Universiteit Utrecht

Opleiding Natuur- en Sterrenkunde

The Entropy and Temperature of Black Hole and Cosmological Event Horizons

Bachelor Thesis

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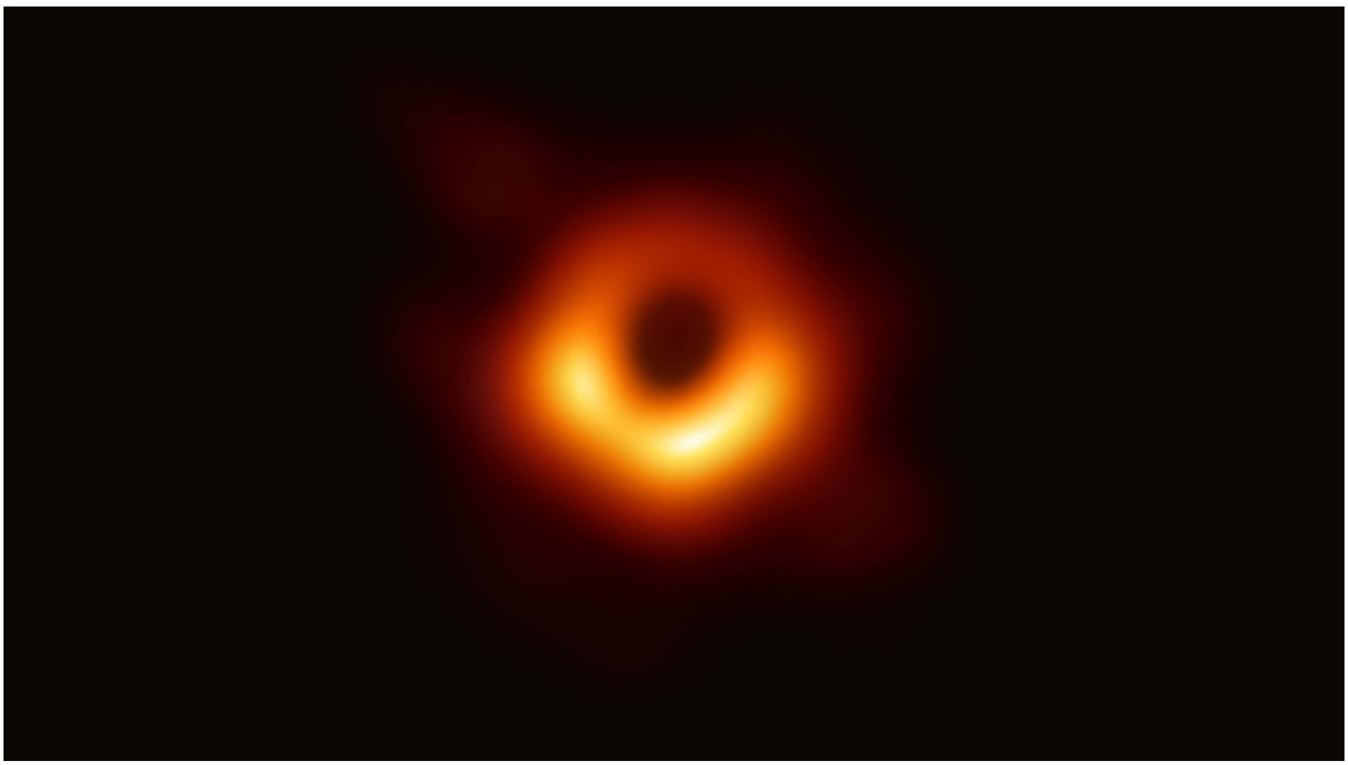

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January 15, 2020

Abstract

In this thesis the same Hawking temperature and Hawking-Bekenstein entropy of the black hole and cosmological event horizon are derived as was done by Hawking in the 1970s. Mass functions for the Schwarzschild, de Sitter and Schwarzschild-de Sitter spacetime are derived. Using the similarity between the derived laws of black hole mechanics with the laws of thermodynamics a temperature and entropy can be associated with both event horizons through a Wick rotation. For both event horizons the temperature is proportional to $\frac{1}{2\pi}$ the surface gravity and the entropy therefore is $\frac{1}{4}$ the area of the event horizon.

On the front page the first ever picture of a black hole taken by the Event Horizon Telescope is displayed[1], the supermassive black hole is located in the center of the galaxy M87. Its mass is estimated to be 6.5 billion times the Sun's mass and it is the strongest evidence today of the existence of black holes.

Contents

1 Introduction

When black holes were first discovered in the early 20th century, it was hard to believe that they could be real. They were mathematically infinitely dense infinitesimally small points that curve spacetime so much that not even light can escape. If a ball falls in a black hole, to an outside observer it appears as if time is slowing down asymptotically for the ball: it never enters the black hole and freezes on the event horizon as it redshifts towards what the human eye cannot see. The ball however, simply experiences spacetime as if it was flat and enters the black hole without ever noticing when it can never go back. From the moment the ball enters the event horizon, its fate is determined as it will inevitably hurdle down towards the center of the black hole known as the singularity. In the reference frame of the observer the ball (which carries information) is forever lost in the black hole. This violates keystone principles of quantum mechanics and causes big problems, this is known as the information paradox[2].

Half a century later black holes became again a very hot topic. In 1973 Bekenstein[3] explained intuitively that there has to be a connection between the area of the event horizon A and the information lost in the black hole S. Hawking[4] showed in 1975 that this information indeed could be stored on the event horizon of the black hole. He did this by showing that a black hole radiates black-body radiation with a specific temperature known as the Hawking temperature T_H, this was done using particle creation in quantum field theory. In this thesis the same temperature is derived using a technique called Wick rotation. In comparison with the laws of thermodynamics this then leads us to the proportionality constant between entropy and black hole area.

Next to the discovery of black hole event horizons, cosmological event horizons may be even more mysterious. These horizons appear in spacetimes with a positive cosmological constant Λ, i.e. a vacuum energy density. A spacetime with a positive cosmological constant is known as a de Sitter spacetime. The cosmological constant acts in such a way that it curves the fabric of spacetime with constant curvature. This means that de Sitter spacetime is locally flat, but as you look further away spacetime will be curved more and more. This curvature stretches spacetime in such a way that it will take light coming from further destinations a longer time to reach you. Eventually, this curvature becomes so large that not even light can escape from its grasp. This happens at a two-sphere at constant radius known as the cosmological event horizon. Einstein showed us of course that the curvature of spacetime is perceived by us as gravitational acceleration. This means that as light, or other particles, approach us from distant places in an asymptotically de Sitter Universe, spacetime is accelerating away from us and therefore slowing the light down. This acceleration is also known as the expansion of the Universe. This acceleration of further objects was also experimentally observed by Hubble closely after Einstein published his theory of gravity, meaning that we most likely live in an asymptotically de Sitter Universe.

The information paradox can be extended to also include cosmological event horizons by considering a ball being thrown into it. Similar to the black hole horizon, the observer whose cosmological event horizon we are considering will think time is slowing down for the ball as it approaches the horizon. The ball however experiences spacetime locally so doesn't notice a thing when it passes the horizon, therefore information again seems to be lost. Seeing the similarity between the black hole and cosmological event horizon we might be wondering if similar techniques can also be applied to both. In 1977 Hawking published a paper[5] discussing the thermodynamics of the cosmological event horizons using the same approach as used for black hole horizons. We will derive the same entropy and temperature using similar techniques used for the black hole horizon.

In the following section we will give a short introduction to the mathematics of general relativity. Next we will consider the Schwarzschild black hole in flat spacetime in detail. In the fourth section the cosmological event horizon will be considered and in the final section we will consider both horizons in one spacetime.

2 The Einstein Equations

To study black holes it is essential that we understand the mathematics to describe curved spacetime. The goal of this section is to give a brief understanding of the mathematics of general relativity. This will be done by informally deriving all components of the Einstein field equations:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab}. \tag{1}$$

We will be denoting the metric tensor by g_{ab} with signature +2. First we consider the Einstein field equations without a cosmological constant Λ, i.e. flat spacetime. The first two terms on the left hand side arise from the Riemann curvature tensor:

$$R^c{}_{dab} = \partial_a \Gamma^c_{bd} - \partial_b \Gamma^c_{ad} + \Gamma^c_{ae}\Gamma^e_{bd} - \Gamma^c_{be}\Gamma^e_{ad}, \tag{2}$$

where

$$\Gamma^b_{ac} = \frac{1}{2}g^{bd}(\partial_a g_{cd} + \partial_c g_{da} - \partial_d g_{ac}), \tag{3}$$

are the Christoffel symbols. Clearly there is a lot of new notation that needs to be explained. In the following we will closely follow multiple sections of Sean Carroll's "*Spacetime and Geometry, An Introduction to General Relativity*"[6].
A convenient way to understanding the Christoffel symbols is through the introduction of the covariant derivative operator ∇. The covariant derivative is the curved spacetime generalization of the partial derivative. The reason we need a new operator for taking derivatives in curved spacetime is that the partial derivative is coordinate dependent. As we will see, we will be working in different coordinate systems hence we need an operator that takes derivates independent of the coordinates used. In the following we derive an expression for the covariant derivative in an intuitive way. We will see that equations (2) and (3) will simply follow from the conditions we impose.

We require the covariant derivative to obey (i) the Leibniz rule and (ii) linearity, these are the most key algebraic identities need for differentiation. Mathematically this means that the covariant derivative can always be written as a partial derivative plus some linear transformation to make the result covariant. This linear transformation will be a set of n matrices $(\Gamma_a)^b{}_c$, one matrix for every coordinate a. These matrices are known as the connection coefficients and we will drop the parentheses from now on. We therefore have for the covariant derivative of a vector

$$\nabla_a V^b = \partial_a V^b + \Gamma^b_{ac} V^c. \tag{4}$$

Similarly, the covariant derivative of a covector $\nabla_a W_b = \partial_a W_b + \tilde{\Gamma}^c_{ab} W_c$. One can confirm that the connection coefficients and the partial derivative do not transform as tensors, but the covariant derivative as constructed above does.
Next we impose the conditions that the covariant derivative (iii) commutes with contractions and (iv) reduces to the partial derivative when acting on scalars. By calculating $\nabla_a(W_c V^c)$ and imposing the conditions just mentioned we conclude that $\Gamma^b_{ac} = -\tilde{\Gamma}^b_{ac}$. Therefore, the covariant derivative of a covector becomes

$$\nabla_a W_b = \partial_a W_b - \Gamma^c_{ab} W_c. \tag{5}$$

Lastly, we require (v) the connection to be torsion free, that is $\Gamma^b_{[ac]} = 0$ and we require (vi) the covariant derivative to be metric compatible. This gives us our unique expression for the connection coefficients in equation (3), known as the Christoffel symbols. Note that this expression is uniquely defined by the conditions we imposed. Different conditions lead to different connections, but in this paper we will be only using the Christoffel symbols.

To put this result to use, we need a quantity that measures the curvature of spacetime. In flat space, a vector that is parallel transported[1] along a closed loop will remain unchanged. In curved space however, this is not true in general as can be seen in Figure 1. One can imagine that how much the vector has changed after being parallel transported gives a measure of how curved spacetime is. Note that the commutator of two covariant derivatives computes the difference between parallel transporting a vector twice, but in different orders . When acting on a vector, we can compute this commutator: $[\nabla_a, \nabla_b]V^c = R^c{}_{dab}V^d - 2\Gamma^d_{[ab]}\nabla_d V^c$. Here the Riemann tensor is defined as in equation (2). Recall that we are using the Christoffel symbols, therefore the connection is torsion free. We can contract the Riemann tensor to form the Ricci tensor $R_{ab} = R^c{}_{acb}$, and the trace of the Ricci tensor is the Ricci scalar $R = R^c{}_c$.

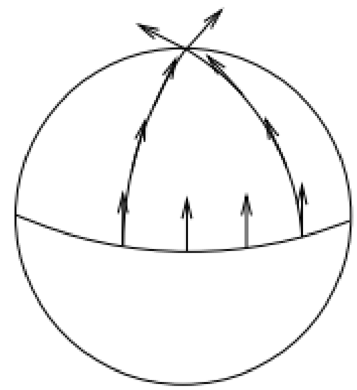

Figure 1: *Parallel transporting a vector on the surface of a sphere.*

The Riemann tensor is uniquely defined by the connection used.
We are using the Christoffel symbols and these are uniquely defined by the metric tensor. Therefore for our purposes the Riemann tensor is uniquely defined by the metric tensor. The first two terms in the Einstein field equations are the Ricci tensor and the Ricci scalar, which are simply different contractions of the Riemann tensor and therefore describe the unique curvature of spacetime.

The T_{ab} on the right hand side of the Einstein equation is the energy-momentum tensor. It contains information about non-gravitational force fields, i.e. the distribution of matter and radiation. If we now look at the Einstein field equations it is still hard to notice what is going on. Written down explicitly the equations are $g^{cd}(\delta^e_a\delta^f_b - \frac{1}{2}g^{ef}g_{ab})(\partial_c\partial_{[f}g_{d]e} + \partial_e\partial_{[d}g_{f]c}) = 8\pi T_{ab}$ from which we see that it is a set of second-order partial differential equations of the metric tensor. Because the metric tensor is symmetric, there are 10 independent components of the metric tensor. Therefore the Einstein field equations are a set of 10 second-order partial differential equations of the metric tensor; matter tells spacetime curve, and curvature tells matter to accelerate. We assume throughout this paper that $T_{ab} = 0$. Thankfully for us, we do not need to solve this differential equation because we will be using well known solutions.

If we now add the cosmological constant Λ into the mix we essentially are adding a constant energy density throughout the Universe. This can be most easily seen by adding $-\Lambda g_{ab}$ to both sides of equation (1), we can then interpret the new term as some gravitational energy-momentum tensor given by $T^G_{ab} = -\frac{\Lambda}{8\pi}g_{ab}$. Using the properties of the energy-momentum tensor we can then define a constant energy-density throughout the Universe: $\rho = \frac{\Lambda}{8\pi}$.

[1]Parallel transporting a vector is the concept of keeping a vector constant while moving it along a path.

3 The Schwarzschild Black Hole Event Horizon

We would like to understand the thermodynamic behaviour of the black hole event horizon. It is conceptually easier to understand the event horizon of a black hole in flat space so we will begin with only the black hole event horizon. The Schwarzschild metric describes a nonrotating neutral black hole in flat spacetime and it was discovered by Karl Schwarzschild in 1916. It is the most general spherically symmetric vacuum solution of the Einstein field equations with vanishing cosmological constant. The metric line element is given by[7]

$$ds^2 = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2, \tag{6}$$

here M is the mass parameter of the Schwarzschild black hole, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ the two-sphere line element and (t, r, θ, ϕ) denote the spherical coordinate system. We can identify singularities in the metric by identifying components of the metric that blow up. In these coordinates there are two singularities; a true singularity at the origin, and a coordinate singularity at $r = 2M$ known as the black hole event horizon.

Radial null geodesics (i.e. light rays) satisfy $\frac{dr}{dt} = \pm(1-\frac{2M}{r})$, which is solved by $t = \pm(r + 2M\log|\frac{2M}{r} - 1|) = \pm r^*$, where r^* is the tortoise coordinate. Note that $+$ denotes outgoing and $-$ denotes ingoing geodesics. We can clearly see that t tends to infinity as r tends to $2M$, i.e. as seen by an outside observer it takes an infalling particle infinite coordinate time to reach the event horizon.

The singularity at $r = 2M$ is only a coordinate singularity because it can be removed by choosing different coordinates. The singularity at the origin cannot be removed and is therefore a true singularity. The problem with our current coordinates is that the time is measured on the observers clock. He can never see a particle pass the horizon and therefore his time coordinate goes to infinity. The particle however, can surely pass the event horizon but this means that time has gone beyond infinity causing for a contradiction. We can define a new time coordinate by considering Eddington-Finkelstein coordinates. In these coordinates, the time coordinate is measured by incoming or outgoing radial null geodesics, i.e. light rays. We will consider the incoming Eddington-Finkelstein coordinates defined as $v = t + r^*$. In these coordinates the metric becomes:

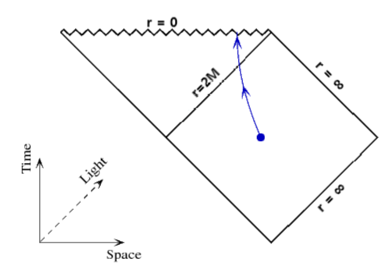

Figure 2: *Penrose diagram for Schwarzschild spacetime.*

$$ds^2 = -(1-\frac{2M}{r})dv^2 + 2dvdr + r^2 d\Omega^2. \tag{7}$$

Clearly there is no longer a singularity at $r = 2M$ because no components of the metric blow up there. In Figure 2 the Penrose diagram of this spacetime is displayed. A Penrose diagram gives a two-dimensional display of the causal structure of the entire spacetime. We can clearly see that the event horizon is the boundary between timelike geodesics inevitably ending in the singularity or spatial infinity.

The event horizon is a two-sphere located at $r = 2M$ and it has some interesting properties. Similarly to the discussion in the previous section for the cosmological event horizon, we could consider the black hole event horizon. As we might guess there is also a gravitational acceleration associated with this event horizon with respect to some observer. Using the close connection of Killing horizons and event horizons we can calculate this acceleration as will be done below. A Killing horizon is a null hypersurface whose normal vectors are also Killing vectors. A null hypersurface is a surface whose normal vectors are also null, meaning that they have vanishing norm. Killing vectors are vectors that satisfy Killing's equation:

$$\nabla^{(a}\xi^{b)} = 0. \tag{8}$$

Consider a hypersurface at constant radius $S = S(x)$. It has normal vector $n = f\partial^a S \partial_a = fg^{ar}\partial_a$. Here $f = f(x)$ is a scalar function for normalization purposes. S is a null hypersurface when its normal vector at every point is a null vector, i.e. when $n_a n^a = f^2 g^{rr}$ equals zero. Clearly this becomes zero when $r = 2M$, therefore the event horizon of a Schwarzschild black hole is a null hypersurface.
Next we would like to check whether the event horizon is also a Killing horizon, i.e. if $\xi \propto n$ at the horizon for some Killing vector ξ. Note that from before we have $n = fg^{ar}\partial_a = f((1 - \frac{2M}{r})\partial_r + \partial_v)$ which when evaluated at the horizon becomes $n = f\partial_v$.
The Schwarzschild metric has a timelike Killing vector, namely $K = \gamma_t \partial_t = \gamma_t \partial_v$. Note that the normalization of this vector depends on γ_t. Clearly $K \propto n$ at the horizon and therefore the event horizon of a Schwarzschild black hole is also a Killing horizon.
To define the surface gravity, we use a result often used in literature. Namely that we can choose the function f such that $n^a \nabla_a n^b = 0$ at the horizon[8]. Because the horizon is also a Killing horizon, we can use $n \propto K$ to write this as

$$K^a \nabla_a K^b = \kappa_H K^b, \tag{9}$$

evaluated at the event horizon. Here κ_H is the surface gravity of the black hole event horizon and K a timelike Killing vector. The value of κ depends on the normalization of K and therefore we require $\lim_{r\to\infty} K_a K^a = -1$ for asymptotically flat spacetimes, i.e. $\gamma_t = 1$. Note that we used incoming Eddington-Finkelstein coordinates throughout this derivation. It is important to use coordinates that are well-defined on the event horizon when evaluating equation (9) otherwise one finds trivial results. To see why, we need to know what the surface gravity in asymptotically flat spacetime means.

Consider for the following static observers, i.e. the four-velocity U^a is proportional to the timelike Killing vector K^a with proportionality function $V(x)$. The four-velocity is normalized such that $U_a U^a = -1$ and therefore the proportionality function is $V = \sqrt{-K_a K^a}$, it ranges from zero at the horizon to one at infinity. Imagine two static observers, observer 1 is closer to the horizon than observer 2. Observer 1 emits a photon with conserved energy $E = -p_a K^a$ and observer 2 measures its frequency as $\omega = -p_a U^a = EV^{-1}$. Similarly, observer 1 emits a photon that will be observed by observer 2 with wavelength $\lambda_2 = \frac{V_2}{V_1}\lambda_1$. When observer 2 is at infinity (i.e. $V_2 = 1$) the observed wavelength is $\lambda_2 = \lambda_1 / V_1$. Now it is clear that we can interpret V as a redshift factor.
However, a static observer near a black hole will typically not be moving on a geodesic; it needs to apply an acceleration to remain static. The four-acceleration $a^a = U^b \nabla_b U^a$ can be expressed in terms of the redshift factor: $a_a = \nabla_a \log V$ with normalization $a = \sqrt{a_a a^a} =$

$V^{-1}\sqrt{\nabla_a V \nabla^a V}$. This tends to infinity at the horizon, i.e. it takes an infinite acceleration to stay on a static trajectory at the horizon. However, an observer at infinity will observe this acceleration to be redshifted by a factor V. This redshifted acceleration as seen by a static observer at infinity is known as the surface gravity; $\kappa_H = Va = \sqrt{\nabla_a V \nabla^a V}$.

So the surface gravity in asymptotically flat spacetimes is the acceleration needed to remain static at the event horizon as measured by an observer at spatial infinity. In the original Schwarzschild coordinates (t, r, θ, ϕ) the acceleration at the event horizon as seen by an outside observer is zero because for an outside observer everything that enters the black hole freezes before it reaches the event horizon. Therefore the observer will measure zero acceleration at the event horizon. Coordinates that are non-singular at the horizon such as Eddington-Finkelstein coordinates resolve this problem in an obvious way. Note that in Eddington-Finkelstein coordinates the only nonzero component of K^a is the v component, we then have for the left hand side in equation (9); $K^v \nabla_v K^v = K^v \Gamma^v_{vc} K^c = \Gamma^v_{vv} = -\frac{1}{2} g^{vr} \partial_r g_{vv} = \frac{1}{4M} K^v$. Therefore we conclude that the surface gravity for the Schwarzschild black hole is

$$\kappa_H = \frac{1}{4M}. \tag{10}$$

To see why the surface gravity is interesting for the thermodynamics of black holes we need to calculate the mass function for a Schwarzschild black hole. This function relates characteristic quantities of the spacetime with the mass parameter of the black hole.

In curved spacetime the notion of mass is not well defined, but in spacetimes with a global timelike Killing vector we can construct a conserved[2] energy current $R^a_b K^b$ that can be integrated over the spacelike hypersurface S, this is known as the Komar integral associated with the timelike Killing vector K:

$$E_S = \frac{1}{4\pi} \int_S R^a_b K^b dS_a. \tag{11}$$

Here, $\frac{1}{4\pi}$ is for normalization and $dS_a = t_a \sqrt{\gamma^{(3)}} dS$ where t_a is a timelike unit normal vector to S and $\gamma^{(3)}$ the determinant of the induced metric tensor on S. By using the identity[3] $\nabla_b \nabla^b \xi^a = -R^a_b \xi^b$, Stokes' theorem and Killing's equation we can write this expression as a surface integral:

$$E_S = -\frac{1}{4\pi} \int_{\partial S} \nabla^b K^a dA_{ab}. \tag{12}$$

Here ∂S is a closed surface that encloses S and $dA_{ab} = t_a s_b \sqrt{\gamma^{(2)}} dA$ where t_a and s_b unit normal vectors to ∂S and $\gamma^{(2)}$ the determinant of the induced metric tensor on ∂S. Note that if we have S be the entire spacetime, then ∂S becomes a closed surface at spatial infinity. E_R then represents the total energy contained in spacetime. Because a Schwarzschild black hole is static and spherically symmetric S is spacelike, therefore the normal vectors must have normalization $t_a t^a = -1$ and $s_b s^b = +1$. This normalization defines the normal vectors as $t^a = ((1 - \frac{2M}{r})^{-1/2}, 0, 0, 0)$ and $n^b = (0, (1 - \frac{2M}{r})^{1/2}, 0, 0)$. By noting that $\sqrt{\gamma^{(2)}} = r^2 \sin\theta$ one can easily verify that equation (12) simplifies to $E_S = M$ where M is the same mass

[2] See Appendix A.1.

[3] See Appendix A.2 for a proof.

parameter as in the Schwarzschild line element (equation (6)). Thus, the energy contained in the entire Schwarzschild spacetime is the same as the mass of the black hole, which is exactly what we would expect since $T_{ab} = 0$.
Alternatively, we could consider the spacelike region S_∞ outside of the event horizon. Now there are two boundaries, one at spatial infinity ∂S_∞ and one at the event horizon ∂H. Clearly the integral over the surface at spatial infinity gives M as shown above, but the other term requires a little more attention. It can be interpreted as the negative of the energy contained inside the event horizon. Negative because the normal vectors of ∂H will point towards spatial infinity and not the center of the black hole.
As shown before, the event horizon of the Schwarzschild black hole at $r = 2M$ is a Killing horizon and the Killing vector that is normal to this horizon is $K = \partial_t$. Therefore, we can choose the normal vectors to ∂H such that $t_a = K_a$ and $t_a s^a = -1$[9]. By equation (12) we then have for the integral over ∂B: $-\frac{1}{4\pi}\int_{\partial H} K_a \nabla^b K^a s_b \sqrt{\gamma^2} dA = -\frac{1}{4\pi}\kappa_H A_H$. Here $A_H = \int_{\partial H} \sqrt{\gamma^2} dA = 16\pi M^2$ is the area of the black hole event horizon. For the second equality we used the definition of the surface gravity (equation (9)) and that the surface gravity is constant across the event horizon, known as the zeroth law of black hole mechanics[9]. Clearly this simplifies to $-M$. The energy contained in S_∞ is the sum of the energy trapped between the surfaces ∂S_∞ and ∂H. Therefore, the mass contained in S_∞ is trivially zero, which is exactly what we expect because we are considering vacuum solutions.
The result that is most useful to us is that the energy contained inside H (the spacelike region enclosed by the event horizon) can be written as

$$E_H = M = \frac{1}{4\pi}\kappa_H A_H. \tag{13}$$

For reasons that will become clear later, we would like to calculate the differential of this equation. By computing the variations of the Komar integrals explicitly, variation of M is $\delta M = (8\pi)^{-1}\kappa_H \delta A_H$[9]. However, the same result can be achieved in a different way. Note that $A_H = \pi\kappa_H^{-2}$, and therefore $d\kappa_H = -(2\pi)^{-1}\kappa_H^3 dA_H$. Then the differential of equation (13) becomes:

$$dM = \frac{1}{8\pi}\kappa_H dA_H, \tag{14}$$

We have differentials instead of variations because we first evaluated the integrals and then calculated the differential, whilst when calculating the variation we do it the other way around. This equation is known as the first law of black hole mechanics. Note the similarity with the first law of thermodynamics $dE = TdS$, the work terms are zero because we are considering a static neutral black hole.
Hawking showed that a classical black hole's area can never decrease[10], i.e.

$$\frac{dA}{dt} \geq 0. \tag{15}$$

This is known as the second law of black hole mechanics. There is a similar law to this equation; the second law of thermodynamics. It says that the total entropy of a system never decreases. The zeroth law of thermodynanamics can be formulated such that it says that the temperature is constant across the surface of a body in thermal equilibrium. This is similar to the zeroth law of black hole dynamics[9], which states that the surface gravity is constant across the event horizon of a black hole. The similarity between the laws of black hole mechanics and the laws of thermodynamics imply temperature to be proportional to surface gravity and entropy to be proportional to the are of the event horizon. This would resolve the the information paradox in some sense; information is never lost, simply stored on the event horizon forever. Bekenstein discussed this relation between the area and entropy of a black hole in detail in [3].

Hawking showed[4] that black holes radiate black-body radiation with the Hawking temperature through particle creation. We will derive the same equation through a Wick rotation. A Wick rotation connects the cyclic imaginary time-coordinate with the inverse temperature of a thermodynamic system. This is done through a connection between statistical mechanics and quantum mechanics. To see where this relation comes from we can consider the average value of an observable A of a large number of harmonic oscillators with temperature T:

$$\langle A \rangle = \sum_n A_n e^{-\frac{E_n}{T}}, \tag{16}$$

here A_n is the observables value in the n^{th} state and E_n the energy of the n^{th} state.

Consider now a single quantum harmonic oscillator $|\phi\rangle = \sum_n |n\rangle$ in a superposition of uniform basis states $|n\rangle$. The probability amplitude that $|\phi\rangle$ evolves to an arbitrary state $|A\rangle = \sum_n A_n |n\rangle$ is

$$\langle A| e^{-iHt} |\phi\rangle = \sum_n A_n e^{-iE_n t}. \tag{17}$$

Comparing these equations shows the relation between imaginary time and inverse temperature.

We will now perform a Wick rotation on equation (6). Consider the transformation $\tau = -it$; the metric then becomes $ds^2 = +(1 - \frac{2M}{r})d\tau^2 + (1 - \frac{2M}{r})dr^2$. For the sake of convenience we will neglect the two-sphere part of the metric. A Wick rotation gives a way to relate cyclic imaginary time to the temperature of a thermodynamic system. Hence we make our imaginary time cyclic on the interval $\tau \in [0, T^{-1}]/\sim$, the tilde denotes that the interval is periodic. The surface gravity is defined on the event horizon, so we will have to do the same for the temperature. Define the new radial coordinate $\chi = r - 2M$ and expand to lowest order around the event horizon. The metric up to first order is then $ds^2 \approx \chi(2M)^{-1}d\tau^2 + 2M\chi^{-1}d\chi^2$. To make the metric more familiar we define a new coordinate $\rho^2 = 8M\chi$, the metric then becomes $ds^2 \approx \rho^2 \frac{d\tau^2}{16M^2} + d\rho^2$. This is simply a canonical metric and to avoid conic singularities we must require $\frac{\tau}{4M} \in [0, 2\pi]/\sim$. Comparing with our previous compactification gives us Hawking's famous result

$$T_H = \frac{\kappa_H}{2\pi} = \frac{1}{8\pi M}. \tag{18}$$

That is, a Schwarzschild black hole radiates black-body radiation with temperature $T_H = (8\pi M)^{-1}$. Comparing the equation(14) and the first law of thermodynamics. We conclude that the entropy of a Schwarzschild black hole is proportional to one fourth of the black hole event horizon area:

$$S_H = \frac{A_H}{4} = 4\pi M^2. \tag{19}$$

We will discuss this result in the final section after we have also derived a similar relation for the cosmological event horizon.

4 The Cosmological Event Horizon

As we saw in the last chapter, it takes quite some calculations before we get to the entropy and temperature of a black hole. In this section, we will go through similar calculations but for a different spacetime. The metric we are considering is the de Sitter metric:

$$ds^2 = -(1 - \frac{r^2}{l^2})dt^2 + (1 - \frac{r^2}{l^2})^{-1}dr^2 + r^2 d\Omega^2. \tag{20}$$

where $l = \sqrt{\frac{3}{\Lambda}}$. Even though there is no black hole in de Sitter spacetime, there is a coordinate singularity at $r = l$, called the cosmological event horizon. Radial null geodesics satisfy $\frac{dr}{dt} = \pm(1 - \frac{r^2}{l^2})$, which is solved by $t = \pm\frac{l}{2}\log\frac{|r+l|}{|r-l|} = \pm r^*$. In these coordinates, outgoing null geodesics as seen by an observer inside the cosmological horizon will take infinite coordinate time to reach the cosmological horizon.

To see that there are no true singularities in de Sitter we have to define new coordinates. By going to incoming Eddington-Finkelstein coordinates $v = t + r^*$ we can write the metric as

$$ds^2 = -(1 - \frac{r^2}{l^2})dv^2 + 2dvdr + r^2 d\Omega^2. \tag{21}$$

Clearly there no longer are any singularities. As before, we will first verify that the cosmological horizon is a Killing horizon and then compute the surface gravity associated with it. The Penrose diagram for this spacetime is shown in Figure 3. We can again clearly see that the event horizon separates two regions of spacetime.

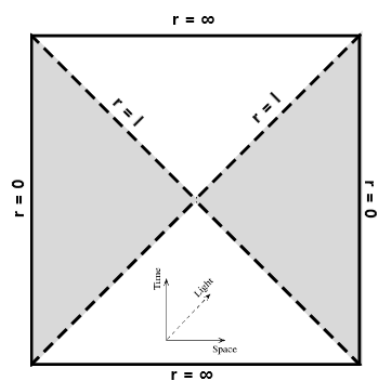

Figure 3: Penrose diagram for de Sitter spacetime.

Consider a two-sphere of constant radius $S(x) = r$. A vector normal to $S(x)$ is $n = f\partial^a S\partial_a = fg^{ar}\partial_a$. The two-sphere is a null hypersurface when its normal vectors have vanishing norm, i.e. when $n_a n^a = f^2 g^{rr}$ equals zero. Clearly this happens at the cosmological horizon $r = l$ and therefore the cosmological horizon of de Sitter spacetime is a null hypersurface.

To show that the cosmological horizon is also a Killing horizon we need to show that $\xi \propto n$ at the horizon for some Killing vector ξ. Note that from before we have $n = f((1 - \frac{r^2}{l^2})\partial_r - \partial_u)$ which when evaluated at the horizon becomes $n = -f\partial_u$. De Sitter has no globally defined timelike Killing vector; the Killing vector $K = \gamma_t\partial_t = \gamma_t\partial_u$ is timelike in the region enclosed by the horizon, null at the horizon and spacelike beyond the horizon. As will be clear when calculating the mass of de Sitter, we will only be looking at the spacetime region enclosed by the cosmological horizon where K is always timelike. Clearly $K \propto n$ at the horizon and therefore the cosmological event horizon of a de Sitter universe is also a Killing horizon.

We now do the same as in the previous section, we choose the normalization function f such that $n^a\nabla_a n^b = 0$ at the horizon. Because the horizon is also a Killing horizon we can use $n \propto K$ to write this as

$$K^a\nabla_a K^b = \kappa_C K^b, \tag{22}$$

evaluated at the cosmological horizon. Here κ_C is the surface gravity of the cosmological event horizon and K a timelike Killing vector. Clearly the value of κ_C depends on the normalization of K, therefore we require $\lim_{r\to 0} K_a K^a = -1$ for de Sitter, i.e. $\gamma_t = 1$.

In Eddington-Finkelstein coordinates, the only nonzero component of the timelike Killing vector is K^v, then the left hand side in equation 22 is $K^v \nabla_v K^v = \Gamma^v_{vv} = -\frac{1}{2} g^{vr} \partial_r g_{vv} = -\frac{1}{l} K^v$. Therefore the surface gravity for the cosmological horizon of de Sitter spacetime is $\kappa_C = -\frac{1}{l}$. Similarly to the discussion for Schwarzschild, the surface gravity can be interpreted as the acceleration needed to remain static at the cosmological event horizon as measured by an observer at the origin. Because we are using incoming Eddington-Finkelstein coordinates, our time coordinate v is the time experienced by an incoming null geodesic. The gravitational acceleration of spacetime due to the cosmological constant is therefore negative with respect to incoming null geodesics, so we will also get a negative result for the surface gravity.
If we would have done the same calculation with outgoing Eddington-Finkelstein coordinates ($u = t - r^*$) we would have gotten the same result but positive. To keep things consistent with the black hole case we will keep considering ingoing Eddington-Finkelstein coordinates, however it is important to keep in mind where this sign difference comes from. This means that the surface gravity of the cosmological event horizon in a de Sitter spacetime is

$$\kappa_C = -\frac{1}{l} \tag{23}$$

As said before, there is no globally defined timelike Killing vector in de Sitter, the Killing vector $K = \partial_t$ is only timelike in the region enclosed by the cosmological horizon. This means that we cannot take the Komar integral of the current $R^a_b K^b$ over all of spacetime because that would cause for divergence issues for regions outside of the cosmological horizon. We therefore need another way to define the mass of de Sitter. An observer in de Sitter only has access to the region enclosed by the cosmological horizon and can never exit this region. Therefore it seems logical to only consider the flux of the energy current in the region that is accessible to the observer. The energy contained in the region enclosed by the cosmological horizon C is

$$E_C = -\frac{1}{4\pi} \int_{\partial C} \nabla^b K^a dA_{ab}, \tag{24}$$

where all quantities are as defined for Schwarzschild and ∂C denotes the cosmological horizon. The cosmological horizon is a Killing horizon for the Killing vector $K = \partial_t$ as shown above. Therefore we can choose the normal vectors to ∂C such that $t_a = K_a$ and $t_a s^a = -1$. Then equation (24) can be written as

$$\begin{aligned} E_C &= -\frac{1}{4\pi} \int_{\partial C} K_a \nabla^b K^a s_b \sqrt{\gamma^{(2)}} dA, \\ &= -\frac{1}{4\pi} \kappa_C A_C, \end{aligned} \tag{25}$$

where $A_C = \int_{\partial C} \sqrt{\gamma^{(2)}} dA = 4\pi l^2$ is the area of the cosmological event horizon. This simplifies to $E_C = -l$, so the mass of de Sitter is inversely proportional to the cosmological constant. By using the same method as before, we can calculate the differential of equation (25). Note that $A_C = 4\pi \kappa_C^{-2}$, and therefore $d\kappa_C = -(8\pi)^{-1} \kappa_C^3 dA_C$. The first law of cosmological event horizons is therefore

$$dE_C = -dl = -\frac{1}{8\pi} \kappa_C dA_C \tag{26}$$

This equation is very similar to the differential formula we derived for Schwarzschild. Having already seen the similarity between the first law of black hole mechanics and the first law of thermodynamics we might ask ourselves if there is a similar relation for the cosmological event horizon. Next we will calculate the temperature associated with the cosmological event horizon by performing a Wick rotation as was done for Schwarzschild.
We will again omit the 2-sphere part of the metric and compactify the imaginary time coordinate such that $\tau \in [0, T^{-1}]/\sim$. The de Sitter metric then becomes an Euclidean metric given by $ds^2 = +(1 - \frac{r^2}{l^2})d\tau^2 + (1 - \frac{r^2}{l^2})^{-1}dr^2$. We are interested in the behaviour near the cosmological horizon, therefore we make the substitution $l^2 - r^2 = \chi^2$ and expand to lowest order in χ around $r = l$. The metric near the cosmological horizon is then $ds^2 \approx \chi^2 l^{-2} d\tau^2 + d\chi^2$. Which can be written as the canonical metric of a two-sphere $ds^2 = dr^2 + r^2 d\phi^2$ if we identify $\phi = \tau l^{-1}$. We require $\phi \in [0, 2\pi]/\sim$ such that there are no conic singularities. This implies that $\tau \in [0, 2\pi l]/\sim$ and comparing this with the compactification of τ gives us the temperature of the cosmological event horizon

$$T_C = -\frac{\kappa_c}{2\pi} = \frac{1}{2\pi l}. \tag{27}$$

This result is the same as found by Hawking[5]. In combination with the first law of thermodynamics $dE = TdS$ we find that this implies that the entropy of the cosmological horizon is

$$S_C = \frac{A_C}{4} = \pi l^2, \tag{28}$$

similar to the Schwarzschild black hole. This entropy can be interpreted as the information that was lost by particles that went across the horizon. Although we have derived an expression for this entropy of the cosmological event horizon indirectly, a full proof in quantum gravity has not been discovered.

5 The Schwarzschild Black Hole in asymptotically de Sitter spacetime

In the previous sections we discussed spacetimes with either a black hole or cosmological event horizon. In this section we consider a spacetime with both horizons, known as the Schwarzschild-de Sitter metric:

$$ds^2 = -(1 - \frac{2M}{r} - \frac{r^2}{l^2})dt^2 + (1 - \frac{2M}{r} - \frac{r^2}{l^2})^{-1}dr^2 + r^2 d\Omega^2, \quad (29)$$

where M the black hole mass and $l = \sqrt{\frac{3}{\Lambda}}$ (which is no longer the radius of the cosmological event horizon). For $0 < M < \frac{l}{3\sqrt{3}}$ there are two coordinate singularities, namely the two positive roots of $(1 - \frac{2M}{r} - \frac{r^2}{l^2})$. They are rather messy to write down so we will denote the smaller root by r_H and the larger root by r_C, they represent respectively the black hole event horizon and the cosmological event horizon. One interesting property of these horizons is that if r_H increases, then r_C decreases and thus bringing the horizons closer together as M increases. When $M = \sqrt{\frac{3}{\Lambda}}$ the two horizons coincide, this is known as the Nariai solution.
Next to the two positive roots, there is also a negative root which we will denote by r_{--}. Radial null geodesics obey $\frac{dr}{dt} = \pm(1 - \frac{2M}{r} - \frac{r^2}{l^2})$, which can be integrated by using partial fractions and the roots given above, the solution is $t = \pm l^2(A_1 \log|r - r_H| + A_2 \log|r - r_C| + A_3 \log|r - r_{--}|) = \pm r^*$. Here the A's are constants determined by M and l. This simply shows that null geodesics that approach the black hole or cosmological horizon from the region in between the horizons will never reach the horizons as seen by an observer with $r_H < r < r_C$.
Just as we did for the individual horizons, we would like to calculate the surface gravity of both horizons. The incoming Eddington-Finkelstein coordinates for Schwarzschild de Sitter are obtained by replacing the t coordinate by $v = t + r^*$, then the metric becomes

$$ds^2 = -(1 - \frac{2M}{r} - \frac{r^2}{l^2})dv^2 + 2dvdr + r^2 d\Omega^2. \quad (30)$$

This metric only has a true singularity at the origin. The Penrose diagram for this spacetime is given in Figure 4. This time there are two event horizons and therefore there are multiple separated regions as spacetime can be extended horizontally infinitely many times.

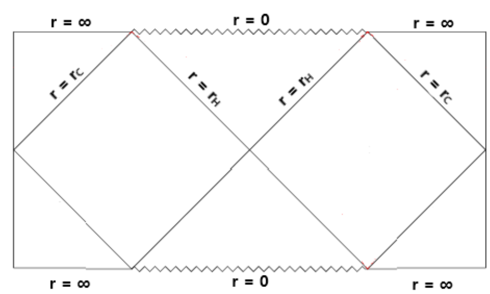

Figure 4: Penrose diagram for Schwarzschild-de Sitter spacetime.

We can quickly confirm that both event horizons are null hypersurfaces by computing their normal vectors $l^2 = f^2(1 - \frac{2M}{r} - \frac{r^2}{l^2})$, this is clearly zero at both horizons. Note as well that $l = fg^{ar}\partial_a = f((1 - \frac{2M}{r} - \frac{r^2}{l^2})\partial_r + \partial_v)$ which when evaluated at both horizons simplifies to $l = f\partial_v$. Since $K = \gamma_t\partial_v$ is a Killing vector, both event horizons are clearly Killing horizons. In the previous sections we were very lucky with the normalization of the timelike Killing vector because $\gamma_t = 1$ in both cases. In the Schwarzschild de Sitter case we will need a different normalization as the ones from before do not make sense in this spacetime. In the previous sections we unintentionally chose to normalize with respect to the place at which an observer does not require any acceleration to stay there. In Schwarzschild and de Sitter these locations are respectively at spatial infinity and the origin. Similarly, we would like to find where this happens in Schwarzschild de Sitter. The radius of the two-sphere at which the effects of the cosmological expansion and the black hole attraction balance out exactly is $r_g = (Ml^2)^{1/3}$. The normalization constant of K is then simply $\gamma_t = (1 - \frac{2M}{r_g} - \frac{r_g^2}{l^2})^{-1/2} = (1 - (\frac{27M^2}{l^2})^{1/3})^{-1/2}$. The timelike Killing vector for Schwarzschild de Sitter is now normalized such that $\lim_{r \to r_g} K^2 = -1$.

Following the same reasoning as in the previous sections, we can write down the defining equation for the surface gravity of both horizons:

$$K^a \nabla_a K^b = \kappa_{H,C} K^b, \tag{31}$$

where the surface gravity is associated with the event horizon on which the equation is evaluated. Here K is the unique timelike Killing vector normalized as discussed above. Note that we can write the g_{vv} component of the metric tensor as $\frac{(r-r_H)(r-r_C)(r-r_{--})}{rl^2}$. In Eddington-Finkelstein coordinates the left hand side of equation (31) becomes $K^v \nabla_v K^v = \gamma_t^2 \Gamma^v_{vv} = \frac{1}{2}\gamma_t^2 g^{vr} \partial_r (-g_{vv})$. Evaluating this expression at the black hole and cosmological event horizon gives us the surface gravity of both horizons[11]:

$$\kappa_H = \gamma_t \frac{(r_C - r_H)(r_H - r_{--})}{2r_H l^2}, \qquad \kappa_C = \gamma_t \frac{(r_H - r_C)(r_C - r_{--})}{2r_C l^2}. \tag{32}$$

Our definition of the surface gravity is negative because we have used incoming Eddington-Finkelstein coordinates, therefore the acceleration near the cosmological event horizon is negative resulting in a negative surface gravity.

We now would like an expression for the mass of the Schwarzschild de Sitter spacetime. The Killing vector K is only timelike in the region between the two horizons, therefore we will take the Komar integral over the spacelike region S enclosed by the cosmological and black hole event horizon. The energy contained in this region is then

$$E_S = -\frac{1}{4\pi} \int_{\partial H} \nabla^b K^a dA_{ab} - \frac{1}{4\pi} \int_{\partial C} \nabla^b K^a dA_{ab}, \tag{33}$$

where all quantities are as before. Note that the surface ∂H points outwards and ∂C points inwards. Therefore the first integral is the negative of the energy contained inside the black hole and the second integral is the energy contained inside of the cosmological horizon. We have already evaluated both expressions in the previous sections and therefore the mass function for Schwarzschild de Sitter is:

$$E_S = -\frac{1}{4\pi} \kappa_H A_H - \frac{1}{4\pi} \kappa_C A_C = -M_H + M_C. \tag{34}$$

Which is exactly what we would expect. As in the previous sections we can relate the differential of the mass function to the first law of thermodynamics. This is known as the first law of black hole mechanics with a cosmological constant

$$dE_S = -\frac{1}{8\pi} \kappa_H dA_H - \frac{1}{8\pi} \kappa_C dA_C. \tag{35}$$

Next we would like to compare this expression with the first law of thermodynamics. We can't just state our previous results because our metric has changed, and therefore the calculations will be different. First we will perform a Wick rotation to calculate a temperature associated with both horizons.
We will ignore the two-sphere part of the metric as usual. By going to the imaginary time coordinate $\tau = -it$ and rewriting the metric components we can write the metric as: $ds^2 = -\frac{(r-r_H)(r-r_C)(r-r_{--})}{rl^2}d\tau^2 - \frac{rl^2}{(r-r_H)(r-r_C)(r-r_{--})}dr^2$. Written down like this it is directly clear where the horizons are. First we consider the black hole event horizon. Define $\chi = (r - r_H)$ and expand up to first order around the black hole event horizon, we have: $ds^2 \approx A_H \chi d\tau^2 + \frac{1}{A_H \chi} d\chi^2$, here $A_H = \frac{(r_C - r_H)(r_H - r_{--})}{r_H l^2}$. By defining $\rho^2 = \frac{4\chi}{A_H}$ we can write this metric in a more familiar form: $ds^2 = \rho^2(\frac{A_H d\tau}{2})^2 + d\rho^2$. We now identify $\frac{A_H \tau}{2}$ as $\phi \in [0, 2\pi]/\sim$ such that there are no conic singularities. Comparing with our compactification of the imaginary time we end with the Hawking temperature of the black hole

$$T_H = \frac{A_H}{4\pi} = \frac{\kappa_H}{2\pi\gamma_t}, \tag{36}$$

where κ_H as in equation (32). This is the temperature an observer would measure as the temperature of the black hole event horizon. Note that the normalization constant in the expression of the surface gravity cancels out giving us the same result as Hawking[4].
We can do exactly the same for the cosmological event horizon, but now define $\chi = (r - r_C)$ and expand up to first order around the cosmological event horizon. Similar calculations will give

$$T_C = \frac{A_C}{4\pi} = \frac{\kappa_C}{2\pi\gamma_t}, \tag{37}$$

where $A_C = \frac{(r_H - r_C)(r_H - r_{--})}{r_C l^2}$ and κ_C as in equation (32). Comparing equation (35) with the modified second law of thermodynamics $-T_H dS_H + T_C dS_C$ we find that the temperature we found in equation (37) actually is the negative of the temperature of the thermodynamic system we are considering. Therefore the actual measured temperature of the cosmological event horizon will be the negative of T_C, giving us the same relation as in [5]. We find the same proportionality constant between the area of a horizon and the entropy that is stored on it as before

$$S_H = \frac{A_H}{4}, \qquad S_C = \frac{A_C}{4}. \tag{38}$$

We have found that the entropy is proportional to one fourth the area of the event horizon in question. Similar results have been found in string theory in the black hole case, but for the cosmological event horizon the result is not confirmed. This relation implies that information must be stored on the event horizon forever resolving the information paradox. Bekenstein showed[12] that black holes attain the maximum amount of entropy-to-energy ratio possible; black holes contain a lot of information. However, by the no hair theorem a black hole can only have three characteristic properties, its mass, spin and charge. This means that even if information is stored on the event horizon, it is impossible to reverse engineer the dynamics of the black hole to figure out what information fell into it. However, Hawking has shown[4] that black holes radiate black-body radiation with the Hawking temperature causing them to evaporate. This again causes information to be destroyed, however recent developments in string theory might have resolved the information paradox once and for all.

A Appendix

A.1 Conservation of $R^a_b\xi^b$

In this appendix we will show that the current $R^a_b\xi^b$ is conserved. Consider the Bianchi identity

$$\nabla_{[a}R^e{}_{bc]d} = 0. \tag{39}$$

Contracting this expression yields the contracted Bianchi identity

$$\nabla_a R^a_b = \frac{1}{2}\nabla_b R. \tag{40}$$

Using the contracted Bianchi identity and Killing's equation (equation (8)) we can now show the conservation of the current.

$$\begin{aligned}\nabla_a(R^a_b\xi^b) = (\nabla_a\xi^b)R^a_b + \xi^b(\nabla_a R^a_b),\\ = \frac{1}{2}\xi^b\nabla_b R,\\ = 0.\end{aligned} \tag{41}$$

For the second equality follows from the symmetric and antisymmetric properties of respectively R^a_b and $\nabla_a\xi^b$. For the last equality we used that the directional derivative of the Ricci scalar along a Killing vector vanishes. This follows from the contraction of equation (46):

$$\nabla_a\nabla_b\xi^a = R_{ba}\xi^a, \tag{42}$$

with the Bianchi identity. Together with Killing's equation one finds

$$\xi^b\nabla_b R = 0, \tag{43}$$

which confirms the last equality in equation (41).

A.2 Identity $\nabla_b\nabla^b\xi^a = -R^a_b\xi^b$

In this appendix we will prove a crucial identity used for the Komar integral. For any vector V we have

$$\begin{aligned}\nabla_{[a}\nabla_b V_{c]} =\\ \frac{1}{6}([\nabla_a,\nabla_b]V_c + [\nabla_b,\nabla_c]V_a + [\nabla_c,\nabla_a]V_b),\\ = \frac{1}{6}(R_{cdab}V^d + R_{adbc}V^d + R_{bdca}V^d),\\ = \frac{1}{6}(R_{cdab} + R_{adbc} + R_{bdca})V^d,\\ = 0.\end{aligned} \tag{44}$$

In the second equality we used $[\nabla_a,\nabla_b]V^c = R^c{}_{dab}V^d$ because the torsion tensor is zero for the Christoffel connection. In the last equality we used the first Bianchi identity $R_{a[bcd]} = 0$ which can be verified after some algebra by plugging in the definition of the Riemann tensor.

If we now take for the vector V a Killing vector ξ we have:

$$\begin{aligned}\nabla_{[a}\nabla_b\xi_{c]} &= \frac{1}{6}(\nabla_a\nabla_b\xi_c - \nabla_a\nabla_c\xi_b + \nabla_b\nabla_c\xi_a \\ &\quad -\nabla_b\nabla_a\xi_c + \nabla_c\nabla_a\xi_b - \nabla_c\nabla_b\xi_a), \\ &= \frac{1}{3}(\nabla_a\nabla_b\xi_c + \nabla_b\nabla_c\xi_a - \nabla_c\nabla_b\xi_a), \\ &= \frac{1}{3}(\nabla_a\nabla_b\xi_c + [\nabla_b,\nabla_c]\xi_a).\end{aligned} \tag{45}$$

In the second equality we used Killings equation $\nabla_{(a}\xi_{b)} = 0$. However, as we saw in equation 44 this equals zero, therefore we have:

$$\begin{aligned}\nabla_a\nabla_b\xi_c &= -[\nabla_b,\nabla_c]\xi_a, \\ &= -R_{adbc}\xi^d, \\ &= R_{cbad}\xi^d.\end{aligned} \tag{46}$$

In the last equality we used the skew symmetry and interchange symmetry of the Riemann tensor. Clearly we have $\nabla_b\nabla_c\xi^a = R^a{}_{cbd}\xi^d$, we can now multiply by g^{bc} to get the desired result.

$$\nabla_b\nabla^b\xi^a = -R^{ba}{}_{bd}\xi^d = -R^a_b\xi^b, \tag{47}$$

where we used the definition of the Ricci tensor.

References

[1] K. Akiyama and et. al, *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole* (2019), 10.3847/2041-8213/ab0ec7.

[2] S. Hawking, Physical Review D **72**, 084013 (2005), 10.1103/PhysRevD.72.084013.

[3] J. Bekenstein, Physical Review D **7**, 2333 (1973), 10.1103/PhysRevD.7.2333.

[4] S. Hawking, Communications in Mathematical Physics **43**, 199 (1975), 10.1007/BF02345020.

[5] G. Gibbons and S. Hawking, Physical Reviews D **15**, 2738 (1977), 10.1103/PhysRevD.15.2738.

[6] S. Carroll, *Spacetime and Geometry, An Introduction to General Relativity* (Pearson New International Edition, 2014).

[7] K. Schwarzschild, *On the gravitational field of a mass point according to einstein's theory* (1999), physics/9905030.

[8] P. Townsend, arXiv (1997), gr-qc/9707012.

[9] J. Bardeen, B. Carter, and S. Hawking, Communications in Mathemtical Physics **31**, 161 (1973), 10.1007/BF01645742.

[10] S. Hawking, Communications in Mathematical Physics **25**, 152 (1972), 10.1007/BF01877517.

[11] R. Bousso and S. W. Hawking, Physical Review D **54**, 6312–6322 (1996), 10.1103/physrevd.54.6312.

[12] J. Bekenstein, Physical Review D **23**, 287 (1981), 10.1103/PhysRevD.23.287.