Numerical Analysis: HW 2

1a)

$E_1: a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = a_{1,n+1}$

$E_2: a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = a_{2,n+1}$

$\vdots$

$E_n: a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = a_{n,n+1}$

Augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2,n+1} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{n,n+1} \end{array}\right]$$

Perform Gaussian Forward Elim:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \dots & a_{2n} & a_{2,n+1} \\ 0 & 0 & \dots & a_{nn} & a_{n,n+1} \end{array}\right]$$

From class, we know:

mult/div : $\frac{2n^3+3n^2-5n}{6}$

add/sub : $\frac{n^3-n}{3}$

Perform Backward Elim:

$m_{n-1,n} = \frac{a_{n-1,n}}{a_{nn}} \to 1 \text{ div}, \dots, m_{1n} = \frac{a_{1n}}{a_{nn}} = 1 \text{ div} = \underset{n-1 \text{ div}}{2 \text{ div}}$

so $(i-1)$ mult/div

$R_{n-1} \to R_{n-1} - m_{n-1,n}\cdot R_n \quad \dots \quad R_1 \to R_1 - m_{1n}\cdot R_n$

2 mult and 2 add/sub $\to$ # rows $n-1$ sub

so $(i-1)$ add/sub

$\Rightarrow$ $(i-1)$ mult/div

$$\left[\begin{array}{ccc|c} \hat{a}_{11} & 0 \dots & 0 & a_{1,n+1} \\ 0 & \hat{a}_{22} \dots & 0 & a_{2,n+1} \\ 0 & 0 \dots & \hat{a}_{nn} & a_{n,n+1} \end{array}\right]$$

After, get solution by

$x_i = \frac{\hat{a}_{i,n+1}}{\hat{a}_{ii}}, \ i = 1, \dots, n \to n$ divisions

$\Rightarrow$ mult/div is $(i-1) + (i-1) = 2(i-1)$

$$\frac{2n^3+3n^2-5n}{6} + n + \underbrace{\sum_{i=2}^{n} 2(i-1)}$$

$$\sum_{i=2}^{n} 2i - 2 = 2\sum_{i=2}^{n} i - 2\sum_{i=2}^{n} 1 = 2\left(\frac{n(n+1)}{2}\right) - 2n \ ✓$$

$$\frac{2n^3+3n^2-5n}{6} + n + \frac{2n^2+2n}{2} - 2n$$

$$\frac{2n^3+3n^2-5n}{6} \cancel{+n} + n^2 + \cancel{n - 2n} = \frac{2n^3}{3} + \frac{n^2}{2} + n^2 - \frac{5n}{6} = \boxed{\frac{n^3}{3} + \frac{3n^2}{2} - \frac{5n}{6}}$$

$\Rightarrow$ add/sub is $(i-1)$

$$\frac{n^3-n}{3} + \underbrace{\sum_{i=2}^{n} (i-1)}$$

$$\sum_{i=2}^{n}(i-1) = \sum_{i=2}^{n} i - \sum_{i=2}^{n} 1 = \frac{n(n+1)}{2} - n$$

$$= \frac{n^3-n}{3} + \frac{n^2+n}{2} - n$$

$$= \frac{n^3}{3} - \frac{n}{3} + \frac{n^2}{2} + \frac{n}{2} - n$$

$$= \boxed{\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}} \ ✓$$

Verified ✓

1b.

```
n=[3,10,50,100]

%Gaussian Elimination + back substitution

    %mult/div
    Gmultdiv=(n.^3/3)+(n.^2)-(n/3)
    %add/sub
    Gaddsub=(n.^3/3)+(n.^2/2)-(5*n/6)

%Hybrid

    %mult/div
    Hmultdiv=(n.^3/3)+((3*n.^2)/2)-(5*n/6)
    %add/sub
    Haddsub=(n.^3/3)+((n.^2)/2)-(5*n/6)

    %print out a table with the counts
    T=table(n', Gmultdiv', Gaddsub', Hmultdiv', Haddsub')
    T.Properties.VariableNames = {'n' 'Gmultdiv' 'Gaddsub' 'Hmultdiv'
'Haddsub'}
```

Output:

```
>> Gtable

T =

  4×5 table

     n      Gmultdiv       Gaddsub        Hmultdiv       Haddsub
    ___    __________    ___________    ___________    ___________

      3            17             11             20             11
     10           430            375            475            375
     50         44150          42875          45375          42875
    100     3.433e+05     3.3825e+05     3.4825e+05     3.3825e+05
```

1c.

```
function h = hybrid(A)

% Function to perform hybrid elimination (Gaussian forward and backward
elimination)
% on the augmented matrix A of an n-by-n linear system of equations
%
% Input: A = n-by-(n+1) augmented matrix
```

```
%
% Output: x = n-by-1 solution vector


%input matrix
A = [1 1 0 3 4; 2 1 -1 1 1; 3 -1 -1 2 -3; -1 2 3 -1 4];

% Determine size of system (n-by-n)
n = size(A,1);

% Gaussian (forward) elimination
for i = 1:n-1
    % Switch rows (if necessary)
    for p = 2:n
        % Moves row with zero in A(i,i) entry to last row
        if ( A(i,i) == 0 )
            temp   = A(i,:);
            A(i,:) = A(p,:);
            A(p,:) = temp;
        end
    end

    % Perform forward elimination
    for j = i+1:n %iterate forward
        mu     = A(j,i)/A(i,i); %find multiplying constant m
        A(j,:) = A(j,:) - mu*A(i,:); %Perform the row manipulation Rn->Rn-
m*Rn-1
    end

end

% Perform backward elimination
for i = n:-1:2 %iterate backward
    for j=i-1:-1:1
        mu     = A(j,i)/A(i,i); % same as forward above
        A(j,:) = A(j,:) - mu*A(i,:);
    end
end

% Initialize solution vector
x = zeros(n,1);

% Find entries of solution vector
x(n) = A(n,n+1)/A(n,n);
for i = n-1:-1:1
    x(i) = A(i,n+1)/A(i,i);
end

%display matrix A
A
%display solution matrix
x
```

```
Output:

>> hybrid

A =

     1     0     0     0    -1
     0    -1     0     0    -2
     0     0     3     0     0
     0     0     0   -13   -13


x =

    -1
     2
     0
     1

Where x1=-1, x2=2, x3=0, and x4=1
```

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2a) $(2+1)i_1 + 2i_2 = 12+10$ $\qquad 3i_1 + 2i_2 = 22$

$(2+1)i_1 + 4i_3 = 12 \quad \Rightarrow \quad 3i_1 + 4i_3 = 12$

$i_1 - i_2 - i_3 = 0 \qquad i_1 - i_2 - i_3 = 0$

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 22 \\ 3 & 0 & 4 & 12 \\ 1 & -1 & -1 & 0 \end{array}\right] \begin{array}{l} R_2 \to \\ R_2 - \frac{3}{3}R_1 \end{array} \left[\begin{array}{ccc|c} 3 & 2 & 0 & 22 \\ 0 & -2 & 4 & -10 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{l} R_3 \to \\ R_3 - \frac{1}{3}R_1 \end{array} \left[\begin{array}{ccc|c} 3 & 2 & 0 & 22 \\ 0 & -2 & 4 & -10 \\ 0 & -\frac{5}{3} & -1 & -\frac{22}{3} \end{array}\right] \begin{array}{l} R_3 \to \quad \frac{5}{6} \\ R_3 - \frac{-\frac{5}{3}}{-2} R_2 \end{array} \left[\begin{array}{ccc|c} 3 & 2 & 0 & 22 \\ 0 & -2 & 4 & -10 \\ 0 & 0 & -\frac{13}{3} & 1 \end{array}\right]$$

$-\frac{13}{3} i_3 = 1 \qquad -2i_2 + 4\left(-\frac{3}{13}\right) = -10 \qquad 3i_1 + 2\left(\frac{59}{13}\right) + 0 = 22 \quad -\frac{118}{13}$

$i_3 = -\frac{3}{13} \qquad i_2 = \frac{-118}{26} = \frac{59}{13} \qquad i_1 = \frac{168}{39} = \frac{56}{13}$

b) $(0.001 + 0.012)i_1 + 3.33 i_2 = 22 \qquad 0.0013 i_1 + 3.33 i_2 = 22$

$(0.001 + 0.012)i_1 + 4.00 i_3 = 12 \Rightarrow 0.0013 i_1 + 4.00 i_3 = 12$

$i_1 - i_2 - i_3 = 0 \qquad i_1 - i_2 - i_3 = 0$

$$\left[\begin{array}{ccc|c} 0.013 & 3.33 & 0 & 22 \\ 0.013 & 0 & 4.00 & 12 \\ 1 & -1 & -1 & 0 \end{array}\right] \begin{array}{l} R_2 \to \\ R_2 - \frac{0.013}{0.013} R_1 \end{array} \left[\begin{array}{ccc|c} 0.013 & 3.33 & 0 & 22 \\ 0 & -3.33 & 4.00 & -10 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{l} R_3 \to \\ R_3 - \frac{1}{0.013} R_1 \end{array} \left[\begin{array}{ccc|c} 0.013 & 3.33 & 0 & 22 \\ 0 & -3.33 & 4.00 & -10 \\ 0 & -257 & -1 & -169 \times 10 \end{array}\right] \begin{array}{l} R_3 \to \\ R_3 - \frac{-257}{-3.33} \cdot R_2 \end{array} \left[\begin{array}{ccc|c} 0.013 & 3.33 & 0 & 22 \\ 0 & -3.33 & 4.00 & -10 \\ 0 & 0 & -309 & -918 \end{array}\right]$$

$-309 i_3 = -918 \qquad -3.33 i_2 + 4.00(2.97) = -10 \qquad 0.013 i_1 + 3.33(6.57) = 22$

$i_3 = 2.97 \qquad i_2 = 6.57 \qquad i_1 = 9.37$

c) Partial Pivoting

$\max\{|a_{11}|, |a_{21}|, |a_{31}|\} = |a_{31}| = 1$ Swap

$$\begin{array}{l} R_1 \leftrightarrow R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0.0013 & 0 & 4.00 & 12 \\ 0.0013 & 3.33 & 0 & 22 \end{array}\right] \begin{array}{l} R_2 \to \\ R_2 - \frac{0.0013}{1} \cdot R_1 \\ R_3 \to \\ R_3 - \frac{0.0013}{1} R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0.0013 & 4.00 & 12 \\ 0 & 3.33 & 0.0013 & 22 \end{array}\right]$$

$\max\{|a_{22}|, |a_{32}|\} = a_{32} = 3.33$ swap

$R_2 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 3.33 & 0.0013 & 22 \\ 0 & 0.0013 & 4.00 & 12 \end{array}\right] \xrightarrow{R_3 \to R_3 - \frac{0.0013}{3.33}R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 3.33 & 0.0013 & 22 \\ 0 & 0 & 4.00 & 11.9 \end{array}\right]$

$4.00 i_3 = 11.9$ $\quad 3.33 i_2 + 0.0013(2.97) = 22$ $\quad i_1 - 6.60 - 2.97 = 0$

$i_3 = 2.97$ $\quad i_2 = 6.60$ $\quad i_1 = 9.57$

Much better result because the numbers are smaller to deal with and so the error due to chopping/rounding is less.

3. $\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 6 & 8 & 5 \\ 6 & \alpha & 10 & 5 \end{array}\right]$ $S_1 = \max\{|a_{11}|, |a_{12}|, |a_{13}|\} = 3$

$S_2 = \max\{|a_{21}|, |a_{22}|, |a_{23}|\} = 8$ $\quad |\alpha| < 10$

$S_3 = \max\{|a_{31}|, |a_{32}|, |a_{33}|\} = 10$

largest no swap $\to$ $\frac{|a_{11}|}{S_1} = \frac{2}{3}$ $\quad \frac{|a_{21}|}{S_2} = \frac{1}{2}$ $\quad \frac{|a_{31}|}{S_3} = \frac{3}{5}$

$R_2 \to R_2 - \frac{4}{2}R_1$
$R_3 \to R_3 - \frac{6}{2}R_1$

$\begin{array}{c} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & \alpha-3 & 1 & 2 \end{array}\right] \end{array} \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array}$ $\quad \frac{|a_{22}|}{S_2} = \frac{4}{8} = \frac{1}{2}$ $\quad \frac{|a_{32}|}{S_3} = \frac{|\alpha-3|}{10}$ must be less than $\frac{1}{2}$

The value of α must be (a) 6 because $\frac{|\alpha-3|}{10}$ must be less than $\frac{1}{2}$ to not exchange rows.

$\frac{(6)-3}{10} = 0.3$ ✓ $\quad \frac{(9)-3}{10} = 0.6$ × $\quad \frac{|(-3)-3|}{10} = 0.6$ ×