Homework 5, Due: Thursday, 2/14

This assignment is due on **Thursday, February 14**, by 11:59 PM. Your assignment should be well-organized, typed (or neatly written and scanned) and saved as a .pdf for submission on Canvas. You must show all of your work to receive full credit. For problems requiring the use of MATLAB code, remember to include your output in the main body of your solution .pdf and to also submit your .m-files on Canvas as a part of your completed assignment. Your code should be appropriately commented to receive full credit.

Problems

$\boxed{1}$ Consider the Successive Over-Relaxation (SOR) iterative method for solving linear systems of the form $\mathsf{A}\mathbf{x} = \mathbf{b}$.

(a) (6 points) Write a MATLAB **function** implementing the SOR method with relaxation parameter ω. Name your function `SOR.m`. List the input and output variables of your function, and insert comments to describe what each line of the code does. You may use `jacobi.m` or your `gauss_seidel.m` from HW4 as a starting point for modification. Note that your `SOR.m` function should take in ω as an input variable, so that it can be easily changed to different values depending on the problem.

Demonstrate that your function works by using it to approximate the solution to

$$\begin{aligned} 3x_1 - \ \ x_2 + \ \ x_3 &= 5 \\ 3x_1 + 6x_2 - 2x_3 &= 1 \\ 3x_1 + 3x_2 + 7x_3 &= 3 \end{aligned}$$

with relaxation parameter $\omega = 1.2$, starting with $\mathbf{x}^{(0)} = \mathbf{0}$ and stopping when

$$||\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}||_\infty < 10^{-6}.$$

Report the solution and the number of iterations the algorithm took to converge.

(b) (4 points) Explore the effects of using different values of ω in your SOR code from part (a). In particular, experiment with your `SOR.m` function on the linear system from part (a) using at least 5 different values of ω between 0 and 2. Discuss your findings, noting any differences in solving the linear system from part (a) using the same initial guess $\mathbf{x}^{(0)}$ and the same stopping criterion, including the number of iterations it takes the algorithm to converge.

(c) (4 points) Does the SOR algorithm converge for all choices of $0 < \omega < 2$ in part (b)? Explain why or why not, using $\rho(\mathsf{T}_\omega)$ in your argument.

$\boxed{2}$ Consider the linear system

$$\begin{aligned} 10x_1 - \quad x_2 \qquad\quad &= 9 \\ -x_1 + 10x_2 - \quad 2x_3 &= 7 \\ -2x_2 + 10x_3 &= 6 \end{aligned}$$

(a) (4 points) Find (by hand) the first two iterations of the SOR method with $\omega = 1.1$ for the linear system, using $\mathbf{x}^{(0)} = \mathbf{0}$.

(b) (4 points) Show that $\rho(\mathsf{T}_\omega) = 0.1$, where T_ω is the iteration matrix associated with the SOR method when $\omega = 1.1$.

(c) (4 points) Use the SOR method with $\omega = 1.1$ and $\mathbf{x}^{(0)} = \mathbf{0}$ to approximate the solution to the linear system to within 10^{-5} in the ℓ_∞ norm. You may do this numerically using your function `SOR.m` from Problem 1. Report your solution and the number of iterations it took the algorithm to converge.

(d) (4 points) Find the optimal value of ω for applying the SOR method to this linear system. Repeat part (c) using the optimal value of ω, and discuss how this speeds up convergence of the SOR method in this case.

$\boxed{3}$ (4 points) Consider the 3×3 matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Determine if A has a set of 3 linearly independent eigenvectors by analytically (i.e., by hand) computing its eigenvalues and associated eigenvectors.

$\boxed{4}$ (4 points) Use the Geršgorin Circle Theorem to show that a strictly diagonally dominant matrix must be nonsingular.

Note: For any of the above problems for which you use MATLAB to help you solve, you must submit your code/.m-files as part of your work. Your code must run in order to receive full credit. If you include any plots, make sure that each has a title, axis labels, and readable font size, and include the final version of your plots as well as the code used to generate them.