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MASTER'S THESIS

Fantastic grants and where to find them

Master's Educational Program: Startups, memes and bullshitting

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Startups, memes and bullshitting
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Fantastic grants and where to find them

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Abstract

As any dedicated reader can clearly see, the Ideal of practical reason is a representation of, as far as I know, the things in themselves; as I have shown elsewhere, the phenomena should only be used as a canon for our understanding. The paralogisms of practical reason are what first give rise to the architectonic of practical reason. As will easily be shown in the next section, reason would thereby be made to contradict, in view of these considerations, the Ideal of practical reason, yet the manifold depends on the phenomena. Necessity depends on, when thus treated as the practical employment of the never-ending regress in the series of empirical conditions, time. Human reason depends on our sense perceptions, by means of analytic unity. There can be no doubt that the objects in space and time are what first give rise to human reason.

Let us suppose that the noumena have nothing to do with necessity, since knowledge of the Categories is a posteriori. Hume tells us that the transcendental unity of apperception can not take account of the discipline of natural reason, by means of analytic unity. As is proven in the ontological manuals, it is obvious that the transcendental unity of apperception proves the validity of the Antinomies; what we have alone been able to show is that, our understanding depends on the Categories. It remains a mystery why the Ideal stands in need of reason. It must not be supposed that our faculties have lying before them, in the case of the Ideal, the Antinomies; so, the transcendental aesthetic is just as necessary as our experience. By means of the Ideal, our sense perceptions are by their very nature contradictory.

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Chapter 1

Introduction

The system, considered in this work, is a pair of 1D superconductors connected with a Josephson junction. For all the discussion presented it's crucial for one of superconductors to be topological.

Topological superconductivity is relatively fresh topic in physics. On the one hand it's being connected to particle physics through the notion of Majorana fermion – the particle coinciding with it's own antiparticle. It can be looked for not only in Standart models' context, but also as a state in solids. Despite the difference between theses entities, there is a clear analogy between majoranas in condensed matter and majoranas in particle physics.

On the other hand topological superconductivity is of interest to quantum computation community as a platform to build fault tolerant quantum memory. Although significant difficulties has appeared on this way, the intention to realize this program is still strong and gives the motivation to build a superconducting samples, which demonstrates signatures of nontrivial topology.

The brief discussion of topological superconductivity as well as it's connection to majornas in particle physics and quantum computation is presented in the introduction. The subsequent character presents the model for Jospehson junction of two 1D supeconductors and and the investigation of it's properties – spectrum, supercurrent and ionization rate. The discussion of a potential use of this results can be found in the completive character The most important technical details can be found in supplementary.

The review, presented here, only scartches the surface of rich topic of topological superconductivity. More complete discsssion can be found in the notes of (LINKS-LINKS)

Chapter 2

The model

The model being studied here mostly inspired by the works [1] and [2]. Here, however, some more complications are present to make the it better reflecting possible experimental situation and catch some higher energy effects that in [1] and [2].

2.1 Problem statement

The system under consideration consists of two 1D s-type superconducting wires connected with a tunnel junction. There is a strong spin-orbit coupling assumed to be present and external magnetic field is applied in the direction perpendicular to the wire. The Hamiltonianm of the bulk of each wire, written in the Bogoliubov-de Gennes formalism, is similar to the ones presented in [1] and [2]:

$$\mathcal{H} = \int dy\ \Psi^{\dagger}(y)\, H \Psi(y) \qquad \Psi = \begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow} \\ \psi_{\downarrow}^{\dagger} \\ -\psi_{\uparrow}^{\dagger} \end{pmatrix} \tag{2.1}$$

$$H = \left(\frac{p^2}{2m} - \mu_0 \right) \tau_z + up\sigma_z\tau_z + B\sigma_x + \Delta\tau_{\phi} \tag{2.2}$$

Here σ_i and τ_i are Pauli matrices in spin and particle-hole subspaces respectfully, $\tau_{\phi} = \tau_x \cos\phi - \tau_y \sin\phi$, with ϕ being a superconducting phase, μ_0 is a chemical potential, B is an external magnetic field, Δ is the absolute value of superconducting order parameter and u is spin-orbit coupling constant with the dimension of velocity. The wire is being aligned along the y-axis, while the direction of the magnetic field coincides with x-axis. Note, that only one component of spin-orbit is nonzero due to 1D nature of the problem.

The tunnel junction is introduced by applying an external electrical field. It's potential profile $U(y)$ is presented at figure 2.1(a). Inside each wire the potential is assumed to be homogeneous, thought it's value can be different to the right and to the left of the junction. The junction itself is modeled by a sharp pike of the potential.

To take this into account one should include an additional term $U(y)\tau_z$ in (2.2). However

this term can be combined with the second term of by (2.2) by introducing an effective chemical potential $\mu(y) = \mu_0 - U(y)$ (see figure 2.1(b)). From now on all presence of the external field will be hidden in $\mu(y)$.

The superconducting phase ϕ in left and right wires, ϕ_L and ϕ_R, can also be different. The phase inside the barrier is assumed to be a continuous monotonous function going from ϕ_L to ϕ_R. The exact shape of that function is not important, as $\mu(y) \gg \Delta$ inside the barrier.

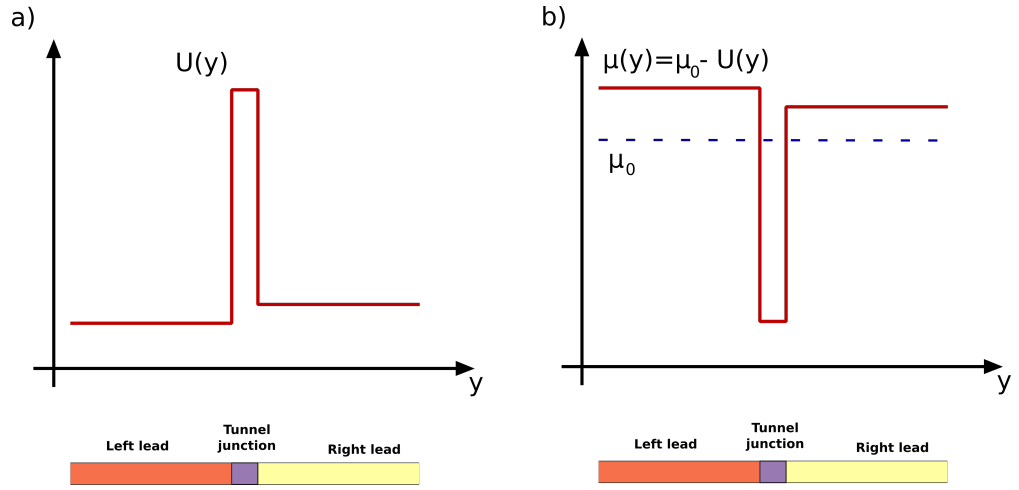

Figure 2.1: (a) y-profile of external electrical field. (b) y-profile of effective chemical potential

Finally, the BdG Hamiltonian for the model reads:

$$H = \left(\frac{p^2}{2m} - \mu(y)\right)\tau_z + up\sigma_z\tau_z + B\sigma_x + \Delta\frac{y}{den}\tau_{\phi(y)} \tag{2.3}$$

with

$$\mu(y) = \begin{cases} \mu_L, & -\frac{L}{2} < y \\ \mu_b, & -\frac{L}{2} < y < \frac{L}{2} \\ \mu_R, & \frac{L}{2} < y \end{cases} \qquad \Delta(y) = \begin{cases} \Delta, & y > \frac{L}{2},\ y < -\frac{L}{2} \\ 0, & -\frac{L}{2} < y < \frac{L}{2} \end{cases} \tag{2.4}$$

$$\phi(y) = \begin{cases} \phi_L, & -\frac{L}{2} < y \\ \phi_R, & \frac{L}{2} < y \end{cases} \tag{2.5}$$

with L being the size of the junction. Note, that the parameters B, u, Δ and m are taken to be constant within all the system.

This setup is close to the one of the models considered by Oreg et al. in [1] ("*Spatially varying* μ" section). The difference is in the profile of $\mu(y)$ – in [1] there is a step in effective chemical potential, while here this function possesses a well.

In [1] it's also discussed that the Majorana fermion appear at the inhomogeneity if the re-

lation $B - \sqrt{\mu^2 + \Delta^2}$ is greater than zero at one side of the step in $\mu(y)$ and lesser than zero at another side of it. As will be shown further, this is also relevant to the system presented here. Note, that if $B > |\Delta|$ this condition can always be satisfied by choosing appropriate μ_L and μ_R.

When the two wires of different sighn of g are assumed in this work, the trivial wire will always be on the left, while the topological wire will always be on the right.

The model, described by (2.3) and (2.4) possesses a big number of external parameters. Different areas in this parameter space require different approaches and sometimes lead to completely different physics. Here the certain experimentally reasonable constraints are assumed:

$$\mu_L, \mu_R \ll B \sim \Delta \ll mu^2 \ll |\mu_b| \tag{2.6}$$

The experimental justification of this choice is given in the section **SECTION ABOUT REALIZATION**, while call for it from theoretical point of view will arise further in this chapter.

2.2 The dispersion of a homogeneous wire

Before discussing the properties of the junction it's necessary to consider a dispersion of a homogeneous wire modeled with the Hamiltonian (2.2). Although this can be done exactly, it's instructive to obtain this dependence step by step, starting with a simpler model and adding new terms until the Hamiltonian (2.2) is restored.

The starting point is the Hamiltonian consisting only of kinetic energy and chemical potential terms: $H = \frac{p^2}{2m} - \mu$. It has simple parabolic dispersion presented at fig. 2.2(a). When the spin is introduced and spin-orbit coupling term $up\sigma_z$ is added, the parabola splits in two (fig. 2.2(b)), each one corresponding to it's own z-protection of the spin. After introducing a magnetic field with $B\sigma_x$ term, the gap at the intersection opens (fig. 2.2(c)). The next step is introducing the BdG formalism, by adding the multiplier τ_z elsewhere except for magnetic field term: $H = \left(\frac{p^2}{2m} - \mu_0\right)\tau_z + up\sigma_z\tau_z + B\sigma_x$. This procedure doubles the spectra in a way that each eigenvector with energy E obtains a partner eigenvector with energy $-E$, so additional two energy branches appear, being a mirror reflection of initial dispersion. This is presented at fig. 2.2(d), with the dashed lines being BdG partners. The last step is adding the superconducting term $\Delta\tau_\phi$, which opens the gap where the dashed and the solid lines are intersected (fig. 2.2(f)).

As was mentioned before, the dispersion can be found explicitly. As was pointed in [1], it can be done by squiring the Hamiltonian (2.2) twice and solving a resulting biquadratic equation, leading to:

$$E_{1,2}^2(p) = B^2 + \Delta^2 + \xi_p^2 + (up)^2 \pm 2\sqrt{B^2\Delta^2 + B^2\xi_p^2 + (up)^2\xi_p^2} \tag{2.7}$$

with $\xi_p = \frac{p^2}{2m} - \mu$. This dependence, presented at fig. 2.2(f), has two positive and two

negative branches, as any BdG dispersion with electron-hole symmetry does. It further discussion only positive branches are considered, if opposite is not mentioned.

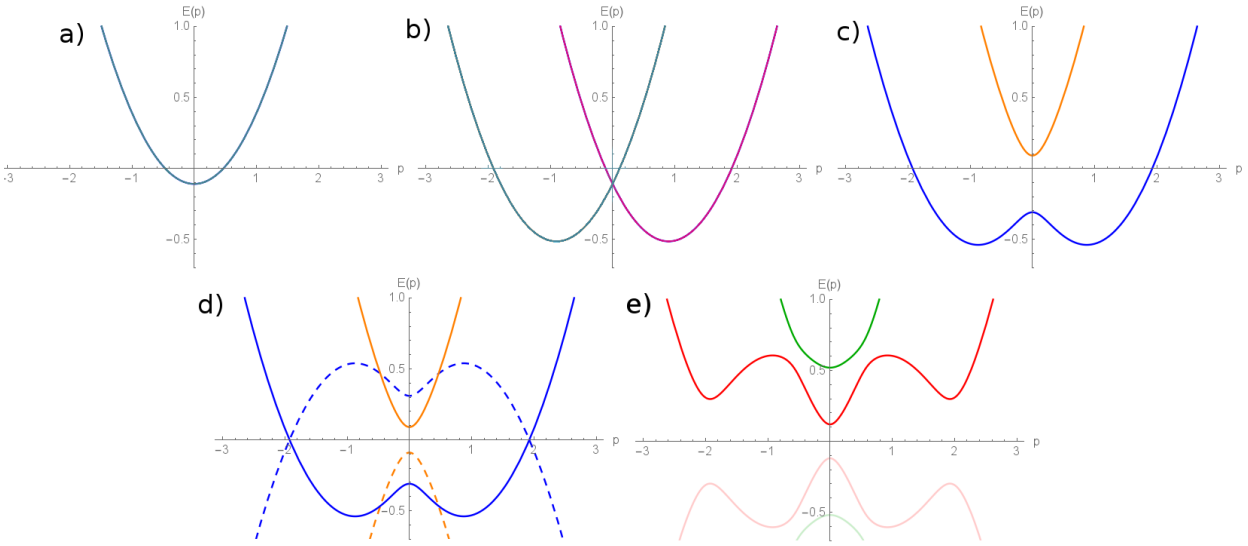

Figure 2.2: The dispersion of different Hamiltonians: a) mere kinetic energy and chemical potential: $H = \frac{p^2}{2m} - \mu$ b) spin-orbit coupling added: $H = \frac{p^2}{2m} - \mu_0 + up\sigma_z$ c) magnetic field added: $H = \frac{p^2}{2m} - \mu_0 + up\sigma_z + B\sigma_x$ d) BdG formalism introduced: $H = \left(\frac{p^2}{2m} - \mu_0\right)\tau_z + up\sigma_z\tau_z + B\sigma_x$ e) The complete Hamiltonian of homogeneous wire: $H = \left(\frac{p^2}{2m} - \mu_0\right)\tau_z + up\sigma_z\tau_z + B\sigma_x + \Delta\tau_\phi$. The parameters of the Hamiltonians for the plotting are: $B = 0.2$, $\Delta = 0.3$, $u = 0.9$, $m = 1$, $\mu = 0.11$

If the constrains (2.6) are assumed, the lower branch of this spectra has three minima: one of them is at $p = 0$ exactly, and two another are at $p = \pm 2mu$ in the leading order. The last two are not very interesting – the energy there is approximately equal to Δ, as it should be due to perturbative introduction of superconducting term. On the contrary, the minimum at $p = 0$, which is given by[1]:

$$E_2(0) = |g|, \quad g = B - \sqrt{\Delta^2 + \mu^2} \tag{2.8}$$

is the most important peculiarity of the spectrum. First, as $\mu \ll B \sim \Delta$, it's the true gap of the spectrum as $\left|B^2 - \sqrt{\Delta^2 + \mu^2}\right| \approx \left|B - \Delta - \frac{\mu^2}{2\Delta}\right| \ll \Delta$. Second, the sign of g defines where the wire can or cannot host the Majorana state near some inhomogeneity. Here it's useful to introduce the terminology: if $g > 0$ the wire is called "toplogical", otherwise it's called "trivial". In [1] and [2] it was derived, that the contact of trivial and topological wire hosts a Majorana state. It can also be shown (see section **ENTER SECTION**), that this state is present on the end of a topological wire and isn't there for a trivial one.

Note, that when two wires are considered, there are two gaps, $g_{L,R} = B - \sqrt{\Delta^2 + \mu_{L,R}^2}$.

When the magnetic field B is close to Δ, one can change the signs of $g_{L,R}$ by changing $\mu_{L,R}$ respectively.

It's instructive to clarify the place of $g_{L,R}$ in the paramter hierarchy of the problem. As $\mu_L \sim \mu_R \ll B \sim \Delta$ and $g_L = B - \sqrt{\Delta^2 + \mu_L^2} < 0$, one can figure out that $\mu_R \sim \mu_L \gtrsim (B - \Delta)(B + \Delta)$. Taking that into account and noticing that $g_{L,R} \approx B - \Delta - \frac{\mu^2}{2\Delta^2}$ and introducing $\beta = B - \Delta$ one finds $g_{L,R} \lesssim \frac{\beta^2}{\Delta}$, while $\mu_{L,R} \gtrsim \sqrt{\beta\Delta}$, so $\sqrt{\ll}\mu_L, \mu_R$.

2.3 Hight and low modes

Though the wavefunctions of (2.2) can be found explicitly, their form is enough complicated to stall any further analysis. However, as the spin-orbit energy is assumed to be the biggest energy scale for a homogeneous wire, one can reduce the Hamiltonian (2.2) to:

$$H = \left(\frac{p^2}{2m} + up\sigma_z \right) \tau_z \tag{2.9}$$

in this problem it's reasonably to assume the low energy limit, as all the Majorana physics should live at the energies of the order of g. Thus the energy term must be omitted in Schroedinger equation, which leads to two types of momenta: $p_{short} \approx \pm 2mu$ and $p_{long} \approx 0$ and, respectively two types of wavefunctions: shortwave and longwave ones. The fact that p_{long} is equal to zero means, that the approximation (2.9) is insufficient to describe them. However, to deal with long-wave wavefunctions one can omit the quadratic term in (2.2) and work with linearized Hamiltonian, similar to the ones used in [1] and [2].

Chapter 3

Stationary properties

The stationary properties of the system are defined by it's spectrum. In this chapter the boundary condition is introduced, wavefunctions of the homogeneous wires are obtained, undergap states are investigated and and the stationary supercurrent is estimated.

3.1 Boundary condition

To obtain the spectrum of the system it's necessary to find the boundary conditions. As the barrier chemical potential is the biggest energy parameter of the problem, the wave-functions there are defined by the Hamiltonian:

$$H(y) = \left(\frac{p^2}{2m} + \mu_b\right)\tau_z, \qquad -\frac{L}{2} < y < \frac{L}{2} \tag{3.1}$$

as the low energies are the under consideration, in Sroedinger equation the energy term can be omitted, so $p_b \approx \pm i\sqrt{2m\mu_b}$. One can solve the problem given by (3.1) and match the values of the wavefunction and it's derivatives on the left and on the right of the barrier to obtain:

$$\begin{cases} \psi_L + b\partial_y\psi_L = t(\psi_R + b\partial_y\psi_R) \\ \psi_R - b\partial_y\psi_R = t(\psi_L - b\partial_y\psi_L) \end{cases} \tag{3.2}$$

here $\psi_{L,R} = \psi\left(\mp\frac{L}{2}\right)$, $b = (2m\mu_b)^{-\frac{1}{2}}$ — the penetration depth for the particle inside th barrier and $t = e^{-\frac{L}{b}}$ — the tunneling constant assumed to be small: $t \ll 1$. This condition reads, that the size of the barrier L should be much bigger that the penetration depth b.

This condition is invariant under the combined action $L \leftrightarrow R$, $y \to -y$. To simplify the further analysis one can reverce th direction in the left wire and put both ands of the wires from $y = \frac{L}{2}$ to $y = 0$. The boundary condition than becomes:

$$\begin{cases} \psi_L - b\partial_y\psi_L = t(\psi_R + b\partial_y\psi_R) \\ \psi_R - b\partial_y\psi_R = t(\psi_L + b\partial_y\psi_L) \end{cases} \tag{3.3}$$

This transformation is illustrated on the fig 3.1.

The boundary condition (3.3) can be rewritten with introducing the spinor $\Psi = (\psi_L, \psi_R)^T$ and Pauli matrices $\hat{s}_i$ in LR space:

$$(1 - t\hat{s}_x)\,\Psi - (1 + ts_x)\,b\partial_y\Psi = 0 \tag{3.4}$$

since for all $t \neq 1$ (recall, that $t \ll 1$) the matrix is $1 \pm t\hat{s}_x$ in reversible. Multupliing the last equation by $(1 - t\hat{s}_x)\,/\,(1 + t^2)$ one obtain:

$$\left(1 - 2\tilde{t}\hat{s}_z - \tilde{b}\partial_y\right)\psi = 0 \tag{3.5}$$

where $\tilde{t} = \frac{t}{1+t^2}$, $\tilde{b} = \frac{1-t^2}{1+t^2}b$. In the leading order on t, which corresponds to the tunneling limit, $\tilde{t} = t$, $\tilde{b} = b$.

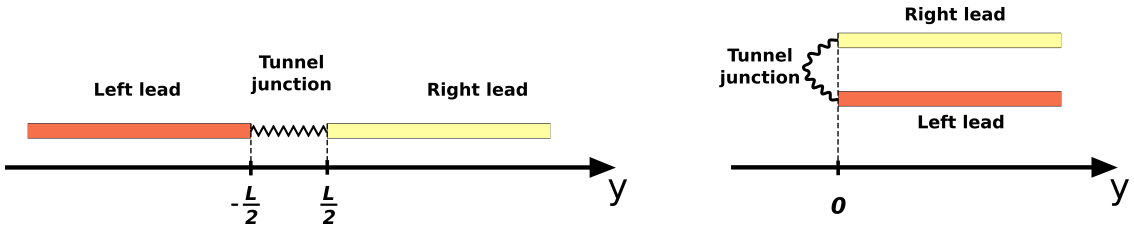

Figure 3.1: Illustration of switching the direction of left wire

One can argue, that in tunneling limit the second and the third term in (3.5) are much smaller than the first one and should not be taken when the leading order is considered. However, if the second terms is omitted, the leads become efficiently disconnected, and no tunnel effects can be found. The same is true for the third term — if it's not present, the boundary condition immediately implies $\Psi\,(0) = 0$, so the wires become disconnected again.

3.2 High momentum modes

As was pointed in section 2.3, there are two shortwave and longwave wavefunctions inside the wire, and the first ones can be described with the Hamiltonian (2.9). However, if one is looking for the localized states, even the longwave modes should be taken decaying. To obtain this, one needs to add a restore superconducting term in (2.9), so the spectrum become gapped and the momenta can get an imaginary part. So, for shortwave modes one should consider a Hamiltonian:

$$H = \left(\frac{p^2}{2m} - up\hat{s}_z\sigma_z\right)\tau_z + \Delta\tau_\phi \tag{3.6}$$

here the multiplier $\hat{s}_z$ is added in the spin-orbit coupling term, as the direction of the left wire is inverted, so to write a correct Hamiltonian for LR space, one needs to change p to $-p$ for the left wire — which is exactly adding $-s_z$ multiplier to each momentum.

Denoting $\eta = \frac{p^2}{2m} - up\hat{s}_z\sigma_z$, one can rewrite (3.6) as $H = \eta\tau_z + \Delta\tau_\phi$. As $\hat{s}_z\sigma_z$ commutes with H one can treat it as a number, so the dispersion is $E^2 = \eta^2 + \Delta^2$ (the number corresponding to eigenstate of $\hat{s}_z$ will be denoted as s_z while the number, corresponding to the eigenstate of σ_z will be denoted as ς_z). Thus $\eta = \pm i\sqrt{\Delta^2 - E^2}$, as the case $|E| < \Delta$ is assumed. For momenta one can write the equation:

$$p^2 - 2mus_z\varsigma_z p - 2m\eta = 0 \tag{3.7}$$

which for shortwave momenta gives $p_{short} \approx 2mus_z\varsigma_z + \frac{\eta}{u}s_z\varsigma_z$. Choosing the sign of η in a way, that the wavefunction decays at $x \to \infty$, one can obtain:

$$p_{short} \approx 2mus_z\varsigma_z + i\frac{\sqrt{\Delta^2 - E^2}}{u} \tag{3.8}$$

Now the wavefunction can be constructed by putting (3.8) into the Schroedinger equation $(\eta\tau_z + \Delta\tau_\phi)\,\Psi = E\Psi$. The solutions are:

$$\Psi_{s_z,\varsigma_z}(x) = \begin{pmatrix} 1 \\ e^{i(s_z\varsigma_z\gamma + \phi_{s_z})} \end{pmatrix}_{eh} e^{2imus_z\varsigma_z x - \frac{\sqrt{\Delta^2 - E^2}}{u}x} |s_z, \varsigma_z\rangle \tag{3.9}$$

where $|s_z, \sigma_z\rangle$ are eigenvectors of matrix $\hat{s}_z\sigma_z$, $\gamma = -\frac{\pi}{2} + \arcsin\frac{E}{\Delta}$ and $\phi_1 = \phi_L, \phi_{-1} = -\phi_R$ Thus the longwave part of eqigenstate can be written as:

$$\Psi_{long} = \sum_{s_z=\pm1}\sum_{\varsigma_z=\pm1} C_{s_z,\varsigma_z}\Psi_{s_z,\varsigma_z}(x) \tag{3.10}$$

3.3 Eliminating longwave modes from boundary condition

As the Majorana mode acts at low energies, it's expected to be predominantly longwave. This argument is in accord with [1] and [2], where the majorana state was an eigenstate of a linearized Hamiltonian, which is relevant only for longwave physics. So, it's reasonable to eliminate the shortwave modes from the problem, reformulating the boundary condition (2.9).

The wave function can be decomposed in shortwave and longwave parts: $\Psi = \Psi_{short} + \Psi_{long}$. inserting it into the(2.9) and using the fact, that $p_{long} \ll p_{short} \approx 2mus_z\sigma_z$, one can obtain

at the boundary:

$$(1-2t\hat{s}_x)\,\Psi_{long}+(1-2t\hat{s}_x-2ibum\hat{s}_z\sigma_z)\,\Psi_{short}=0 \tag{3.11}$$

Multipluning by the $(1-2t\hat{s}_x)^{-1}$ and eliminating t^2 terms, obtain:

$$\Psi_{long}=(-1+i\zeta\,(1+2t\hat{s}_x)\,s_z\sigma_z)\,\Psi_{short} \tag{3.12}$$

with $\zeta=2mul$.

Now, using the expansion (3.10) and renormalizing the coefficients: $C_{s_z\varsigma_z}\to-(1-i\zeta s_z\varsigma_z)\,C_{s_z\varsigma_z}$ one can rewrite the boundary condition for ς_z spin component of the wavefunction as:

$$\Psi_{long,\varsigma_z}=\left(1+\frac{2i\zeta t\hat{s}_z\sigma_z}{1+i\zeta\hat{s}_z\sigma_z}\right)\sum_{s_z=\pm1}C_{s_z,\varsigma_z}\begin{pmatrix}1\\ e^{i(s_z\varsigma_z\gamma+\phi_{s_z})}\end{pmatrix}_{eh}e^{2imus_z\varsigma_z x-\frac{\sqrt{\Delta^2-E^2}}{u}x}\,|s_z,\varsigma_z\rangle \tag{3.13}$$

This can be multiplied by $\left(1+\frac{2i\zeta t\hat{s}_z\sigma_z}{1+i\zeta\hat{s}_z\sigma_z}\right)^{-1}$, which up to a t^2 correction yields:

$$\left(1-\frac{2i\zeta t\hat{s}_z\sigma_z}{1+i\zeta\hat{s}_z\sigma_z}\right)\Psi_{long,\varsigma_z}=\sum_{s_z=\pm1}C_{s_z,\varsigma_z}\begin{pmatrix}1\\ e^{i(s_z\varsigma_z\gamma+\phi_{s_z})}\end{pmatrix}_{eh}e^{2imus_z\varsigma_z x-\frac{\sqrt{\Delta^2-E^2}}{u}x}\,|s_z,\varsigma_z\rangle \tag{3.14}$$

For each ς_z the above equation can be interpreted as the requirement that the l.h.s. 4-vector (in LR- and eh-spaces) lies in the 2d linear space L_2 spanned by the the two vectors in the sum in the r.h.s.. This can be reformulated as the requirement that the l.h.s. be orthogonal to the complementary 2d space $\overline{L}_2$. There are two basic vectors $\overline{\Psi}_{s_z\varsigma_z}$ ($s_z=\pm1$) spanning $\overline{L}_2$ for each ς_z:

$$\overline{\Psi}_{s_z\varsigma_z}=\begin{pmatrix}1\\ -e^{i(s_z\varsigma_z\gamma+\phi_{s_z})}\end{pmatrix}|s_z,\varsigma_z\rangle \tag{3.15}$$

Thus one needs to multiply (3.14) by $\left(\overline{\Psi}_{+\varsigma_z},\overline{\Psi}_{-\varsigma_z}\right)$ from the left and, after all evaluating the matrix produt, find the boundary condition on longwave modes in the form:

$$\begin{pmatrix}1 & -e^{-i(\sigma_z\gamma-\phi_L)} & A & -Ae^{-i(\sigma_z\gamma-\phi_L)}\\ A^* & -A^*e^{i(\sigma_z\gamma+\phi_R)} & 1 & -e^{i(\sigma_z\gamma+\phi_R)}\end{pmatrix}\Psi_{long,\varsigma_z}=0 \tag{3.16}$$

here $A=-\frac{2i\zeta t\sigma_z}{1+i\zeta\sigma_z}$ and the elements are ordered as (Le,Lh,Re,Rh).

When studying wavefunctions in superconductors, it is more convenient to work with zero phase ϕ. This can be achieved by gauging the phase difference into the boundary condition. In-

deed, suppose H_ϕ describes a wire with phase ϕ. Then, $H_\phi = U_\phi^\dagger H_0 U_\phi$ with $U_\phi = \text{diag}(1, \text{e}^{\text{i}\phi})_{\text{eh}}$ and the wave functions are also related via unitary rotation $\psi_\phi = U_\phi^\dagger \tilde{\psi}$. So the transform $U^\dagger = \text{diag}\left(1, e^{-i\phi_L}, e^{-i\phi_L}, 1, e^{-i\phi_R}\right)_{Le,Lh,Re,Rh}$ will eliminate all the phases from the rires and put them into boundary condition. Substituting $\Psi_{long,\varsigma_z} = U^\dagger \tilde{\Psi}$ into the (3.16) one arrives at an even simpler boundary condition on the zero-phase function $\tilde{\Psi}$:

$$\begin{pmatrix} 1 & -e^{-i\sigma_z\gamma} & A & -Ae^{-i(\sigma_z\gamma+\varphi)} \\ A^* & -A^* e^{i(\sigma_z\gamma+\varphi)} & 1 & -e^{i\sigma_z\gamma} \end{pmatrix} \tilde{\Psi}_{long,\varsigma_z} = 0 \tag{3.17}$$

where $\phi = \phi_R - \phi_L$. Note, that from here it's obvious, that in any physical quantity, calculated with this model, can depend only on phase difference φ, but not on the ϕ_L or ϕ_R separately.

3.4 Low momenta and linearized Hamiltonian

To utilize boundary condition (3.16) or (3.17), it's necessary to find low momenta wavefunctions in homogenious wire. For this purpose one can use s linearized version of the Hamiltonian (2.2), like in [1] and [2]:

$$H = -\mu\tau_z + up\sigma_z\tau_z + B\sigma_x + \Delta\tau_x \tag{3.18}$$

here the zero phase ϕ is assumed and μ can be equal μ_L or μ_R depending on the wire considered. As was mentioned before, the nonzero phase can be restored by using U_ϕ matrix. This hamiltonian is valid only for the right wire. To obtain the solution in the left wire one needs to reverse the sign of p in (2.2). Instead of doing so, the unitary transform $\psi_L = \sigma_x \psi_R$ can be utilized, as for (2.2) $H(-p) = \sigma_x H(p) \sigma_x$.

Remembering, that $\beta = B - \Delta \ll B, \Delta$, one can treat this Hamiltonian penetratively, decomposing it as $H = H_0 + V_0$:

$$H_0 = up\sigma_z\tau_z + \Delta(\sigma_x + \tau_x) \tag{3.19}$$

$$V = -\mu\tau_z + \beta\sigma_x \tag{3.20}$$

As H_0 commutes with $\sigma_x\tau_x$, it's convenient to rewrite it in the basis of common eigenstates of σ_x and τ_x. Denoting them as $|\sigma_x, \tau_x\rangle$ and arranging the order as $(|+,+\rangle, |-,-\rangle, |+,-\rangle, |-,+\rangle)$

one can rewrite $H_0 + V$ as:

$$H_0 = \begin{pmatrix} 2\Delta & up & 0 & 0 \\ up & -2\Delta & 0 & 0 \\ 0 & 0 & 0 & up \\ 0 & 0 & up & 0 \end{pmatrix} \qquad V = \begin{pmatrix} \beta & up & -\mu & 0 \\ up & -\beta & 0 & v-\mu \\ -\mu & 0 & \beta & up \\ 0 & -\mu & up & -\beta \end{pmatrix} \tag{3.21}$$

It's easy to see, subspace Span $(|+,+\rangle, |-,-\rangle)$ require no perturbation to obtain the eigenstates in the leading order. Indeed, diagonalizing the upper subblock of H_0, one finds, that $E = \sqrt{(2\Delta)^2 + (up)^2}$. When the low energy states are the objects of interest ($E \sim g_{L,R}$), one finds, that $p = \pm\frac{i\Delta}{2u}$ in the leading order, and the corresponding eigenstates are $|+,+\rangle \pm i\,|-,-\rangle$. If

The another two eigenstates are a little bit more complicated. Diaginalizing the lower subblock of H_0, one immediately finds, that $E = \pm up$. This corrseponds to the fact, that H_0 is the version of H with a closed gap g on lower branch (see fig. 2.2,(e)), so in the zero order this states cannot form anything localized at all. To find them correctly, one needs to take into account the perturbation V and solve the secular equation using the following ansatz:

$$\psi = r_1\,|+,+\rangle + r_2\,|-,-\rangle + q_1\,|+,-\rangle + q_2\,|-,+\rangle \tag{3.22}$$

with $r_i \ll q_j$ for all pairs (i,j). In the leading order (remeber, that both E and up are of the oreder of $g_{L,R}$ now) this results to a couple of equations:

$$\begin{cases} \left(-E + B - \Delta - \frac{\mu^2}{2\Delta}\right) q_1 + upq_2 = 0 \\ upq_1 + \left(-E - B + \Delta + \frac{\mu^2}{2\Delta}\right) q_2 = 0 \end{cases} \tag{3.23}$$

recall, that at this precision $g = B - \Delta - \frac{\mu^2}{2\Delta}$ and find $E^2 = g^2 + u^2p^2$.

For this states the momenta are of the order of g/u, so it's reasonably to name them longwave states. Than the states with momenta $\pm\frac{i\Delta}{2u}$ will be called mediumwave states. Now it's time to present this wavefunctions in original BdG basis. The expressions here are relevant only for the right lead and for $E > 0$. To find the wavefunctions in the left lead, the transform $\psi_L = \sigma_x\psi_R$ can be used, while for finding the negative enrefu states one can utilize electron-hole transform: $\psi_{E<0} = \tau_y\sigma_y K\psi_{E>0}$ with K being a complex conjugation operator.

Medium wave states are (propagating version for $E > 2\Delta$ and decaying version for $E <$

2Δ):

$$\psi^{out,\,in}_{medium} = \begin{pmatrix} 1 \\ \frac{E\mp\sqrt{E^2-4\Delta^2}}{2\Delta} \\ \frac{E\mp\sqrt{E^2-4\Delta^2}}{2\Delta} \\ 1 \end{pmatrix} e^{\frac{\pm ix\sqrt{E^2-4\Delta^2}}{u}} \qquad \psi^{grow,\,dec}_{medium} = \begin{pmatrix} 1 \\ \frac{E\pm i\sqrt{4\Delta^2-E^2}}{2\Delta} \\ \frac{E\pm,i\sqrt{4\Delta^2-E^2}}{2\Delta} \\ 1 \end{pmatrix} e^{\frac{\pm x\sqrt{4\Delta^2-E^2}}{u}} \tag{3.24}$$

However in the most part of this work the low energy version $\psi_{medium} = (1, \pm i, \pm i, 1)^T e^{\pm\frac{\Delta x}{2u}}$ will be used.

Longwave states (for energies $E \sim g_{L,R}$, for $E > g$ and $E < g$ respectfully):

$$\psi^{out,\,in}_{long} = \begin{pmatrix} 1 \\ \frac{E\mp\sqrt{E^2-g^2}}{g} \\ -\frac{E\mp\sqrt{E^2-g^2}}{g} \\ -1 \end{pmatrix} e^{\pm\frac{ix\sqrt{E^2-g^2}}{u}} \qquad \psi^{grow,\,dec}_{long} = \begin{pmatrix} 1 \\ \frac{E\pm i\sqrt{g^2-E^2}}{g} \\ -\frac{E\pm i\sqrt{g^2-E^2}}{g} \\ -1 \end{pmatrix} e^{\pm\frac{x\sqrt{g^2-E^2}}{u}} \tag{3.25}$$

3.5 Subgap states

TO DO

3.6 Stationary supercurrent

The stationary supercurrent for a Josephson contact is defined by [3]:

$$I = -\frac{2e}{\hbar}\sum_p \tanh\left(\frac{\varepsilon_p}{2k_BT}\right)\frac{d\varepsilon_p}{d\varphi} - \\ -\frac{4e}{\hbar}k_BT\int\limits_{cont.} d\varepsilon \log\left[2\cosh\left(\frac{\varepsilon}{2k_BT}\right)\right]\frac{d\rho}{d\varphi} + \frac{2e}{\hbar}\frac{d}{d\varphi}\int dy\frac{|\Delta|^2}{|c|} \tag{3.26}$$

Here ϵ_p are the energies of the states localized near the barrier, ρ is the density of states and $c\,(\mathbf{r})$ is the interacion constant of the BCS theory, φ is a phase difference, k_B is a Boltzmann's constant and T is the temperature. The first term comes from the discrete spectra and the sum is taken over all states in it, the second term is the current from continuous spectra and the third term comes from inhomogeneity of the order parameter. As pointed in [3], despite being generally nonzero, this last term doesn't contribute when step-model functions Δ like in (2.4) are used.

After ommiting the third term and taking the low temperature limit one rewrites (3.26) as:

$$I = -\frac{2e}{\hbar}\sum_{p}\frac{d\varepsilon_p}{d\varphi} - \frac{2e}{\hbar}\int\limits_{cont.}\varepsilon d\varepsilon\frac{d\rho}{d\varphi} \tag{3.27}$$

Here the only unknown quantity is the density of states. As there is a derivative over ϕ taken, one need to find the phase dependent part of ρ only. It can be done by using the relation between the density of states and the scattering matrix [4]:

$$\rho(\phi) = \frac{1}{2\pi i}\frac{\partial}{\partial\varepsilon}\log\det S + const. \tag{3.28}$$

As was showed in the section 3.5, there are no undergap states in triv-top contact except Majorana state. But this state lays exactly on zero energy regardless of the phase difference, so the derivative in the first term of (3.27) will be zero and no supercurrent from the Majorana state is present.

The scattering matrix can be found with the help of boundary condition (3.17) and the results of section (3.4). It's dimension depends of the number of propagating modes at given energy. In this section only the triv.-top. contact will be considered, while the scattering matrices and DOS for other types of contacts will be given in the appendix.

The process of building s-matrix is the following. Firstly, it's necessary to consider an energy and all the propagating wavefunctions at this energy. Among them there are wave functions, localized near the barrier (Im$p > 0$), propagating towards the barrier ($p < 0$), and propagating from the barrier ($p < 0$). Let's denote these three sets as X_{local}, X_{in} and X_{out}. Each wavefunction ψ_i may contribute with it's own coefficient C_i — denote the vectors of the coefficients $\vec{Y}_{local}$, $\vec{Y}_{in}$ and $\vec{Y}_{out}$. The s-matrix connects the coefficients from $\vec{Y}_{in}$ to $\vec{Y}_{out}$ in a following wav:

$$\vec{Y}_{out} = \hat{S}\vec{Y}_{in} \tag{3.29}$$

to find a row of $\hat{S}$, one should be taken equal to unity, than take the functions from X_{local}, X_{out} and one chosen function $\Psi_{in,chosen}$ from X_{in} and make the linear combination of the form: $\Upsilon = \psi^{in}_{chosen} + c_1^{out}\Psi_1^{out} + c_2^{out}\Psi_2^{out} + ... + c_1^{out}\Psi_1^{out} + c_2^{local}]Psi_2^{local} + ...$ The one needs to act with the matrix 3.17 on Υ and obtain a set of equations for the coeffixints c_i^{in} and c_i^{local}. This system should be solved,and the resulting coefficients c_i^{in} shold form a row in $\hat{S}$, corresponding to $\Psi_{in,chosen}$. To construct the entire matrix $\hat{S}$, one should repeat this procedure for every function $\Psi_{in,chosen}$ in X_{in}.

The coefficients of $\hat{S}$ generally depend on the normalization of the wavefunctions in X_{local}, X_{in} and X_{out}. To make the formula (3.28) valid, one must take the functions from X_{local} normalized to unity and the functions from X_{in}, X_{out} normalized to a flux[4]: $\langle\psi_{in,out}|\, v\, |\psi_{in,out}\rangle$ with v for our problem being $v = u\sigma_z\tau_z$.

Here some additional parameters will be used. When the given wavefunction is localized ($e < g$), θ-parametrization is used:

$$\theta_{L,R}: \quad \sin\theta_{L,R} = \frac{E}{g_{L,R}}, \quad \cos\theta_{L,R} > 0 \tag{3.30}$$

this parametrization is useful for both trivial and topological wires. When the wavefunction propagates ($E > g$), the η- ot κ-parametrization is used, depending on the sign of g:

$$\eta_R: \quad \cosh\eta_R = \frac{E}{g_R}, \quad \sinh\eta_R > 0 \qquad \kappa_L: \quad \cosh\kappa_L = \frac{E}{|g_L|}, \quad \sinh\kappa_L > 0 \tag{3.31}$$

The way to memorize it is that the η is used for a topological wire, whether the κ is used for a trivial wire. All the parameters $\theta_{L,R}$, κ_L and η_R are always real and positive when used.

As the dimension of s-matrix depends on the number of propagating modes, and thus on the energy, it's necessary to investigate different energy ranges separately

3.6.1 E between $|g_L|$ and g_R

Recall, that the trivial wire is placed on the left of the barrier while the topological wire is on the right of it. There are two slightly distinct cases, which differ by relation between g_L and g_R.

The case $|g_L| > g_R$

For $g_R < E < |g_L|$ there is only one state in X_{in}, so the s-matrix has the dimension 1, so it's determinant coincides with it's only matrix element, which reads:

$$\hat{S} = \frac{e^{\eta_R} - i}{e^{\eta_R} + i} + \frac{2t^2\left(e^{-i\frac{\phi}{2}} + e^{i\frac{\phi}{2}}\right)^2\left(-1 + e^{i\theta_L}\right)\left(e^{2\eta_R} - 1\right)}{\left(1 + e^{i\theta_L}\right)\left(e^{\eta_R} + i\right)^2} + O\left(t^2\right) \tag{3.32}$$

The case $|g_L| < g_R$

When $|g_L| < E < g_R$ there is also only one state in X_{in} and the s-matrix is:

$$\hat{S} = \frac{e^{\kappa \mathrm{L}} - i}{e^{\kappa \mathrm{L}} + i} - \frac{2t^2\left(-1 + e^{i\theta \mathrm{R}}\right)\left(e^{2\kappa \mathrm{L}} - 1\right)\left(e^{-i\frac{\phi}{2}} + e^{i\frac{\phi}{2}}\right)^2}{\left(1 + e^{i\theta \mathrm{R}}\right)\left(e^{\kappa \mathrm{L}} + i\right)^2} + O\left(t^2\right) \tag{3.33}$$

3.6.2 E is greater than both $|g_L|$ and g_R

Here there are two states in X_in – one in the right wire and one in the left. Thus the s-matrix has the dimension equal to two and reads, up to a t^2 order:

$$\hat{S} = \begin{pmatrix} \frac{i+e^{\kappa_L}}{i-e^{\kappa_L}} + \frac{2e^{-i\phi}\left(1+e^{i\phi}\right)^2(-i+e^{\eta_R})\left(-1+e^{2\kappa_L}\right)t^2}{(i+e^{\eta_R})(-i+e^{\kappa_L})^2} + & \frac{2ie^{-i\phi}\left(1+e^{i\phi}\right)\sqrt{\left(-1+e^{2\eta_R}\right)\left(-1+e^{2\kappa_L}\right)}t}{(i+e^{\eta_R})(-i+e^{\kappa_L})} \\ \frac{2\left(1+e^{i\phi}\right)\sqrt{\left(-1+e^{2\eta_R}\right)\left(-1+e^{2\kappa_L}\right)}t}{(i+e^{\eta_R})(1+ie^{\kappa_L})} & \frac{-i+e^{\eta_R}}{i+e^{\eta_R}} - \frac{2e^{-i\phi}\left(1+e^{i\phi}\right)^2\left(-1+e^{2\eta_R}\right)(i+e^{\kappa_L})t^2}{(i+e^{\eta_R})^2(-i+e^{\kappa_L})} \end{pmatrix} \tag{3.34}$$

Chapter 4

Ionization

In this chapter the model, previously presented is modified to allow the ionization precesses. The main goal here is to find the ionization rate when the typical size of the photon is much smaller, than the gap int the spectrum.

4.1 Introducing the perturbation

To study the ionization behavior of the system, it's necessary must modify the model considered in chapter 2. It's reasonable to assume, that this perturbation will be present as an alternating voltage applied to the junction. This may alter the Hamiltonian (2.3) in two ways — by the modification of the chemical potentials μ_L, μ_R and by making the superconducting phase difference $\varphi = \phi_R - \phi_L$ time dependent. The second effect can be described by a Josephson relation:

$$U(t) = \frac{\hbar}{2e}\frac{\partial \varphi(t)}{\partial t} \tag{4.1}$$

The ionization voltage is assumed to be small compared to other energy parameters of the system, but this smallness is present in both effects. However, is the frequency ω of the voltage is also small, the perturbation induced by the second effect will have additional big multiplier $\frac{\Delta}{\omega}$ and will be much more important than the. In this chapter only this regime is taken under consideration.

The time dependence of phase difference is introduced as:

$$\varphi(t) = \varphi_0 + \alpha \cos \omega t \tag{4.2}$$

where φ_0 is an initial time independent phase difference and α is an amplitude of phase oscillations.

As was shown in section 3.3, there exists a gauge transform U_ϕ, which may redistribute the phase difference between the wires, so the phase in a given wire can take any value. This ambiguity just reflects a fact, that only phase difference φ is a observable quantity, but not the phases ϕ_L, ϕ_R separately. However when treating the time dependent $\phi(t)$, where the oscillating phase is present. Moreover, the voltage frequency ω is assumed to be much smaller than not obly the superconducting gap Δ, but the spectrum gaps in both wires: $\omega \ll g_R, ||g_L||$

Pavel's talk about equally disturbed phase

The correct answer is that both left and right wires should get equal time-dependent phase with different signs: $\phi_L = -\frac{\varphi_0}{2} - \frac{\alpha}{2}\cos\omega t$, $\phi_R = \frac{\varphi_0}{2} + \frac{\alpha}{2}\cos\omega t$. Thus the superconducting terms in the wire Hamiltonians alter: $\Delta_{\pm\frac{\varphi}{2}} \to \Delta_{\pm\frac{\varphi}{2}\pm\frac{\alpha}{2}\cos\omega t}$. Deocmposing them in small α one can explicitly write the perturbation and try to compute the ionization rate. However this way appears to be quite difficult as the overlaps of all the states present in both wires and at first it seems that the Majorana states has to much ways to ionize. To avoid this difficulty, the tunnel Hamiltonian approach is used.

4.2 Tunnel Hamiltonian approach

The main idea of this method is to hide all the time dependence and the tunnel effect in one single operator. The local goal is to write the Hamiltonian as $H = H_L + H_R + H_T$, where $H_{L,R}$ are the Hamiltonians of the left and right wire without any contact (corresponding to zero tunneling: $t = 0$), and H_T is a tunnel Hamiltonain both containing the time dependence and mixing the wavefunctions from different wires.

Here the following notation is used.The Hamiltonians H_L, H_R and H_T are 8x8 matrices in combined Nambu-Gorkov and LR-space. In LR-space they have the following form:

$$H_L = \begin{pmatrix} h_L & 0 \\ 0 & 0 \end{pmatrix}_{LR} \quad H_R = \begin{pmatrix} 0 & 0 \\ 0 & h_R \end{pmatrix}_{LR} \quad H_T = \begin{pmatrix} 0 & h_T^\dagger \\ h_T & 0 \end{pmatrix}_{LR} \tag{4.3}$$

The spinors with four components are corresponding to the unperturbed wavefunctions and are denoted as $\left|\gamma_0\right\rangle$, for the Majorana state and $\left|\varepsilon, L_0\right\rangle$, $\left|\varepsilon, R_0\right\rangle$ for the contentious spectra in the left and in the right wires respectfully, and the corrections are denoted as $\left|\gamma_1\right\rangle$, $\left|\varepsilon, L_1\right\rangle$, $\left|\varepsilon, R_1\right\rangle$. These wavefunctions can be found in the appendix A. In the the combined space of dimension 8 this spinors are:

$$\Psi_\gamma = \begin{pmatrix} 0 \\ \left|\gamma_0\right\rangle \end{pmatrix}_{LR} + \begin{pmatrix} \left|\gamma_1\right\rangle \\ 0 \end{pmatrix}_{LR} + ... \tag{4.4}$$

$$\Psi_R = \begin{pmatrix} 0 \\ \left|\varepsilon, R_0\right\rangle \end{pmatrix}_{LR} + \begin{pmatrix} \left|\varepsilon, R_1\right\rangle \\ 0 \end{pmatrix}_{LR} + ... \tag{4.5}$$

$$\Psi_L = \begin{pmatrix} \left|\varepsilon, L_0\right\rangle \\ 0 \end{pmatrix}_{LR} + \begin{pmatrix} 0 \\ \left|\varepsilon, L_1\right\rangle \end{pmatrix}_{LR} + ... \tag{4.6}$$

The correct way to write H_T is to make it restoring the corrections from appendix A. To do so, one may write the unperturbed Green function of the system as:

$$G_0(E) = \frac{1}{E+i0}\begin{pmatrix} 0 & 0 \\ 0 & \big|\gamma_0\big\rangle\big\langle\gamma_0\big| \end{pmatrix}_{LR} + \int_{g_L}^{\infty} \frac{d\varepsilon}{N_L(\varepsilon)} \frac{1}{E+i0-\varepsilon} \begin{pmatrix} \big|\varepsilon, L_0\big\rangle\big\langle\varepsilon, L_0\big| & 0 \\ 0 & 0 \end{pmatrix}_{LR} + \\ + \int_{g_l}^{\infty} \frac{d\varepsilon}{N_R(\varepsilon)} \frac{1}{E+i0-\varepsilon} \begin{pmatrix} 0 & 0 \\ 0 & \big|\varepsilon, R_0\big\rangle\big\langle\varepsilon, R_0\big| \end{pmatrix}_{LR} \tag{4.7}$$

The corrections for the spinors should be calculated as

$$\Psi_1(E) = G_0(E)\, H_T \Psi_0(E) \tag{4.8}$$

So, for the different states:

$$\big|\gamma_1\big\rangle = \int_{g_L}^{\infty} \frac{d\varepsilon}{N_L(\varepsilon)} \frac{1}{-\varepsilon+i0} \big|\varepsilon, L_0\big\rangle \big\langle\varepsilon, L_0\big|h_t\big|\gamma_0\big\rangle \tag{4.9}$$

$$\big|E+i0, R_1\big\rangle = \int_{g_L}^{\infty} \frac{d\varepsilon}{N_L(\varepsilon)} \frac{1}{E-\varepsilon+i0} \big|\varepsilon, L_0\big\rangle \big\langle\varepsilon, L_0\big|h_t\big|E\, R_0\big\rangle \tag{4.10}$$

$$\big|E+i0, L_1\big\rangle = \frac{1}{E+i0}\big|\gamma_0\big\rangle\big\langle\gamma_0\big|h_t^{\dagger}\big|E, L_0\big\rangle + \\ + \int_{g_l}^{\infty} \frac{d\varepsilon}{N_R(\varepsilon)} \frac{1}{E-\varepsilon+i0} \big|\varepsilon\, R_0\big\rangle \big\langle\varepsilon, R_0\big|h_t^{\dagger}\big|E, L_0\big\rangle \tag{4.11}$$

Now, multiplying the third equation by $\big\langle\gamma_0\big|$ and $\big\langle\epsilon, R_0\big|$, find:

$$\big\langle\gamma_0\big|E+i0, L_1\big\rangle = \frac{1}{E+i0}\big\langle\gamma_0\big|h_t^{\dagger}\big|E, L_0\big\rangle \tag{4.12}$$

$$\big\langle\epsilon, R_0\big|E+i0, L_1\big\rangle = \frac{1}{E-\epsilon+i0}\big\langle\epsilon, R_0\big|h_t^{\dagger}\big|E, L_0\big\rangle \tag{4.13}$$

The l.h.s. of both equations above can be found explicitly with the help of appendix A. The result yields:

$$\big\langle\gamma_0\big|h_t^{\dagger}\big|E, L_0\big\rangle = 4\sqrt{ug_R}\zeta^2 t\left(e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}}\right) f\left(\frac{E}{|g_L|}\right) \tag{4.14}$$

$$\big\langle\epsilon, R_0\big|h_t^{\dagger}\big|E, L_0\big\rangle = -16u\zeta^2 t\left(e^{i\frac{\phi}{2}} + e^{-i\frac{\phi}{2}}\right) f\left(\frac{E}{|g_L|}\right) f\left(\frac{\varepsilon}{g_R}\right) \tag{4.15}$$

where $f(x) = \sqrt{x^2-1}\left(x+\sqrt{x^2-1}\right)$. The fact, that all energy dependences here are described by a single function $f(x)$ insinuates that, maybe it's possible to make this calculations in some more beautiful way.

4.3 Ionization rate

Appendix A

Wavefunctions for the stationary contact

Here the eigenstates of the junction are presented in leading and subleading order of the tunneling constant t. They are obtained with the methods from the chapter 3 and written in the notation from the section 4.2. Only low and medium momenta parts are presented here, as it's sufficient for the section 4.2, which uses the formulas for from this appendix. The states are normalized as:

$$\left\langle \gamma_0 \middle| \gamma_0 \right\rangle = 1 \quad \left\langle \epsilon, R_0 \middle| \varepsilon, R_0 \right\rangle = N_R(\epsilon)\,\delta(\epsilon-\varepsilon) \quad \left\langle \epsilon, L_0 \middle| \varepsilon, L_0 \right\rangle = N_L(\epsilon)\,\delta(\epsilon-\varepsilon) \tag{A.1}$$

where

$$N_L(\varepsilon) = \frac{4\pi u\sqrt{\varepsilon^2-g_L^2}\left(e^{2\kappa_L(\varepsilon)}+1\right)^2}{\varepsilon} \quad N_R(\varepsilon) = \frac{4\pi u\sqrt{\varepsilon^2-g_R^2}\left(e^{2\eta_R(\varepsilon)}+1\right)^2}{\varepsilon} \tag{A.2}$$

The diefinition of the $\eta_{L,R}$, η_L and κ_R is given in the 3.6. The indexes L, R near the spinors are relate to the wire, where this spinor is present.This reference can be used either in $|g_L| < g_R$ or in $|g_L| > g_R$ cases.

The Majorana state is:

$$\left\langle x \middle| \gamma_0 \right\rangle = \frac{1}{2}\sqrt{\frac{g_R}{u}} \begin{pmatrix} -1 \\ i \\ -i \\ 1 \end{pmatrix}_R e^{-\frac{g_R x}{u}} \tag{A.3}$$

$$\left\langle x \middle| \gamma_1 \right\rangle = \frac{1}{2}\sqrt{\frac{g_R}{u}}\zeta t\left(e^{i\frac{\phi}{2}}+e^{-i\frac{\phi}{2}}\right)\left[\begin{pmatrix} -i \\ 1 \\ 1 \\ -i \end{pmatrix}_L e^{-\frac{2\Delta x}{u}} - i\zeta \begin{pmatrix} -i \\ -1 \\ 1 \\ i \end{pmatrix}_L e^{-\frac{|g_L|x}{u}}\right] \tag{A.4}$$

Continuous states from the right wire are:

$$\langle x|E,\, R_0\rangle =$$

$$\left(-ie^{\eta_R(E)}-1\right)\begin{pmatrix}-1\\ -e^{\eta_R(E)}\\ e^{\eta_R(E)}\\ 1\end{pmatrix}_R e^{-\frac{ix\sqrt{E^2-g_R^2}}{u}}+\left(e^{\eta_R(E)}+i\right)\begin{pmatrix}-e^{\eta_R(E)}\\ -1\\ 1\\ e^{\eta_R(E)}\end{pmatrix}_R e^{\frac{ix\sqrt{E^2-g_R^2}}{u}} \quad \text{(A.5)}$$

$$\langle x|E\,R_1\rangle\Big|_{E<g_L} = t\zeta\left(e^{2\eta_R(E)}-1\right)\left(e^{i\frac{\phi}{2}}+e^{-i\frac{\phi}{2}}\right)\times$$
$$\times\left[\begin{pmatrix}-i\\ 1\\ 1\\ -i\end{pmatrix}_L - e^{-\frac{2\Delta x}{u}} - \frac{2i\zeta}{(1+e^{-i\theta_L(E)})}\begin{pmatrix}-ie^{-i\theta_L(E)}\\ -1\\ 1\\ ie^{-i\theta_L(E)}\end{pmatrix}_L e^{\frac{-x\sqrt{g_L^2-E^2}}{u}}\right] \quad \text{(A.6)}$$

$$\langle x|E\,R_1\rangle\Big|_{E>g_L} = t\zeta\left(e^{2\eta_R(E)}-1\right)\left(e^{i\frac{\phi}{2}}+e^{-i\frac{\phi}{2}}\right)\times$$
$$\times\left[\begin{pmatrix}-i\\ 1\\ 1\\ -i\end{pmatrix}_L e^{-\frac{2\Delta x}{u}} - \frac{2i\zeta}{(1+ie^{-\kappa_l(E)})}\begin{pmatrix}e^{-\kappa_L(E)}\\ -1\\ 1\\ -e^{-\kappa_L(E)}\end{pmatrix}_L e^{\frac{ix\sqrt{E^2-g_L^2}}{u}}\right] \quad \text{(A.7)}$$

Continuous states from the left wire are:

$$\langle x|\varepsilon, L_0\,\rangle =$$

$$\left(e^{\kappa_L(\varepsilon)}-i\right)\begin{pmatrix}1\\ -e^{\kappa_L(\varepsilon)}\\ e^{\kappa_L(\varepsilon)}\\ -1\end{pmatrix}_L e^{+i\frac{\sqrt{\varepsilon^2-g_L^2}}{u}x}+\left(-1+ie^{\kappa_L(\varepsilon)}\right)\begin{pmatrix}e^{\kappa_L(\varepsilon)}\\ -1\\ 1\\ -e^{\kappa_L(\varepsilon)}\end{pmatrix}_L e^{-i\frac{\sqrt{\varepsilon^2-g_L^2}}{u}x} \quad \text{(A.8)}$$

$$\left\langle x\middle|\varepsilon, L_1\right\rangle\Big|_{\varepsilon<g_R} = \zeta t\left(e^{2\kappa_L(\varepsilon)}-1\right)\left(e^{i\frac{\phi}{2}}+e^{-i\frac{\phi}{2}}\right)\times$$
$$\times\left[\frac{2i\zeta}{\left(-1+e^{i\theta_R(\varepsilon)}\right)}\begin{pmatrix}-1\\ ie^{i\theta_R(\varepsilon)}\\ -ie^{i\theta_R(\varepsilon)}\\ 1\end{pmatrix}_R e^{-x\frac{\sqrt{g_r^2-\varepsilon^2}}{u}}+\begin{pmatrix}1\\ -i\\ -i\\ 1\end{pmatrix}_R e^{-\frac{2\Delta x}{u}}\right] \quad \text{(A.9)}$$

$$\left\langle x\middle|\varepsilon, L_1\right\rangle\Big|_{\varepsilon>g_R} = \zeta t\left(e^{2\kappa_L(\varepsilon)}-1\right)\left(e^{i\frac{\phi}{2}}+e^{-i\frac{\phi}{2}}\right)\times$$
$$\times\left[\frac{2it\zeta}{\left(-1+ie^{-\eta_R(\varepsilon)}\right)}\begin{pmatrix}-1\\ -e^{-\eta_R(\varepsilon)}\\ e^{-\eta_R(\varepsilon)}\\ 1\end{pmatrix}_R e^{\frac{ix\sqrt{E^2-g_R^2}}{u}}+\begin{pmatrix}1\\ -i\\ -i\\ 1\end{pmatrix}_R e^{-\frac{2\Delta x}{u}}\right] \quad \text{(A.10)}$$

Appendix B

Multiphoton ionization

This appendix is focused on high-order perturbation theory, ionization rates in particular.

B.1 Basics about Green's functions

The starting point is the general setup of a discrete bound state subject to a weak and slow perturbation $V(t)$.The bound state energy is zero. The goal is to obtain the ionization rate. The time evolution of the wave function obeys the Schroedinger equation:

$$i\frac{\partial}{\partial t}\Psi = (H_0 + V)\Psi \tag{B.1}$$

In the absence of perturbations, the solution is $\Psi(t) = \Psi_0$ (since $E_0 = 0$ it is literally time-independent). For further analysis it's convenient to consider the unperturbed retarded Green's function $G^R(E)$ defined so that:

$$G^R(E)(E + i0 - H_0) = 1 \tag{B.2}$$

If the bound state is normalized, $\langle\gamma|\gamma\rangle$ =1 and the continuous spectrum is normalized according to $\langle E|E'\rangle = N(E)\delta(E - E')$ with some reasonably nice $N(E)$ then:

$$\mathbb{I} = |\gamma\rangle\langle\gamma| + \int \frac{|E\rangle\langle E|}{N(E)}dE \tag{B.3}$$

Similarly, H_0 and G^R in the energy representation:

$$H_0 = \int \frac{|E\rangle\langle E|}{N(E)}EdE \qquad G^R(\epsilon) = \frac{|\gamma\rangle\langle\gamma|}{\epsilon + i0} + \int \frac{|E\rangle\langle E|}{(\epsilon + i0 - E)N(E)}dE \tag{B.4}$$

Integrals over E include all states in the continuous spectrum. Where the latter is degenerate, a summation over the degenerate states must be carried out. From now on the "R" index for retarded Green's function will be omitted.

In time representation the Green's function obeys:

$$\left(i\frac{\partial}{\partial t_2} - H_0\right) G(t_2, t_1) = \delta(t_2 - t_1) \tag{B.5}$$

One can formally introduce the Green's function in energy representation with two arguments as a Fourier-transform of $G(t_2, t_1)$:

$$G\left(E_2, E_1\right) = \iint e^{iE_2 t_2 - iE_1 t_1} G(t_2, t_1) dt_1 dt_2 \tag{B.6}$$

As H_0 is time independent, $G\left(t_2, t_1\right) = G\left(t_2 - t_1, 0\right)$, so:

$$G\left(E_2, E_1\right) = 2\pi\delta\left(E_2 - E_1\right) G\left(E_1\right) \tag{B.7}$$

One can also find, that for any eigenstate $|E_0\rangle$

$$G\left(t\right)|E_0\rangle = -ie^{-iE_0 t}\theta_H\left(t\right)|E_0\rangle \tag{B.8}$$

With the help of Green's function the Schroedinger equation (B.1) can be solved as:

$$\Psi(t) = \Psi_0 + \int G_0(t - t')V(t')\Psi_0(t')dt' + \iint G_0(t - t')V(t')G_0(t' - t'')V(t'')\Psi_0(t'')dt'dt'' + \dots \tag{B.9}$$

where the Ψ_0 is unperturbed wavefunction. This equation can be rewriteen by the introducion

B.2 Fermi Golden Rule (first order)

Consider first the lowest order to the Fermi Golden rule by only keeping the linear term in V. Let the Fourier decomposition of V be:

$$V(t) = \sum_n V_n e^{-i\omega_n t} \tag{B.10}$$

(Hermiticity demands $V(t) = V(t)^*$ so that $V_n = V_{-n}^*$). Suppose there is an unperturbed continuous spectrum parameterized by E, with $\langle E|E'\rangle = f(E)\delta(E - E')$. The first step is to calculate

$\langle E|\Psi(t)\rangle$, i.e. the quantum amplitude of being in state $|E\rangle$ at time t. It is:

$$\langle E|\Psi(t)\rangle = \langle E|\int G_0(t-t')V(t')\Psi_0(t')dt'\rangle = \int \langle E|G_0(t-t')V(t')|\Psi_0\rangle dt' =$$
$$= \int\int \langle E|G_0(t-t')|E'\rangle\langle E'|V(t')|\Psi_0\rangle dt' \frac{dE'}{N(E)}$$
$$= \int e^{-i(t-t')E}\theta_H(t-t')\langle E|V(t')|\Psi_0\rangle dt' =$$
$$= e^{-iEt}\sum_n \int e^{it'(E-\omega_n)}\theta_H(t-t')\langle E|V_n|\Psi_0\rangle dt' \quad \text{(B.11)}$$

Assuming that the perturbation was turned on at $t' = 0$ the integral over t' is taken from t_0 to t and gives:

$$\langle E|\Psi(t)\rangle = e^{-iEt}\sum_n \langle E|V_n|\Psi_0\rangle \frac{e^{it(E-\omega)}-1}{i(E-\omega_n)} \quad \text{(B.12)}$$

Thus, the probability density of being in the state $|E\rangle$ is (omit all frequencies except the resonant on should be ommited since the contributions from non-resonant frequencies can be neglected at long times):

$$\rho(E) = |\langle E|V_n|\Psi_0\rangle|^2 \frac{4\sin^2\frac{t(E-\omega_n)}{2}}{(E-\omega_n)^2} \quad \text{(B.13)}$$

so that the probability of being in a state between E and $E+\delta E$ is at large times:

$$P(E+\delta E, E) = |\langle E|V_n|\Psi_0\rangle|^2 \int_E^{E+\delta E} \frac{dE}{N(E)} \frac{4\sin^2\frac{t(E-\omega_n)}{2}}{(E-\omega_n)^2} =$$
$$= 2t|\langle E|V_n|\Psi_0\rangle|^2 \int_E^{E+\delta E} \frac{d(tE/2)}{N(E)} \frac{4\sin^2\frac{t(E-\omega_n)}{2}}{t^2(E-\omega_n)^2} =$$
$$= 2t|\langle E|V_n|\Psi_0\rangle|^2 \frac{\pi}{N(E)}\theta_H(\omega_n - E)\theta_H(E+\delta E-\omega_n) \quad \text{(B.14)}$$

In other words, the probability density at large t behaves exactly like the δ-function:

$$\rho(E) = 2\pi|\langle E|V_n|\Psi_0\rangle|^2\delta(E-\omega_n) \quad \text{(B.15)}$$

which is the well-known Fermi Golden rule. (Note, that the probability is $dP = \rho(E)dE/f(E)$ – do not forget the normalization). Thus, the ionization rate (i.e. P/t) for a single photon is expressed

as:

$$I = \frac{2\pi|\langle E|V_n|\Psi_0\rangle|^2}{f(E)} \tag{B.16}$$

B.3 Fermi Golden Rule (high order)

Tho treat the higher-order perturbation theory it's convenient to rewrite (B.9) as:

$$\Psi(t) = \Psi_0 + \int\int G_0(t-t')W(t',t'')\Psi_0(t'')dt'dt'' \tag{B.17}$$

where W incorporates all powers of perturbation theory:

$$W = V + VGV + VGVGV + \dots \tag{B.18}$$

The perturbation V in energy space is:

$$\begin{aligned} V(E',E) \equiv \int V(t',t)e^{it'E'-iEt}dt'dt = \int V(t)e^{i(E'-E)t}dt = \\ = \sum_n V_n \int e^{i(E'-E-\omega_n)t}dt = \sum_n V_n 2\pi\delta(E'-E-\omega_n) \end{aligned} \tag{B.19}$$

The second term of the perturbation theory $W_2 = VGV$ in energy representation reads:

$$\begin{aligned} W_2(E_2,E_1) = \int V(E_2,E)G(E,E')V(E',E_1)\frac{dEdE'}{(2\pi)^2} = \\ \sum_{nm} V_n 2\pi\delta(E_2-E-\omega_n)G^R(E)V_m 2\pi\delta(E-E_1-\omega_m)\frac{dE}{2\pi} = \\ \sum_{nm} V_n G^R(E_1+\omega_m)V_m 2\pi\delta(E_2-E_1-\omega_n-\omega_m) \end{aligned} \tag{B.20}$$

Very similarly, the N-th-order term is:

$$W_N(E_2,E_1) = \sum_{n_1,\dots,n_N} 2\pi\delta\left(E_2-E_1-\sum_{i=1}^{N}\omega_{n_i}\right)V_{n_N}\prod_{j=1}^{N-1}G^R\left(E_1+\sum_{s=1}^{j}\omega_{n_s}\right)V_{n_j} \tag{B.21}$$

The above expression sums over all processes involving N photons. Note how the Green's functions contains the cumulative energy of all photons absorbed by the time. The δ-function at the start of the expression indicates that the energy is changed by $\sum\omega_{n_i}$. It makes more sense to sort processes within W not by photon count N but rather by the total energy gained, since the total ionization rate should be dominated by ionization to the lowest accessible continuum state.

Defining this activation energy as $\mathcal{E}$, write:

$$W_{\mathcal{E}}(E_2, E_1) = 2\pi\delta(E_2 - E_1 - \mathcal{E})w_{\mathcal{E}}(E_1) \tag{B.22}$$

$$w_{\mathcal{E}}(E) = \sum_{\{n_i\}:\mathcal{E}} V_{n_N} \prod_{j=1}^{N-1} G^R \left(E + \sum_{s=1}^{j} \omega_{n_s} \right) V_{n_j} \tag{B.23}$$

The summation is over all sets of photons n_i that total an energy of $\mathcal{E}$, i.e. such sets that $\sum_i^N \omega_{n_i} = \mathcal{E}$. The photon number N itself depends on the particular photon set n_i. The total composite perturbation W can be written as a sum of $W_{\mathcal{E}}$ with different $\mathcal{E}$. However, only one term is relevant — the one with $\mathcal{E} = \min |g_L|, |g_L|$. (If the elementary frequency quantum ω is large enough, this is replaced by the lowest $\mathcal{E}$ that surpasses the minimum gap).

Now the Fermi Golden rule result (B.16) can be rederived for $W_{\mathcal{E}}$. It's easy ti see that W has the same energy structure as the single-photon perturbation V. Both operators simply shift the energy. Thus, one may simply put $W_{\mathcal{E}}$ into the Fermi Golden rule instead of V to get the full-theory results:

$$I = \frac{2\pi |\langle \mathcal{E} | w_{\mathcal{E}} | \Psi_0 \rangle|^2}{f(\mathcal{E})} \tag{B.24}$$

Thus to find the ionization rate one should calculate $w_{\mathcal{E}}$ from (B.23).

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