1a) Please see the SOR method MATLAB code in the file named SOR.m. I input these variables:

```
A=[3 -1 1; 3 6 -2; 3 3 7];
b=[5;1;3];
x0=[0;0;0];
w=1.2;
tol=0.000001;
maxiter=100;
[x,iter] = SOR(A,b,x0,w,tol,maxiter)
```

And got back this output where x is the solution vector and iter is the maximum number of iterations it took the algorithm to converge:

```
>> HW5_script

x =

   1.469135979927580
  -0.555555695339397
   0.037037002444266


iter =

    21
```

b) I used these values for omega: 0.1, 0.5, 1.0, 1.5, 1.9 which produced the following output:

```
>> HW5_script
```

For omega = 0.1

```
x =

   1.469135213678013
  -0.555546191756907
   0.037037048619889


iter =

   137
```

For omega = 0.5

```
x =

   1.469136777659948
  -0.555555612344203
   0.037036296060309


iter =
```

```
    24
```

For omega = 1.0

```
x =

   1.469135995626869
  -0.555555703628854
   0.037037017715137


iter =

    10
```

For omega = 1.5

```
x =

   1.0e+21 *

  -1.464147513806442
   2.036080682058773
  -0.573925632861893


iter =

   150
```

For omega = 1.9

```
x =

   1.0e+63 *

  -1.235803812457661
   2.212892977860248
  -1.204491819835551


iter =

   150
```

There are many differences in the solution and number of iterations it took the algorithm to converge for the different omega values. For example, the omega values that are closer to the limit values for omega, 0 and 2, seem to take a lot longer to converge. This is especially true for the omega values that are closer to 2 in which the algorithm takes much longer to converge than the rest of the omega values I input and as a result, the solutions are very off. The omega value 1 took the least amount of iterations, 10, to converge and has the most accurate solution.

1c. The SOR algorithm does not converge for all choices of omega between 0 and 2. This is because rho of Tw is greater than 1 as shown below:

1c. $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ $L = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ -3 & -3 & 0 \end{bmatrix}$ $U = \begin{bmatrix} 0 & 1 & -3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\underline{w=1.9}$

$(D-wL) = \begin{bmatrix} 3 & 0 & 0 \\ 5.7 & 6 & 0 \\ 5.7 & 5.7 & 7 \end{bmatrix}$ $(D-wL)^{-1} = \begin{bmatrix} 0.333333 & 0 & 0 \\ -0.316667 & 0.166667 & 0 \\ -0.013571 & -0.135714 & 0.142857 \end{bmatrix}$

$(1-w)D+wU = \begin{bmatrix} -2.7 & 1.9 & -5.7 \\ 0 & -5.4 & 3.8 \\ 0 & 0 & -6.3 \end{bmatrix}$ $(D-wL)^{-1} \cdot (1-w)D+wU = \underbrace{\begin{bmatrix} -0.9 & 0.633333 & -1.9 \\ 0.855 & -1.50167 & 2.43833 \\ 0.036643 & 0.707071 & -1.33836 \end{bmatrix}}_{T_w \text{ matrix}}$

Eigenvalues of T_w matrix are:

$-0.350417+0.342162i$, $-0.350417-0.342162i$, -3.03919

$|\lambda_1| = \sqrt{(-0.350417)^2+(0.342162)^2}$ $|\lambda_2| = \sqrt{(-0.350417)^2+(-0.342162)^2}$ $|\lambda_3| = 3.03919$

$= 0.489762$ $= 0.489762$

So,

$\rho(T_w) = 3.03919$ which is greater than 1

Therefore,

SOR doesn't converge for all choices of $0 < w < 2$

2a.

2a) $\left[\begin{array}{ccc|c} 10 & -1 & 0 & 9 \\ -1 & 10 & -2 & 7 \\ 0 & -2 & 10 & 6 \end{array}\right]$

$$x^{(k)} = (D-\omega L)^{-1}[(1-\omega)D + \omega U]x^{(k-1)} + (D-\omega L)^{-1}\omega b$$

$$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x^{(1)} = (D-\omega L)^{-1}[(1-\omega)D + \omega U]x^{(0)} + (D-\omega L)^{-1}\omega b$$

$x^{(0)} = 0$ so $x^{(1)} = (D-\omega L)^{-1}\omega b$

$$(D-\omega L) = \begin{bmatrix} 10 & 0 & 0 \\ -1.1 & 10 & 0 \\ 0 & -2.2 & 10 \end{bmatrix} \quad (D-\omega L)^{-1} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.011 & 0.1 & 0 \\ 0.00242 & 0.022 & 0.1 \end{bmatrix}$$

$$(D-\omega L)^{-1}\omega = \begin{bmatrix} 0.11 & 0 & 0 \\ 0.0121 & 0.11 & 0 \\ 0.002662 & 0.0242 & 0.11 \end{bmatrix} \quad (D-\omega L)^{-1}\omega b = \begin{bmatrix} 0.99 \\ 0.8789 \\ 0.853358 \end{bmatrix} = x^{(1)}$$

$$x^{(2)} = (D-\omega L)^{-1}[(1-\omega)D + \omega U]x^{(1)} + (D-\omega L)^{-1}\omega b$$

$$(1-\omega)D + \omega U = \begin{bmatrix} -1 & 1.1 & 0 \\ 0 & -1 & 2.2 \\ 0 & 0 & -1 \end{bmatrix} \quad (D-\omega L)^{-1} \cdot [(1-\omega)D+\omega U] = \begin{bmatrix} -0.1 & 0.11 & 0 \\ -0.011 & -0.0879 & 0.22 \\ -0.00242 & -0.019338 & -0.0516 \end{bmatrix}$$

T_ω matrix

$$(D-\omega L)^{-1}[(1-\omega)D+\omega U]x^{(1)} + (D-\omega L)^{-1}\omega b = \begin{bmatrix} 0.987679 \\ 0.97849345 \\ 0.789933 \end{bmatrix} = x^{(2)}$$

2b.

2b) From part a, we know that:

$$T_\omega = \begin{bmatrix} -0.1 & 0.11 & 0 \\ -0.011 & -0.0879 & 0.22 \\ -0.00242 & -0.019338 & -0.0516 \end{bmatrix} \qquad \begin{array}{l} \omega = 1.1 \\ 0 < \omega < 2 \text{ so true } \checkmark \end{array}$$

$$\rho(T_\omega) \geq |\omega - 1| \Rightarrow \rho(T_\omega) \geq |1.1 - 1| \Rightarrow \rho(T_\omega) \geq 0.1$$

Find eigenvalues: $\det(A - \lambda I) = 0$ where $A = T_\omega$

$$\begin{vmatrix} -0.1-\lambda & 0.11 & 0 \\ -0.011 & -0.0879-\lambda & 0.22 \\ -0.00242 & -0.019338 & -0.0516-\lambda \end{vmatrix} = 0$$

$$= (-0.1-\lambda)\cdot\begin{vmatrix} -0.0879-\lambda & 0.22 \\ -0.019338 & -0.0516-\lambda \end{vmatrix} - 0.11 \begin{vmatrix} -0.011 & 0.22 \\ -0.00242 & -0.0516-\lambda \end{vmatrix} + 0$$

$$= (-0.1-\lambda)\big((-0.0879-\lambda)(-0.0516-\lambda) - (-0.019338)(0.22)\big) - 0.11\big((-0.011)(-0.0516-\lambda) - ((-0.00242)(0.22))\big)$$

$$\Rightarrow -(\lambda + 0.1)(\lambda^2 + 0.1395\lambda + 0.01) = 0 \quad \text{So } \lambda = -0.1 \quad \text{complex}$$

$$|\lambda| = |0.1| \Rightarrow \rho(T_\omega) = 0.1 \checkmark$$

2c. The following is the output where x is the solution vector and iter is the maximum number of times it took the algorithm to converge.

```
>> HW5_script

x =

   0.995789772828388
   0.957894940328669
   0.791579010306630


iter =

     7
```

2d. The following is the output where x is the solution vector and iter is the maximum number of times it took the algorithm to converge.

```
>> HW5_script

x =

   0.995789409632900
   0.957894716954159
   0.791578945197880


iter =

     5
```

The number of iterations reduced by two, from 7 to 5, when using the optimal value for omega, which was calculated to be 1.01282. It speeds up convergence in this case because you are using the omega value that can quickly reach the desired solution.

2d) We can use this eq. $\omega = \dfrac{2}{1+\sqrt{1-\rho(T_{GS})}}$ if matrix is Symmetric positive definite and tridiagonal, which it is ✓

$T_{GS} = (D-L)^{-1}U$ where D, L, U used from part (a)

$$D-L = \begin{bmatrix} 10 & 0 & 0 \\ -1 & 10 & 0 \\ 0 & -2 & 10 \end{bmatrix} \quad (D-L)^{-1} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.01 & 0.1 & 0 \\ 0.002 & 0.02 & 0.1 \end{bmatrix} \quad (D-L)^{-1}U = \begin{bmatrix} 0 & 0.1 & 0 \\ 0 & 0.01 & 0.2 \\ 0 & 0.002 & 0.04 \end{bmatrix}$$

Find eigenvalues: $\det(A-\lambda I)=0$ where $A = T_{GS}$

$$0 = \begin{vmatrix} -\lambda & 0.1 & 0 \\ 0 & 0.01-\lambda & 0.2 \\ 0 & 0.002 & 0.04-\lambda \end{vmatrix} \Rightarrow 0 - 0.002\begin{vmatrix} -\lambda & 0 \\ 0 & 0.2 \end{vmatrix} + (0.04-\lambda)\begin{vmatrix} -\lambda & 0.1 \\ 0 & 0.01-\lambda \end{vmatrix}$$

$$\Rightarrow -0.002(-0.2\lambda - 0) + (0.04-\lambda)(-0.01\lambda + \lambda^2) - 0)$$

$\Rightarrow -\lambda^2(\lambda - 0.05) = 0$ $\lambda = 0$ and $\lambda = 0.05$ so $\rho(T_{GS}) = 0.05 < 1$ ✓

$$\omega = \frac{2}{1+\sqrt{1-0.05}} = \boxed{1.01282}$$

3.

3. $$A = \begin{bmatrix}1&3&3\\-3&-5&-3\\3&3&1\end{bmatrix} \Rightarrow \quad \det(A-\lambda I)=0 \quad \begin{vmatrix}1-\lambda&3&3\\-3&-5-\lambda&-3\\3&3&1-\lambda\end{vmatrix}=0$$

$$3\cdot\begin{vmatrix}3&3\\-5-\lambda&-3\end{vmatrix} - 3\cdot\begin{vmatrix}1-\lambda&3\\-3&-3\end{vmatrix} + (1-\lambda)\begin{vmatrix}1-\lambda&3\\-3&-5-\lambda\end{vmatrix}=0$$

$$3(-9-(-15-3\lambda)) - 3((-3+3\lambda)+9) + (1-\lambda)((1-\lambda)(-5-\lambda)+9)=0$$

$$-(\lambda-1)(\lambda^2+4\lambda+4)=0, \quad (-\lambda-1)(\lambda+2)(\lambda+2)=0$$

$$\lambda = 1, -2, -2$$

For $\lambda = 1$:

$$\left[\begin{array}{ccc|c}0&3&3&0\\-3&-6&-3&0\\3&3&0&0\end{array}\right] \xrightarrow{R_1\leftrightarrow R_3} \left[\begin{array}{ccc|c}3&3&0&0\\-3&-6&-3&0\\0&3&3&0\end{array}\right] \xrightarrow[\frac{1}{3}R_3]{\frac{1}{3}R_1} \left[\begin{array}{ccc|c}1&1&0&0\\-3&-6&-3&0\\0&1&1&0\end{array}\right] \xrightarrow{R_2\to R_2+3R_1} \left[\begin{array}{ccc|c}1&1&0&0\\0&-3&-3&0\\0&1&1&0\end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|c}1&1&0&0\\0&1&1&0\\0&1&1&0\end{array}\right] \xrightarrow{R_3\to R_3-R_2} \left[\begin{array}{ccc|c}1&1&0&0\\0&1&1&0\\0&0&0&0\end{array}\right] \xrightarrow{R_1\to R_1-R_2} \left[\begin{array}{ccc|c}1&0&-1&0\\0&1&1&0\\0&0&0&0\end{array}\right]$$

$x_1 - x_3 = 0$

$x_2 + x_3 = 0$

$x_3 =$ free

$$x_3\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

For $\lambda = -2$:

$$\begin{bmatrix}3&3&3\\-3&-3&-3\\3&3&3\end{bmatrix} \xrightarrow[-\frac{1}{3}R_2,\ \frac{1}{3}R_3]{\frac{1}{3}R_1} \left[\begin{array}{ccc|c}1&1&1&0\\1&1&1&0\\1&1&1&0\end{array}\right] \xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-R_1} \left[\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

$x_2 =$ free

$x_3 =$ free

$x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$ so, $$x_2\begin{bmatrix}-1\\1\\0\end{bmatrix} + x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

x_2

x_3

$$\left\{\underset{v_1}{\begin{bmatrix}1\\-1\\1\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}-1\\1\\0\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}-1\\0\\1\end{bmatrix}}\right\} \xrightarrow{\text{Row Reduce}} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

v_1, v_2, v_3 are a set of linearly independent eigenvectors because there is a pivot in every column, meaning the solution vector is 0. $x_1 = x_2 = x_3 = 0$

4. A strictly diagonally dominant matrix means that its diagonal elements are larger in magnitude than the sum of the elements in its row. So, the center of each Gersgorin Circle has a larger magnitude than the radius of the circle. Because of this, a circle cannot be at the origin, or zero, meaning that zero cannot be an eigenvalue of the matrix and the matrix is nonsingular.