We discussed relations and functions. A relation R from A to B is a subset $R \subseteq A \times B$. A relation on $A \times A$ can be reflexive, symmetric or transitive. A relation which is reflexive, symmetric and transitive is called an equivalence relation. An equivalence relation on $A \times A$ partitions A into equivalence classes.

A relation $R \subseteq A \times B$ is functional if

- For each $a \in A$, there is an element $(a, b) \in R$.
- If $(a, b) \in R$ and $(a, b') \in R$, then $b = b'$.

Special kinds of functions are one-to-one and onto.

If $f : A \to B$ is one-to-one then $|A| \leq |B|$.

If $g : A \to B$ is onto then $|A| \geq |B|$.

We counted the number of functions of various kinds from a finite set A to a finite set B.

1. Let $A = \{a, b, c\}$ and $B = \{a, c, e, g\}$.

   a) How many relations between A and B are there? List three of them.

   b) How many relations between B and A are there? List three of them.

   c) List three relations which are both from A to B and from B to A. How many such things are there?

2. Let $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   Define an equivalence relation on D for which $\{1, 5, 7\}$ is an equivalence class.

3. Let $E = \{0, 1, 2\}$

   a) Define a relation X on E, $X \subseteq E \times E$, which is symmetric and transitive, but not reflexive.

   b) Define a relation Y on E, $Y \subseteq E \times E$, which is symmetric and reflexive, but not transitive.

   c) Define a relation Z on E, $Z \subseteq E \times E$, which is reflexive and transitive, but not symmetric.

   d) Define a relation W on E, $W \subseteq E \times E$, which is not reflexive, not symmetric, and not transitive.

   [Note, there are many of each type.]

4. Let S be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by

   $$S = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid n = 2^k m, k \in \mathbb{Z}\}$$

   a) Show that S is an equivalence relation.

   b) Find the equivalence class of 0.

   c) Find the equivalence class of 500.

   d) Find the equivalence class of -500.

   e) How many equivalence classes are finite?

5. Let S' be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$S' = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid n = 2^k m, k \in \mathbb{N}\}$$

   a) Is S' symmetric?

   b) Is S' reflexive?

   c) Is S' transitive?

   d) What is the equivalence class of 500?

6. Let M be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$M = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid n + m \text{ is even.}\}$$

   Is M an equivalence relation? If so prove it. If not, state which of the properties is violated, and how.

   If M is an equivalence relation, how many equivalence classes are there?

7. Let N be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$N = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid n + m \text{ is odd.}\}$$

   Is N an equivalence relation? If so prove it. If not, state which of the properties is violated, and how.

   If N is an equivalence relation, how many equivalence classes are there?

8. Let $X = \{1, 3, 5, 7, 9\}$ and $Y = \{2, 4, 6, 8\}$.

   a) How many functions are there $X \to Y$?

   b) How many functions are there $Y \to X$?

   c) How many one-to-one functions are there $X \to Y$?

   d) How many one-to-one functions are there $Y \to X$?

   e) How many onto functions are there $X \to Y$?

   f) How many onto functions are there $Y \to X$?

   [We did not work out a formula for onto functions, but this is a small enough example that we can just work out by hand.]

9. Let $X = \{1, 3, 5, 7, 9\}$ and $Y = \{0, 2, 4, 6, 8\}$.

   a) Give a function from X to Y which is onto but not one-to-one.

   b) Give a function from X to Y which is one-to-one but not onto.

   c) Give a function from X to Y which is both one-to-one and not onto.

   d) Give a function from X to Y which is neither one-to-one nor onto.