CS 3133 Foundations of Computer Science
C term 2018

Solutions for Homework 2

READING: Chapters 3, 4, 5, 18.

1. Exercise 2 on page 97.

   Solution:

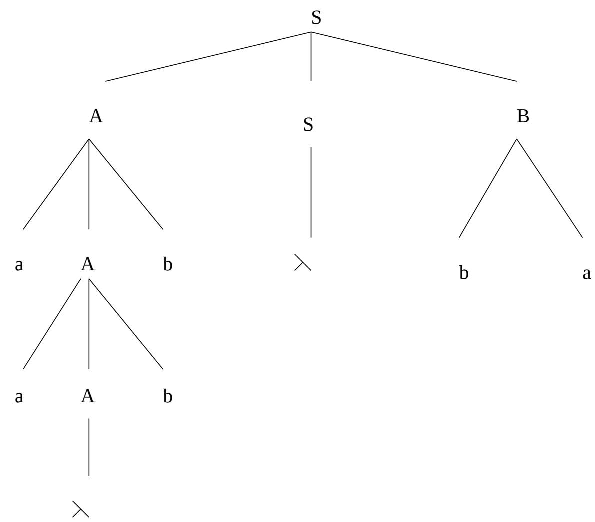
   (a) The following is a leftmost derivation of $aabbba$:

$$\begin{aligned} S &\Rightarrow ASB \\ &\Rightarrow aAbSB \\ &\Rightarrow aaAbbSB \\ &\Rightarrow aabbSB \\ &\Rightarrow aabbB \\ &\Rightarrow aabbba \end{aligned}$$

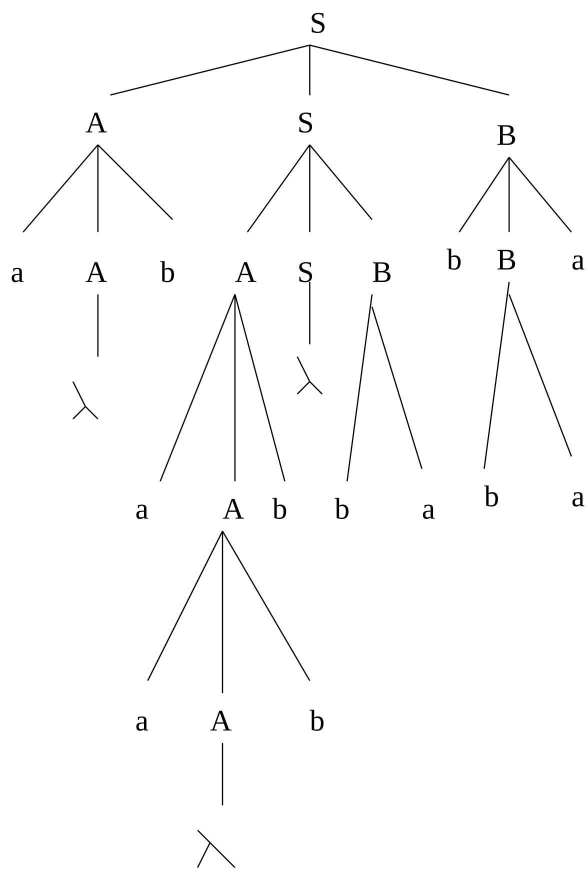
   (b) The following is a rightmost derivation of $abaabbbabbaa$:

$$\begin{aligned} S &\Rightarrow ASB \\ &\Rightarrow ASbBa \\ &\Rightarrow ASbbaa \\ &\Rightarrow AASBbbaa \\ &\Rightarrow AASbabbaa \\ &\Rightarrow AAbabbaa \\ &\Rightarrow AaAbbabbaa \\ &\Rightarrow AaaAbbbabbaa \\ &\Rightarrow Aaabbbabbaa \\ &\Rightarrow aAbaabbbabbaa \\ &\Rightarrow abaabbbabbaa \end{aligned}$$

(c) Derivation tree for (a):



Derivation tree for (b):



(d)

$$L(G) = \{a^{n_1}b^{n_1} \dots a^{n_k}b^{n_k}b^{m_1}a^{m_1} \dots b^{m_l}a^{m_l} | n_i, m_j, l \geq 0, k \geq 0, k \leq l\}$$

(15 points)

2. Exercise 4 on page 98.

Solution:

(a) The following is a leftmost derivation that generates the given tree

DT:

$$\begin{aligned} S &\Rightarrow AB \\ &\Rightarrow aAB \\ &\Rightarrow aaB \\ &\Rightarrow aaAB \\ &\Rightarrow aaaB \\ &\Rightarrow aaab \end{aligned}$$

(b) The following is a rightmost derivation that generates the given tree DT:

$$\begin{aligned} S &\Rightarrow AB \\ &\Rightarrow AAB \\ &\Rightarrow AAb \\ &\Rightarrow Aab \\ &\Rightarrow aAab \\ &\Rightarrow aaab \end{aligned}$$

(c) There are 20 derivations that generate DT. (20 points)

3. Exercise 11 on page 99.

   Solution: The following is a grammar over $\{a, b\}$ whose language is exactly $\{a^m b^i a^n | i = m + n\}$:

$$\begin{aligned} S &\rightarrow AB \mid \lambda \\ A &\rightarrow aAb \mid \lambda \\ B &\rightarrow bBa \mid \lambda \end{aligned}$$

   (15 points)

4. Show by induction that for every natural number n

$$0^2 + 1^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   Solution:

   Basis: It is true for $n = 0$.

   Inductive Hypothesis: Assume that it is true for all values $k = 0, 1, \ldots, n$, i.e.

$$\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

Inductive Step: We need to show that it is true for $n+1$, i.e.

$$\sum_{i=0}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}.$$

Indeed,

$$\sum_{i=0}^{n+1} i^2 = \sum_{i=0}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 =$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)(2n^2+7n+6)}{6} =$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}.$$

(15 points)

5. Let G be the grammar

$$\begin{aligned} &S \to ASB|\lambda \\ &A \to a \\ &B \to b. \end{aligned}$$

(a) What is $L(G)$?

(b) Prove formally (so using induction on the length of the derivations) that $L(G)$ is the set given in (a).

Solution:

(a) $L(G) = \{a^n b^n | n \geq 0\}$.

(b) First we show that $L(G) \subset \{a^n b^n | n \geq 0\}$. For this purpose we will show by induction on the length of the derivations that $n(a) + n(A) = n(b) + n(B)$ and that in all the strings in the derivation the a-s and A-s form a prefix of the string and the b-s and B-s form a suffix of the string.

Basis: Derivations of length 0, so S. True. Remark: Sometimes you have to start with derivations of length 1 as basis. Here the statement is true for $n = 0$ so we can start with $n = 0$ as basis, but sometimes this is not case.

Inductive Hypothesis: We assume that this statement is true for all strings w that can be obtained by n rule applications, so $S \overset{n}{\Rightarrow} w$.

Inductive Step: We have to show that the statement is true for all strings w that can be obtained by $n+1$ rule applications, so $S \overset{n+1}{\Rightarrow} w$. Once again the key step is to reformulate the derivation to apply the inductive hypothesis. The derivation of w can be written $S \overset{n}{\Rightarrow} w' \Rightarrow w$. By the Inductive Hypothesis we know that the statement is true for w', so $n_{w'}(a) + n_{w'}(A) = n_{w'}(b) + n_{w'}(B)$ (say $= k$) and the a-s and A-s form a prefix of w' and the b-s and B-s form a suffix of w'. But these obviously remain true when we apply one more rule:

Rule	$n_w(a) + n_w(A)$	$n_w(b) + n_w(B)$
$S \to ASB$	$k+1$	$k+1$
$S \to \lambda$	k	k
$A \to a$	k	k
$B \to b$	k	k

Next we show that $\{a^n b^n | n \geq 0\} \subset L(G)$. Indeed,

$$\begin{aligned} S &\overset{n}{\Rightarrow} \underbrace{A \dots A}_{n} S \underbrace{B \dots B}_{n} \\ &\Rightarrow \underbrace{A \dots A}_{n} \underbrace{B \dots B}_{n} \\ &\overset{n}{\Rightarrow} \underbrace{a \dots a}_{n} \underbrace{B \dots B}_{n} \\ &\overset{n}{\Rightarrow} \underbrace{a \dots a}_{n} \underbrace{b \dots b}_{n} \end{aligned}$$

(20 points)

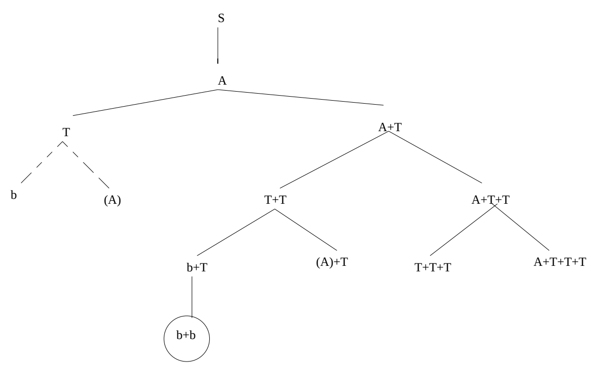
6. In this problem we consider the grammar of arithmetic expressions AE, so

$$\begin{array}{lll} AE: & V = & \{S, A, T\} \\ & \Sigma = & \{b, +, (,)\} \\ & P: & 1. S \to A \\ & & 2. A \to T \\ & & 3. A \to A + T \\ & & 4. T \to b \\ & & 5. T \to (A) \end{array}$$

Build the search tree constructed by the breadth-first top-down parsing algorithm for the string $b + b$.

Solution:

The breadth-first top-down search tree:



(15 points)