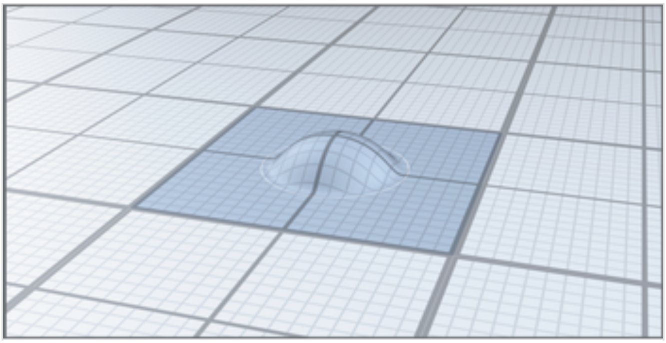
Bump function

In mathematics, a **bump function** (also called a **test function**) is a function $f : \mathbb{R}^n \to \mathbb{R}$ on a Euclidean space $\mathbb{R}^n$ which is both smooth (in the sense of having continuous derivatives of all orders) and compactly supported. The set of all bump functions with domain $\mathbb{R}^n$ forms a vector space, denoted $\mathrm{C}_0^\infty(\mathbb{R}^n)$ or $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^n)$. The dual space of this space endowed with a suitable topology is the space of distributions.

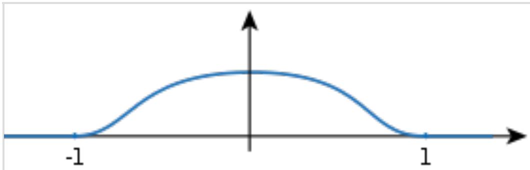


The graph of the bump function $(x, y) \in \mathbb{R}^2 \mapsto \Psi(r)$, where $r = (x^2 + y^2)^{1/2}$ and $\Psi(r) = e^{-1/(1-r^2)} \cdot \mathbf{1}_{\{|r|<1\}}$.

Examples

The function $\Psi : \mathbb{R} \to \mathbb{R}$ given by

$$\Psi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right), & \text{if } x \in (-1, 1) \\ 0, & \text{if } x \in \mathbb{R} \setminus \{(-1, 1)\} \end{cases}$$



The 1d bump function $\Psi(x)$.

is an example of a bump function in one dimension. It is clear from the construction that this function has compact support, since a function of the real line has compact support if and only if it has bounded closed support. The proof of smoothness follows along the same lines as for the related function discussed in the Non-analytic smooth function article. This function can be interpreted as the Gaussian function $\exp(-y^2)$ scaled to fit into the unit disc: the substitution $y^2 = 1/(1 - x^2)$ corresponds to sending $x = \pm 1$ to $y = \infty$.

A simple example of a (square) bump function in n variables is obtained by taking the product of n copies of the above bump function in one variable, so

$$\Phi(x_1, x_2, \dots, x_n) = \Psi(x_1)\Psi(x_2)\cdots\Psi(x_n).$$

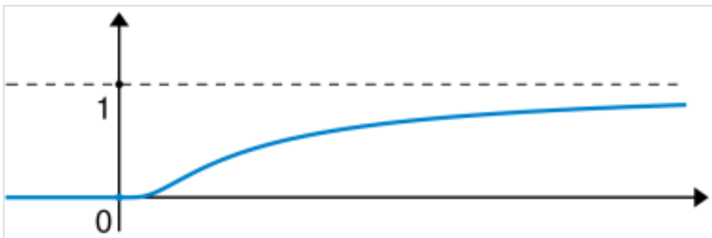
A radially symmetric bump function in n variables can be formed by taking the function $\Psi_n : \mathbb{R}^n \to \mathbb{R}$ defined by $\Psi_n(\mathbf{x}) = \Psi(|\mathbf{x}|)$. This function is supported on the unit ball centered at the origin.

Smooth transition functions

Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases}$$

defined for every real number x.



The non-analytic smooth function $f(x)$ considered in the article.

The function

$$g(x) = \frac{f(x)}{f(x) + f(1-x)}, \qquad x \in \mathbb{R},$$

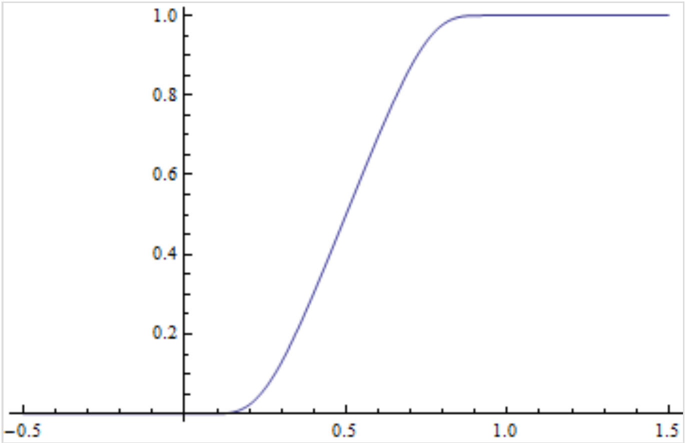
has a strictly positive denominator everywhere on the real line, hence g is also smooth. Furthermore, $g(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x \geq 1$, hence it provides a smooth transition from the level 0 to the level 1 in the unit interval $[0, 1]$. To have the smooth transition in the real interval $[a, b]$ with $a < b$, consider the function

$$\mathbb{R} \ni x \mapsto g\Big(\frac{x-a}{b-a}\Big).$$

For real numbers $a < b < c < d$, the smooth function

$$\mathbb{R} \ni x \mapsto g\Big(\frac{x-a}{b-a}\Big)\, g\Big(\frac{d-x}{d-c}\Big)$$

equals 1 on the closed interval $[b, c]$ and vanishes outside the open interval (a, d), hence it can serve as a bump function.



The smooth transition g from 0 to 1 defined here.

Caution must be taken since, as example, taking $\{a=-1\} < \{b=c=0\} < \{d=1\}$, leads to:

$$q(x) = \frac{1}{1 + e^{\frac{1-2|x|}{x^2-|x|}}}$$

which is not an infinitely differentiable function (so, is not "smooth"), so the constraints $a < b < c < d$ must be strictly fulfilled.

Some interesting facts about the function:

$$q(x, a) = \frac{1}{1 + e^{\frac{a(1-2|x|)}{x^2-|x|}}}$$

Are that $q\left(x, \frac{\sqrt{3}}{2}\right)$ make smooth transition curves with "almost" constant slope edges (behaves like inclined straight lines on a non-zero measure interval).

A proper example of a smooth Bump function would be:

$$u(x) = \begin{cases} 1, \text{if } x = 0, \\ 0, \text{if } |x| \geq 1, \\ \frac{1}{1+e^{\frac{1-2|x|}{x^2-|x|}}}, \text{otherwise}, \end{cases}$$

A proper example of a smooth transition function will be:

$$w(x) = \begin{cases} \frac{1}{1+e^{\frac{2x-1}{x^2-x}}} & \text{if } 0 < x < 1, \\ 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq 1, \end{cases}$$

where could be noticed that it can be represented also through Hyperbolic functions: