Exercises for Lectures 10 and 11

These lectures considered functions and cardinality.

$|A| < |B|$: There is a one-to-one function from A to B.

or

There is an onto function from B to A.

$|A| = |B|$: $|A| \leq |B|$ and $|B| \leq |A|$

or

There is a one-to-one onto function from A to B.

For any set X, and any function $f : X \to \mathcal{P}(X)$, we defined the set $C_f = \{x \in X \mid x \notin f(x)\}$ and used it to show that there is no onto function from X to $\mathcal{P}(X)$.

We described Cantor's diagonal argument (a variation on this) to show that $\mathbb{R}$ is uncountable.

We showed that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all countably infinite.

We showed that $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$ are both uncountable.

We showed that if A and B are countably infinite, then $A \cup B$ and $A \times B$ are both countably infinite.

We showed that if you have countably infinite collection of countably infinite sets, A_1, A_2, A_3, A_4, A_5, $\ldots$, then the set

$$\bigcup_{i=1}^{\infty} A_i$$

is countably infinite.

But noted that then the set

$$A_1 \times A_2 \times A_3 \times \cdots$$

is uncountable.

Exercises

1. Definite a function $\mathcal{P}(\mathbb{N}) \to \mathbb{N}$ which is onto. What does this say about the cardinalities of $\mathcal{P}(\mathbb{N})$ and $\mathbb{N}$?

2. Definite a function $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ which is one-to-one. What does this say about the cardinalities of $\mathcal{P}(\mathbb{N})$ and $\mathbb{N}$?

3. Define a function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ for which $C_f = \mathbb{N}$. Show that the set $\mathbb{N}$ is not the image of any element in $\mathbb{N}$ under f.

4. Define a function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ for which $C_f = \emptyset$. Show that the set $\emptyset$ is not the image of any element in $\mathbb{N}$ under f.

5. Define a function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ for which $C_f = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

6. Let $X = \{n \in \mathbb{N} \mid n \text{ is evenly divisible by 1 million}\}$. Define a function which shows that X is countably infinite.

   Is this compatible with the fact that $X \subseteq \mathbb{N}$?

7. Let $Y = \{n \in \mathbb{N} \mid \text{the decimal representation of } n \text{ contains no 7.}\}$. Define a function which shows that X is countably infinite.

8. Let $Z = \{n \in \mathbb{R} \mid \text{the decimal representation of } n \text{ contains only 4's and 7's.}\}$. Define a function which shows that X is uncountable.

9. Find a one-to-one onto function with domain $A \times (B \times C)$ and target $(A \times B) \times C$.

10. Label each of the following as finite, countable, or uncountable.

    (a) The set of subsets of $\mathbb{Q}$.

    (b) $\mathcal{P}(\mathbb{R})$.

    (c) The set of functions with domain $\mathbb{Z} \times \mathbb{Z}$ and target $\{0, 1\} \times \{0, 1\}$.

    (d) The set of functions with domain $\{0, 1\} \times \{0, 1\}$ and target $\mathbb{Z} \times \mathbb{Z}$.

    (e) the set of positive prime integers.

    (f) the set of finite subsets of $\mathbb{Z}$. [Hint: Write it as a countable union of finite sets.]

    (g) The set of non-increasing infinite sequences of elements of $\mathbb{N}$, that is sequences

    $$\{a_k\}_{k \in \mathbb{N}}$$

    such that $a_{k+1} \leq a_k$ for all k.

11. **Challenging: Don't do this unless you found the problems above were too easy.**

    Let

    $\{X \in \mathcal{P}(\mathbb{R}) \mid \text{X is the set of real roots of some polynomial with rational coeficients.}\}$

    Find three elements of X, each with different cardinalities.

    Show that X is countably infinite.