + \dots + c_1^{out}\Psi_1^{out} + c_2^{local}]\Psi_2^{local} + \dots$ . The one needs to act with the matrix 3.17 on  $\Upsilon$  and obtain a set of equations for the coefficients  $c_i^{in}$  and  $c_i^{local}$ . This system should be solved, and the resulting coefficients  $c_i^{in}$  should form a row in  $\hat{S}$ , corresponding to  $\Psi_{in,chosen}$ . To construct the entire matrix  $\hat{S}$ , one should repeat this procedure for every function  $\Psi_{in,chosen}$  in  $X_{in}$ .

The coefficients of  $\hat{S}$  generally depend on the normalization of the wavefunctions in  $X_{local}$ ,  $X_{in}$  and  $X_{out}$ . To make the formula (3.40) valid, one must take the functions from  $X_{local}$  normalized to unity and the functions from  $X_{in}$ ,  $X_{out}$  normalized to a flux[17]:  $\langle \psi_{in,out} | v | \psi_{in,out} \rangle$  with  $v$  for our problem being  $v = u\sigma_z\tau_z$ .

Here some additional parameters will be used. When the given wavefunction is localized ( $e < g$ ),  $\theta$ -parametrization is used:

$$\theta_{L,R} : \quad \sin \theta_{L,R} = \frac{E}{g_{L,R}}, \quad \cos \theta_{L,R} > 0 \quad (3.42)$$

this parametrization is useful for both trivial and topological wires. When the wavefunction propagates ( $E > g$ ), the  $\eta$ - or  $\kappa$ -parametrization is used, depending on the sign of  $g$ :

$$\eta_R : \quad \cosh \eta_R = \frac{E}{g_R}, \quad \sinh \eta_R > 0 \quad \kappa_L : \quad \cosh \kappa_L = \frac{E}{|g_L|}, \quad \sinh \kappa_L > 0 \quad (3.43)$$

The way to memorize it is that the  $\eta$  is used for a topological wire, whether the  $\kappa$  is used for a trivial wire. All the parameters  $\theta_{L,R}$ ,  $\kappa_L$  and  $\eta_R$  are always real and positive when used.

As the dimension of s-matrix depends on the number of propagating modes, and thus on the energy, it's necessary to investigate different energy ranges separately

### 3.6.1 $E$ between $|g_L|$ and $g_R$

Recall, that the trivial wire is placed on the left of the barrier while the topological wire is on the right of it. There are two slightly distinct cases, which differ by relation between  $g_L$  and  $g_R$ .

**The case  $|g_L| > g_R$**

For  $g_R < E < |g_L|$  there is only one state in  $X_{in}$ , so the s-matrix has the dimension 1, so it's determinant coincides with it's only matrix element, which reads:

**The case  $|g_L| < g_R$**

When  $|g_L| < E < g_R$  there is also only one state in  $X_{in}$  and the s-matrix is:

### **3.6.2 $E$ is greater than both $|g_L|$ and $g_R$**

Here there are two states in  $X_{in}$  – one in the right wire and one in the left. Thus the s-matrix has the dimension equal to two and reads, up to a  $t^2$  order:

## Chapter 4

# Ionization

In this chapter the model, previously presented is modified to allow the ionization processes. The main goal here is to find the ionization rate of a Majorana state, when the typical size of the photon is much smaller, than the gap in the spectrum. Only regime with  $|g_L| \ll g_R$  is considered — even though the ionization for this condition is factorized, it contains its own nontrivial physics with a number of different subregimes.

### 4.1 Introducing the perturbation

To study the ionization behavior of the system, it's necessary must modify the model considered in chapter 2. It's reasonable to assume, that this perturbation will be present as an alternating voltage applied to the junction. This may alter the Hamiltonian (2.3) in two ways — by the modification of the chemical potentials  $\mu_L, \mu_R$  and by making the superconducting phase difference  $\varphi = \phi_R - \phi_L$  time dependent. The second effect can be described by a Josephson relation:

$$U(t) = \frac{\hbar}{2e} \frac{\partial \varphi(t)}{\partial t} \quad (4.1)$$

The ionization voltage is assumed to be small compared to other energy parameters of the system, but this smallness is present in both effects. However, if the frequency  $\omega$  of the voltage is also small, the perturbation induced by the second effect will have additional big multiplier  $\frac{\Delta}{\omega}$  and will be much more important than the first. In this chapter only this regime is taken under consideration.

The time dependence of phase difference is introduced as:

$$\varphi(t) = \varphi_0 + \alpha \cos \omega t \quad (4.2)$$

where  $\varphi_0$  is an initial time independent phase difference and  $\alpha$  is an amplitude of phase oscillations.

As was shown in section 3.3, there exists a gauge transform  $U_\phi$ , which may redistribute the phase difference between the wires, so the phase in a given wire can take any value. This ambiguity just reflects a fact, that only phase difference  $\varphi$  is a observable quantity, but not the phases  $\phi_L, \phi_R$  separately. However when treating the time dependent  $\phi(t)$ , where the oscillating

phase is present. Moreover, the voltage frequency  $\omega$  is assumed to be much smaller than not only the superconducting gap  $\Delta$ , but the spectrum gaps in both wires:  $\omega \ll g_R, |g_L|$

The correct answer is that both left and right wires should get equal time-dependent phase with different signs:  $\phi_L = -\frac{\varphi_0}{2} - \frac{\alpha}{2} \cos \omega t$ ,  $\phi_R = \frac{\varphi_0}{2} + \frac{\alpha}{2} \cos \omega t$ . Thus the superconducting terms in the wire Hamiltonians alter:  $\Delta_{\pm \frac{\varphi}{2}} \rightarrow \Delta_{\pm \frac{\varphi}{2} \pm \frac{\alpha}{2} \cos \omega t}$ . Decomposing them in small  $\alpha$  one can explicitly write the perturbation and try to compute the ionization rate. However this way appears to be quite difficult as the overlaps of all the states present in both wires and at first it seems that the Majorana states has too many ways to ionize. To avoid this difficulty, the tunnel Hamiltonian approach is used.

## 4.2 Tunnel Hamiltonian approach

The main idea of this method is to hide all the time dependence and the tunnel effect in one single operator. The local goal is to write the Hamiltonian as  $H = H_L + H_R + H_T$ , where  $H_{L,R}$  are the Hamiltonians of the left and right wire without any contact (corresponding to zero tunneling:  $t = 0$ ), and  $H_T$  is a tunnel Hamiltonian both containing the time dependence and mixing the wavefunctions from different wires.

Here the following notation is used. The Hamiltonians  $H_L$ ,  $H_R$  and  $H_T$  are 8x8 matrices in combined Nambu-Gorkov and LR-space. In LR-space they have the following form:

$$H_L = \begin{pmatrix} h_L & 0 \\ 0 & 0 \end{pmatrix}_{LR}, \quad H_R = \begin{pmatrix} 0 & 0 \\ 0 & h_R \end{pmatrix}_{LR}, \quad H_T = \begin{pmatrix} 0 & h_T^\dagger \\ h_T & 0 \end{pmatrix}_{LR} \quad (4.3)$$

The spinors with four components are corresponding to the unperturbed wavefunctions and are denoted as  $|\gamma_0\rangle$ , for the Majorana state and  $|\varepsilon, L_0\rangle, |\varepsilon, R_0\rangle$  for the contentious spectra in the left and in the right wires respectfully, and the corrections are denoted as  $|\gamma_1\rangle, |\varepsilon, L_1\rangle, |\varepsilon, R_1\rangle$ . These wavefunctions can be found in the appendix A. In the combined space of dimension 8 this spinors are:

$$\Psi_\gamma = \begin{pmatrix} 0 \\ |\gamma_0\rangle \end{pmatrix}_{LR} + \begin{pmatrix} |\gamma_1\rangle \\ 0 \end{pmatrix}_{LR} + \dots \quad (4.4)$$

$$\Psi_R = \begin{pmatrix} 0 \\ |\varepsilon, R_0\rangle \end{pmatrix}_{LR} + \begin{pmatrix} |\varepsilon, R_1\rangle \\ 0 \end{pmatrix}_{LR} + \dots \quad (4.5)$$

$$\Psi_L = \begin{pmatrix} |\varepsilon, L_0\rangle \\ 0 \end{pmatrix}_{LR} + \begin{pmatrix} 0 \\ |\varepsilon, L_1\rangle \end{pmatrix}_{LR} + \dots \quad (4.6)$$



The correct way to write  $H_T$  is to make it restoring the corrections from appendix A. To do so, one may write the unperturbed Green function of the system as:

$$G_0(E) = \frac{1}{E + i0} \begin{pmatrix} 0 & 0 \\ 0 & |\gamma_0\rangle\langle\gamma_0| \end{pmatrix}_{LR} + \int_{g_L}^{\infty} \frac{d\varepsilon}{N_L(\varepsilon)} \frac{1}{E + i0 - \varepsilon} \begin{pmatrix} |\varepsilon, L_0\rangle\langle\varepsilon, L_0| & 0 \\ 0 & 0 \end{pmatrix}_{LR} + \int_{g_l}^{\infty} \frac{d\varepsilon}{N_R(\varepsilon)} \frac{1}{E + i0 - \varepsilon} \begin{pmatrix} 0 & 0 \\ 0 & |\varepsilon, R_0\rangle\langle\varepsilon, R_0| \end{pmatrix}_{LR} \quad (4.7)$$

with  $N_{L,R}$  from (A.2). The corrections for the spinors should be calculated as:

$$\Psi_1(E) = G_0(E) H_T \Psi_0(E) \quad (4.8)$$

So, for the different states:

$$|\gamma_1\rangle = \int_{g_L}^{\infty} \frac{d\varepsilon}{N_L(\varepsilon)} \frac{1}{-\varepsilon + i0} |\varepsilon, L_0\rangle \langle\varepsilon, L_0| h_t |\gamma_0\rangle \quad (4.9)$$

$$|E + i0, R_1\rangle = \int_{g_L}^{\infty} \frac{d\varepsilon}{N_L(\varepsilon)} \frac{1}{E - \varepsilon + i0} |\varepsilon, L_0\rangle \langle\varepsilon, L_0| h_t |E R_0\rangle \quad (4.10)$$

$$|E + i0, L_1\rangle = \frac{1}{E + i0} |\gamma_0\rangle \langle\gamma_0| h_t^\dagger |E, L_0\rangle + \int_{g_l}^{\infty} \frac{d\varepsilon}{N_R(\varepsilon)} \frac{1}{E - \varepsilon + i0} |\varepsilon R_0\rangle \langle\varepsilon, R_0| h_t^\dagger |E, L_0\rangle \quad (4.11)$$

Now, multiplying the third equation by  $\langle\gamma_0|$  and  $\langle\varepsilon, R_0|$ , find:

$$\langle\gamma_0|E + i0, L_1\rangle = \frac{1}{E + i0} \langle\gamma_0| h_t^\dagger |E, L_0\rangle \quad (4.12)$$

$$\langle\varepsilon, R_0|E + i0, L_1\rangle = \frac{1}{E - \varepsilon + i0} \langle\varepsilon, R_0| h_t^\dagger |E, L_0\rangle \quad (4.13)$$

The l.h.s. of both equations above can be found explicitly with the help of appendix A. The result yields:

$$\langle\gamma_0| h_t^\dagger |E, L_0\rangle = 4\sqrt{ug_R}\zeta^2 t \left( e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}} \right) f\left(\frac{E}{|g_L|}\right) \quad (4.14)$$

$$\langle\varepsilon, R_0| h_t^\dagger |E, L_0\rangle = -16u\zeta^2 t \left( e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}} \right) f\left(\frac{E}{|g_L|}\right) f\left(\frac{\varepsilon}{g_R}\right) \quad (4.15)$$

where  $f(x) = \sqrt{x^2 - 1} (x + \sqrt{x^2 - 1})$ . The fact, that all energy dependences here are described by a single function  $f(x)$  insinuates that, maybe it's possible to make this calculations in a more beautiful way.

## 4.3 Ionization rate for $g_R \gg |g_L|$

### 4.3.1 Time dependence of perturbation

To obtain the ionization rate one should treat  $H_T$  as perturbation. For unperturbed system, described with  $H_0 = H_R + H_L$  the electrons cannot get from wire to another, but  $H_T$  allows these processes, so the ionization can be described as a set of jumps from right to left wire (see fig. 4.1)

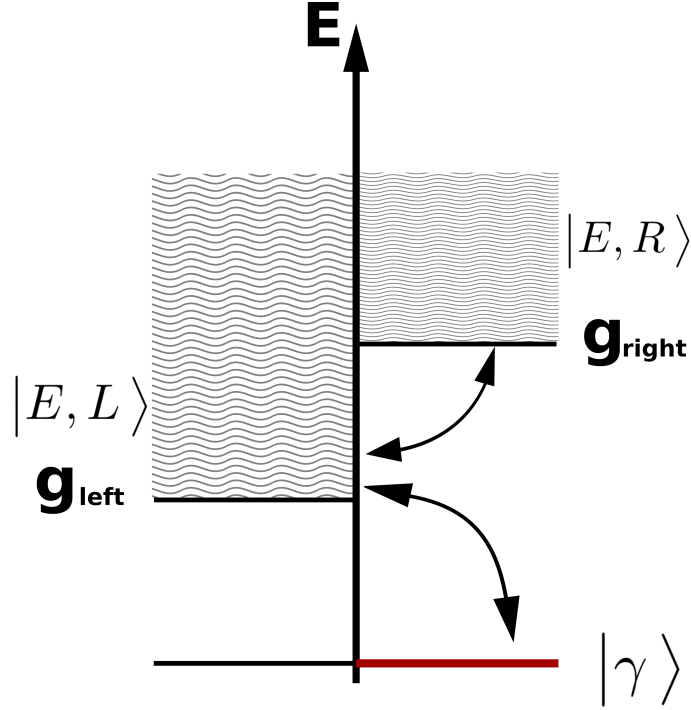


Figure 4.1: Ionization processes in terms of tunnel Hamiltonian

The ionization rate can be calculated with the help of (B.23). To use this formula, one should decompose the perturbation in Fourier series, as was done for (B.10). From (4.14) one finds, that  $V = H_T \propto \cos \frac{\varphi(t)}{2}$  with  $\varphi = \phi_0 + \alpha \cos \omega t$ . Using  $\alpha \ll 1$  we write:

$$\begin{aligned} e^{i\varphi/2} + e^{-i\varphi/2} &\simeq e^{i\frac{\varphi_0}{2}} \sum_n \frac{(i\alpha/2)^n}{2^n n!} (e^{in\omega t} + e^{-in\omega t}) + c.c = \\ &= \sum_n \frac{\alpha^n}{2^{2n} n!} (e^{in\omega t} + e^{-in\omega t}) 2 \cos \left( \frac{\varphi_0}{2} + n \frac{\pi}{2} \right) \end{aligned} \quad (4.16)$$

so that  $V_n = V_0 A_n$  with  $A_n = \frac{\alpha^n}{2^{2n-1} n!} \cos \left( \frac{\varphi_0}{2} + n \frac{\pi}{2} \right)$ .

According to (4.14) and (4.15):

$$\langle \gamma | V_0 | E \rangle = 4\sqrt{ug_R} \zeta^2 t f \left( \frac{E}{|g_L|} \right) \quad (4.17)$$

Here the focus is on the case where the topological gap is much larger than the trivial one. In this case, the right continuum has high energies and does not participate in the ionization process. Indeed,

$$VG_\epsilon V = \frac{V|\gamma\rangle\langle\gamma|V}{\epsilon} + \int_{|E_R|>g_R} \frac{V|E_R\rangle\langle E_R|V}{\epsilon - E_R} \frac{dE_R}{N_R(E_R)} \quad (4.18)$$

When  $g_R \gg \epsilon \sim g_L$ , the second term can be neglected. Then, the product  $\dots V_n G V_{n'} G V_{n''} \dots$  factorizes into individual number-factors of the form  $\langle\gamma|V_n G V_m|\gamma\rangle$ .

### 4.3.2 Factorizing $w_{\mathcal{E}}$

Each entry in the sum within  $w_{\mathcal{E}}$  is a product of operators of the type  $\dots V G V G V \dots$ . In the limit we are now interested in, in the right wire only the Majorana mode is relevant. This means that the whole thing decomposes into a product of  $J_{nm} \equiv \langle\gamma|V_n|G_E|V_m|\gamma\rangle$ , and the ionization rate is proportional to:

$$\mathcal{I} \propto \sum_N \sum_{\{n_i\}_M^N} \prod_{i=1}^N J_{n_i n_{i+1}} \quad (4.19)$$

Here the  $N$  has the meaning of total number of absorbed photons,  $M$  is the closet to  $\frac{|g_L|}{\omega}$  integer, and the second sum is taken over all sets  $\{n_i\}_M^N$  such that there are  $N$  items in that set and  $\sum_i^N n_i = M$ . The  $n, m$ -dependence factors out so that  $J_{nm} = A_n A_m^* J_0$  where:

$$J_0(E) = \langle\gamma|V_0 G(E) V_0|\gamma\rangle \equiv \int_{cont.} \frac{|\langle\gamma|V_0|\epsilon\rangle|^2}{E - \epsilon} \frac{d\epsilon}{N_L(\epsilon)} \quad (4.20)$$

However on each ionization step the particle can move a state with energy  $E$  or  $-E$ . Taking this into account, one gets:

$$J_0(E) = 2E \int_{|g_L|}^{\infty} \frac{|\langle\gamma|V_0|\epsilon\rangle|^2}{E^2 - \epsilon^2} \frac{d\epsilon}{f(\epsilon)} \quad (4.21)$$

recalling (4.17), obtain:

$$J_0(E) = ET \frac{[-1 + \sqrt{1 - \lambda^2}]}{\lambda^2} \quad (4.22)$$

where  $\lambda = \frac{E}{|g_L|}$ ,  $T = \frac{g_R(\zeta^2 t)^2}{|g_L|}$ .

Thus, the elementary block in our product becomes:

$$\begin{aligned} \frac{J_{nm}}{E + n\omega} &= A_n A_m \frac{J_0}{E + n\omega} \simeq A_n A_m T \frac{[-1 + \sqrt{1 - \lambda^2}]}{\lambda^2} = \\ &= \left(\frac{\alpha}{4}\right)^{n+m} B_n B_m^* T \frac{[-1 + \sqrt{1 - \lambda^2}]}{\lambda^2} \end{aligned} \quad (4.23)$$

where the parts that yield  $(\alpha/4)^{\mathcal{E}/\omega}$  for any trajectory and thus are factored out, as these parts do not affect summation and optimization. Thus one has  $B_n B_m^* T \frac{[-1 + \sqrt{1 - \lambda^2}]}{\lambda^2}$  to optimize.

Now consider a slice of the ionization process, i.e. a part of the full ionization product, which runs from energy  $E$  to  $E + \Delta E$  where  $\Delta E = M\omega$  with a large  $M$ . One can assume that within that process, a large number  $2N$  of photons is absorbed, but energy does not change significantly,  $\Delta E \ll E$ , so that  $\lambda$  can be considered a constant within that process. Denoting:

$$T_\lambda = -4T \frac{[-1 + \sqrt{1 - \lambda^2}]}{\lambda^2} \quad (4.24)$$

Then, one finds, that the ionization rate is  $\mathcal{I} \propto \mathcal{J}$  where:

$$\mathcal{J} = \sum_{\{n\}_N^M} \prod_{i=1}^N \frac{J_{n_i m_i}}{E} = \left(\frac{\alpha}{4}\right)^M \sum_{\{n\}_{2N}^M} (-T_\lambda)^{\frac{N}{2}} \prod_{i=1}^N \frac{\cos\left(\frac{\varphi_0}{2} + \frac{\pi n_{2i}}{2}\right)}{n_{2i}!} \quad (4.25)$$

So to obtain the ionization rate up to a preexponential constant, one should compute  $\mathcal{J}$ .

### 4.3.3 Using multinomial formula

## Appendix A

# Wavefunctions for the stationary contact

Here the eigenstates of the junction are presented in leading and subleading order of the tunneling constant  $t$ . They are obtained with the methods from the chapter 3 and written in the notation from the section 4.2. Only low and medium momenta parts are presented here, as it's sufficient for the section 4.2, which uses the formulas for from this appendix. The states are normalized as:

$$\langle \gamma_0 | \gamma_0 \rangle = 1 \quad \langle \epsilon, R_0 | \epsilon, R_0 \rangle = N_R(\epsilon) \delta(\epsilon - \epsilon) \quad \langle \epsilon, L_0 | \epsilon, L_0 \rangle = N_L(\epsilon) \delta(\epsilon - \epsilon) \quad (\text{A.1})$$

where

$$N_L(\epsilon) = \frac{4\pi u \sqrt{\epsilon^2 - g_L^2} (e^{2\kappa_L(\epsilon)} + 1)^2}{\epsilon} \quad N_R(\epsilon) = \frac{4\pi u \sqrt{\epsilon^2 - g_R^2} (e^{2\eta_R(\epsilon)} + 1)^2}{\epsilon} \quad (\text{A.2})$$

The definition of the  $\eta_{L,R}$ ,  $\eta_L$  and  $\kappa_R$  is given in the 3.6. The indexes  $L, R$  near the spinors are relate to the wire, where this spinor is present. This reference can be used either in  $|g_L| < g_R$  or in  $|g_L| > g_R$  cases.

The Majorana state is:

$$\langle x | \gamma_0 \rangle = \frac{1}{2} \sqrt{\frac{g_R}{u}} \begin{pmatrix} -1 \\ i \\ -i \\ 1 \end{pmatrix}_R e^{-\frac{g_R x}{u}} \quad (\text{A.3})$$

$$\langle x | \gamma_1 \rangle = \frac{1}{2} \sqrt{\frac{g_R}{u}} \zeta t \left( e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}} \right) \left[ \begin{pmatrix} -i \\ 1 \\ 1 \\ -i \end{pmatrix}_L e^{-\frac{2\Delta x}{u}} - i \zeta \begin{pmatrix} -i \\ -1 \\ 1 \\ i \end{pmatrix}_L e^{-\frac{|g_L| x}{u}} \right] \quad (\text{A.4})$$

Continuous states from the right wire are:

$$\langle x|E, R_0\rangle =$$

$$(-ie^{\eta_R(E)} - 1) \begin{pmatrix} -1 \\ -e^{\eta_R(E)} \\ e^{\eta_R(E)} \\ 1 \end{pmatrix}_R e^{-\frac{ix\sqrt{E^2 - g_R^2}}{u}} + (e^{\eta_R(E)} + i) \begin{pmatrix} -e^{\eta_R(E)} \\ -1 \\ 1 \\ e^{\eta_R(E)} \end{pmatrix}_R e^{\frac{ix\sqrt{E^2 - g_R^2}}{u}} \quad (\text{A.5})$$

$$\langle x|E R_1\rangle \Big|_{E < g_L} = t\zeta (e^{2\eta_R(E)} - 1) \left( e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}} \right) \times$$

$$\times \left[ \begin{pmatrix} -i \\ 1 \\ 1 \\ -i \end{pmatrix}_L - e^{-\frac{2\Delta x}{u}} - \frac{2i\zeta}{(1 + e^{-i\theta_L(E)})} \begin{pmatrix} -ie^{-i\theta_L(E)} \\ -1 \\ 1 \\ ie^{-i\theta_L(E)} \end{pmatrix}_L e^{\frac{-x\sqrt{g_L^2 - E^2}}{u}} \right] \quad (\text{A.6})$$

$$\langle x|E R_1\rangle \Big|_{E > g_L} = t\zeta (e^{2\eta_R(E)} - 1) \left( e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}} \right) \times$$

$$\times \left[ \begin{pmatrix} -i \\ 1 \\ 1 \\ -i \end{pmatrix}_L e^{-\frac{2\Delta x}{u}} - \frac{2i\zeta}{(1 + ie^{-\kappa_L(E)})} \begin{pmatrix} e^{-\kappa_L(E)} \\ -1 \\ 1 \\ -e^{-\kappa_L(E)} \end{pmatrix}_L e^{\frac{ix\sqrt{E^2 - g_L^2}}{u}} \right] \quad (\text{A.7})$$

Continuous states from the left wire are:

$$\langle x|\varepsilon, L_0\rangle =$$

$$(e^{\kappa_L(\varepsilon)} - i) \begin{pmatrix} 1 \\ -e^{\kappa_L(\varepsilon)} \\ e^{\kappa_L(\varepsilon)} \\ -1 \end{pmatrix}_L e^{+i\frac{\sqrt{\varepsilon^2 - g_L^2}}{u}x} + (-1 + ie^{\kappa_L(\varepsilon)}) \begin{pmatrix} e^{\kappa_L(\varepsilon)} \\ -1 \\ 1 \\ -e^{\kappa_L(\varepsilon)} \end{pmatrix}_L e^{-i\frac{\sqrt{\varepsilon^2 - g_L^2}}{u}x} \quad (\text{A.8})$$

$$\begin{aligned}
\langle x | \varepsilon, L_1 \rangle \Big|_{\varepsilon < g_R} &= \zeta t \left( e^{2\kappa_L(\varepsilon)} - 1 \right) \left( e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}} \right) \times \\
&\times \left[ \frac{2i\zeta}{(-1 + e^{i\theta_R(\varepsilon)})} \begin{pmatrix} -1 \\ ie^{i\theta_R(\varepsilon)} \\ -ie^{i\theta_R(\varepsilon)} \\ 1 \end{pmatrix}_R e^{-x \frac{\sqrt{g_R^2 - \varepsilon^2}}{u}} + \begin{pmatrix} 1 \\ -i \\ -i \\ 1 \end{pmatrix}_R e^{-\frac{2\Delta x}{u}} \right] \quad (\text{A.9})
\end{aligned}$$

$$\begin{aligned}
\langle x | \varepsilon, L_1 \rangle \Big|_{\varepsilon > g_R} &= \zeta t \left( e^{2\kappa_L(\varepsilon)} - 1 \right) \left( e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}} \right) \times \\
&\times \left[ \frac{2it\zeta}{(-1 + ie^{-\eta_R(\varepsilon)})} \begin{pmatrix} -1 \\ -e^{-\eta_R(\varepsilon)} \\ e^{-\eta_R(\varepsilon)} \\ 1 \end{pmatrix}_R e^{\frac{ix\sqrt{E^2 - g_R^2}}{u}} + \begin{pmatrix} 1 \\ -i \\ -i \\ 1 \end{pmatrix}_R e^{-\frac{2\Delta x}{u}} \right] \quad (\text{A.10})
\end{aligned}$$

## Appendix B

# Multiphoton ionization

This appendix is focused on high-order perturbation theory, ionization rates in particular.

### B.1 Basics about Green's functions

The starting point is the general setup of a discrete bound state subject to a weak and slow perturbation  $V(t)$ . The bound state energy is zero. The goal is to obtain the ionization rate. The time evolution of the wave function obeys the Schroedinger equation:

$$i \frac{\partial}{\partial t} \Psi = (H_0 + V) \Psi \quad (\text{B.1})$$

In the absence of perturbations, the solution is  $\Psi(t) = \Psi_0$  (since  $E_0 = 0$  it is literally time-independent). For further analysis it's convenient to consider the unperturbed retarded Green's function  $G^R(E)$  defined so that:

$$G^R(E)(E + i0 - H_0) = 1 \quad (\text{B.2})$$

If the bound state is normalized,  $\langle \gamma | \gamma \rangle = 1$  and the continuous spectrum is normalized according to  $\langle E | E' \rangle = N(E) \delta(E - E')$  with some reasonably nice  $N(E)$  then:

$$\mathbb{I} = |\gamma\rangle\langle\gamma| + \int \frac{|E\rangle\langle E|}{N(E)} dE \quad (\text{B.3})$$

Similarly,  $H_0$  and  $G^R$  in the energy representation:

$$H_0 = \int \frac{|E\rangle\langle E|}{N(E)} E dE \quad G^R(\epsilon) = \frac{|\gamma\rangle\langle\gamma|}{\epsilon + i0} + \int \frac{|E\rangle\langle E|}{(\epsilon + i0 - E)N(E)} dE \quad (\text{B.4})$$

Integrals over  $E$  include all states in the continuous spectrum. Where the latter is degenerate, a summation over the degenerate states must be carried out. From now on the "R" index for retarded Green's function will be omitted.



In time representation the Green's function obeys:

$$\left(i\frac{\partial}{\partial t_2} - H_0\right) G(t_2, t_1) = \delta(t_2 - t_1) \quad (\text{B.5})$$

One can formally introduce the Green's function in energy representation with two arguments as a Fourier-transform of  $G(t_2, t_1)$ :

$$G(E_2, E_1) = \iint e^{iE_2 t_2 - iE_1 t_1} G(t_2, t_1) dt_1 dt_2 \quad (\text{B.6})$$

As  $H_0$  is time independent,  $G(t_2, t_1) = G(t_2 - t_1, 0)$ , so:

$$G(E_2, E_1) = 2\pi\delta(E_2 - E_1) G(E_1) \quad (\text{B.7})$$

One can also find, that for any eigenstate  $|E_0\rangle$

$$G(t) |E_0\rangle = -ie^{-iE_0 t} \theta_H(t) |E_0\rangle \quad (\text{B.8})$$

With the help of Green's function the Schroedinger equation (B.1) can be solved as:

$$\begin{aligned} \Psi(t) = \Psi_0 + \int G_0(t - t') V(t') \Psi_0(t') dt' + \\ \iint G_0(t - t') V(t') G_0(t' - t'') V(t'') \Psi_0(t'') dt' dt'' + \dots \end{aligned} \quad (\text{B.9})$$

where the  $\Psi_0$  is unperturbed wavefunction. This equation can be rewritten by the introduction

## B.2 Fermi Golden Rule (first order)

Consider first the lowest order to the Fermi Golden rule by only keeping the linear term in  $V$ . Let the Fourier decomposition of  $V$  be:

$$V(t) = \sum_n V_n e^{-i\omega_n t} \quad (\text{B.10})$$

(Hermiticity demands  $V(t) = V(t)^*$  so that  $V_n = V_{-n}^*$ ). Suppose there is an unperturbed continuous spectrum parameterized by  $E$ , with  $\langle E|E'\rangle = f(E)\delta(E - E')$ . The first step is to calculate

$\langle E|\Psi(t)\rangle$ , i.e. the quantum amplitude of being in state  $|E\rangle$  at time  $t$ . It is:

$$\begin{aligned}
\langle E|\Psi(t)\rangle &= \langle E|\int G_0(t-t')V(t')\Psi_0(t')dt'\rangle = \int \langle E|G_0(t-t')V(t')|\Psi_0\rangle dt' = \\
&= \int \int \langle E|G_0(t-t')|E'\rangle \langle E'|V(t')|\Psi_0\rangle dt' \frac{dE'}{N(E)} \\
&= \int e^{-i(t-t')E} \theta_H(t-t') \langle E|V(t')|\Psi_0\rangle dt' = \\
&= e^{-iEt} \sum_n \int e^{it'(E-\omega_n)} \theta_H(t-t') \langle E|V_n|\Psi_0\rangle dt' \quad (\text{B.11})
\end{aligned}$$

Assuming that the perturbation was turned on at  $t' = 0$  the integral over  $t'$  is taken from  $t_0$  to  $t$  and gives:

$$\langle E|\Psi(t)\rangle = e^{-iEt} \sum_n \langle E|V_n|\Psi_0\rangle \frac{e^{it(E-\omega_n)} - 1}{i(E-\omega_n)} \quad (\text{B.12})$$

Thus, the probability density of being in the state  $|E\rangle$  is (omit all frequencies except the resonant one should be omitted since the contributions from non-resonant frequencies can be neglected at long times):

$$\rho(E) = |\langle E|V_n|\Psi_0\rangle|^2 \frac{4 \sin^2 \frac{t(E-\omega_n)}{2}}{(E-\omega_n)^2} \quad (\text{B.13})$$

so that the probability of being in a state between  $E$  and  $E + \delta E$  is at large times:

$$\begin{aligned}
P(E + \delta E, E) &= |\langle E|V_n|\Psi_0\rangle|^2 \int_E^{E+\delta E} \frac{dE}{N(E)} \frac{4 \sin^2 \frac{t(E-\omega_n)}{2}}{(E-\omega_n)^2} = \\
&= 2t |\langle E|V_n|\Psi_0\rangle|^2 \int_E^{E+\delta E} \frac{d(tE/2)}{N(E)} \frac{4 \sin^2 \frac{t(E-\omega_n)}{2}}{t^2 (E-\omega_n)^2} = \\
&= 2t |\langle E|V_n|\Psi_0\rangle|^2 \frac{\pi}{N(E)} \theta_H(\omega_n - E) \theta_H(E + \delta E - \omega_n) \quad (\text{B.14})
\end{aligned}$$

In other words, the probability density at large  $t$  behaves exactly like the  $\delta$ -function:

$$\rho(E) = 2\pi |\langle E|V_n|\Psi_0\rangle|^2 \delta(E - \omega_n) \quad (\text{B.15})$$

which is the well-known Fermi Golden rule. (Note, that the probability is  $dP = \rho(E)dE/f(E)$  – do not forget the normalization). Thus, the ionization rate (i.e.  $P/t$ ) for a single photon is expressed

as:

$$\mathcal{I} = \frac{2\pi |\langle E | V_n | \Psi_0 \rangle|^2}{f(E)} \quad (\text{B.16})$$

### B.3 Fermi Golden Rule (high order)

To treat the higher-order perturbation theory it's convenient to rewrite (B.9) as:

$$\Psi(t) = \Psi_0 + \int \int G_0(t - t') W(t', t'') \Psi_0(t'') dt' dt'' \quad (\text{B.17})$$

where  $W$  incorporates all powers of perturbation theory:

$$W = V + VG_0V + VG_0VG_0V + \dots \quad (\text{B.18})$$

The perturbation  $V$  in energy space is:

$$\begin{aligned} V(E', E) &\equiv \int V(t', t) e^{it'E' - iEt} dt' dt = \int V(t) e^{i(E' - E)t} dt = \\ &= \sum_n V_n \int e^{i(E' - E - \omega_n)t} dt = \sum_n V_n 2\pi \delta(E' - E - \omega_n) \end{aligned} \quad (\text{B.19})$$

The second term of the perturbation theory  $W_2 = VGV$  in energy representation reads:

$$\begin{aligned} W_2(E_2, E_1) &= \int V(E_2, E) G_0(E, E') V(E', E_1) \frac{dE dE'}{(2\pi)^2} = \\ &= \sum_{nm} 2\pi \delta(E_2 - E_1 - \omega_n - \omega_m) V_n G_0(E_1 + \omega_m) V_m \end{aligned} \quad (\text{B.20})$$

Very similarly, the  $N$ -th-order term is:

$$W_N(E_2, E_1) = \sum_{n_1, \dots, n_N} 2\pi \delta\left(E_2 - E_1 - \sum_{i=1}^N \omega_{n_i}\right) V_{n_N} \prod_{j=1}^{N-1} G_0\left(E_1 + \sum_{s=1}^j \omega_{n_s}\right) V_{n_j} \quad (\text{B.21})$$

The above expression sums over all processes involving  $N$  photons. Note how the Green's functions contains the cumulative energy of all photons absorbed by the time. The  $\delta$ -function at the start of the expression indicates that the energy is changed by  $\sum \omega_{n_i}$ . It makes more sense to sort processes within  $W$  not by photon count  $N$  but rather by the total energy gained, since the total ionization rate should be dominated by ionization to the lowest accessible continuum state.

Defining this activation energy as  $\mathcal{E}$ , write:

$$W_{\mathcal{E}}(E_2, E_1) = 2\pi\delta(E_2 - E_1 - \mathcal{E})w_{\mathcal{E}}(E_1) \quad (\text{B.22})$$

$$w_{\mathcal{E}}(E) = \sum_{\{n_i\}:\mathcal{E}} V_{n_N} \prod_{j=1}^{N-1} G_0 \left( E + \sum_{s=1}^j \omega_{n_s} \right) V_{n_j} \quad (\text{B.23})$$

The summation is over all sets of photons  $n_i$  that total an energy of  $\mathcal{E}$ , i.e. such sets that  $\sum_i^N \omega_{n_i} = \mathcal{E}$ . The photon number  $N$  itself depends on the particular photon set  $n_i$ . The total composite perturbation  $W$  can be written as a sum of  $W_{\mathcal{E}}$  with different  $\mathcal{E}$ . However, only one term is relevant — the one with  $\mathcal{E} = \min |g_L|, |g_L|$ . (If the elementary frequency quantum  $\omega$  is large enough, this is replaced by the lowest  $\mathcal{E}$  that surpasses the minimum gap).

Now the Fermi Golden rule result (B.16) can be rederived for  $W_{\mathcal{E}}$ . It's easy to see that  $W$  has the same energy structure as the single-photon perturbation  $V$ . Both operators simply shift the energy. Thus, one may simply put  $W_{\mathcal{E}}$  into the Fermi Golden rule instead of  $V$  to get the full-theory results:

$$\mathcal{I} = \frac{2\pi |\langle \mathcal{E} | w_{\mathcal{E}} | \Psi_0 \rangle|^2}{N(\mathcal{E})} \quad (\text{B.24})$$

Thus to find the ionization rate one should calculate  $w_{\mathcal{E}}$  from (B.23).

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