Causal Directed Acylic Graphs

introduction

Wouter van Amsterdam

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Day 2 intro: Causal Directed Acyclic Graphs and Structural Causal Models

Today's lectures

- introduce 1.5 new framework based on
  - causal Directed Acyclic Graphs (DAGs)
  - Structral Causal Models (SCMs)
- counterfactuals and Pearl's Causal Hierarchy of questions
- lectures will follow Pearl's book Causality Pearl (2009), specifically chapters 3 (DAGs) and 7 (SCMs)

Causal inference frameworks

What are they for?

Mathematical language to

- define *causal* quantities
- express *assumptions*
- derive how to *estimate* causal quantities

Causal inference frameworks

Why learn more than one?

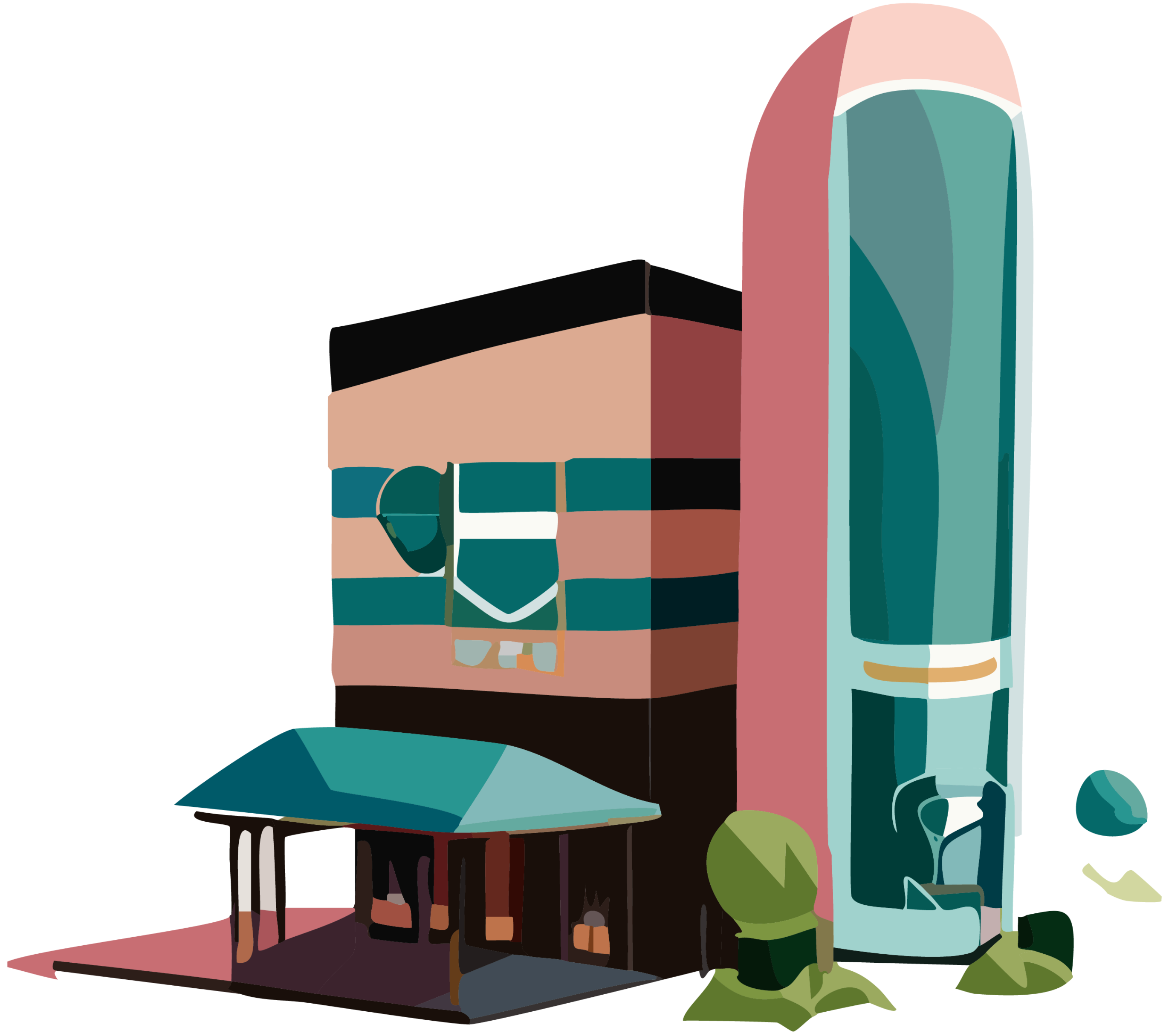
- On day 1 we learned about the Potential Outcomes framework
  - Defines causal effects in terms of (averages of) *individual potential outcomes*
  - Estimation requires assumptions of (conditional) exchangeability and positivity / overlap and consistency
- There isn't only 1 way to think about causality, find one that '*clicks*'
- Now we will learn another framework: *Structural Causal Models* and *causal graphs*
  - causal relations and manipulations of *variables*
  - Developed by different people initially - Judea Pearl, Peter Spirtes, Clark Glymour
  - SCM approach is broader in that it can define more different types of causal questions
- Equivalence: given the same data and assumptions, get the same estimates

Lecture 1 & 2 topics

- motivating examples for DAGs
- what are DAGs
- causal inference with DAGs
  - what is an intervention
  - DAG-structures: confounding, mediation, colliders
  - d-separation
  - back-door criterion

Motivating examples

Example task: are hospital deliveries good for babies?



Example task: are hospital deliveries good for babies?

- You’re a data scientist in a children’s hospital
- Have data on
  - delivery location (home or hospital)
  - neonatal outcomes (good or bad)
  - pregnancy risk (high or low)
- Question: do hospital deliveries result in better outcomes for babies?

Observed data

percentage of good neonatal outcomes

		location	
		home	hospital
risk	low	648 / 720 = 90%	19 / 20 = 95%
	high	40 / 80 = 50%	144 / 180 = 80%

- better outcomes for babies delivered in the hospital for *both risk groups*

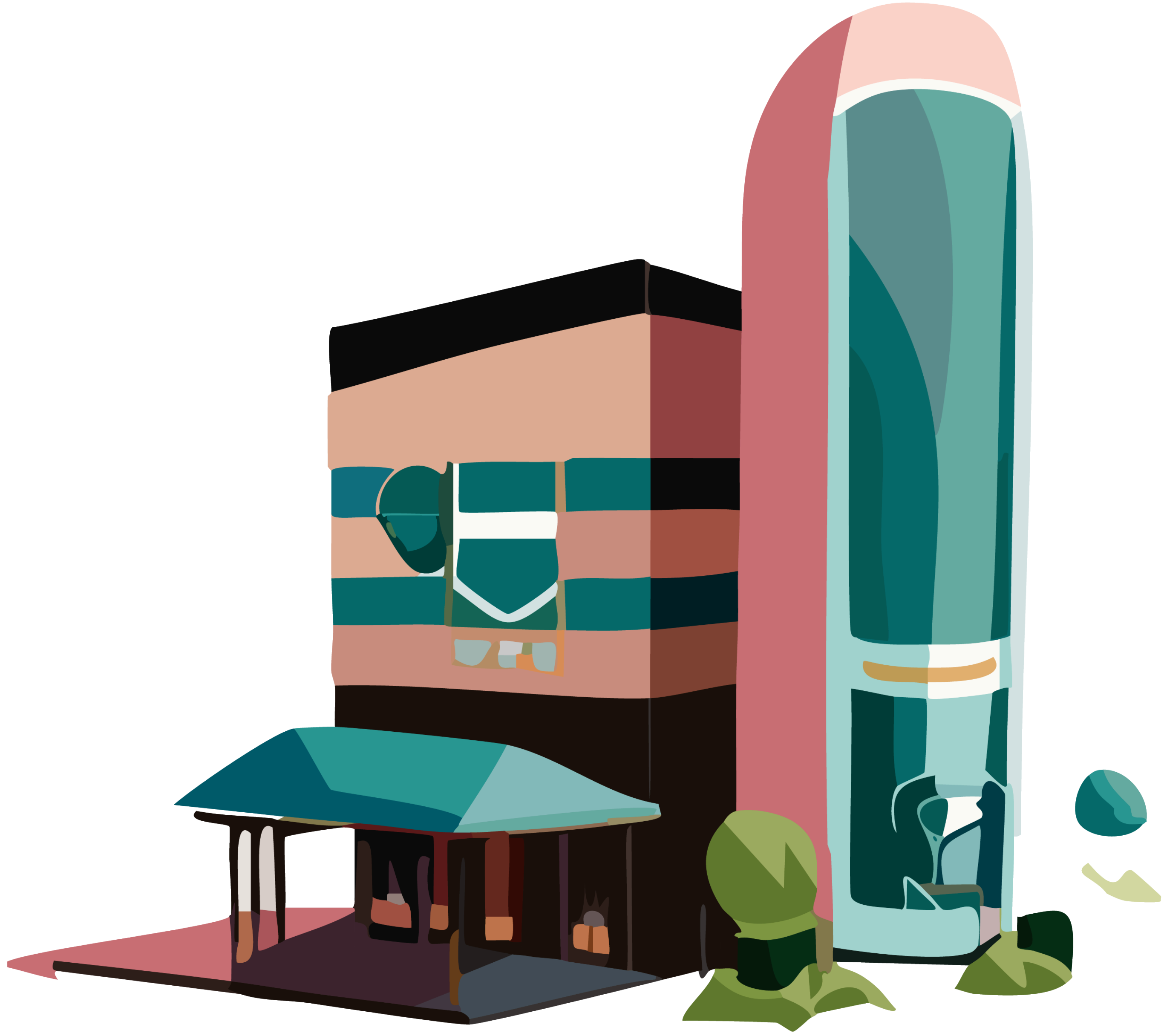
Observed data

		location	
		home	hospital
risk	low	648 / 720 = 90%	19 / 20 = 95%
	high	40 / 80 = 50%	144 / 180 = 80%
	marginal	688 / 800 = 86%	163 / 200 = 81.5%

- better outcomes for babies delivered in the hospital for *both risk groups*
- but not better *marginal* (‘overall’)
- how is this possible?
- what is the correct way to estimate the effect of delivery location?

New question: hernia

- for a patient with a hernia, will they be able to walk sooner when recovering at home or when recovering in a hospital?
- observed data: location, recovery, bed-rest



Observed data 2

		location	
		home	hospital
bedrest	no	648 / 720 = 90%	19 / 20 = 95%
	yes	40 / 80 = 50%	144 / 180 = 80%
	marginal	688 / 800 = 86%	163 / 200 = 81.5%

- more bed rest in hospital
- what is the correct way to estimate the effect of location?

How to unravel this?

- we got two questions with exactly the same data
- in one example, ‘stratified analysis’ seemed best
- in the other example, ‘marginal analysis’ seemed best
- need a language to formalize this *differentness*
- with *Directed Acyclic Graphs* we can make our decision

Causal Directed Acyclic Graphs

diagram that represents our assumptions on causal relations

1. nodes are variables
2. arrows (directed edges) point from cause to effect

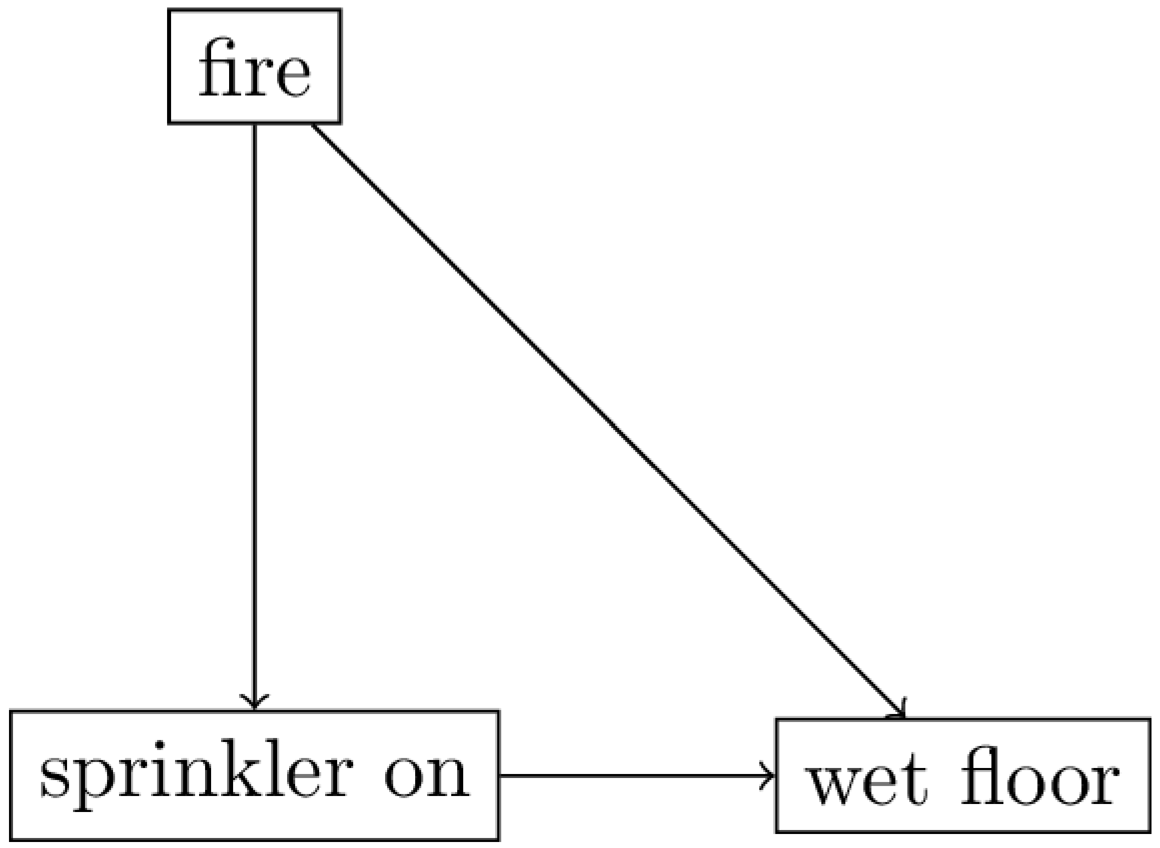


Figure 1: Directed Acyclic Graph

- when used to convey causal assumptions, DAGs are ‘causal’ DAGs[1]

1. this is not the only use of DAGs (see day 4)

Making DAGs for our examples:

The pregnancy DAG

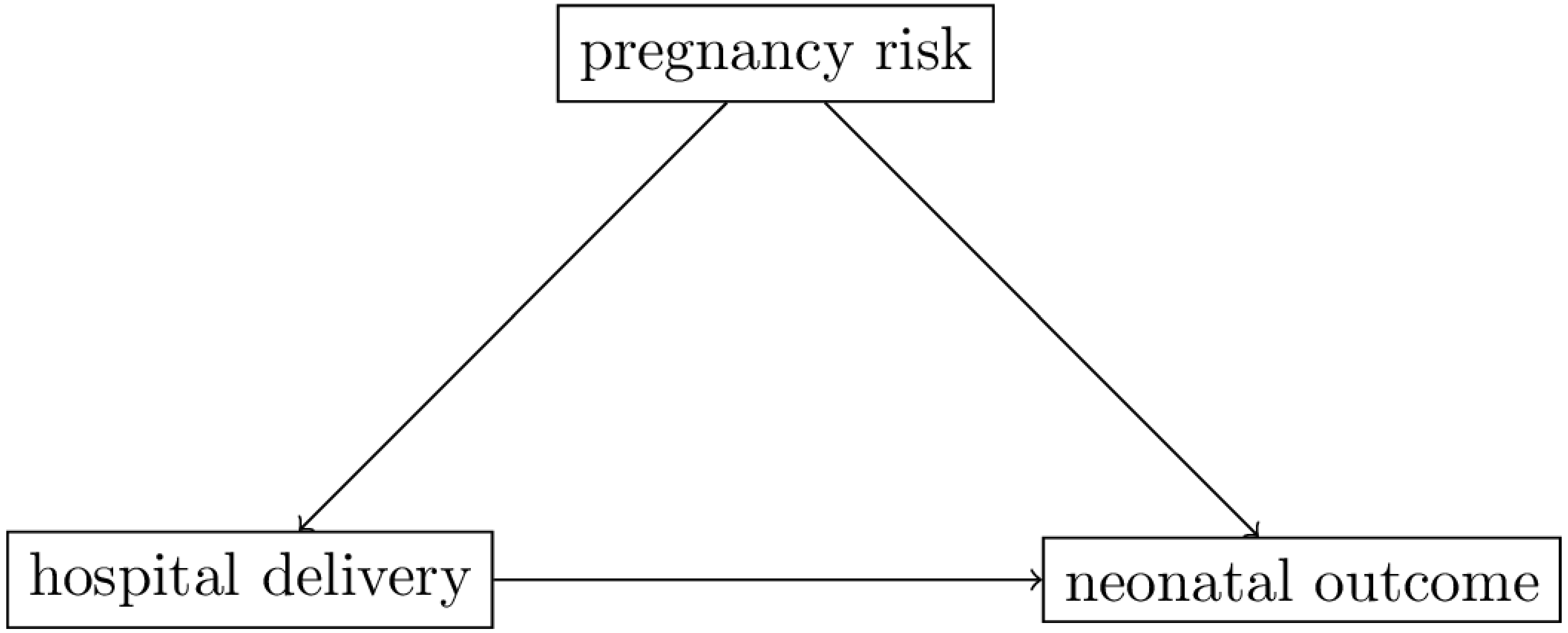


Figure 2

- assumptions:
  - women with high risk of bad neonatal outcomes (`pregnancy risk`) are referred to the hospital for delivery
  - hospital deliveries lead to better outcomes for babies as more emergency treatments possible
  - both `pregnancy risk` and `hospital delivery` cause `neonatal outcome`
- the *other variable* `pregnancy risk` is a common cause of the treatment (`hospital delivery`) and the outcome (this is called a confounder)

Making DAGs for our examples:

The hernia DAG

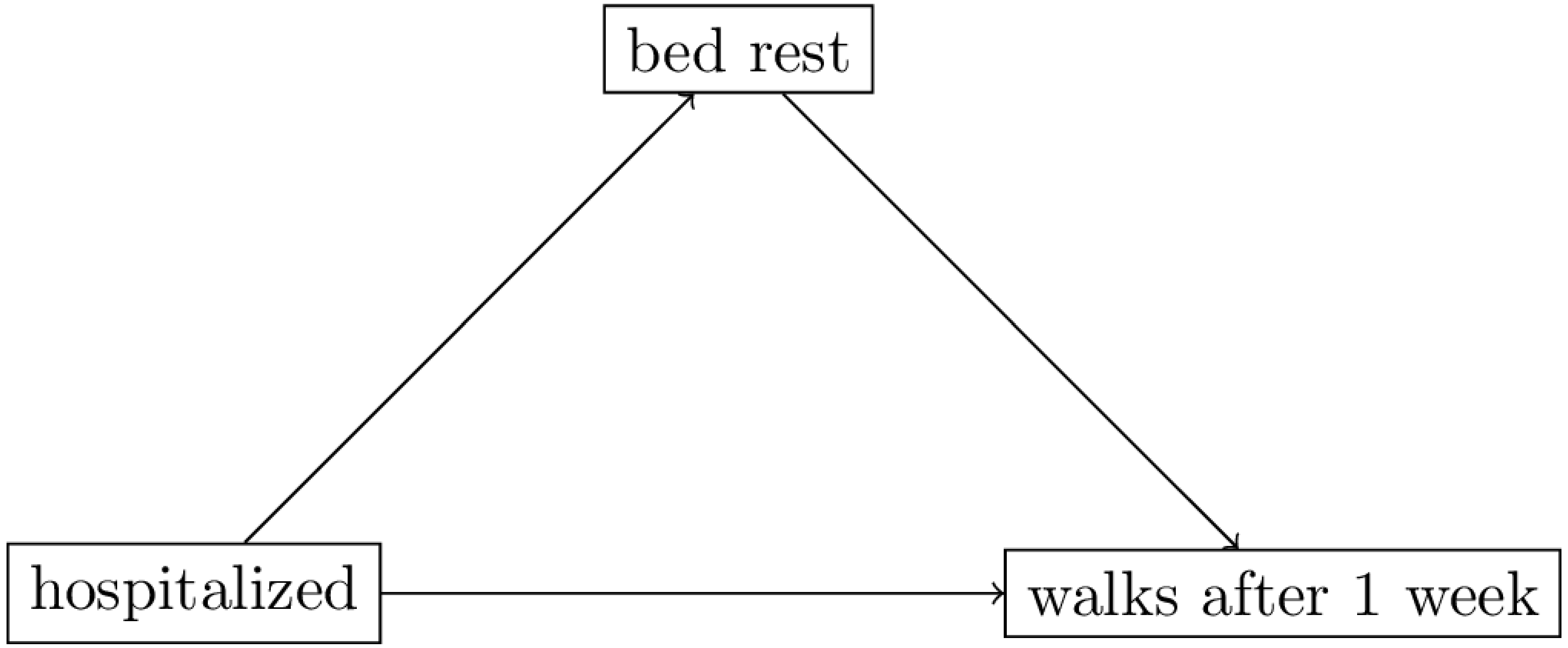


Figure 3

- assumptions:
  - patients admitted to the hospital keep more `bed rest` than those who remain at home
  - `bed rest` leads to lower recovery times thus less walking patients after 1 week
- the *other variable* `bed rest` is a *mediator* between the treatment (`hospitalized`) and the outcome

Causal DAGs to the rescue

- the *other variable* was:
  - a **common cause (confounder)** of the treatment and outcome in the pregnancy example
  - a **mediator** between the treatment and the outcome in the hernia example
- using our background knowledge we could see *something* is different about these examples
- this insight prompted us to a different analysis
- next: ground this in causal theory and see implications for analysis

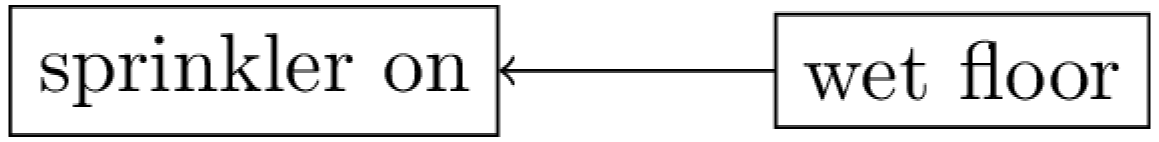
DAG definitions and properties

DAGs convey two types of assumptions:

causal direction and conditional independence

1. causal direction: what causes what?

Figure 4: DAG 1



DAG 2

- read Figure 4 as
  - `sprinkler on` **may** (or may not) cause `wet floor`
  - `wet floor` **cannot** cause `sprinkler on`

Basic DAG patterns: fork

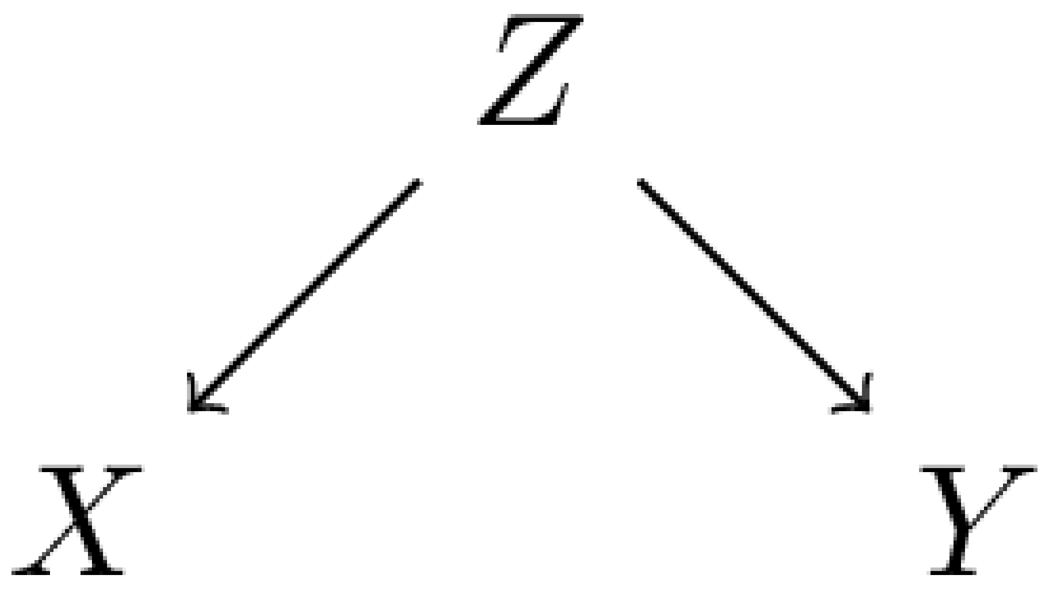


Figure 5: fork / confounder

- Z *causes* both X and Y (common cause / confounder)
- Z = sun rises, X = rooster crows, Y = temperature rises
- $X \not\perp Y$ (i.e. X and Y are dependent)
- $X \perp Y|Z$ (conditioning on the sun rising, the rooster crowing has no information on the temperature)
- $Z \rightarrow X$ is a *back-door*: a path between X and Y that starts with an arrow into X
- *typically want to adjust for Z (see later 6.4)*

Basic DAG patterns: chain

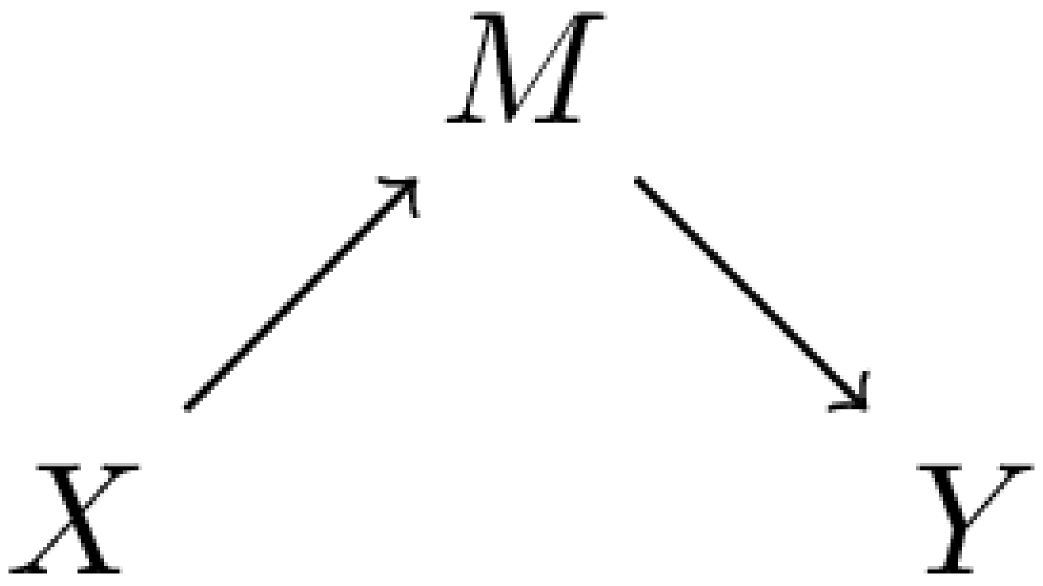


Figure 6: chain / mediation

- M *mediates* effect of X on Y
- X: student signs up for causal inference course, M: student studies causal inference, Y: student understands causal inference
- $X \not\perp Y$ (i.e. X and Y are dependent)
- $X \perp Y | M$
- *typically do not want to adjust for M when estimating total effect of X on Y*

Basic DAG patterns: collider

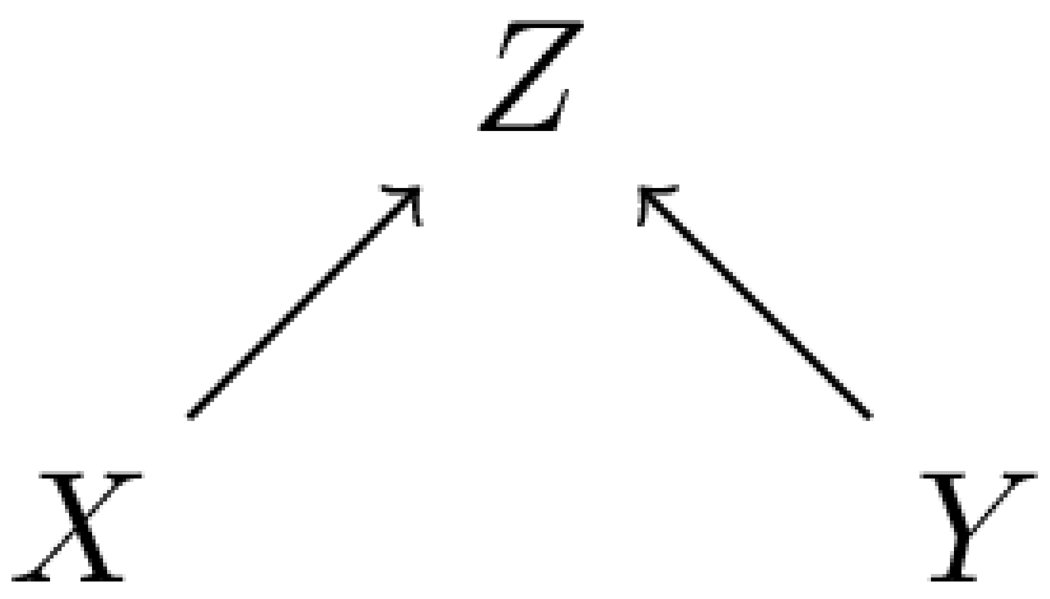
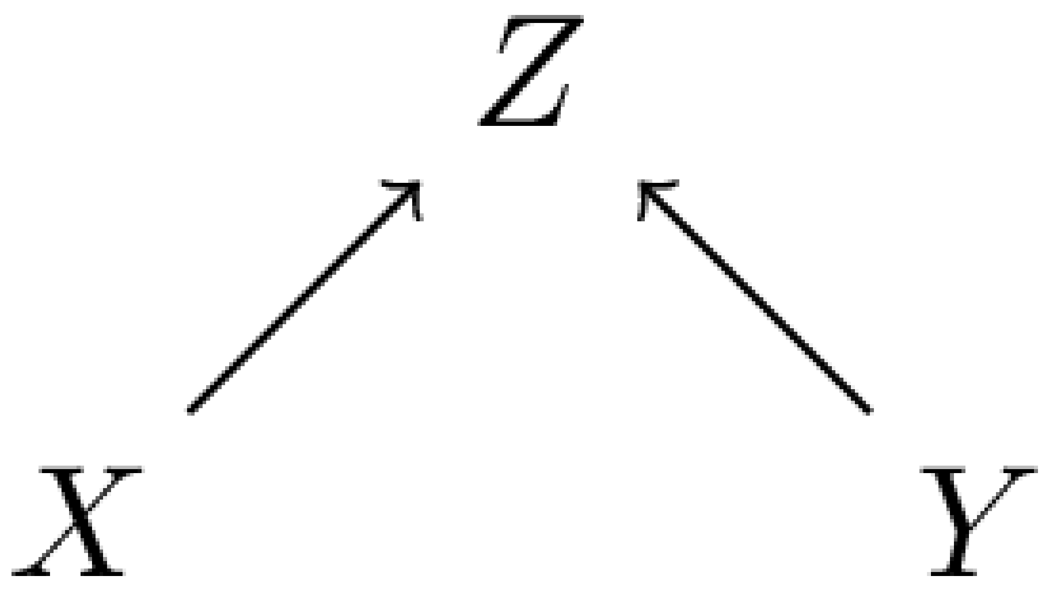


Figure 7: collider

- X and Y *both cause* Z
- $X \perp Y$ (but *NOT* when conditioning on Z)
- *often do not want to condition on* Z *as this induces a correlation between* X *and* Y

Collider bias - Tinder



(a) collider

Figure 8:

$$
\begin{aligned}
\text{intelligent} &\sim U[0,1] \\
\text{attractive} &\sim U[0,1] \\
\text{on tinder} &= I_{\text{intelligent+attractive<1}}
\end{aligned}
$$

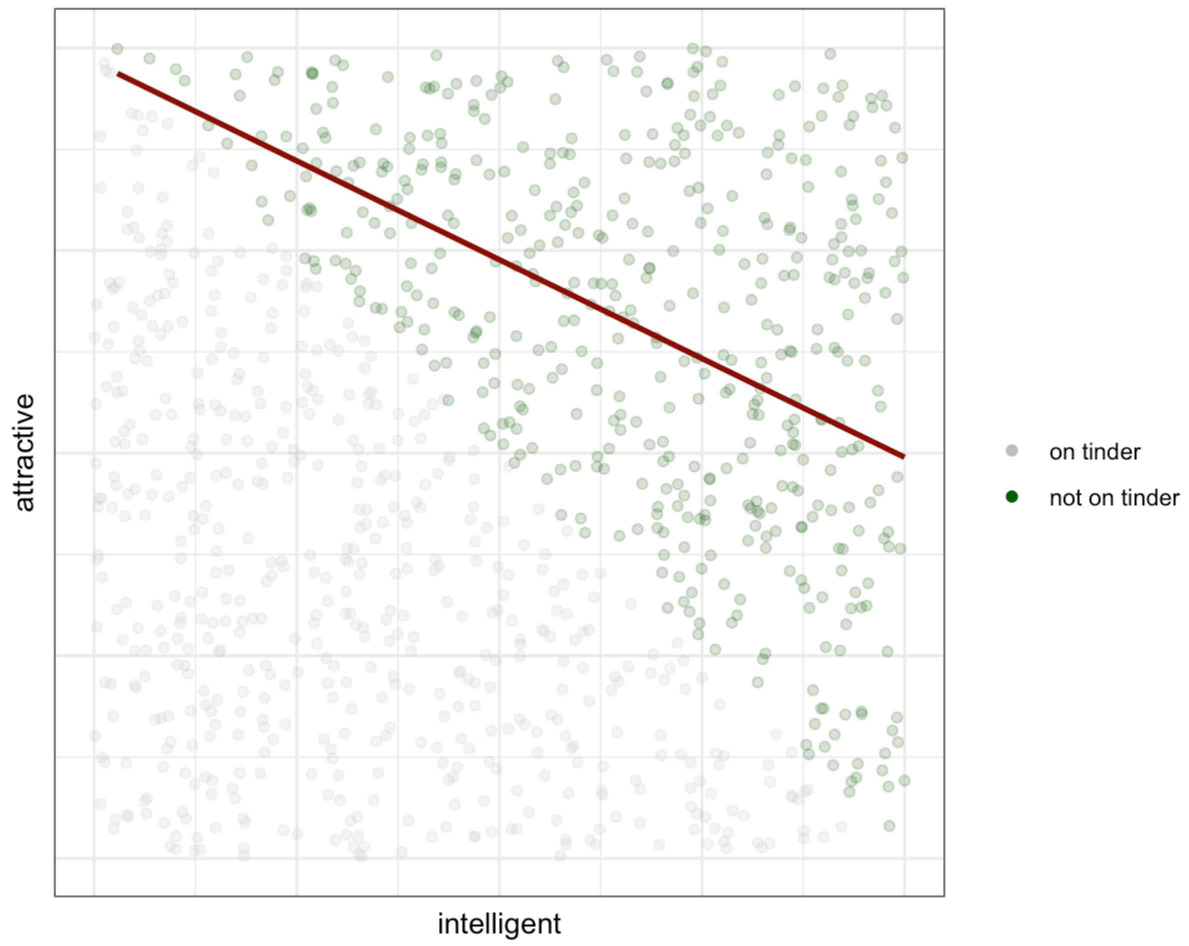


Figure 9

Conditioning on a collider creates dependence of its parents

- may not be too visible: doing an analysis in a selected subgroup is a form of (‘invisible’) conditioning
- e.g. when selecting only patients in the hospital
  - being admitted to the hospital is a collider (has many different causes, e.g. traffic accident or fever)
  - usually only one of these is the reason for hospital admission
  - the causes for hospital admission now seem anti-correlated

DAGs convey two types of assumptions:

causal direction and conditional independence

1. conditional indepence (e.g. exclusion of influence / information)

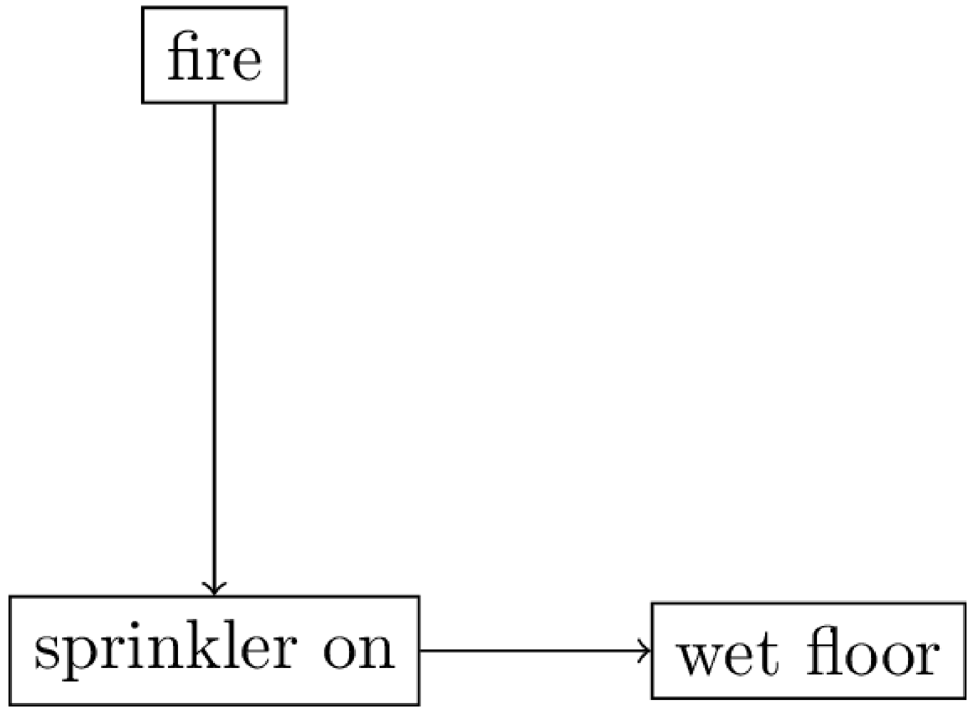


Figure 10: DAG 1

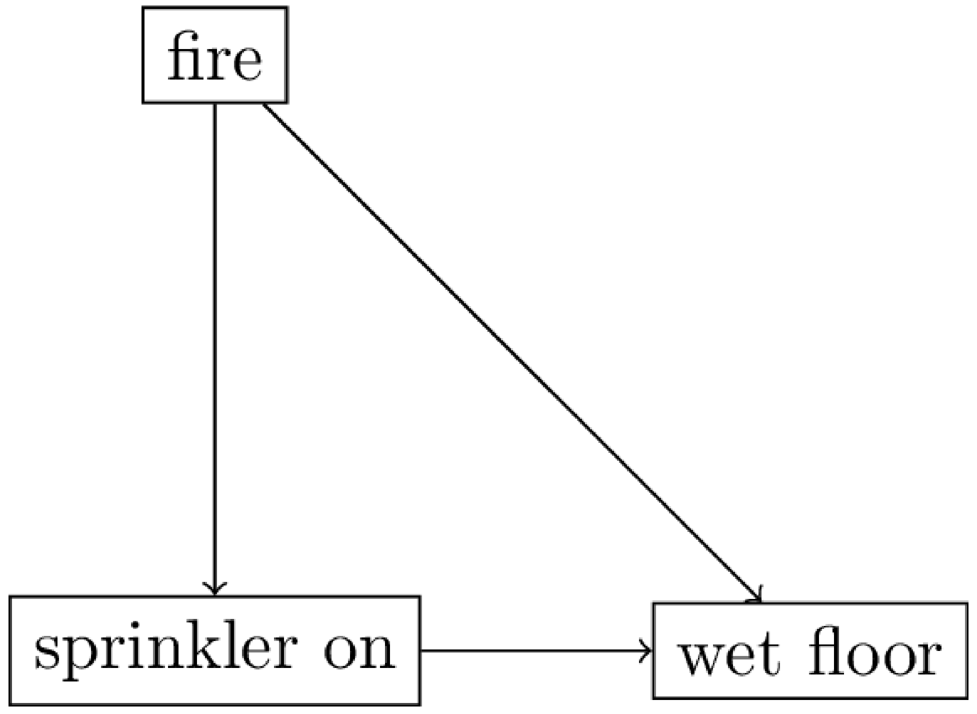


Figure 11: DAG 2

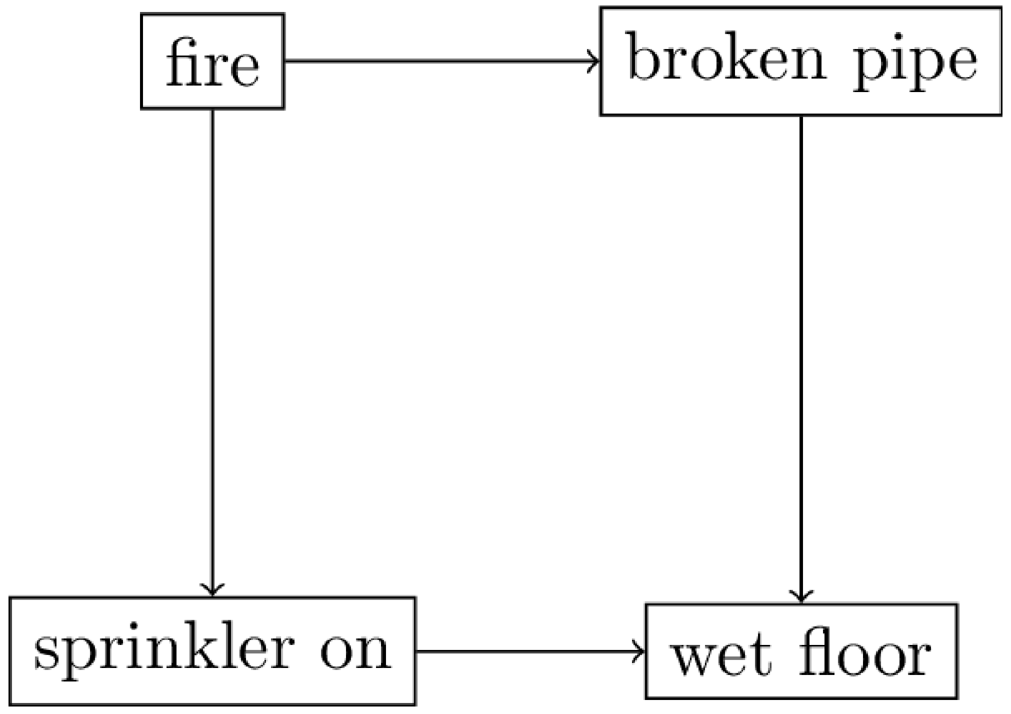


Figure 12: DAG 3

- Figure 10 says `fire` can **only** cause `wet floor` through `sprinkler on`
  - this implies `fire` is independent of `wet floor` given `sprinkler on` and can be tested!
- Figure 11 says *there may be other ways through which* `fire` *causes* `wet floor`
  - Figure 11 is thus a *weaker* assumption than Figure 10
- Figure 12 is also compatible with Figure 11

DAGs are ‘non-parametric’

They relay what variable ‘listens’ to what, but not in what way

$$\begin{tikzcd} & X \arrow[d] \\ T \arrow[r] & Y \end{tikzcd}$$

DAG

- this DAG says Y is a function of X, T and external noise U_Y, or:
- $Y = f_Y(X, T, U_Y)$
- in the next lecture we’ll talk more about these ‘structural equations’

DAGs are ‘non-parametric’

They relay what variable ‘listens’ to what, but not in what way

$$\begin{tikzcd} & X \arrow[d] \\ T \arrow[r] & Y \end{tikzcd}$$

DAG

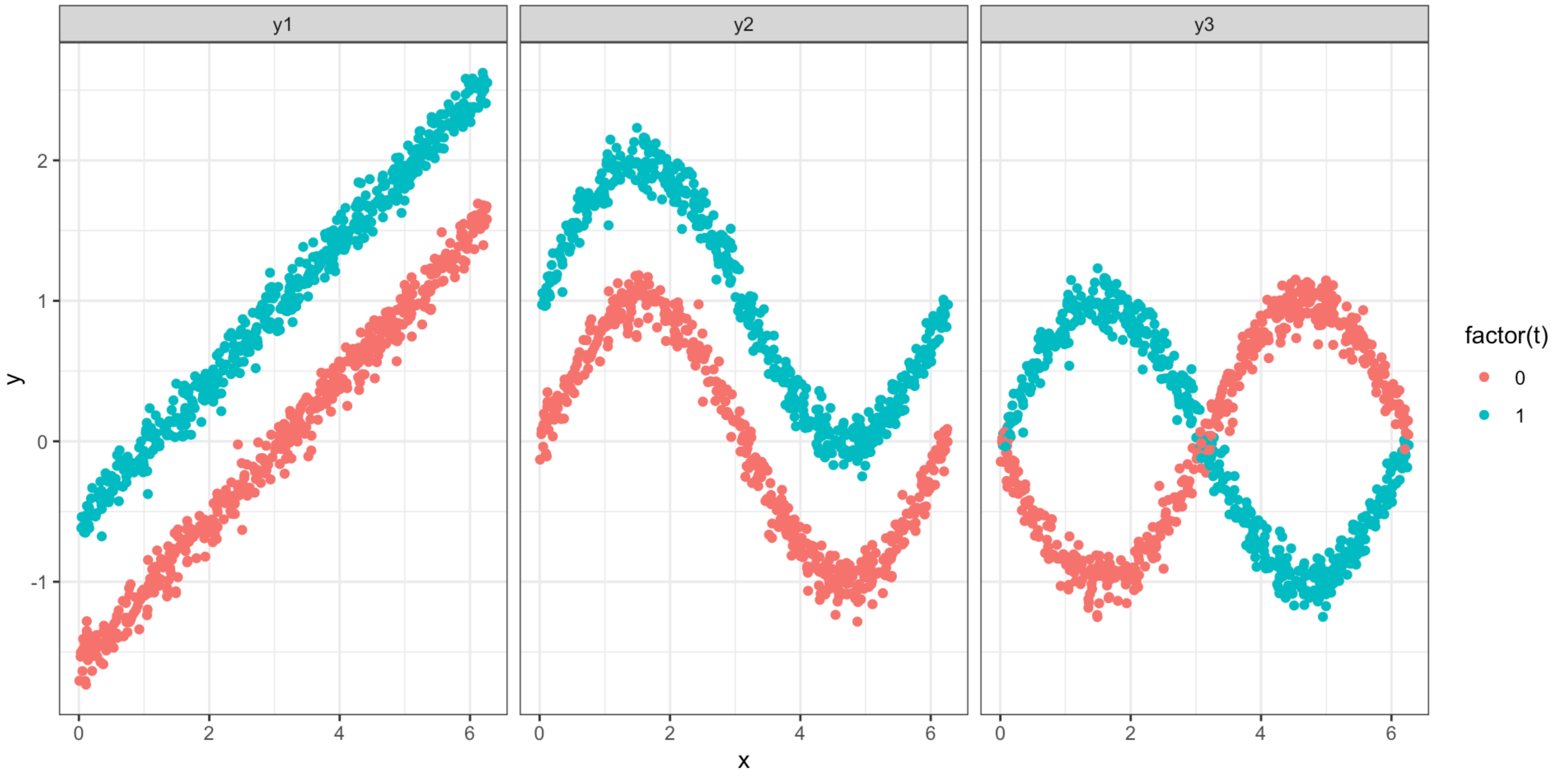


Figure 13: Three datasets with the same DAG

1. $Y = T + 0.5(X - \pi) + \epsilon$ (linear)
2. $Y = T + \sin(X) + \epsilon$ (non-linear additive)
3. $Y = T * \sin(X) - (1 - T)\sin(x) + \epsilon$ (non-linear + interaction)

Mini Quiz

Google Form https://bit.ly/dagquiz

From Directed Acyclic Graphs to causality

The DAG definition of an intervention

assume this is our DAG for a situation and we want to learn the effect T has on Y

- in the graph, intervening on variable T means removing all incoming arrows
- this assumes such a *modular* intervention is possible: i.e. leave everything else unaltered

$$\begin{tikzcd}
W \arrow[d] & Z \arrow[dl] \arrow[d] \\
T \arrow[r] & Y
\end{tikzcd}$$

Figure 14: observational data

$$\begin{tikzcd}
W & Z \arrow[d] \\
T \arrow[r] & Y
\end{tikzcd}$$

Figure 15: intervened DAG

- which means T does not *listen* to other variables anymore, but is set at a particular value, like in an experiment
- imagining this scenario requires a well-defined treatment variable (akin to consistency)

Intervention example: hospital deliveries

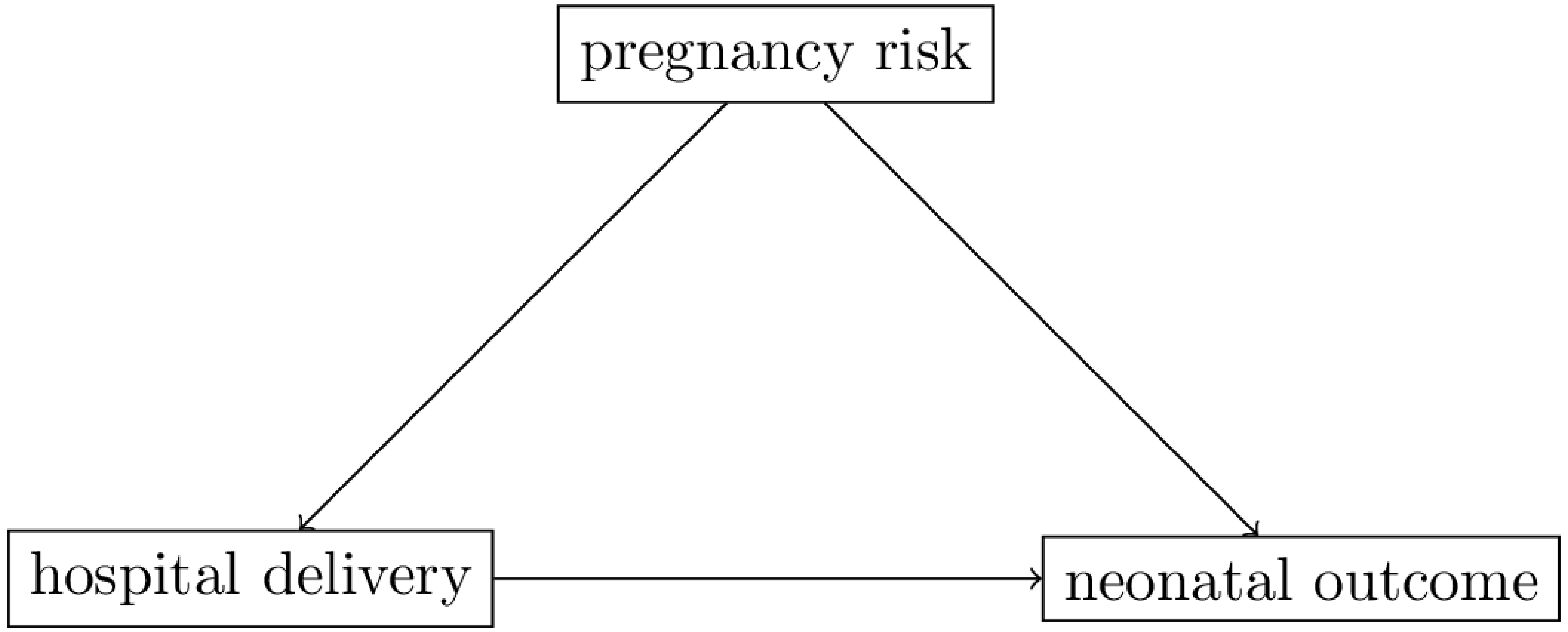


Figure 16: observ(ational /ed) data: hospital delivery depends on pregnancy outcome risk

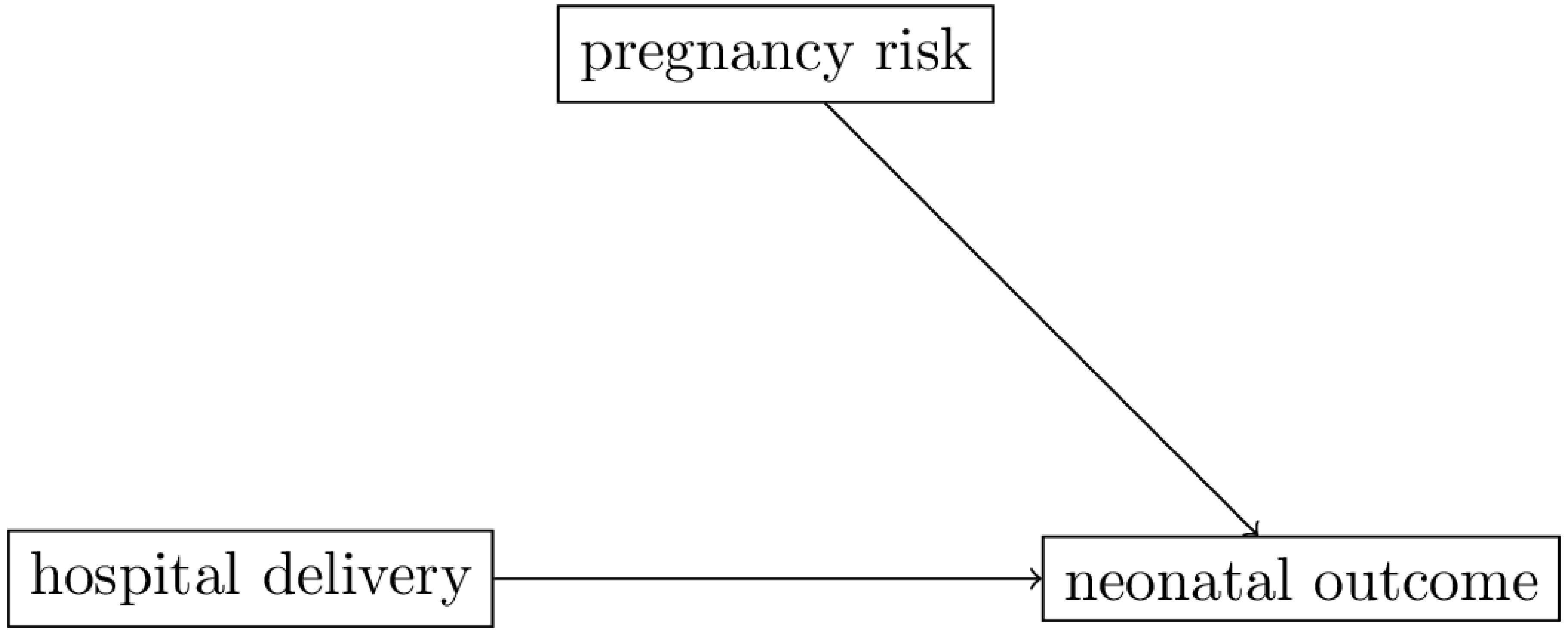


Figure 17: hypothetical situation: send all pregnancies to hospital or home, regardless of risk

- this is called **graph surgery** because we *cut* all the arrows going to the treatment (hospital delivery)

From graph to data

- we now have a *graphical* definition of an intervention, how to map this onto data?

All we need is basic probability applied to the DAG

- product rule: $P(A, B) = P(A|B)P(B)$
- sum rule: $P(A) = \sum_B P(A|B)P(B)$
- total probability: $P(A|C) = \sum_B P(A|B, C)P(B|C)$

See the preporatory math lecuture

DAGs imply a causal factorization of the joint distribution

- assume these variables T: treatment, Y: outcome, Z: 'other' variable
- the product rule allows us to write this joint in many (9) different *factorizations*, $P(Y,T,Z) =$
  - $P(Y|T,Z)P(T,Z)$
  - $P(Z|T,Y)P(T,Y)$
  - $P(Y|T,Z)P(T|Z)P(Z)$
  - $\ldots$
- whereas all of these are correct, knowing the DAG, one of these is *special*: the *causal factorization*

DAGs imply a causal factorization of the joint distribution

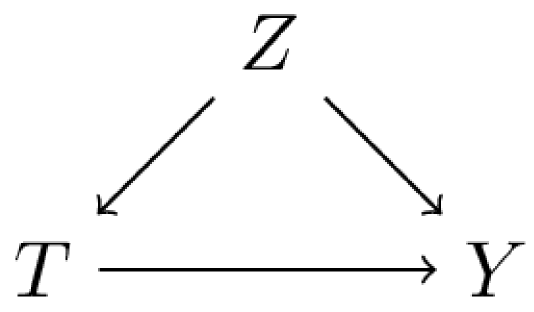


Figure 18: observational data

$$
\begin{aligned}
P(Y,T,Z) &= P(Y|T,Z)P(T,Z) \\
&= P(Y|T,Z)P(T|Z)P(Z)
\end{aligned}
$$

- 2 times the product rule

- If this looks complicated: just follow the arrows, starting with variables with no incoming arrows

Intervention as graph surgery

Why is the causal factorization special?

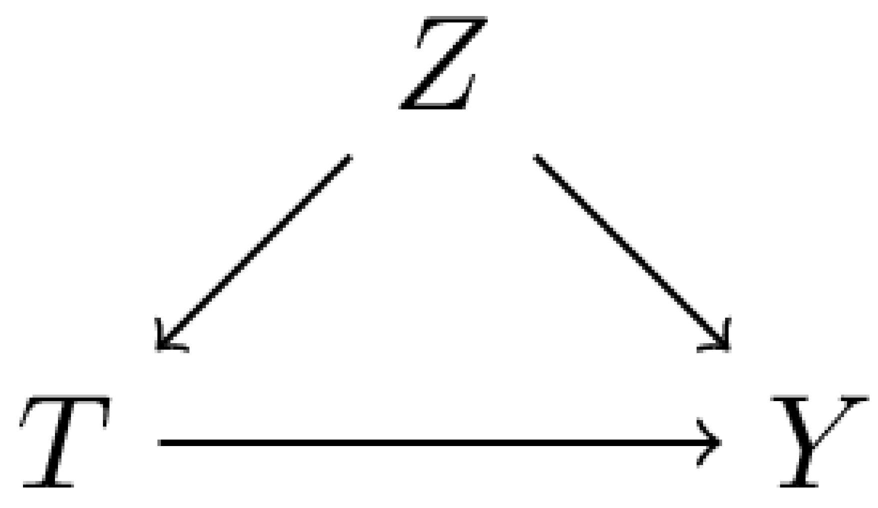


Figure 19: observational data

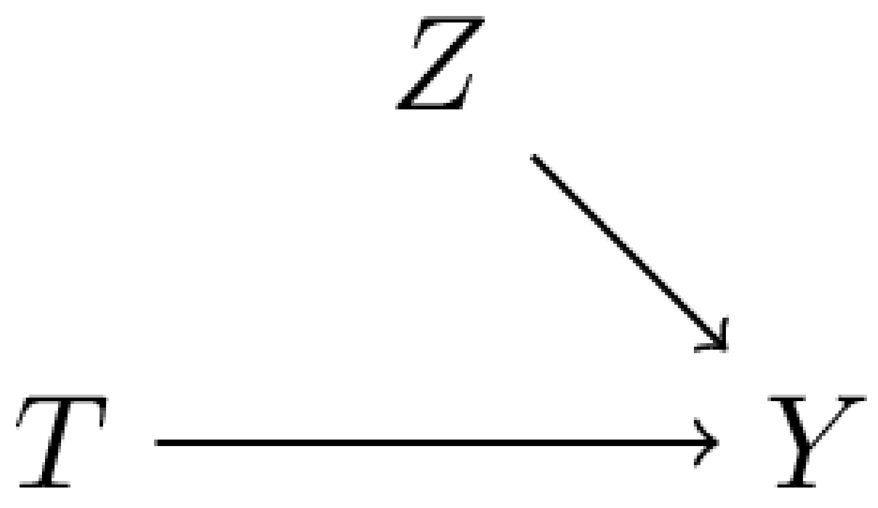


Figure 20: intervened DAG

$$P_{\text{obs}}(Y,T,Z) = P(Y|T,Z){\color{red}P(T|Z)}P(Z)$$

$$P_{\text{int}}(Y,T,Z) = P(Y|T,Z){\color{green}P(T)}P(Z)$$

- in the *causal factorization*, *intervening* on T means changing only one of the conditionals in the factorization, the others remain the same
- this is what is meant with a *modular intervention*

Intervention as graph surgery

Connection with probabilities

- the *conditional distribution* of Y given T is denoted as $P(Y|T)$ (‘*seeing*’)
- the *causal effect* of T on Y is denoted $P(Y|\text{do}(T))$, which is Y given T in the graph where all arrows coming in to T are removed (‘*doing*’)
- we compute this from the *truncated* factorization, which comes from the *causal factorization* by removing $P(T|Z)$:
  - *causal factorization*: $P(Y|T,Z)P(T|Z)P(Z)$
  - truncated factorization: $P(Y|T,Z)P(Z)$

Intervention as graph surgery

Changed distribution

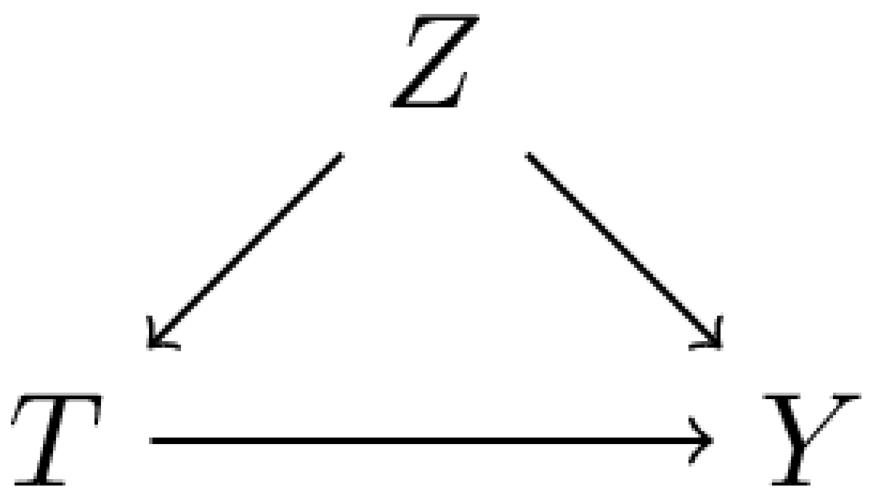


Figure 21: observational data

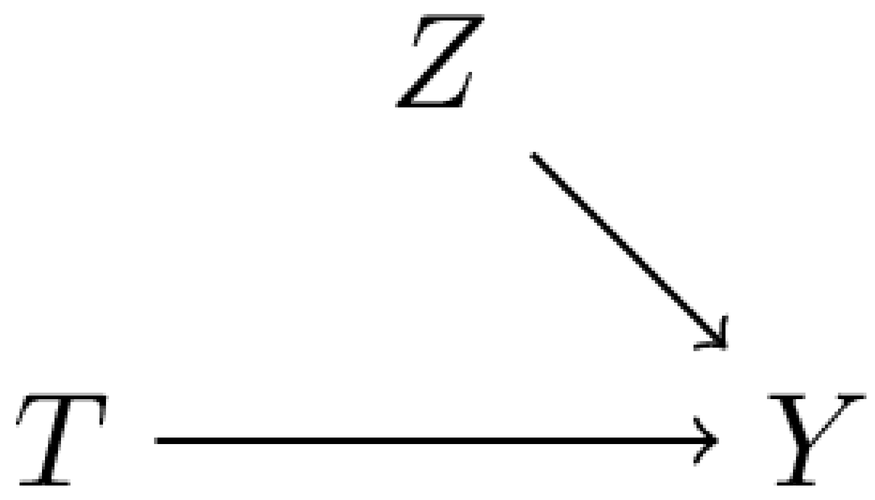


Figure 22: intervened DAG

$$
\begin{aligned}
P_{\text{obs}}(Y,T,Z) &= P(Y|T,Z)P(T|Z)P(Z) \\
P_{\text{obs}}(Y|T) &= \sum_z P(Y|T,Z=z)P(Z=z|T)
\end{aligned}
$$

$$
\begin{aligned}
P_{\text{int}}(Y,T,Z) &= P(Y|T,Z)P(T)P(Z) \\
P_{\text{int}}(Y|T) &= \sum_z P(Y|T,Z=z)P(Z=z|T) \\
&= \sum_z P(Y|T,Z=z)P(Z) \\
&= P(Y|\text{do}(T))
\end{aligned}
$$

Intervention as graph surgery - changed distribution

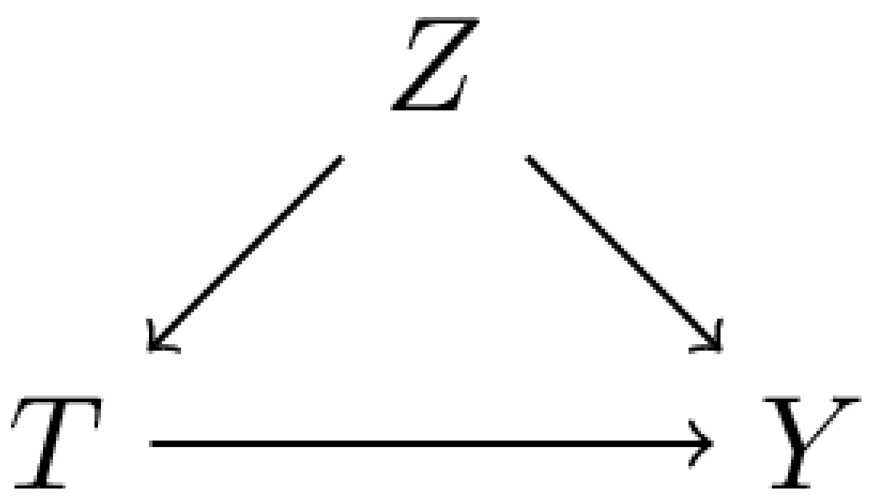


Figure 23: observational data

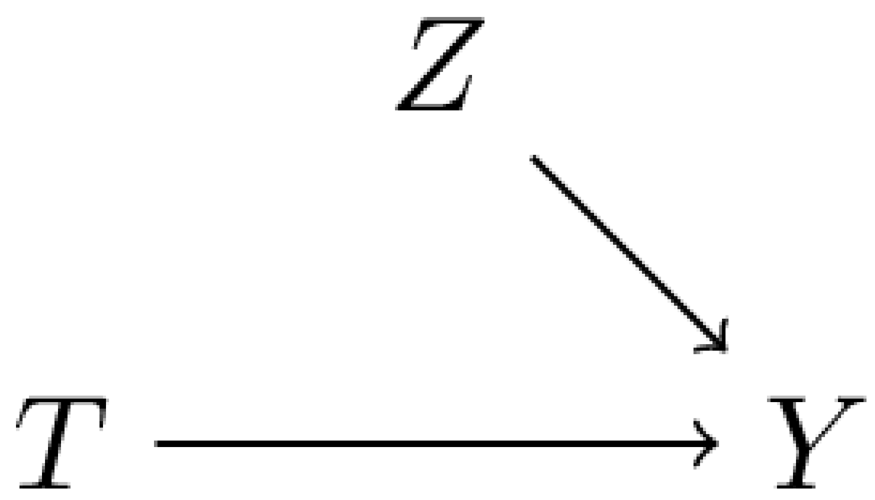


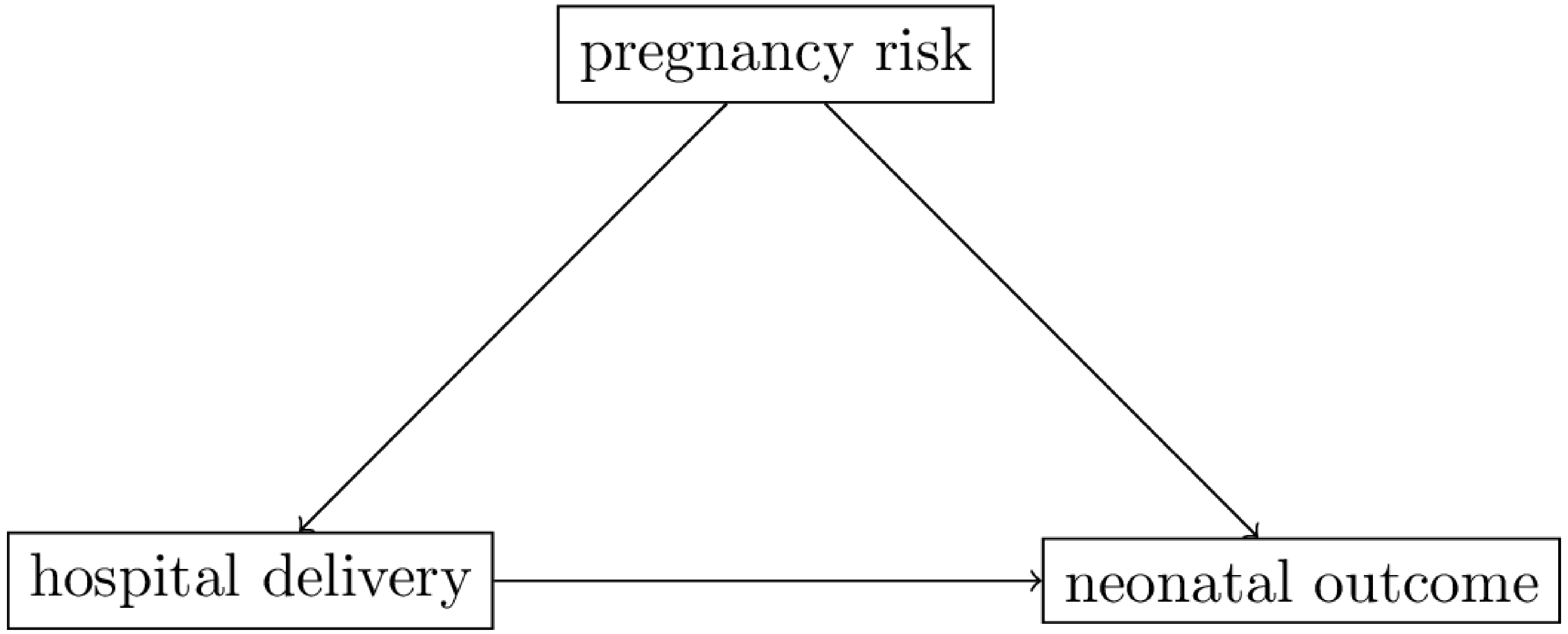
Figure 24: intervened DAG

$$P_{\text{obs}}(Y|T) = \sum_z P(Y|T, Z=z)P(Z=z|T) \qquad P_{\text{int}}(Y|T) = \sum_z P(Y|T, Z=z)P(Z=z) \tag{1}$$

- in P_{obs}, $P(Z|T) \neq P(Z)$
- in P_{int}, $P(Z|T) = P(Z)$
- thereby $P_{\text{obs}}(Y|T) \neq P_{\text{int}}(P(Y|T)) = P(Y|\text{do}(T))$
- **seeing is not doing**
- **looking at Equation 1, we can compute these from P_{obs}!** (this is what is called an *estimand*)

Back to example 1

Seeing



DAG

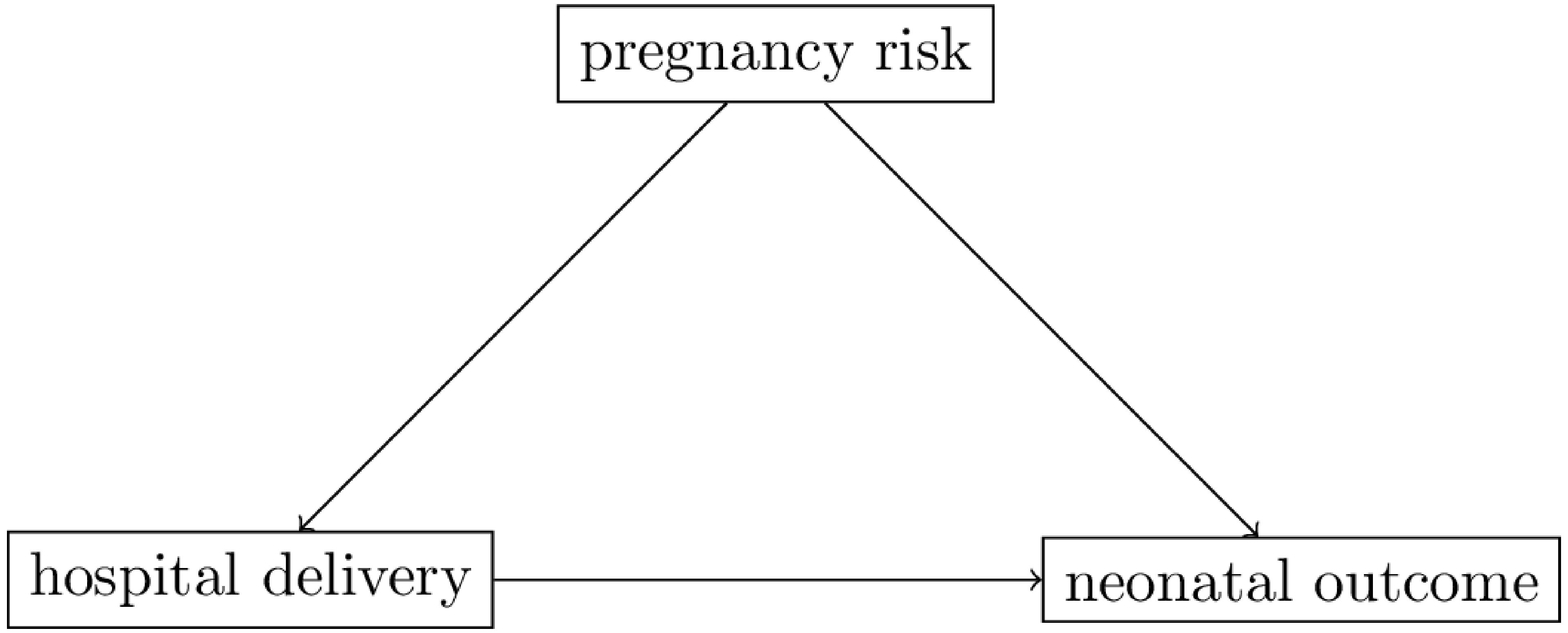
		location	
		home	hospital
risk	low	648 / 720 = 90%	19 / 20 = 95%
	high	40 / 80 = 50%	144 / 180 = 80%
	marginal	688 / 800 = 86%	163 / 200 = 81.5%

- seeing: $P(\text{outcome}|\text{location}) = \sum_{\text{risk}} P(\text{outcome}|\text{location}, \text{risk}) P(\text{risk}|\text{location})$
- $P(\text{risk} = \text{low}|\text{location} = \text{hospital}) = 10\%$
- $P(\text{risk} = \text{low}|\text{location} = \text{home}) = 90\%$

$$P(\text{outcome}|\text{location} = \text{hospital}) = 95 * 0.1 + 80 * 0.9 = 81.5\%$$
$$P(\text{outcome}|\text{location} = \text{home}) = 90 * 0.9 + 50 * 0.1 = 86\%$$

- **conclusion**: deliveries in the hospital had worse neonatal outcomes

Back to example 1



DAG

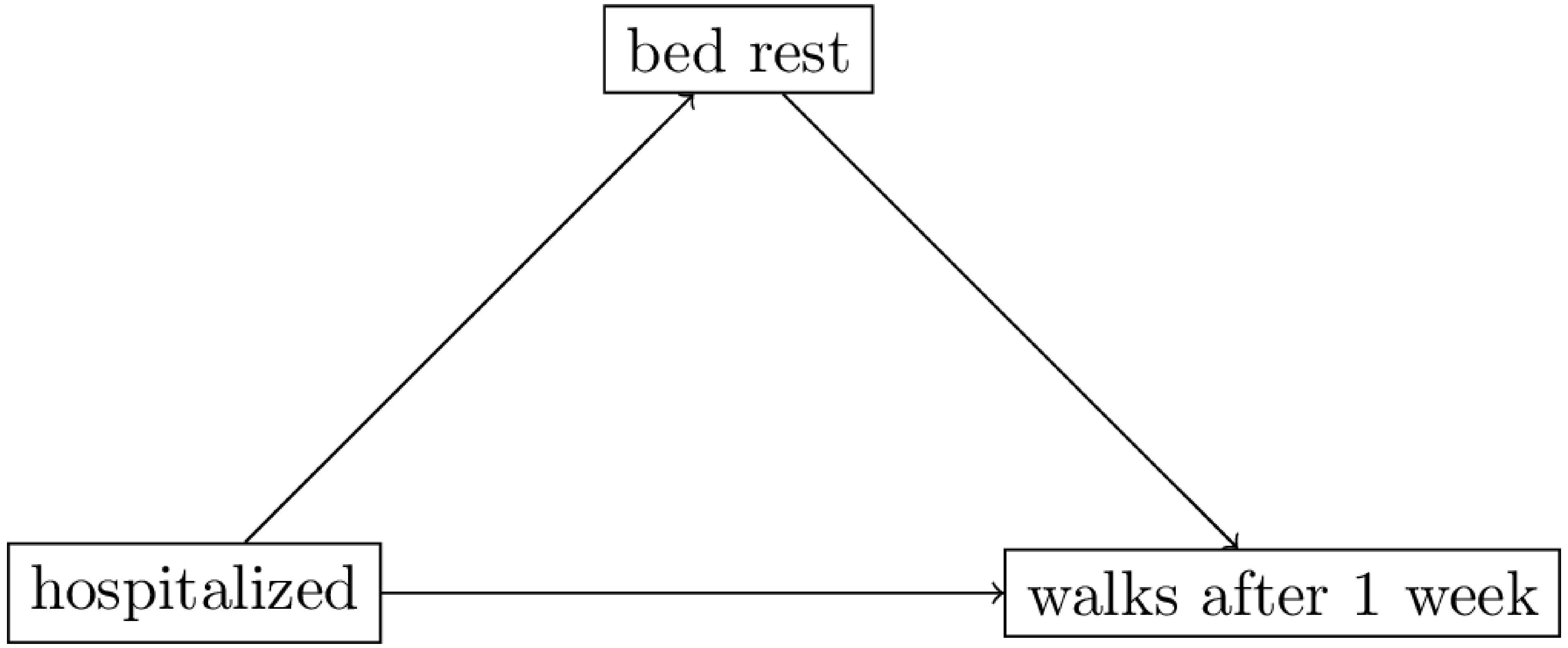
		location	
		home	hospital
risk	low	648 / 720 = 90%	19 / 20 = 95%
	high	40 / 80 = 50%	144 / 180 = 80%
	marginal	688 / 800 = 86%	163 / 200 = 81.5%

- estimand: $P(\text{outcome}|\text{do}(\text{location})) = \sum_{\text{risk}} P(\text{outcome}|\text{location}, \text{risk})P(\text{risk})$
- $P(\text{risk} = \text{low}) = 74\%$

$$
\begin{aligned}
P(\text{outcome}|\text{do}(\text{hospital})) &= 95 * 0.74 + 80 * 0.26 = 91.1\% \\
P(\text{outcome}|\text{do}(\text{home})) &= 90 * 0.74 + 50 * 0.26 = 79.6\%
\end{aligned}
$$

- **conclusion**: sending all deliveries to the hospital leads to better neonatal outcomes

Back to example 2



DAG

- removing all arrows going in to T results in the same DAG
- so $P(Y|T) = P(Y|\text{do}(T))$
- i.e. use the marginals

The gist of observational causal inference

is to take data we have to make inferences about data from a different distribution (i.e. the intervened-on distribution)

$$\begin{tikzcd} W \arrow[d] & Z \arrow[dl] \arrow[d] \\ T \arrow[r] & Y \end{tikzcd}$$

Figure 25: observational data: data we have

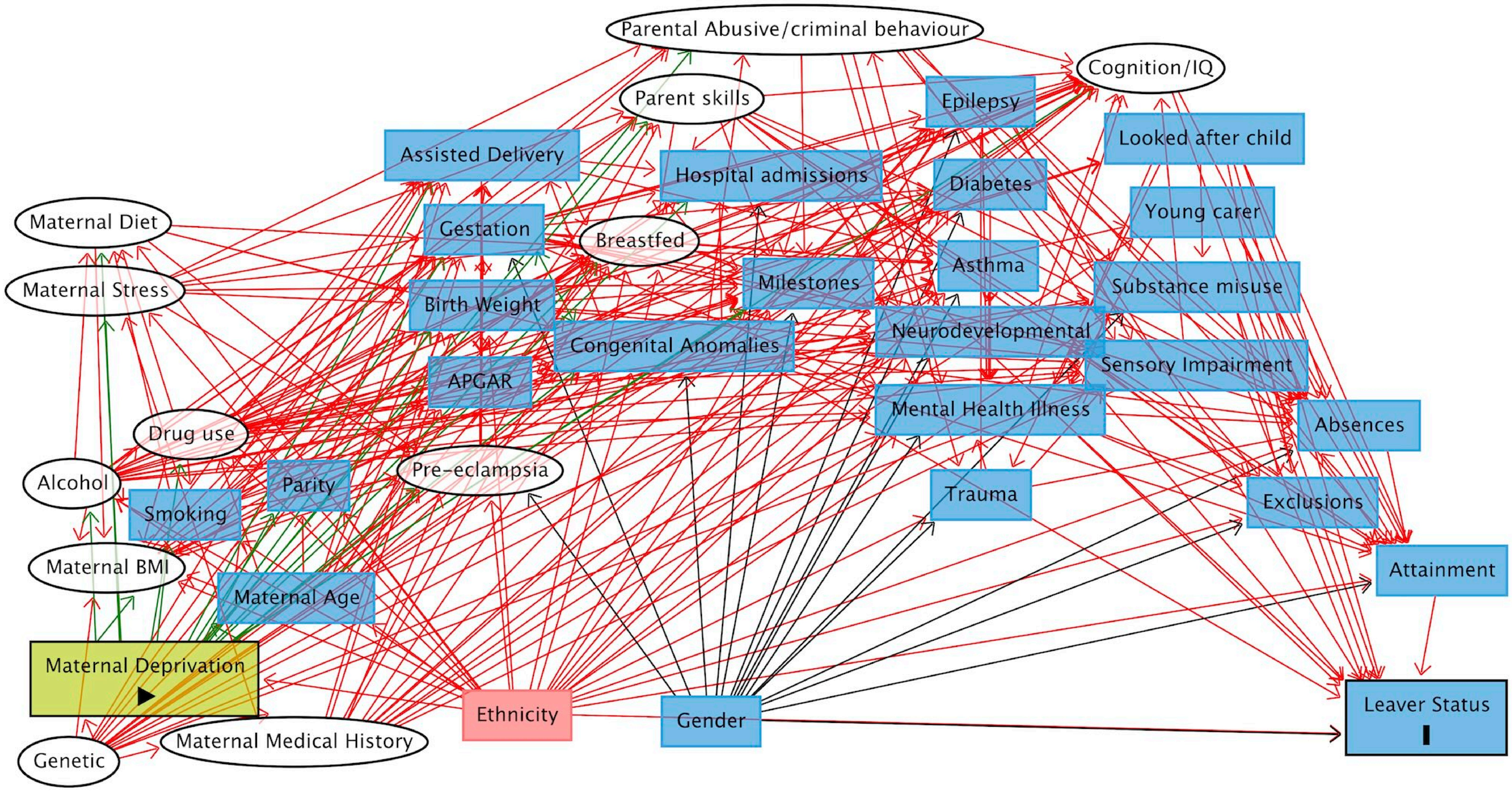
$$\begin{tikzcd} W & Z \arrow[d] \\ T \arrow[r] & Y \end{tikzcd}$$

Figure 26: intervened DAG: what we want to know

- causal inference frameworks provide a language to express assumptions
- based on these assumptions, the framework tells us whether such an inference is possible
  - this is often referred to as *is the effect identified*
- and provide formula(s) for how to do so based on the observed data distribution (*estimand(s)*)
- (one could say this is essentially assumption-based extrapolation, some researchers think this entire enterprise is anti-scientific)
- not yet said: *how* to do statistical inference to estimate the estimand (much can still go wrong here)
  - can also be part of identification, see the following lecture on SCMs

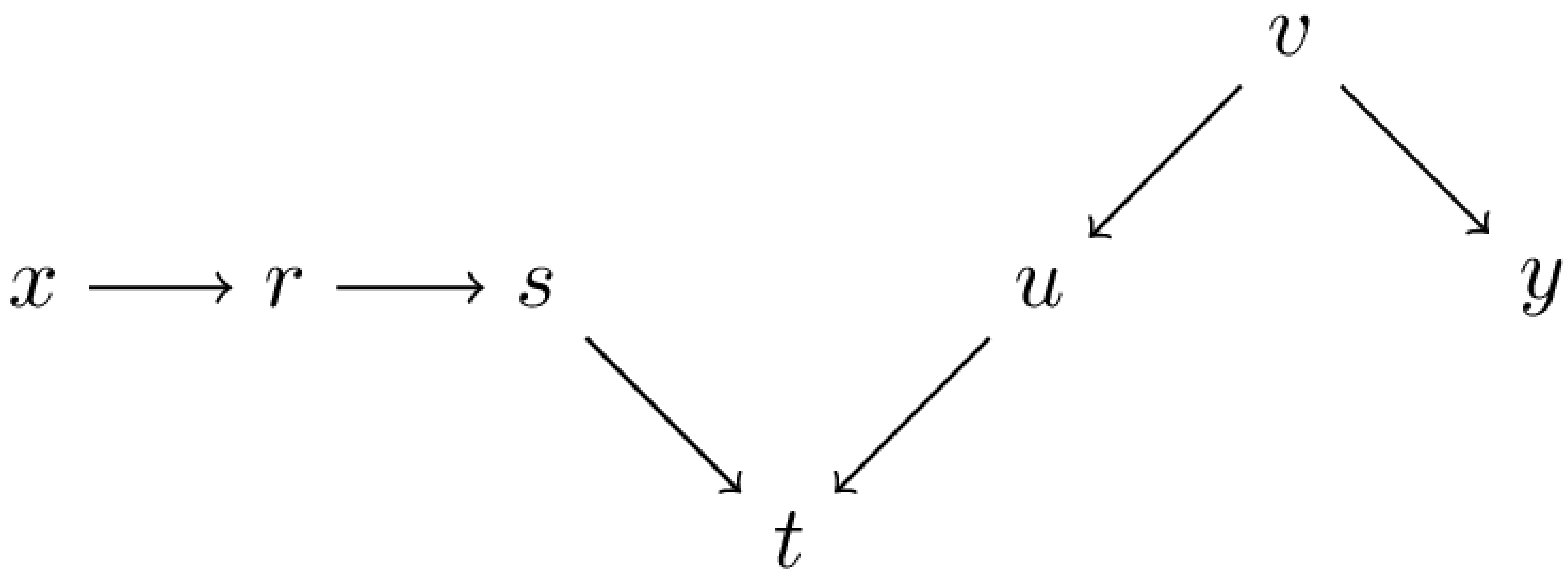
Beyond toy examples: d-separation and back-door criterion

When life gets complicated / real



Bogie, James; Fleming, Michael; Cullen, Breda; Mackay, Daniel; Pell, Jill P. (2021). Full directed acyclic graph.. PLOS ONE. Figure. https://doi.org/10.1371/journal.pone.0249258.s003

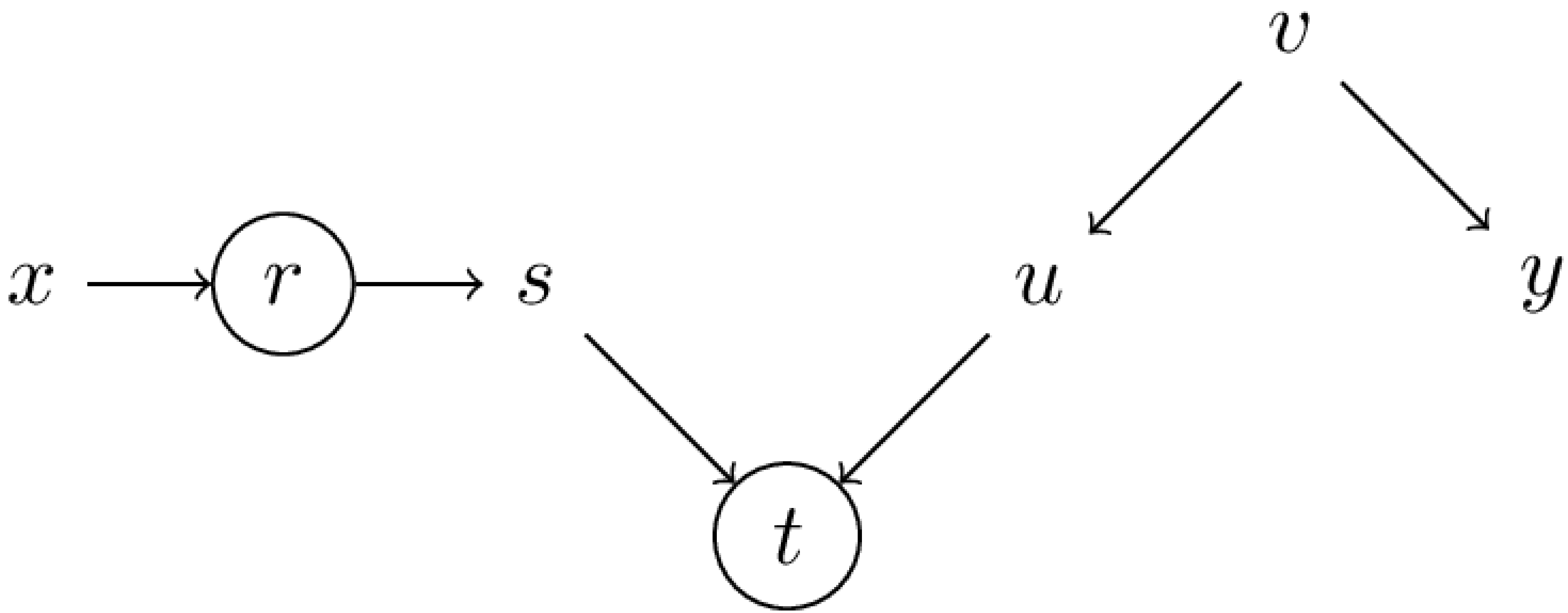
d-separation (directional-separation)



paths

- a *path* is a set of nodes connected by edges ($x \dots y$)
- a *directed-path* is a path with a constant direction ($x \dots t$)
- an *unblocked-path* is a path without a collider ($t \dots y$)
- a *blocked-path* is a path with a collider (s, t, u)
- *d(irectional)-separation* of x, y means there is no unblocked path between them

d-separation when conditioning



paths with conditioning variables r, t

- conditioning on variable:
  - when variable is a collider: *opens a path* (t opens s, t, u etc.)
  - otherwise: *blocks a path* (e.g. r blocks x, r, s)
- conditioning *set* $Z = \{r, t\}$: set of conditioning variables

The back-door criterion and adjustment

Definition 3.3.1 (Back-Door) (for pairs of variables)

A set of variables Z satisfies the *back-door* criterion relative to an ordered pair of variables (X, Y) in a DAG if:

1. no node in Z is a descendant of X *(e.g. mediators)*
2. Z blocks every path between X and Y that contains an arrow into X

Theorem 3.2.2 (Back-Door Adjustment)

If a set of variables Z satisfies the back-door criterion relative to (X, Y), then the causal effect of X on Y is identifiable and is given by the formula

$$P(y|\mathrm{do}(x)) = \sum_z P(y|x, z)P(z) \qquad (2)$$

Did we see this equation before?

- Yes! When computing the effect of hospital deliveries on neonatal outcomes Equation 1
- DAGs tell us what to adjust for
- automatic algorithms tell use whether an estimand exists and what it is
- several point-and-click websites for making DAGs that implement these algorithms:
  - dagitty.net
  - causalfusion.net

How about positivity

- backdoor adjustment with z requires computing $P(y|x,z)$
- by the product rule:

$$P(y|x,z) = \frac{P(y,x,z)}{P(x,z)}$$

- this division is only defined when $P(x,z) > 0$
- which is the same as the positivity assumption from Day 1 in Potential Outcomes

References

Pearl, Judea. 2009. *Causality*. Cambridge University Press.