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Ex 5.1 i) Input: n unsorted elements, $a_1, a_2, \cdots, a_n$

Output: all the a_i, $1 \le i \le n$ in increasing order

```
for i ← 2 to n do
    left ← 1
    right ← i-1
    while left < right do
        middle ← ⌊(left+right)/2⌋
        if a_i > a_middle then left ← middle+1;
        else right ← middle;
    end while
    temp ← a_i
    for k ← 0 to i-left-1 do
        a_{i-k} ← a_{i-k-1}
    end for
    a_left ← temp
end for
return (a_1, a_2, ..., a_n) in increasing order.
```

ii) Insertion sort:

4 7 3 8 1 5 4 2 1
3 4 7 8 1 5 4 2 1
3 4 7 8 1 5 4 2 4
1 3 4 7 8 5 4 2 1
1 3 4 5 7 8 4 2 4
1 3 4 4 5 7 8 2 3
1 2 3 4 4 5 7 8 2

Then $1+1+4+1+4+3+2 = 16$ comparison

Binary insertion sort

4 7 3 8 1 5 4 2 1
3 4 7 8 1 5 4 2 2
3 4 7 8 1 5 4 2 2
1 3 4 7 8 5 4 2 3
1 3 4 5 7 8 4 2 3
1 3 4 4 5 7 8 2 3
1 2 3 4 4 5 7 8 4

Then $1+2+2+3+3+3+4 = 18$ comparison

iii) Worset case:

$$2+3+\cdots+n = \frac{n(n+1)}{2} - 1 = O(n^2)$$

iv) When sorting the j th element,

worst case $O(\log_2(j+1)) = O(\log j)$

$$O\left(\sum_{j=1}^{n} \log j\right) = O(n \log n)$$

When n is large, it is significantly large

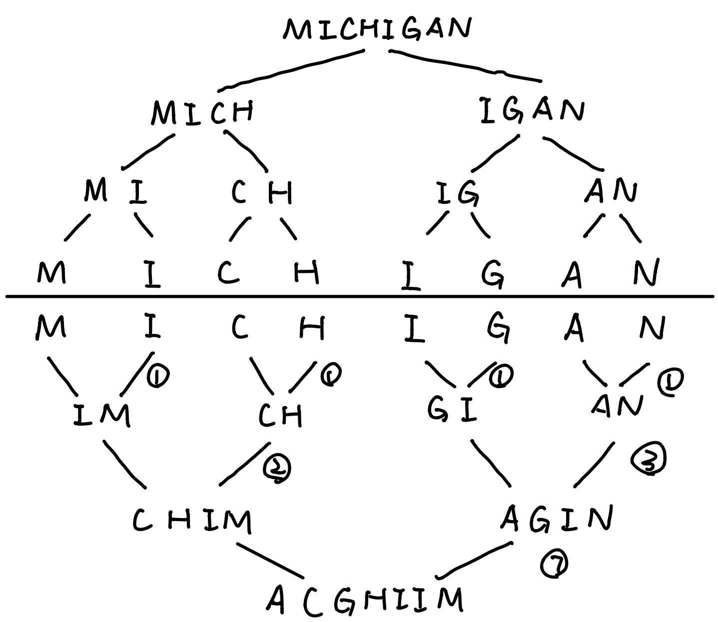
Ex 5.2 Let $a_1 = M$, $a_2 = I$, $a_3 = C$, $a_4 = H$, $a_5 = I$, $a_6 = G$, $a_7 = A$, $a_8 = N$.

i) Input: $L = a_1, a_2, \cdots, a_n$ ($n=9$ for initial input)
Output: L, sorted into elements with their ASCII in nondecreasing order

```
Function MergeSort (L):
    if n>1
        m ← ⌊n/2⌋
        L1 ← a1, ..., am
        L2 ← am+1, ..., an
        L ← Merge (MergeSort (L1), MergeSort (L2))
    end if
    return L
end
```

Input: L_1, L_2, two sorted lists
Output: L a merged list of L_1 and L_2 with elements' ASCII in creasing order

```
Function Merge (L1, L2):
    L ← empty list;
    while L1 and L2 are both nonempty do
        remove the one with smaller ASCII of first element of L1 and L2 from the list it is in and put it at
        the right end of L;
        if removal of this element makes one list empty then
            remove all element from the other list and append to L
        end if
    end while
    return L
end
```



Therefore, $1+1+1+1+2+3+7 = 16$ comparisons.

ii) Input: $a_1, a_2, \cdots, a_8$

Output: all the a_i, $1 \le i \le n$ with ASCII in increasing order

```
for j ← 2 to 8 do
    i ← 1
    while a_j > a_i do i ← i+1
    m ← a_j
    for k ← 0 to j-i-1 do a_{j-k} ← a_{j-k-1}
    a_i ← m
end for
return (a_1, ..., a_n) with their ASCII in increasing order
```

I M C H I G A N ①
C I M H I G A N ①
C H I M I G A N ②
C H I I M G A N ③
C G H I I M A N ②
A C G H I I M N ①
A C G H I I M N ⑧

Therefore, $1+1+2+3+2+1+8 = 18$ comparisons

iii). Input: $a_1, \cdots, a_8$, 8 unsorted elements.

Output: all the a_i, $1 \le i \le 8$ with their ASCII in increasing order

```
for i ← 1 to 7 do
    for j ← 1 to 8-i do
        if a_j > a_{j+1} then interchange a_j and a_{j+1};
    end for
end for
return (a_1, ..., a_8) with their ASCII in increasing order
```

I C H I G A M N ⑦
C H I G A I M N ⑥
C H G A I I M N ⑤
C G A H I I M N ④
C A G H I I M N ③
A C G H I I M N ③

Therefore, $7+6+5+4+3+3 = 28$ comparison

Ex5.3 i) For worst case, it needs 2 addition to eliminate a digit.
Then we need to eliminate $\lfloor \log n \rfloor$ digits.
Therefore the number is $2\lfloor \log n \rfloor$. The complexity is $O(\log n)$.

ii) $n = n_0 + n_1 \times 10^1 + n_2 \times 10^2 + \cdots + n_i \times 10^i$, where n_i is between 0 and 9.

$n \equiv n_0 + n_1 \times 1^1 + n_2 \times 1^2 + \cdots + n_i \times 1^i \equiv n_0 + n_1 + \cdots + n_i \mod 9$

If $n \bmod 9 = 0$, then the iterated integer sum equals 9.
If $n \bmod 9 \neq 0$, then the iterated integer sum equals $n \bmod 9$.

iii) Similarily, we have $n \equiv n_0 + n_1 + \cdots + n_i \mod (b-1)$
Therefore, if $n \bmod (b-1) = 0$, then the iterated integer sum equals $b-1$.
if $n \bmod (b-1) \neq 0$, then the iterated integer sum equals $n \bmod (b-1)$.

Ex5.6 ① $a_n = a_{n-1} + 6a_{n-2}$, $n \geq 2$, $a_0 = 3$, $a_1 = 6$.

$r^2 = r + 6 \Rightarrow r_1 = 3, r_2 = -2$

$$a_n = \frac{6 - 3\times(-2)}{3-(-2)} 3^n + \frac{3\times 3 - 6}{3-(-2)} (-2)^n$$

$$a_n = \frac{12}{5} \cdot 3^n + \frac{3}{5} \cdot (-2)^n$$

② $a_{n+2} = -4a_{n+1} + 5a_n$, $n \geq 0$, $a_0 = 2$, $a_1 = 8$

$r^2 = -4r + 5 \Rightarrow r_1 = 1, r_2 = -5$

$$a_n = \frac{8 - 2\times(-5)}{1-(-5)} 1^n + \frac{2\times 1 - 8}{1-(-5)} (-5)^n$$

$$a_n = 3 - (-5)^n$$

Ex5.8 ① $a_n = 5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$

The associated homogeneous equation is $a_n = 5a_{n-1} - 6a_{n-2}$.

$r^2 = 5r - 6 \Rightarrow r_1 = 2, r_2 = 3$

$a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 3^n$

$F(n) = 42 \cdot 4^n$

$a_n^{Part} = a_0 \cdot 4^n$

$a_0 \cdot 4^n = 5a_0 4^{n-1} - 6a_0 4^{n-2} + 42 \cdot 4^n$

$a_0 = 336$

$a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 3^n + 336 \cdot 4^n$

② $a_n = -5a_n - 6a_{n-2} + 2^n + 3n$

The associated homogeneous equation is $a_n = -5a_n - 6a_{n-2}$

$r^2 = -5r - 6 \Rightarrow r_1 = -2, r_2 = -3$

$a_n = \alpha_1 \cdot (-2)^n + \alpha_2 \cdot (-3)^n$

$F(n) = 2^n + 3n$

$a_n^{Part} = a_0 \cdot 2^n + a_1 + a_2 n$

$a_0 \cdot 2^n + a_1 + a_2 n = -5(a_0 \cdot 2^{n-1} + a_1 + a_2(n-1)) - 6(a_0 \cdot 2^{n-2} + a_1 + a_2(n-2)) + 2^n + 3n$

$a_0 = \frac{1}{5}$, $a_1 = \frac{17}{48}$, $a_2 = \frac{1}{4}$

$a_n = \alpha_1 \cdot (-2)^n + \alpha_2 (-3)^n + \frac{1}{5} \cdot 2^n + \frac{17}{48} + \frac{1}{4} \cdot n$

③ $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$

The associated homogeneous equation is $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$

$r^3 = 7r^2 - 16r + 12 \Rightarrow r_1 = r_2 = 2,\ r_3 = 3$

$a_n = (\alpha_1 + \alpha_2 n)\cdot 2^n + \alpha_3 \cdot 3^n$

$F(n) = n4^n$

$a_n^{part} = (a_0 + a_1 n)4^n$

$(a_0 + a_1 n)4^n = 7(a_0 + a_1(n-1))4^{n-1} - 16(a_0 + a_1(n-2))4^{n-2} + 12(a_0 + a_1(n-3))4^{n-3} + n4^n$

$a_0 = -80,\ a_1 = 16$

$a_n = (\alpha_1 + \alpha_2 n)2^n + \alpha_3 \cdot 3^n + (16n - 80)\cdot 4^n$

Ex 6.1 i) $f(n) = af(\frac{n}{b}) + cn^d$

$$= a\left[af(\tfrac{n}{b^2}) + c(\tfrac{n}{b})^d\right] + cn^d$$

$$= a^2\left[af(\tfrac{n}{b^3}) + c(\tfrac{n}{b^2})^d\right] + ac(\tfrac{n}{b})^d + cn^d$$

$$\vdots$$

$$= a^k f(\tfrac{n}{b^k}) + \sum_{m=1}^{k} a^{m-1} c\left(\frac{n}{b^{m-1}}\right)^d$$

When $n = b^k \Rightarrow k = \log_b n$,

$$f(n) = a^{\log_b n} f(1) + \sum_{m=1}^{\log_b n} cn^d \left(\frac{a}{b^d}\right)^{m-1}$$

Since $a = b^d \Rightarrow \frac{a}{b^d} = 1$ and $a^{\log_b n} = b^{d \log_b n} = n^d$

Therefore, $f(n) = f(1)n^d + cn^d \log_b n$.

ii). $$\lim_{n\to\infty} \frac{f(1)n^d + cn^d \log_b n}{n^d \log n} = \lim_{n\to\infty} \frac{f(1) + c\log_b n}{\log n} = \lim_{n\to\infty} c\frac{\log_b n}{\log n} = \lim_{n\to\infty} c \frac{\log_b n}{\frac{\log_b n}{\log_b 10}} = c\log_b 10 = \text{constant}$$

Therefore, $f(n) = O(n^d \log n)$

iii) $$f(n) = a^{\log_b n} f(1) + \sum_{m=1}^{\log_b n} cn^d \left(\frac{a}{b^d}\right)^{m-1}$$

Since n is a power of b, we can write $n = b^x$. $\Rightarrow \log_b n = x \Rightarrow a^{\log_b n} = a^x$

Also $n^{\log_b a} = (b^{\log_b a})^x = a^x$

Therefore, $a^{\log_b n} = n^{\log_b a}$

$$f(n) = n^{\log_b a} f(1) + c\cdot n^d \frac{1 - (\frac{a}{b^d})^{\log_b n}}{1 - \frac{a}{b^d}}$$

$$= n^{\log_b a} f(1) + c\cdot n^d \frac{b^d - b^d \cdot \frac{a^{\log_b n}}{n^d}}{b^d - a}$$

$$= n^{\log_b a} f(1) + \frac{b^d c}{b^d - a}\cdot n^d - c\cdot n^d \frac{b^d \cdot \frac{n^{\log_b a}}{n^d}}{b^d - a}$$

$$= n^{\log_b a} f(1) + \frac{b^d c}{b^d - a}\cdot n^d - \frac{b^d \cdot c}{b^d - a}\cdot n^{\log_b a}$$

$$= \frac{b^d c}{b^d - a}\cdot n^d + \left(f(1) - \frac{b^d\cdot c}{b^d - a}\right) n^{\log_b a}$$

iv). $a < b^d \Rightarrow \log_b a < d$

$$\lim_{n\to\infty} \frac{C_1 n^d + C_2 n^{\log_b a}}{n^d} = C_1 = \text{constant}$$

Therefore, $f(n) = O(n^d)$

v). $a > b^d \Rightarrow \log_b a > d$

$$\lim_{n\to\infty} \frac{C_1 n^d + C_2 n^{\log_b a}}{n^{\log_b a}} = C_2 = \text{constant}$$

Therefore, $f(n) = O(n^{\log_b a})$

Ex 6.5 Let a_{ij} denote the j th element of A_i,

then for all A_i can be expressed as:

$$\begin{aligned}
A_1 &= \{ a_{11},\ a_{12},\ a_{13},\ a_{14},\ \cdots,\ a_{1n},\ \cdots \} \\
A_2 &= \{ a_{21},\ a_{22},\ a_{23},\ a_{24},\ \cdots,\ a_{2n},\ \cdots \} \\
A_3 &= \{ a_{31},\ a_{32},\ a_{33},\ a_{34},\ \cdots,\ a_{3n},\ \cdots \} \\
A_4 &= \{ a_{41},\ a_{42},\ a_{43},\ a_{44},\ \cdots,\ a_{4n},\ \cdots \} \\
&\vdots
\end{aligned}$$

Then, we can write $\bigcup_{i\in N} A_i = \{a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, a_{41}, a_{32}, a_{23}, a_{14}, \cdots\}$

a_{11} is the first element, for $i+j>2$, a_{ij} is the $[1+2+\cdots+(i+j-2)+j]$ th element.

Therefore, for any a_{ij}, we can find a one-to-one correspondence between a_{ij} and $n=[1+2+\cdots+(i+j-2)+j]$.

Therefore, $\bigcup_{i\in N} A_i$ is countable.

Ex 6.7 ① $f: M\to N$ is surjective.

Since card $M \neq$ card N, by theorem 2.4.21, f is not injective

② $f: M\to N$ is not surjective.

We may find $T \subsetneq N$ such that $f: M\to T$ is surjective.

Also card $T<$ card $N<$ card M. By theorem 2.4.21, $f: M\to T$ is not injective.

Therefore, $f: M\to N$ is not injective.

Hence f is not injective.

Ex7.1 i) Five people A.B.C.D.E. Let AB, BC, CD, DE, EA are friends and others be enemies.
Then there are no 3 mutual friends or 3 mutual enemies.
Therefore $R(3,3)>5$.

ii). Consider one member A. By generalized pigeonhole principle, for the rest nine people, at least five of them are either friends or enemies of A. Let B, C, D, E, F be friends of A. If any two of these five are friends, then they together with A are a group of three mutual friends. If none of these five are friends, four mutual enemies can be found from them.
If B, C, D, E, F are enemies of A, then similarily there are either three mutual enemies or four mutual friends.
Therefore, $R(4,3)\leq 10$.

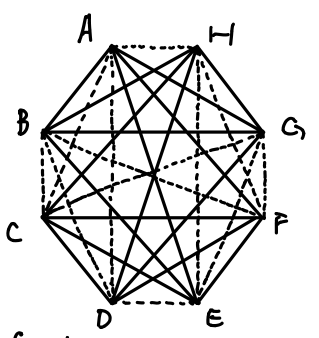
iii) Consider one member A. By generalized pigeonhole principle, for the rest 19 people, at least ten of them are either friends or enemies of A. For the ten people are friends with A, there are 3 mutual friends or 4 mutual enemies. Then together with A, there are 4 mutual friends or 4 mutual enemies. For the ten people are enemies with A, there are 4 mutual friends or 3 mutual enemies. Then together with A, there are 4 mutual friends and 4 mutual enemies.

iv). Among n people, either a pair of friends or all enemies. Then $R(2,n)\leq n$.
Among n-1 people which are all enemies, there are 0 mutual friends and n-1 mutual enemies, $R(2,n)>n-1$
Therefore, $R(2,n)=n$

v). In a group of $R(m-1,n)+R(m,n-1)$ people. Consider one member A. By generalized pigeonhole principle, for the rest $R(m-1,n)+R(m,n-1)-1$ people, suppose $R(m-1,n)\leq R(m,n-1)$, then at least $R(m-1,n)$ of them are either friends or enemies of A. Let them be friends of A. If any m-1 of these $R(m-1,n)$ are friends, then the m-1 people together with A becomes m mutual friends. If n of these $R(m-1,n)$ are enemies, then they form n mutual enemies. If the $R(m-1,n)$ people are enemies of A, since $R(m-1,n)=R(n,m-1)$, we can get the same result. Therefore $R(m,n)\leq R(m-1,n)+R(m,n-1)$.
$R(4,3)\leq R(3,3)+R(4,2)=6+4=10$

vi). In a party of 9 people. Consider one people A. By generalized pigeonhole principle, A has at least 4 enemies or at least 4 friends. First, if A is with 4 friends. If any two of these four people are friends, then these two people together with A forms 3 mutual friends. If none of these four people are friends, then these four people forms 4 mutual enemies.
Second, if A doesn't have 4 friends, i.e. A has five or more enemies. Then at most 4 of these enemies are mutual friends since no one has 4 or more friends. Therefore, at least 2 of these are mutual enemies. These two people, together with A, forms 3 mutual enemies.
Therefore, $R(4,3)\leq 9$.

vii).



friend: — enemy: -----

Neithe 4 mutual friends nor 3 mutual enemies

$R(4,3)>8 \Rightarrow R(4,3)=9$

Ex 7.5 ① $n=1$ $P[A_1]=P[A_1]$, it is true.

② Suppose it is true for $n \le k$.

Then for $n=k+1$,

$$P[A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}] = P[(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}]$$
$$= P[A_1 \cup A_2 \cup \cdots \cup A_k] + P[A_{k+1}] - P[(A_1 \cup A_2 \cup \cdots \cup A_k) \cap A_{k+1}]$$
$$= P[A_1 \cup A_2 \cup \cdots \cup A_k] + P[A_{k+1}] - P[(A_1 \cap A_{k+1}) \cup (A_2 \cap A_{k+1}) \cup \cdots \cup (A_k \cap A_{k+1})]$$
$$= \left(\sum_{1\le i\le k} P[A_i] - \sum_{1\le i<j\le k} P[A_i \cap A_j] + \sum_{1\le i<j<m\le k} P[A_i \cap A_j \cap A_m] - + \cdots + (-1)^{k+1} P[A_1 \cap A_2 \cap \cdots \cap A_k]\right) + P[A_{k+1}]$$
$$- \left(\sum_{1\le i\le k} P[A_i \cap A_{k+1}] - \sum_{1\le i<j\le k} P[(A_i \cap A_{k+1}) \cap (A_j \cap A_{k+1})] + \cdots + (-1)^{k+1} P[(A_1 \cap A_{k+1}) \cap (A_2 \cap A_{k+1}) \cap \cdots \cap (A_k \cap A_{k+1})]\right)$$
$$= \sum_{1\le i\le k+1} P[A_i] - \sum_{1\le i<j\le k} P[A_i \cap A_j] + \sum_{1\le i<j<m\le k} P[A_i \cap A_j \cap A_m] - + \cdots + (-1)^{k+1} P[A_1 \cap A_2 \cap \cdots \cap A_k]$$
$$- \sum_{1\le i\le k} P[A_i \cap A_{k+1}] + \sum_{1\le i<j\le k} P[A_i \cap A_j \cap A_{k+1}] - + \cdots\cdots + (-1)^{k+2} P[A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}]$$
$$= \sum_{1\le i\le k+1} P[A_i] - \sum_{1\le i<j\le k+1} P[A_i \cap A_j] + \sum_{1\le i<j<m\le k+1} P[A_i \cap A_j \cap A_m] - + \cdots + (-1)^{k+2} P[A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1}]$$

Therefore, when $n \le k$ is true, we have $n=k+1$ is also true.

Therefore, the probabilistic inclusion-exclusion principle is true.

Ex 7.7 i) We have $b^{n-1} \equiv 1 \mod n$ ⊛

Since $n-1 = 2^s t$, for $0 \le i \le s$, we have $\frac{n-1}{2^i} = 2^{s-i} t$

Then $b^{\frac{n-1}{2^i}} = b^{2^{s-i} t}$

Let's denote $a_i = b^{\frac{n-1}{2^i}} = b^{2^{s-i} t}$,

then from ⊛, we know $a_0 = b^{n-1} = b^{2^s t} \equiv 1 \mod n$

$a_1 = (a_0)^{\frac{1}{2}} \Rightarrow a_1^2 = a_0 \equiv 1 \mod n \Rightarrow a_1 \equiv \pm 1 \mod n$, which means $b^{\frac{n-1}{2}} = b^{2^{s-1} t} \equiv \pm 1 \mod n$.

If $a_1 \equiv -1 \mod n$, it satisfies.

If $a_1 \equiv 1 \mod n$, similarily, we have $a_2 \equiv \pm 1 \mod n$, which means $b^{\frac{n-1}{4}} = b^{2^{s-2} t} \equiv \pm 1 \mod n$.

Continuing, we have $b^{2^j t} = b^{\frac{n-1}{2^{s-j}}} \equiv -1 \mod n$ or $b^t = b^{\frac{n-1}{2^s}} \equiv 1 \mod n$.

Therefore, n passes Miller's test.

ii). $2047 - 1 = 2046 = 2 \times 1023$

$2 \nmid 2047$

$2^{11} = 2048 \equiv 1 \mod 2047$

$2^{1023} = (2^{11})^{93} \equiv 1^{93} \equiv 1 \mod 2047$

$2047 = 23 \times 89$

Therefore we can find $s=1$, $t=1023$ such that $2047 - 1 = 2^s t$ and $2^t \equiv 1 \mod 2047$.

Therefore, 2047 passes Miller's test to the base 2, but $2047 = 23 \times 89$, which is composite.