# 收敛阶分析

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牛顿法

若 $x^*$ 只是一重零点，有

$$\because \phi(x) = x - \frac{f(x)}{f'(x)}, \phi'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}, x^* = \phi(x^*)$$

$$\therefore \phi'(x^*) = 1 - \frac{(f'(x))^2 - 0}{(f'(x))^2} = 0$$

$$\therefore |x_{k+1} - x^*| = |\phi(x_k) - \phi(x^*)| = |\phi'(x^*)(x_k - x^*) + \frac{\phi''(\xi)}{2!}(x_k - x^*)^2| = |\frac{\phi''(\xi)}{2}||(x_k - x^*)|^2$$

$$\therefore \lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^2} = \frac{\phi(x^*)}{2}$$

所以为二阶收敛，但当初值条件选取不同时，牛顿法收敛次数不同。本次作业当选择靠近零点的初值点时，可以得到三阶收敛性

程序结果如下:

验证牛顿法二阶收敛

初始值 $= 0.10000000000000$

$(x* - x1)/(x* - x0)^2 = 0.06734006734007$

$(x* - x2)/(x* - x1)^2 = 0.00044893398566$

$(x* - x3)/(x* - x2)^2 = 0.00000000000000$

初始值 $= 0.20000000000000$

$(x* - x1)/(x* - x0)^2 = 0.13888888888889$

$(x* - x2)/(x* - x1)^2 = 0.00370381801908$

$(x* - x3)/(x* - x2)^2 = 0.00000007595779$

$(x* - x4)/(x* - x3)^2 = 0.00000000000000$

初始值 $= 0.90000000000000$

$(x* - x1)/(x* - x0)^2 = 0.11920910952829$

$(x* - x2)/(x* - x1)^2 = 0.41181405284336$

$(x* - x3)/(x* - x2)^2 = 0.62889961516554$

$(x* - x4)/(x* - x3)^2 = 0.81180617203253$

$(x* - x5)/(x* - x4)^2 = 0.86370068188097$

$(x* - x6)/(x* - x5)^2 = 0.86601900321691$

$(x* - x7)/(x* - x6)^2 = 0.00000000000000$

初始值 $= 9.00000000000000$

$(x* - x1)/(x* - x0)^2 = 0.08221687836487$

$(x* - x2)/(x* - x1)^2 = 0.12887535780785$

$(x* - x3)/(x* - x2)^2 = 0.20531413429246$

$(x* - x4)/(x* - x3)^2 = 0.33109320287201$

$(x* - x5)/(x* - x4)^2 = 0.52401040399070$

$(x* - x6)/(x* - x5)^2 = 0.74189451289816$

$(x* - x7)/(x* - x6)^2 = 0.85283623341396$

$(x* - x8)/(x* - x7)^2 = 0.86589424779460$

$(x* - x9)/(x* - x8)^2 = 0.00000000000000$

$(x* - x10)/(x* - x9)^2 = -nan(ind)$

判断前两个初始值是否为三阶收敛

初始值 $= 0.10000000000000$

$(x*-x1)/(x*-x0)^3 = 0.67340067340067$

$(x*-x2)/(x*-x1)^3 = -0.66666696871072$

$(x*-x3)/(x*-x2)^3 = 0.00000000000000$

初始值 $= 0.20000000000000$

$(x*-x1)/(x*-x0)^3 = 0.69444444444445$

$(x*-x2)/(x*-x1)^3 = -0.66668724343420$

$(x*-x3)/(x*-x2)^3 = 0.66445825750137$

$(x*-x4)/(x*-x3)^3 = -0.00000000000000$

可以看出在有些特殊的初始值下，牛顿迭代可以达到三阶收敛

可以看出初始值靠近原点时，$\frac{|x_{k+1}|-x^*}{|x_k-x^*|^3}$ 的数量级不变，由误差阶公式定义可以认为其具有三阶误差阶，在初始值原理零点时，可以看出具有二阶误差阶。当迭代接近零点时，也就是每次计算的最后一次或两次迭代，迭代的附近收敛速度很快。程序结果可以看出最后一次迭代收敛阶由 $10^{-1}$ 量级快速减小，并且出现了 nan 结果 (原因可能是上一次误差阶已经过小，并小于 double 的最低精度，因而被看作零，作为分母进行下一步计算得到 nan 结果)

弦截法

设 $x^*$ 为零点，$f(x) \in C_{(2)}^{[a,b]}$ 先证

$$f[x_k, x^*] - f[x_{k-1}, x_k] = \tfrac{f''(\xi)}{2}(x^* - x_{k-1}), \xi \in [a, b]$$

$$f(x) = f(x_k) + (x - x_k)f'(x_k) + \frac{f''(\xi)}{2}(x - x_k)^2, \xi \in [a, b]$$

$$\therefore f[x_k, x^*] - f[x_{k-1}, x_k]$$
$$= \frac{(x^* - x_k)f'(x_k) + \frac{f''(\xi_1)}{2}(x^* - x_k)^2}{x^* - x_k} - \frac{(x_{k-1} - x_k)f'(x_k) + \frac{f''(\xi_2)}{2}(x_{k-1} - x_k)^2}{x_{k-1} - x_k}$$

$$= \frac{f''(\xi_1)(x^* - x_k) - f''(\xi_2)(x_{k-1} - x_k)}{2}$$

$$= \frac{f''(\xi_1)(x^* - x_k) + f''(\xi_2)(x_k - x_{k-1})}{(x^* - x_k) + (x_k - x_{k-1})} \frac{x^* - x_{k-1}}{2}$$

$$\because (f''(\xi_1) - \frac{f''(\xi_1)(x^* - x_k) + f''(\xi_2)(x_k - x_{k-1})}{(x^* - x_k) + (x_k - x_{k-1})})$$

$$* ((f''(\xi_2) - \frac{f''(\xi_1)(x^* - x_k) + f''(\xi_2)(x_k - x_{k-1})}{(x^* - x_k) + (x_k - x_{k-1})}))$$

$$= -\frac{(f''(\xi_1) - f''(\xi_2))^2 (x_k - x_{k-1})(x^* - x_k)}{((x^* - x_k) + (x_k - x_{k-1}))^2} < 0$$

$$\therefore \exists \xi \in [\xi_1, \xi_2], s.t.\ f''(\xi) = \frac{f''(\xi_1)(x^* - x_k) + f''(\xi_2)(x_k - x_{k-1})}{(x^* - x_k) + (x_k - x_{k-1})}$$

$$\therefore f[x_k, x^*] - f[x_{k-1}, x_k] = \frac{f''(\xi)}{2}(x^* - x_{k-1}), \xi \in [a, b]$$

再证弦截法具有 $\frac{\sqrt{5}+1}{2}$ 阶收敛阶

$$|x_{k+1} - x^*| = |x_k - \frac{f(x_k)}{f[x_k, x_{k-1}]} - x^*| = |x_k - x^*|| - \frac{f(x_k) - f(x^*)}{f[x_k, x_{k-1}](x_k - x^*)} + 1|$$

$$= |x_k - x^*||1 - \frac{f[x_k, x^*]}{f[x_k, x_{k-1}]}| = |x_k - x^*||\frac{f[x_k, x_{k-1}] - f[x_k, x^*]}{f[x_k, x_{k-1}]}|$$

$$= |x_k - x^*||x^* - x_{k-1}||\frac{f''(\xi)}{2f[x_k, x_{k-1}]}|$$

$$\therefore \frac{|e_{k+1}|}{|e_k e_{k-1}|} = |\frac{f''(\xi)}{2f[x_k, x_{k-1}]}| = M$$

可知弦截法收敛，设弦截法的收敛阶为 $m$，有

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^m} = \lambda$$

$$\therefore \lim_{k\to\infty} \frac{|e_k|}{|e_{k-1}|^m} = \lambda, \lim_{k\to\infty} \frac{|e_k|^{(m-1)}}{|e_{k-1}|^{m(m-1)}} = \lambda^{m-1}$$

$$\therefore \lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k||e_{k-1}|^{m(m-1)}} = \lambda^m < \infty$$

由 $\frac{|e_{k+1}|}{|e_k e_{k-1}|}$ 收敛可知，$m$ 需要满足条件 $m(m-1) \geq 1$ 即 $m \geq \frac{\sqrt{5}+1}{2}$

通过程序验证弦截法三阶收敛结果如下

验证弦截法法 (sqrt(5) + 1) / 2 阶收敛

初始值 $= -0.10000000000000, 0.10000000000000$

$(x*-x2)/(x*-x1)^{(}(sqrt(5)+1)/2) = 0.00000000000000$

$(x*-x3)/(x*-x2)^{(}(sqrt(5)+1)/2) = -nan(ind)$

初始值 $= -0.20000000000000, 0.20000000000000$

$(x*-x2)/(x*-x1)^{(}(sqrt(5)+1)/2) = 0.00000000000000$

$(x*-x3)/(x*-x2)^{(}(sqrt(5)+1)/2) = -nan(ind)$

初始值 $= -0.20000000000000, 0.90000000000000$

$(x*-x2)/(x*-x1)^{(}(sqrt(5)+1)/2) = 0.06412860593282$

$(x*-x3)/(x*-x2)^{(}(sqrt(5)+1)/2) = 2.43655313924023$

$(x*-x4)/(x*-x3)^{(}(sqrt(5)+1)/2) = 0.00622593890376$

$(x*-x5)/(x*-x4)^{(}(sqrt(5)+1)/2) = 0.16699374938423$

$(x*-x6)/(x*-x5)^{(}(sqrt(5)+1)/2) = 0.00000000000000$

初始值 $= -0.90000000000000, 9.00000000000000$

$(x*-x2)/(x*-x1)^{(}(sqrt(5)+1)/2) = 0.01728509004387$

$(x*-x3)/(x*-x2)^{(}(sqrt(5)+1)/2) = 1.10413451814141$

$(x*-x4)/(x*-x3)^{(}(sqrt(5)+1)/2) = 10.06276279737208$

$(x*-x5)/(x*-x4)^{(}(sqrt(5)+1)/2) = 0.03071422400684$

$(x*-x6)/(x*-x5)^{(}(sqrt(5)+1)/2) = 1.06610833820992$

$(x*-x7)/(x*-x6)^{(}(sqrt(5)+1)/2) = 4.72736085735360$

$(x*-x8)/(x*-x7)^{(}(sqrt(5)+1)/2) = 0.16477132047166$

$(x*-x9)/(x*-x8)^{(}(sqrt(5)+1)/2) = 1.27265953652884$

$(x*-x10)/(x*-x9)^{(}(sqrt(5)+1)/2) = 1.48726468865459$

$(x*-x11)/(x*-x10)^{(}(sqrt(5)+1)/2) = 0.70969838539583$

$(x*-x12)/(x*-x11)^{(}(sqrt(5)+1)/2) = 0.96124828649500$

$(x*-x13)/(x*-x12)^{(}(sqrt(5)+1)/2) = 0.93589650722041$

$(x*-x14)/(x*-x13)^((sqrt(5)+1)/2) = 0.90891746545699$

$(x*-x15)/(x*-x14)^((sqrt(5)+1)/2) = 0.91825943626478$

$(x*-x16)/(x*-x15)^((sqrt(5)+1)/2) = 0.00000000000000$

可以看出在靠近零点附近，迭代收敛速度明显快于远离零点位置处的迭代，远离零点迭代收敛与 $\frac{\sqrt{5}+1}{2}$ 阶相符合，零点附近收敛速度远远快于这一阶

附录：牛顿法计算结果表

牛顿法

初始值 $= 0.10000000000000$

$k = 0xk = +0.10000000000000f(xk) = -0.099666666666677error = -1.00000000000000$

$k = 1xk = -0.00067340067340f(xk) = +0.000673400571617error = 0.10067340067340$

$k = 2xk = +0.00000000020358f(xk) = -0.000000000203587error = 0.00067340087698$

$k = 3xk = +0.00000000000000f(xk) = +0.000000000000007error = 0.00000000020358$

$x0 = 0.00000000000000$

初始值 $= 0.20000000000000$

$k = 0xk = +0.20000000000000f(xk) = -0.197333333333337error = -1.00000000000000$

$k = 1xk = -0.00555555555556f(xk) = +0.005555498399637error = 0.20555555555556$

$k = 2xk = +0.00000011431537f(xk) = -0.000000114315377error = 0.00555566987093$

$k = 3xk = -0.00000000000000f(xk) = +0.000000000000007error = 0.00000011431537$

$k = 4xk = +0.00000000000000f(xk) = +0.000000000000007error = 0.00000000000000$

$x0 = 0.00000000000000$

初始值 $= 0.90000000000000$

$k = 0xk = +0.90000000000000f(xk) = -0.657000000000007error = -1.00000000000000$

$k = 1xk = -2.55789473684211f(xk) = -3.020724887009787error = 3.45789473684211$

$k = 2xk = -2.01291548477783f(xk) = -0.705747458637187error = 0.54497925206427$

$k = 3xk = -1.78166153289696f(xk) = -0.103525116827077error = 0.23125395188088$

$k = 4xk = -1.73404884445769f(xk) = -0.004002991045527error = 0.04761268843926$

$k = 5xk = -1.73205425559277f(xk) = -0.000006896068387error = 0.00199458886492$

$k = 6xk = -1.73205080757917f(xk) = -0.000000000020597error = 0.00000344801360$

$k = 7xk = -1.73205080756888f(xk) = -0.000000000000007error = 0.00000000001030$

$x0 = -1.73205080756888$

初始值 $= 9.00000000000000$

$k = 0xk = +9.00000000000000f(xk) = +234.000000000000007error = -1.00000000000000$

$k = 1xk = +6.07500000000000f(xk) = +68.658890625000007error = 2.92500000000000$

$k = 2xk = +4.16279569705304f(xk) = +19.882716706053917error = 1.91220430294696$

$k = 3xk = +2.94515362368346f(xk) = +5.570198369534327error = 1.21764207336958$

$k = 4xk = +2.21929367109017f(xk) = +1.424242364952357error = 0.72585995259328$

$k = 5xk = +1.85645381615409f(xk) = +0.276253181628647error = 0.36283985493608$

$k = 6xk = +1.74353244757947f(xk) = +0.023192117448447error = 0.11292136857462$

$k = 7xk = +1.73216323531275f(xk) = +0.000224877381347error = 0.01136921226672$

$k = 8xk = +1.73205081851378f(xk) = +0.000000021889807error = 0.00011241679897$

$k = 9xk = +1.73205080756888f(xk) = +0.000000000000007error = 0.00000001094490$

$k = 10xk = +1.73205080756888f(xk) = +0.000000000000007error = 0.00000000000000$

$x0 = 1.73205080756888$

弦截法计算结果

弦截法

初始值 $= -0.10000000000000, 0.10000000000000$

$k = 0xk = -0.10000000000000f(xk) = +0.099666666666677error = -1.00000000000000$

$k = 1xk = +0.10000000000000f(xk) = -0.099666666666677error = -1.00000000000000$

$k = 2xk = +0.00000000000000f(xk) = +0.000000000000007error = 0.10000000000000$

$k = 3xk = +0.00000000000000f(xk) = +0.000000000000007error = 0.00000000000000$

$x0 = 0.00000000000000$

初始值 $= -0.20000000000000, 0.20000000000000$

$k = 0xk = -0.20000000000000f(xk) = +0.197333333333337error = -1.00000000000000$

$k = 1xk = +0.20000000000000f(xk) = -0.197333333333337error = -1.00000000000000$

$k = 2xk = +0.00000000000000f(xk) = +0.000000000000007error = 0.20000000000000$

$k = 3xk = +0.00000000000000f(xk) = +0.000000000000007error = 0.00000000000000$

$x0 = 0.00000000000000$

初始值 $= -0.20000000000000, 0.90000000000000$

$k = 0xk = -0.20000000000000f(xk) = +0.197333333333337error = -1.00000000000000$

$k = 1xk = +0.90000000000000f(xk) = -0.657000000000007error = -1.00000000000000$

$k = 2xk = +0.05407725321888f(xk) = -0.054024539626077error = 0.84592274678112$

$k = 3xk = -0.02171453261222f(xk) = +0.021711119660047error = 0.07579178583110$

$k = 4xk = +0.00001267682833f(xk) = -0.000012676828337error = 0.02172720944054$

$k = 5xk = -0.00000000199161f(xk) = +0.000000001991617error = 0.00001267881994$

$k = 6xk = +0.00000000000000f(xk) = -0.000000000000007error = 0.00000000199161$

$x0 = 0.00000000000000$

初始值 $= -0.90000000000000, 9.00000000000000$

$k = 0 xk = -0.90000000000000 f(xk) = +0.657000000000007 error = -1.00000000000000$

$k = 1 xk = +9.00000000000000 f(xk) = +234.000000000000007 error = -1.00000000000000$

$k = 2 xk = -0.92787441663131 f(xk) = +0.661589635050767 error = 9.92787441663131$

$k = 3 xk = -0.95602314191771 f(xk) = +0.664761052507357 error = 0.02814872528640$

$k = 4 xk = +4.94423329050182 f(xk) = +35.343757458040537 error = 5.90025643241954$

$k = 5 xk = -1.06912505035278 f(xk) = +0.661778294636027 error = 6.01335834085460$

$k = 6 xk = -1.18386792615858 f(xk) = +0.630787219346617 error = 0.11474287580580$

$k = 7 xk = -3.51932543750625 f(xk) = -11.010387398323867 error = 2.33545751134767$

$k = 8 xk = -1.31041673226007 f(xk) = +0.560337683835607 error = 2.20890870524618$

$k = 9 xk = -1.41738797035563 f(xk) = +0.468215851305187 error = 0.10697123809556$

$k = 10 xk = -1.96107699536000 f(xk) = -0.552907997205977 error = 0.54368902500437$

$k = 11 xk = -1.66668566334250 f(xk) = +0.123423017653857 error = 0.29439133201750$

$k = 12 xk = -1.72040886362523 f(xk) = +0.023049660588757 error = 0.05372320028273$

$k = 13 xk = -1.73274581805683 f(xk) = -0.001390857736907 error = 0.01233695443160$

$k = 14 xk = -1.73204374830968 f(xk) = +0.000014118432077 error = 0.00070206974714$

$k = 15 xk = -1.73205080332190 f(xk) = +0.000000008493957 error = 0.00000705501222$

$k = 16 xk = -1.73205080756890 f(xk) = -0.000000000000057 error = 0.00000000424700$

$x0 = -1.73205080756890$