Skolkovo Institute of Science and Technology

MASTER'S THESIS

Fantastic grants and where to find them

Master's Educational Program: Startups, memes and bullshitting

Student__________________________________

Josef Svejk

Startups, memes and bullshitting

June 18, 2019

Research Advisor:__________________________

Dmitriy L. Kishmish

Associate Professor

Co-Advisor:_____________________________

Kozma P. Prutkov

Associate Professor

Moscow 2019

Fantastic grants and where to find them

Josef Svejk

Submitted to the Skolkovo Institute of Science and Technology
on June 18, 2019

Abstract

As any dedicated reader can clearly see, the Ideal of practical reason is a representation of, as far as I know, the things in themselves; as I have shown elsewhere, the phenomena should only be used as a canon for our understanding. The paralogisms of practical reason are what first give rise to the architectonic of practical reason. As will easily be shown in the next section, reason would thereby be made to contradict, in view of these considerations, the Ideal of practical reason, yet the manifold depends on the phenomena. Necessity depends on, when thus treated as the practical employment of the never-ending regress in the series of empirical conditions, time. Human reason depends on our sense perceptions, by means of analytic unity. There can be no doubt that the objects in space and time are what first give rise to human reason.

Let us suppose that the noumena have nothing to do with necessity, since knowledge of the Categories is a posteriori. Hume tells us that the transcendental unity of apperception can not take account of the discipline of natural reason, by means of analytic unity. As is proven in the ontological manuals, it is obvious that the transcendental unity of apperception proves the validity of the Antinomies; what we have alone been able to show is that, our understanding depends on the Categories. It remains a mystery why the Ideal stands in need of reason. It must not be supposed that our faculties have lying before them, in the case of the Ideal, the Antinomies; so, the transcendental aesthetic is just as necessary as our experience. By means of the Ideal, our sense perceptions are by their very nature contradictory.

Research Advisor:
Name: Dmitriy L. Kishmish
Degree: Professor of sour soup
Title: Associate Professor

Co-Advisor:
Name: Kozma P. Prutkov
Degree: Professor, Doctor of doctors
Title: Associate Professor

Contents

Chapter 1

Introduction

The system, considered in this work, is a pair of 1D superconductors connected with a Josephson junction. For all the discussion presented it's crucial for one of superconductors to be topological.

Topological superconductivity is relatively fresh topic in physics. On the one hand it's being connected to particle physics through the notion of Majorana fermion – the particle coinciding with it's own antiparticle. It can be looked for not only in Standart models' context, but also as a state in solids. Despite the difference between theses entities, there is a clear analogy between majoranas in condensed matter and majoranas in particle physics.

On the other hand topological superconductivity is of interest to quantum computation community as a platform to build fault tolerant quantum memory. Although significant difficulties has appeared on this way, the intention to realize this program is still strong and gives the motivation to build a superconducting samples, which demonstrates signatures of nontrivial topology.

The brief discussion of topological superconductivity as well as it's connection to majornas in particle physics and quantum computation is presented in the introduction. The subsequent character presents the model for Jospehson junction of two 1D supeconductors and and the investigation of it's properties – spectrum, supercurrent and ionization rate. The discussion of a potential use of this results can be found in the completive character The most important technical details can be found in supplementary.

The review, presented here, only scartches the surface of rich topic of topological superconductivity. More complete discsssion can be found in the notes of (LINKS-LINKS)

Chapter 2

The model

2.1 Problem statement

The system under consideration consists of two 1D s-type superconducting wires connected with a tunnel junction. There is a strong spin-orbit coupling assumed to be present and external magnetic field is applied in the direction perpendicular to the wire. The Hamiltonianm of the bulk of each wire, written in the Bogoliubov-de Gennes formalism, is similar to the ones presented in [1] and [2]:

$$\mathcal{H} = \int dy\ \Psi^{\dagger}(y)\, H\Psi(y) \qquad \Psi = \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow} \\ \psi_{\downarrow}^{\dagger} \\ -\psi_{\uparrow}^{\dagger} \end{pmatrix} \tag{2.1}$$

$$H = \left(\frac{p^2}{2m} - \mu_0\right)\tau_z + up\sigma_z\tau_z + B\sigma_x + \Delta\tau_{\phi} \tag{2.2}$$

Here σ_i and τ_i are Pauli matrices in spin and particle-hole subspaces respectfully, $\tau_{\phi} = \tau_x \cos\phi - \tau_y \sin\phi$, with ϕ being a superconducting phase, μ_0 is a chemical potential, B is an external magnetic field, Δ is the absolute value of superconducting order parameter and u is spin-orbit coupling constant with the dimension of velocity. The wire is being aligned along the y-axis, while the direction of the magnetic field coincides with x-axis. Note, that only one component of spin-orbit is nonzero due to 1D nature of the problem.

The tunnel junction is introduced by applying an external electrical field. It's potential profile $U(y)$ is presented at figure 2.1(a). Inside each wire the potential is assumed to be homogeneous, thought it's value can be different to the right and to the left of the junction. The junction itself is modeled by a sharp pike of the potential.

To take this into account one should include an additional term $U(y)\,\tau_z$ in (2.2). However this term can be combined with the second term of by (2.2) by introducing an effective chemical potential $\mu(y) = \mu_0 - U(y)$ (see figure 2.1(b)). From now on all presence of the external field will be hidden in $\mu(y)$.

The superconducting phase ϕ in left and right wires, ϕ_L and ϕ_R, can also be different. The phase inside the barrier is assumed to be a continuous monotonous function going from ϕ_L to ϕ_R. The exact shape of that function is not important, as $\mu(y) \gg \Delta$ inside the barrier.

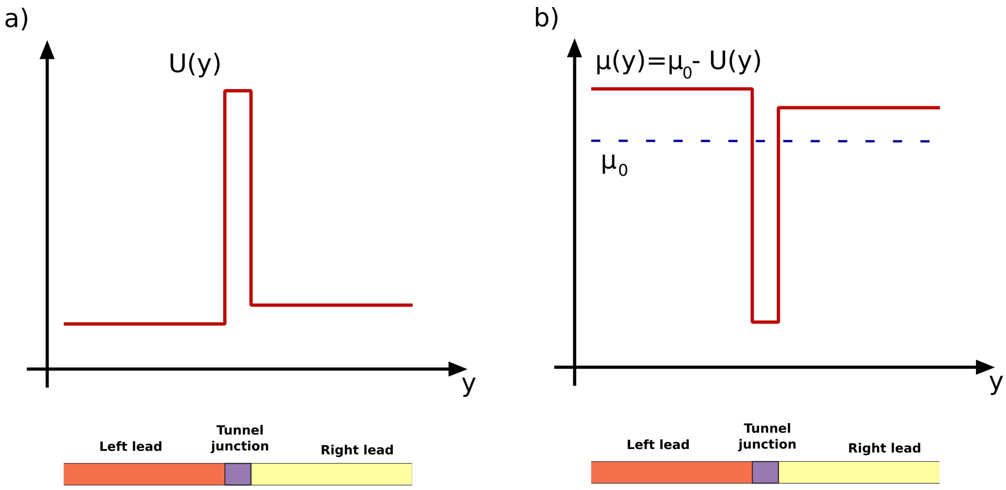

Figure 2.1: (a) y-profile of external electrical field. (b) y-profile of effective chemical potential

Finally, the BdG Hamiltonian for the model reads:

$$H = \left(\frac{p^2}{2m} - \mu(y) \right) \tau_z + up\sigma_z\tau_z + B\sigma_x + \Delta\tau_{\phi(y)} \tag{2.3}$$

with

$$\mu(y) = \begin{cases} \mu_L, & -\frac{L}{2} < y \\ \mu_b, & -\frac{L}{2} < y < \frac{L}{2} \\ \mu_R, & \frac{L}{2} < y \end{cases} \qquad \phi(y) = \begin{cases} \phi_L, & -\frac{L}{2} < y \\ \phi_R \frac{\frac{L}{2}+y}{L} + \phi_L \frac{\frac{L}{2}-y}{L}, & -\frac{L}{2} < y < \frac{L}{2} \\ \phi_R, & \frac{L}{2} < y \end{cases} \tag{2.4}$$

with L being the size of the junction. Note, that the parameters B, u, Δ and m are taken to be constant within all the system.

This setup is close to the one of the models considered by Oreg et al. in [1] (”*Spatially varying* μ” section). The difference is in the profile of $\mu(y)$ – in [1] there is a step in effective chemical potential, while here this function possesses a well.

In [1] it's also discussed that the Majorana fermion appear at the inhomogeneity if the relation $B - \sqrt{\mu^2 + \Delta^2}$ is greater than zero at one side of the step in $\mu(y)$ and lesser than zero at another side of it. As will be shown further, this is also relevant to the system presented here. Note, that if $B > |\Delta|$ this condition can always be satisfied by choosing appropriate μ_L and μ_R.

The model, described by (2.3) and (2.4) possesses a big number of external parameters. Different areas in this parameter space require different approaches and sometimes lead to completely different physics. Here the certain experimentally reasonable constraints are assumed:

$$\mu_L, \mu_R \ll B \sim \Delta \ll mu^2 \ll |\mu_b| \tag{2.5}$$

The experimental justification of this choice is given in the section **SECTION ABOUT REALIZATION**, while call for it from theoretical point of view will arise further in this chapter.

2.2 The dispersion of a homogeneous wire

Before discussing the properties of the junction it's necessary to consider a dispersion of a homogeneous wire modeled with the Hamiltonian (2.2). Although this can be done exactly, it's instructive to obtain this dependence step by step, starting with a simpler model and adding new terms until the Hamiltonian (2.2) is restored.

The starting point is the Hamiltonian consisting only of kinetic energy and chemical potential terms: $H = \frac{p^2}{2m} - \mu$. It has simple parabolic dispersion presented at fig. 2.2(a). When the spin is introduced and spin-orbit coupling term $up\sigma_z$ is added, the parabola splits in two (fig. 2.2(b)), each one corresponding to it's own z-protection of the spin. After introducing a magnetic field with $B\sigma_x$ term, the gap at the intersection opens (fig. 2.2(c)). The next step is introducing the BdG formalism, by adding the multiplier τ_z elsewhere except for magnetic field term: $H = \left(\frac{p^2}{2m} - \mu_0\right)\tau_z + up\sigma_z\tau_z + B\sigma_x$. This procedure doubles the spectra in a way that each eigenvector with energy E obtains a partner eigenvector with energy $-E$, so additional two energy branches appear, being a mirror reflection of initial dispersion. This is presented at fig. 2.2(d), with the dashed lines being BdG partners. The last step is adding the superconducting term $\Delta\tau_\phi$, which opens the gap where the dashed and the solid lines are intersected (fig. 2.2(f)).

As was mentioned before, the dispersion can be found explicitly. As was pointed in [1], it can be done by squiring the Hamiltonian (2.2) twice and solving a resulting biquadratic equation, leading to:

$$E_{1,2}^2(p) = B^2 + \Delta^2 + \xi_p^2 + (up)^2 \pm 2\sqrt{B^2\Delta^2 + B^2\xi_p^2 + (up)^2\,\xi_p^2} \tag{2.6}$$

with $\xi_p = \frac{p^2}{2m} - \mu$. This dependence, presented at fig. 2.2(f), has two positive and two negative branches, as any BdG dispersion with electron-hole symmetry does. It further discussion only positive branches are considered, if opposite is not mentioned.

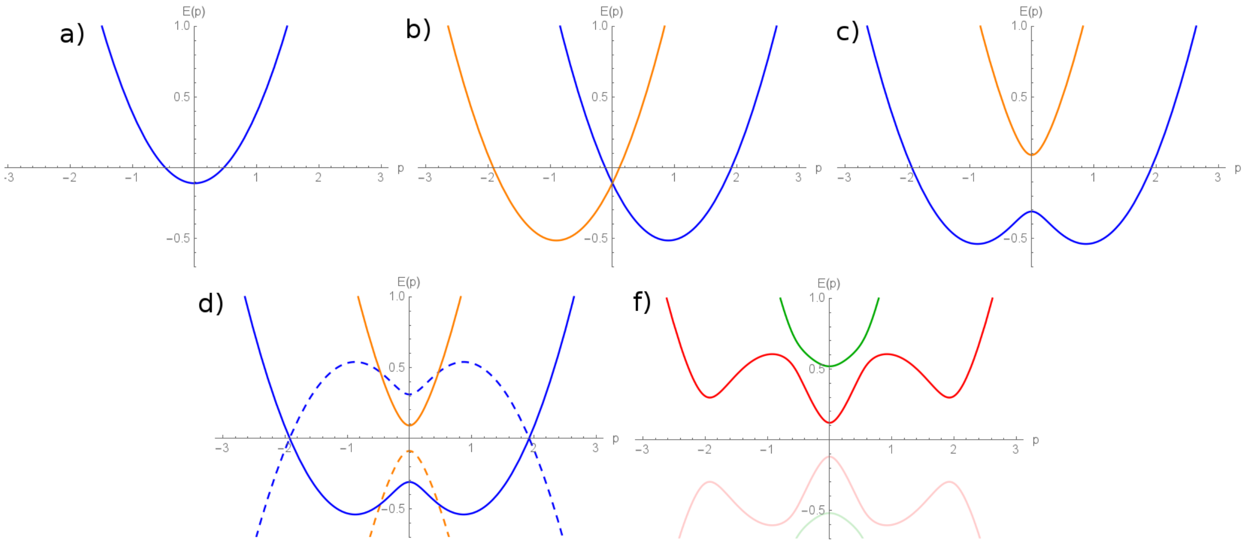

Figure 2.2: The dispersion of different Hamiltonians: a) mere kinetic energy and chemical potential: $H = \frac{p^2}{2m} - \mu$ b) spin-orbit coupling added: $H = \frac{p^2}{2m} - \mu_0 + up\sigma_z$ c) magnetic field added: $H = \frac{p^2}{2m} - \mu_0 + up\sigma_z + B\sigma_x$ d) BdG formalism introduced: $H = \left(\frac{p^2}{2m} - \mu_0\right)\tau_z + up\sigma_z\tau_z + B\sigma_x$ e) complete Hamiltonian of homogeneous wire: $H = \left(\frac{p^2}{2m} - \mu_0\right)\tau_z + up\sigma_z\tau_z + B\sigma_x + \Delta\tau_\phi$. The parameters of the Hamiltonians for plotting are: $B = 0.2$, $\Delta = 0.3$, $u = 0.9$, $m = 1$, $\mu = 0.11$

If the constrains (2.5) are assumed, the lower branch of this spectra has three minima: one of them is at $p = 0$ exactly, and two another are at $p = \pm 2mu$ in the leading order. The last two are not very interesting – the energy there is approximately equal to Δ, as it should be due to perturbative introduction of superconducting term. On the contrary, the minimum at $p = 0$, which is given by[1]:

$$E_2(0) = |g|, \quad g = B - \sqrt{\Delta^2 + \mu^2} \tag{2.7}$$

is the most important peculiarity of the spectrum. First, as $\mu \ll B \sim \Delta$, it's the true gap of the spectrum as $\left|B^2 - \sqrt{\Delta^2 + \mu^2}\right| \approx \left|B - \Delta - \frac{\mu^2}{2\Delta}\right| \ll \Delta$. Second, the sign of g defines where the wire can or cannot host the Majorana state near some inhomogeneity. Here it's useful to introduce the terminology: if $g > 0$ the wire is called "toplogical", otherwise it's called "trivial". In [1] and [2] it was derived, that the contact of trivial and topological wire hosts a Majorana state. It can also be shown (see section **ENTER SECTION**), that this state is present on the end of a topological wire and isn't there for a trivial one.

Note, that when two wires are considered, there are two gaps, $g_{L,R} = B - \sqrt{\Delta^2 + \mu_{L,R}^2}$. When the magnetic field B is close to Δ, one can change the signs of $g_{L,R}$ by changing $\mu_{L,R}$ respectively.

It's instructive to clarify the place of $g_{L,R}$ in the paramter hierarchy of the problem. As $\mu_L \sim \mu_R \ll B \sim \Delta$ and $g_L = B - \sqrt{\Delta^2 + \mu_L^2} < 0$, one can figure out that $\mu_R \sim \mu_L \gtrsim$

$(B-\Delta)(B+\Delta)$. Taking that into account and noticing that $g_{L,R} \approx B - \Delta - \frac{\mu^2}{2\Delta^2}$ and introducing $\beta = B - \Delta$ one finds $g_{L,R} \lesssim \frac{\beta^2}{\Delta}$, while $\mu_{L,R} \gtrsim \sqrt{\beta\Delta}$, so $\sqrt{\ll} \mu_L, \mu_R$.

2.3 Hight and low modes

Though the wavefunctions of (2.2) can be found explicitly, their form is enough complicated to stall any further analysis. However, as the spin-orbit energy is assumed to be the biggest energy scale for a homogeneous wire, one can reduce the Hamiltonian (2.2) to:

$$H = \left(\frac{p^2}{2m} + up\sigma_z\right)\tau_z \tag{2.8}$$

in this problem it's reasonably to assume the low energy limit, as all the Majorana physics should live at the energies of the order of g. Thus the energy term must be omitted in Schroedinger equation, which leads to two types of momenta: $p_{short} \approx \pm 2mu$ and $p_{long} \approx 0$ and, respectively two types of wavefunctions: shortwave and longwave ones. The fact that p_{long} is equal to zero means, that the approximation (2.8) is insufficient to describe them. However, to deal with long-wave wavefunctions one can omit the quadratic term in (2.2) and work with linearized Hamiltonian, similar to the ones used in [1] and [2].

Chapter 3

Stationary properties

3.1 Boundary condition

To obtain the spectrum of the system it's necessary to find the boundary conditions. As the barrier chemical potential is the biggest energy parameter of the problem, the wave-functions there are defined by the Hamiltonian:

$$H(y) = \left(\frac{p^2}{2m} + \mu_b\right)\tau_z, \qquad -\frac{L}{2} < y < \frac{L}{2} \tag{3.1}$$

as the low energies are the under consideration, in Sroedinger equation the energy term can be omitted, so $p_b \approx \pm i\sqrt{2m\mu_b}$. One can solve the problem given by (3.1) and match the values of the wavefunction and it's derivatives on the left and on the right of the barrier to obtain:

$$\begin{cases} \psi_L + b\partial_y\psi_L = t(\psi_R + b\partial_y\psi_R) \\ \psi_R - b\partial_y\psi_R = t(\psi_L - b\partial_y\psi_L) \end{cases} \tag{3.2}$$

here $\psi_{L,R} = \psi\left(\mp\frac{L}{2}\right)$, $b = (2m\mu_b)^{-\frac{1}{2}}$ — the penetration depth for the particle inside th barrier and $t = e^{-\frac{L}{b}}$ — the tunneling constant assumed to be small: $t \ll 1$. This condition reads, that the size of the barrier L should be much bigger that the penetration depth b.

This condition is invariant under the combined action $L \leftrightarrow R$, $y \to -y$. To simplify the further analysis one can reverce th direction in the left wire and put both ands of the wires from $y = \frac{L}{2}$ to $y = 0$. The boundary condition than becomes:

$$\begin{cases} \psi_L - b\partial_y\psi_L = t(\psi_R + b\partial_y\psi_R) \\ \psi_R - b\partial_y\psi_R = t(\psi_L + b\partial_y\psi_L) \end{cases} \tag{3.3}$$

This transformation is illustrated on the fig 3.1.

The boundary condition (3.3) can be rewritten with introducing the spinor $\Psi = (\psi_L, \psi_R)^T$

and Pauli matrices $\hat{s}_i$ in LR space:

$$(1 - t\hat{s}_x)\,\Psi - (1 + ts_x)\,b\partial_y\Psi = 0 \tag{3.4}$$

since for all $t \neq 1$ (recall, that $t \ll 1$) the matrix is $1 \pm t\hat{s}_x$ in reversible. Multupliing the last equation by $(1 - t\hat{s}_x)\,/\,(1 + t^2)$ one obtain:

$$\left(1 - 2\tilde{t}\hat{s}_z - \tilde{b}\partial_y\right)\psi = 0 \tag{3.5}$$

where $\tilde{t} = \frac{t}{1+t^2}$, $\tilde{b} = \frac{1-t^2}{1+t^2}b$. In the leading order on t, which corresponds to the tunneling limit, $\tilde{t} = t, \tilde{b} = b$.

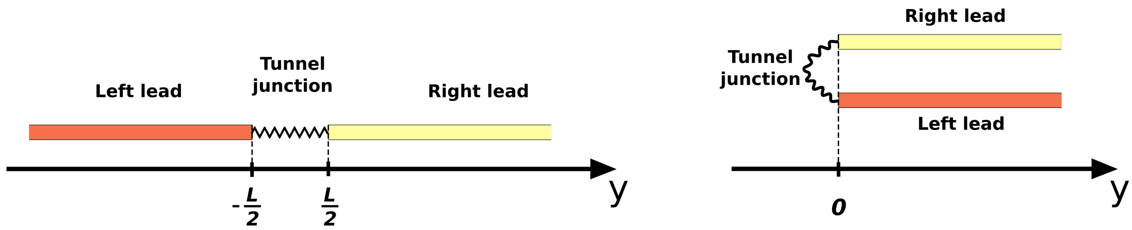

Figure 3.1: Illustration of switching the direction of left wire

One can argue, that in tunneling limit the second and the third term in (3.5) are much smaller than the first one and should not be taken when the leading order is considered. However, if the second terms is omitted, the leads become efficiently disconnected, and no tunnel effects can be found. The same is true for the third term — if it's not present, the boundary condition immediately implies $\Psi(0) = 0$, so the wires become disconnected again.

3.2 High momentum modes

As was pointed in section 2.3, there are two shortwave and longwave wavefunctions inside the wire, and the first ones can be described with the Hamiltonian (2.8). However, if one is looking for the localized states, even the longwave modes should be taken decaying. To obtain this, one needs to add a restore superconducting term in (2.8), so the spectrum become gapped and the momenta can get an imaginary part. So, for shortwave modes one should consider a Hamiltonian:

$$H = \left(\frac{p^2}{2m} - up\hat{s}_z\sigma_z\right)\tau_z + \Delta\tau_\phi \tag{3.6}$$

here the multiplier $\hat{s}_z$ is added in the spin-orbit coupling term, as the direction of the left wire is inverted, so to write a correct Hamiltonian for LR space, one needs to change p to $-p$ for the left wire — which is exactly adding $-s_z$ multiplier to each momentum.

Denoting $\eta = \frac{p^2}{2m} - up\hat{s}_z\sigma_z$, one can rewrite (3.6) as $H = \eta\tau_z + \Delta\tau_\phi$. As $\hat{s}_z\sigma_z$ commutes with H one can treat it as a number, so the dispersion is $E^2 = \eta^2 + \Delta^2$ (the number corresponding to eigenstate of $\hat{s}_z$ will be denoted as s_z while the number, corresponding to the eigenstate of σ_z will be denoted as ς_z). Thus $\eta = \pm i\sqrt{\Delta^2 - E^2}$, as the case $|E| < \Delta$ is assumed. For momenta one can write the equation:

$$p^2 - 2mus_z\varsigma_z p - 2m\eta = 0 \tag{3.7}$$

which for shortwave momenta gives $p_{short} \approx 2mus_z\varsigma_z + \frac{\eta}{u}s_z\varsigma_z$. Choosing the sign of η in a way, that the wavefunction decays at $x \to \infty$, one can obtain:

$$p_{short} \approx 2mus_z\varsigma_z + i\frac{\sqrt{\Delta^2 - E^2}}{u} \tag{3.8}$$

Now the wavefunction can be constructed by putting (3.8) into the Schroedinger equation $(\eta\tau_z + \Delta\tau_\phi)\,\Psi = E\Psi$. The solutions are:

$$\Psi_{s_z,\varsigma_z}(x) = \begin{pmatrix} 1 \\ e^{i(s_z\varsigma_z\gamma + \phi_{s_z})} \end{pmatrix}_{eh} e^{2imus_z\varsigma_z x - \frac{\sqrt{\Delta^2 - E^2}}{u}x}\,|s_z, \varsigma_z\rangle \tag{3.9}$$

where $|s_z, \sigma_z\rangle$ are eigenvectors of matrix $\hat{s}_z\sigma_z$, $\gamma = -\frac{\pi}{2} + \arcsin\frac{E}{\Delta}$ and $\phi_1 = \phi_L$, $\phi_{-1} = -\phi_R$ Thus the longwave part of eqigenstate can be written as:

$$\Psi_{long} = \sum_{s_z=\pm1}\sum_{\varsigma_z=\pm1} C_{s_z,\varsigma_z}\Psi_{s_z,\varsigma_z}(x) \tag{3.10}$$

3.3 Eliminating longwave modes from boundary condition

As the Majorana mode acts at low energies, it's expected to be predominantly longwave. This argument is in accord with [1] and [2], where the majorana state was an eigenstate of a linearized Hamiltonian, which is relevant only for longwave physics. So, it's reasonable to eliminate the shortwave modes from the problem, reformulating the boundary condition (2.8).

The wave function can be decomposed in shortwave and longwave parts: $\Psi = \Psi_{short} + \Psi_{long}$. inserting it into the(2.8) and using the fact, that $p_{long} \ll p_{short} \approx 2mus_z\sigma_z$, one can obtain at the boundary:

$$(1 - 2t\hat{s}_x)\,\Psi_{long} + (1 - 2t\hat{s}_x - 2ibum\hat{s}_z\sigma_z)\,\Psi_{short} = 0 \tag{3.11}$$

Multipluning by the $(1-2t\hat{s}_x)^{-1}$ and eliminating t^2 terms, obtain:

$$\Psi_{long} = \left(-1 + i\zeta\left(1+2t\hat{s}_x\right)s_z\sigma_z\right)\Psi_{short} \tag{3.12}$$

with $\zeta = 2mul$.

Now, using the expansion (3.10) and renormalizing the coefficients: $C_{s_z\varsigma_z} \to -\left(1 - i\zeta s_z\varsigma_z\right)C_{s_z\varsigma_z}$ one can rewrite the boundary condition for ς_z spin component of the wavefunction as:

$$\Psi_{long,\varsigma_z} = \left(1 + \frac{2i\zeta t\hat{s}_z\sigma_z}{1+i\zeta\hat{s}_z\sigma_z}\right)\sum_{s_z=\pm1} C_{s_z,\varsigma_z}\begin{pmatrix}1\\ e^{i(s_z\varsigma_z\gamma+\phi_{s_z})}\end{pmatrix}_{eh} e^{2imus_z\varsigma_z x - \frac{\sqrt{\Delta^2-E^2}}{u}x}\left|s_z,\varsigma_z\right\rangle \tag{3.13}$$

This can be multiplied by $\left(1+\frac{2i\zeta t\hat{s}_z\sigma_z}{1+i\zeta\hat{s}_z\sigma_z}\right)^{-1}$, which up to a t^2 correction yields:

$$\left(1 - \frac{2i\zeta t\hat{s}_z\sigma_z}{1+i\zeta\hat{s}_z\sigma_z}\right)\Psi_{long,\varsigma_z} = \sum_{s_z=\pm1} C_{s_z,\varsigma_z}\begin{pmatrix}1\\ e^{i(s_z\varsigma_z\gamma+\phi_{s_z})}\end{pmatrix}_{eh} e^{2imus_z\varsigma_z x - \frac{\sqrt{\Delta^2-E^2}}{u}x}\left|s_z,\varsigma_z\right\rangle \tag{3.14}$$

For each ς_z the above equation can be interpreted as the requirement that the l.h.s. 4-vector (in LR- and eh-spaces) lies in the 2d linear space L_2 spanned by the the two vectors in the sum in the r.h.s.. This can be reformulated as the requirement that the l.h.s. be orthogonal to the complementary 2d space $\overline{L}_2$. There are two basic vectors $\overline{\Psi}_{s_z\varsigma_z}$ ($s_z = \pm1$) spanning $\overline{L}_2$ for each ς_z:

$$\overline{\Psi}_{s_z\varsigma_z} = \begin{pmatrix}1\\ -e^{i(s_z\varsigma_z\gamma+\phi_{s_z})}\end{pmatrix}\left|s_z,\varsigma_z\right\rangle \tag{3.15}$$

Thus one needs to multiply (3.14) by $\left(\overline{\Psi}_{+\varsigma_z}, \overline{\Psi}_{-\varsigma_z}\right)$ from the left and, after all evaluating the matrix produt, find the boundary condition on longwave modes in the form:

$$\begin{pmatrix}1 & -e^{-i(\sigma_z\gamma-\phi_L)} & A & -Ae^{-i(\sigma_z\gamma-\phi_L)}\\ A^* & -A^*e^{i(\sigma_z\gamma+\phi_R)} & 1 & -e^{i(\sigma_z\gamma+\phi_R)}\end{pmatrix}\Psi_{long,\varsigma_z} = 0 \tag{3.16}$$

here $A = -\frac{2i\zeta t\sigma_z}{1+i\zeta\sigma_z}$ and the elements are ordered as (Le, Lh, Re, Rh).

When studying wavefunctions in superconductors, it is more convenient to work with zero phase ϕ. This can be achieved by gauging the phase difference into the boundary condition. Indeed, suppose H_ϕ describes a wire with phase ϕ. Then, $H_\phi = U_\phi^\dagger H_0 U_\phi$ with $U_\phi = \mathrm{diag}(1, \mathrm{e}^{\mathrm{i}\phi})_{\mathrm{eh}}$ and the wave functions are also related via unitary rotation $\psi_\phi = U_\phi^\dagger\tilde{\psi}$. So the transform $U^\dagger = \mathrm{diag}\left(1, e^{-i\phi_L}, e^{-i\phi_L}, 1, e^{-i\phi_R}\right)_{Le,Lh,Re,Rh}$ will eliminate all the phases from the rires and put them into boundary condition. Substituting $\Psi_{long,\varsigma_z} = U^\dagger\tilde{\Psi}$ into the (3.16) one arrives at an even simpler boundary condition on the zero-phase function $\tilde{\Psi}$:

$$\begin{pmatrix} 1 & -e^{-i\sigma_z\gamma} & A & -Ae^{-i(\sigma_z\gamma+\varphi)} \\ A^* & -A^*e^{i(\sigma_z\gamma+\varphi)} & 1 & -e^{i\sigma_z\gamma} \end{pmatrix} \tilde{\Psi}_{long,\varsigma_z} = 0 \tag{3.17}$$

where $\phi = \phi_R - \phi_L$. Note, that from here it's obvious, that in any physical quantity, calculated with this model, can depend only on phase difference φ, but not on the ϕ_L or ϕ_R separately.

3.4 Low momenta and linearized Hamiltonian

To utilize boundary condition (3.16) or (3.17), it's necessary to find low momenta wavefunctions in homogenious wire. For this purpose one can use s linearized version of the Hamiltonian (2.2), like in [1] and [2]:

$$H = -\mu\tau_z + up\sigma_z\tau_z + B\sigma_x + \Delta\tau_x \tag{3.18}$$

here the zero phase ϕ is assumed and μ can be equal μ_L or μ_R depending on the wire considered. As was mentioned before, the nonzero phase can be restored by using U_ϕ matrix. This hamiltonian is valid only for the right wire. To obtain the solution in the left wire one needs to reverse the sign of p in (2.2). Instead of doing so, the unitary transform $\psi_L = \sigma_x\psi_R$ can be utilized, as for (2.2) $H(-p) = \sigma_x H(p)\sigma_x$.

Remembering, that $\beta = B - \Delta \ll B, \Delta$, one can treat this Hamiltonian penetratively, decomposing it as $H = H_0 + V_0$:

$$H_0 = up\sigma_z\tau_z + \Delta(\sigma_x + \tau_x) \tag{3.19}$$

$$V = -\mu\tau_z + \beta\sigma_x \tag{3.20}$$

As H_0 commutes with $\sigma_x\tau_x$, it's convenient to rewrite it in the basis of common eigenstates of σ_x and τ_x. Denoting them as $|\sigma_x,\tau_x\rangle$ and arranging the order as $(|+,+\rangle, |-,-\rangle, |+,-\rangle, |-,+\rangle)$ one can rewrite $H_0 + V$ as:

$$H_0 = \begin{pmatrix} 2\Delta & up & 0 & 0 \\ up & -2\Delta & 0 & 0 \\ 0 & 0 & 0 & up \\ 0 & 0 & up & 0 \end{pmatrix} \qquad V = \begin{pmatrix} \beta & up & -\mu & 0 \\ up & -\beta & 0 & v-\mu \\ -\mu & 0 & \beta & up \\ 0 & -\mu & up & -\beta \end{pmatrix} \tag{3.21}$$

It's easy to see, subspace $\mathrm{Span}(|+,+\rangle, |-,-\rangle)$ require no perturbation to obtain the eigenstates in the leading order. Indeed, diagonalizing the upper subblock of H_0, one finds, that $E = \sqrt{(2\Delta)^2 + (up)^2}$.

When the low energy states are the objects of interest ($E \sim g_{L,R}$), one finds, that $p = \pm\frac{i\Delta}{2u}$ in the leading order, and the corresponding eigenstates are $|+,+\rangle \pm i\,|-,-\rangle$. If

The another two eigenstates are a little bit more complicated. Diaginalizing the lower sub-block of H_0, one immediately finds, that $E = \pm up$. This corrseponds to the fact, that H_0 is the version of H with a closed gap g on lower branch (see fig. 2.2,(e)), so in the zero order this states cannot form anything localized at all. To find them correctly, one needs to take into account the perturbation V and solve the secular equation using the following ansatz:

$$\psi = r_1\,|+,+\rangle + r_2\,|-,-\rangle + q_1\,|+,-\rangle + q_2\,|-,+\rangle \tag{3.22}$$

with $r_i \ll q_j$ for all pairs (i, j). In the leading order (remeber, that both E and up are of the oreder of $g_{L,R}$ now) this results to a couple of equations:

$$\begin{cases} \left(-E + B - \Delta - \frac{\mu^2}{2\Delta}\right) q_1 + upq_2 = 0 \\ upq_1 + \left(-E - B + \Delta + \frac{\mu^2}{2\Delta}\right) q_2 = 0 \end{cases} \tag{3.23}$$

recall, that at this precision $g = B - \Delta - \frac{\mu^2}{2\Delta}$ and find $E^2 = g^2 + u^2p^2$.

For this states the momenta are of the order of g/u, so it's reasonably to name them long-wave states. Than the states with momenta $\pm\frac{i\Delta}{2u}$ will be called mediumwave states. Now it's time to present this wavefunctions in original BdG basis. The expressions here are relevant only for the right lead and for $E > 0$. To find the wavefunctions in the left lead, the transform $\psi_L = \sigma_x\psi_R$ can be used, while for finding the negative enrefu states one can utilize electron-hole transform: $\psi_{E<0} = \tau_y\sigma_y K\psi_{E>0}$ with K being a complex conjugation operator.

Medium wave states are (propagating version for $E > 2\Delta$ and decaying version for $E < 2\Delta$):

$$\psi_{medium} = \begin{pmatrix} 1 \\ \frac{E\mp\sqrt{E^2-4\Delta^2}}{2\Delta} \\ \frac{E\mp\sqrt{E^2-4\Delta^2}}{2\Delta} \\ 1 \end{pmatrix} e^{\frac{\pm ix\sqrt{E^2-4\Delta^2}}{u}} \qquad \psi_{medium} = \begin{pmatrix} 1 \\ \frac{E\pm i\sqrt{4\Delta^2-E^2}}{2\Delta} \\ \frac{E\pm, i\sqrt{4\Delta^2-E^2}}{2\Delta} \\ 1 \end{pmatrix} e^{\frac{\pm x\sqrt{4\Delta^2-E^2}}{u}} \tag{3.24}$$

However in the most part of this work the low energy version $\psi_{medium} = (1, \pm i, \pm i, 1)^T\, e^{\pm\frac{\Delta x}{2u}}$ will be used.

Longwavestates (for energies $E \sim g_{L,R}$, for $E > g$ and $E < g$ respectfully):

$$\psi_{long} = \begin{pmatrix} 1 \\ \frac{E \mp \sqrt{E^2 - g^2}}{g} \\ -\frac{E \mp \sqrt{E^2 - g^2}}{g} \\ -1 \end{pmatrix} e^{\pm \frac{ix\sqrt{E^2 - g^2}}{u}} \qquad \psi_{long} = \begin{pmatrix} 1 \\ \frac{E \pm i\sqrt{g^2 - E^2}}{g} \\ -\frac{E \pm i\sqrt{g^2 - E^2}}{g} \\ -1 \end{pmatrix} e^{\pm \frac{x\sqrt{g^2 - E^2}}{u}} \tag{3.25}$$

Bibliography

[1] Y. Oreg, G. Refael, and F. von Oppen, “Helical liquids and majorana bound states in quantum wires”, Phys. Rev. Lett. **105** (2010).

[2] R. Lutchyn, J. Sau, and D. Sarma, “Majorana fermions and a topological phase transition in semiconductor-superconductor heterostructures”, Phys. Rev. Lett. **105** (2010).