VE 203 Hw2 周涵凯 518021911039

Ex 1.2 i) $n+0=n$

$n+1=succ(n)$

$n+2=succ(succ(n))$

$\vdots$

$n+m=\underbrace{succ(succ(\cdots succ(n)))}_{m\ succ}=succ^m(n)$

ii) $2+2=succ(succ(2))=succ(3)=4$

iii) First, we need to prove $m+0=0+m$ and $m+1=1+m$

For $m+0=0+m$: ① when $m=0$, $0+0=0+0$, true

② when $m+0=0+m$ holds, $(m+1)+0=m+1=(m+0)+1=(0+m)+1=0+(m+1)$

Therefore, $m+0=0+m$ holds.

For $m+1=1+m$: ① when $m=0$, $0+1=1+0$, which is true from above.

② when $m+1=1+m$ holds, $(m+1)+1=(1+m)+1=1+(m+1)$

Therefore, $m+1=1+m$ holds.

Then for $m+n=n+m$. ① when $n=0$, $m+0=0+m$, true.

② when $m+n=n+m$ holds, $m+(n+1)=(m+n)+1=(n+m)+1=n+(m+1)=n+(1+m)=(n+1)+m$.

Therefore, $m+n=n+m$ holds.

Ex 2.2 ① When $n=1$, $a_1=3\times2^{1-1}+2(-1)^1=1$

② When $n=2$, $a_2=3\times2^{2-1}+2(-1)^2=8$

③ Suppose for $1\leq n\leq k$ $(k\geq2)$, we have $a_n=3\times2^{n-1}+2(-1)^n$,

which means $a_k=3\times2^{k-1}+2(-1)^k$ and $a_{k-1}=3\times2^{(k-1)-1}+2(-1)^{k-1}$.

For $a_{k+1}=a_k+2a_{k-1}$

$$
\begin{aligned}
&=3\times2^{k-1}+2(-1)^k+2\left[3\times2^{(k-1)-1}\right]+2(-1)^{k-1}\Big]\\
&=3\times2^{k-1}+2(-1)^k+3\times2^{k-1}+4(-1)^{k-1}\\
&=2\times3\times2^{k-1}+2(-1)^{k-1}(-1+2)\\
&=3\times2^k+2(-1)^{k-1}\\
&=3\times2^{(k+1)-1}+2(-1)^{k+1}
\end{aligned}
$$

Therefore $a_n=3\times2^{n-1}+2(-1)^n$ holds for all $n>0$.

Ex 2.3 If there exists a non-empty set $S\subset\mathbb{N}$ does not have a least element,

then $0\notin S$, otherwise 0 will be the least element.

Therefore, $0\in\mathbb{N}\setminus S$. Let's denote $A=\mathbb{N}\setminus S$ and A is non-empty because $0\in A$.

If $succ(0)=1\in S$, then $succ(0)=1$ will be the least element in S, therefore, $succ(0)=1\in A$.

Similarily, we can get $succ(1)=2\in A$, $succ(2)=3\in A$, $\cdots$, $succ(n-1)=n\in A$.

Now we want to prove $succ(n)\in A$.

If $succ(n)\in S$, then $succ(n)$ will be the least element in S, therefore, $succ(n)\in A$.

Hence A contains zero and also the successor of every number in A, which mean $A=\mathbb{N}$.

So $S=\emptyset$, which is contradictory to the non-empty set S.

Therefore, every non-empty set $S\subset\mathbb{N}$ has a least element, which means the induction axiom implies the well-ordering principle.

Ex 2.4 We will prove $(1+x)^n \geq nx+1$ first.

① When $n=0$, $(1+x)^0 = 1 \geq 0\cdot x+1$

② Suppose $(1+x)^n \geq nx+1$, then

$(1+x)^{n+1} = (1+x)^n(1+x)$

$\geq (nx+1)(1+x)$ for all $x>-1$

$= nx^2+nx+1+x$

$= (n+1)x+1+nx^2$

$\geq (n+1)x+1$

Therefore, $(1+x)^n \geq nx+1$ hold for all $x>-1$ and $n \in \mathbb{N}$.

Besides, $nx+1 > nx$, then $(1+x)^n \geq nx$

Ex 2.5 ① When $n=1$, we have $1=2^0$

② Suppose that $1, 2, 3, \dots, n$ all can be written as a sum of distinct powers of 2.

If $n+1$ is even, $1 \leq \frac{n+1}{2} \leq n$. Then we can write $\frac{n+1}{2} = \sum_k a_k 2^k$, where a_k is 1 or 0.

Then $n+1 = \sum_k 2a_k 2^k = \sum_k a_k 2^{k+1}$, we are finished.

If $n+1$ is odd, we know that n can be written as $\sum_m a_m 2^m$, where a_m is 1 or 0. And it doesn't contain $2^0=1$ since all other terms are even. Then $n+1 = \sum_m a_m 2^m + 2^0$, which is sum of distinct powers of 2.

Therefore, every $n \in \mathbb{N} \setminus \{0\}$ can be written as a sum of distinct powers of 2.

Ex 2.6 ① When $(a,b)=(0,0)$, we have $5 \mid 0$.

② Suppose $5 \mid (a+b)$, then $a+b=5k$.

Then $(a+2)+(b+3) = a+b+5 = 5(k+1)$, so $5 \mid ((a+2)+(b+3))$

Then $(a+3)+(b+2) = a+b+5 = 5(k+1)$, so $5 \mid ((a+3)+(b+2))$

Therefore, the statement is true.

Ex 2.7 i) not reflective: $(1,1)$ doesn't satisfy.

symmetric: if $x+y=0$, then $y+x=0$.

not transitive: if $(1,-1)$ and $(-1,1)$, then $(1,1)$ doesn't satisfy.

ii) reflective: $2 \mid (a-a) \Leftrightarrow 2 \mid 0$, which is true.

symmetric: if $2 \mid (x-y)$, then $2 \mid -(x-y)$ and $2 \mid (y-x)$.

transitive: if $2 \mid (x-y)$ and $2 \mid (y-z)$, then $2 \mid [(x-y)+(y-z)]$ and $2 \mid (x-z)$

iii) not reflective: $(1,1)$ doesn't satisfy.

symmetric: if $xy=0$, then $yx=0$.

not transitive: if $(1,0)$, $(0,2)$, then $(1,2)$ doesn't satisfy.

iv) not reflective: $(2,2)$ doesn't holds.

symmetric: $x=1$ or $y=1 \Leftrightarrow y=1$ or $x=1$

not transitive: if $(-1,1)$, $(1,2)$, then $(-1,2)$ doesn't hold.

v) reflective: $a=\pm a$ holds.

symmetric: if $x=\pm y$ then $y=\pm x$.

transitive: if $x=\pm y$ and $y=\pm z$, then $x=\pm z$.

vi) not reflective: $(1,1)$ doesn't satisfy.

not symmetric: $(2,1)$ holds but $(1,2)$ doesn't.

not transitive: if $(4,2)$ and $(2,1)$, then $(4,1)$ doesn't satisfy.

vii) reflective: $a^2 \geq 0$

symmetric: if $xy \geq 0$, then $yx \geq 0$

not transitive: $(1,0)$, $(0,-1)$, but $(1,-1)$ doesn't holds.

viii) not reflective: $(0,0)$ doesn't satisfy.

not symmetric: if $(1,2)$, then $(2,1)$ doesn't satisfy.

transitive: if $(1,b)$, (b,c), then $(1,c)$ holds.