**1.**

$f(x) = e^{-x}$

$f'(x) = -e^{-x}$

$f''(x) = e^{-x}$

# Homework 3

#1)

| $x_n$ | $f(x_n)$ |
|---|---|
| $x_0 = 0$ | 1 |
| $x_1 = 0.25$ | 0.778801 |
| $x_2 = 0.75$ | 0.472367 |
| $x_3 = 1$ | 0.367879 |

#1a)

$n = 3$

$$S(x) = \begin{cases} a_0 + b_0(x\ \ ) + c_0(x\ \ )^2 + d_0(x\ \ )^3, & x \in [0, 0.25] \\ a_1 + b_1(x-0.25) + c_1(x-0.25)^2 + d_1(x-0.25)^3 & x \in [0.25, 0.75] \\ a_2 + b_2(x-0.75) + c_2(x-0.75)^2 + d_2(x-0.75)^3 & x \in [0.75, 1] \end{cases}$$

$4 \cdot 3 = 12$ unknown.

$a_0 = f(0) = 1 \quad a_1 = 0.778801 \quad a_2 = 0.472367 \quad a_3 = 0.367879$

$h_0 = x_1 - x_0 = 0.25 \quad h_1 = x_2 - x_1 = 0.5 \quad h_2 = 1 - 0.75 = 0.25$

Linear system:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad x = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1.5 & 0.5 & 0 \\ 0 & 0.5 & 1.5 & 0.25 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.815784 \\ 0.584748 \\ 0 \end{bmatrix}$$

$$\boxed{c_0 = 0 \quad c_1 = 0.465651 \quad c_2 = 0.234615 \quad c_3 = 0}$$

$$b_j = \frac{a_{j+1} - a_j}{h_j} - \frac{h_j(c_{j+1} + 2c_j)}{3}, \quad d_j = \frac{(c_{j+1} - c_j)}{3h_j}$$

$b_0 = \frac{a_1 - a_0}{h_0} - \frac{h_0(c_1 + 2c_0)}{3} = -0.962703 \quad d_0 = \frac{c_1 - c_0}{3h_0} = 0.620868$

$b_1 = \frac{a_2 - a_1}{h_1} - \frac{h_1(c_2 + 2c_1)}{3} = -0.807188 \quad d_1 = \frac{c_2 - c_1}{3h_1} = -0.154024$

$b_2 = \frac{a_3 - a_2}{h_2} - \frac{h_2(c_3 + 2c_2)}{3} = -0.457055 \quad d_2 = \frac{c_3 - c_2}{3h_2} = -0.31282$

$$S(x) = \begin{cases} 1 - 0.962703x + 0.620868x^3, & x \in [0, 0.25] \\ 0.778801 - 0.807188(x-0.25) + 0.465651(x-0.25)^2 \\ - 0.154024(x-0.25)^3, & x \in [0.25, 0.75] \\ 0.472367 - 0.457055(x-0.75) + 0.234615(x-0.75)^2 \\ - 0.31282(x-0.75)^3, & x \in [0.75, 1] \end{cases}$$

#1b) $\int_0^1 S(x) = \int_0^{0.25} S_0(x)\,dx + \int_{0.25}^{0.75} S_1(x)\,dx + \int_{0.75}^{1.0} S_2(x)\,dx$

$$\int_0^{0.25} 1-0.962703x+0.620868x^3\,dx$$
$$1x-\frac{0.962703x^2}{2}+\frac{0.620868x^4}{4}\Big|_0^{0.25}$$
$$=0.220522$$

$$\int_{0.25}^{0.75} 0.778801-0.807188(x-0.25)+0.465651(x-0.25)^2$$
$$-0.154024(x-0.25)^3\,dx$$
$$0.778801x-\frac{0.807188(x-0.25)^2}{2}+\frac{0.465651(x-0.25)^3}{3}$$
$$-\frac{0.154024(x-0.25)^4}{4}\Big|_{0.25}^{0.75}$$
$$=0.305498$$

$$\int_{0.75}^{1} 0.472367-0.457055(x-0.75)+0.234615$$
$$(x-0.75)^2-0.31282(x-0.75)^3\,dx$$
$$=0.104725$$

$$0.220522+0.305498+0.104725$$
$$=0.630745$$

Very close to $1-\frac{1}{e}$ ✓

#1c) $f'(0.5)=-0.606531$

$f''(0.5)=0.606531$

$$S'(x)=-0.807188+2\cdot0.465651(x-0.25)$$
$$+3\cdot-0.154024(x-0.25)^2$$
$$S'(0.5)=-0.603242$$

$$S''(x)=2\cdot0.465651+6\cdot-0.154024(x-0.25)$$
$$S''(0.5)=0.700266$$

They are very close!

**2.**

Please look at MATLAB file called newtonsDD for code and its explanation.

Output:

```
>> X = [1.0, 1.3, 1.6, 1.9, 2.2]

X =

    1.0000    1.3000    1.6000    1.9000    2.2000

>> Y = [0.7651977, 0.6200860, 0.4554022, 0.2818186, 0.1103623]

Y =

    0.7652    0.6201    0.4554    0.2818    0.1104

>> newtonsDD(X, Y)


ans =

    0.7652         0         0         0         0
    0.6201   -0.4837         0         0         0
    0.4554   -0.5489   -0.1087         0         0
    0.2818   -0.5786   -0.0494    0.0659         0
    0.1104   -0.5715    0.0118    0.0681    0.0018
```

4th Lagrange Polynomial:

#2. $P_4(x) = 0.7652 - 0.4837(x-1) - 0.1087(x-1)$
$\cdot(x-1.3) + 0.0659(x-1)(x-1.3)(x-1.6)$
$+ 0.0018(x-1)(x-1.3)(x-1.6)(x-1.9)$

**3. Part A**

#3. Degree 1

$f(x_i) = \theta_1 + \theta_2 x_i$

$i = 1, \ldots, n$

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \\ 1 & x_5 \\ 1 & x_6 \\ 1 & x_7 \\ 1 & x_8 \end{bmatrix} \quad \text{so} \quad X = \begin{bmatrix} 1 & 0.2 \\ 1 & 0.3 \\ 1 & 0.6 \\ 1 & 0.9 \\ 1 & 1.1 \\ 1 & 1.3 \\ 1 & 1.4 \\ 1 & 1.6 \end{bmatrix}_{8\times2} \quad \vec{y} = \begin{bmatrix} 0.050446 \\ 0.098426 \\ 0.33277 \\ 0.72660 \\ 1.0972 \\ 1.5697 \\ 1.8487 \\ 2.5015 \end{bmatrix}_{8\times1}$$

$\theta_1, \theta_2$

$$\hat{\theta} = (X^T X)^{-1} X^T \vec{y}$$

$y = -0.5125 + 1.6655x$

error: $e_i = \sum (y_i - \theta_1 - \theta_2 x_i)^2 = 0.3356$

Degree 2

$f(x_i) = \theta_1 + \theta_2 x_i + \theta_3 x_i^2$

$$X = \begin{bmatrix} 1 & 0.2 & 0.2^2 \\ 1 & 0.3 & 0.3^2 \\ 1 & 0.6 & 0.6^2 \\ 1 & 0.9 & 0.9^2 \\ 1 & 1.1 & 1.1^2 \\ 1 & 1.3 & 1.3^2 \\ 1 & 1.4 & 1.4^2 \\ 1 & 1.6 & 1.6^2 \end{bmatrix}_{8\times3} \quad \vec{y} = \begin{bmatrix} \text{same} \\ \text{as} \\ \text{above} \end{bmatrix}_{8\times1}$$

$y = 0.0851 - 0.3114x + 1.1294x^2$

error: $e_i = \sum (y_i - \theta_1 - \theta_2 x_i - \theta_3 x_i^2)^2 = 0.0024$

Degree 3

$f(x_i) = \theta_1 + \theta_2 x_i + \theta_3 x_i^2 + \theta_4 x_i^3$

$$X = \begin{bmatrix} 1 & 0.2 & 0.2^2 & 0.2^3 \\ 1 & 0.3 & 0.3^2 & 0.3^3 \\ 1 & 0.6 & 0.6^2 & 0.6^3 \\ 1 & 0.9 & 0.9^2 & 0.9^3 \\ 1 & 1.1 & 1.1^2 & 1.1^3 \\ 1 & 1.3 & 1.3^2 & 1.3^3 \\ 1 & 1.4 & 1.4^2 & 1.4^3 \\ 1 & 1.6 & 1.6^2 & 1.6^3 \end{bmatrix}_{8\times4}$$

$\vec{y}$ is same as above $(8\times1)$

$y = -0.0184 + 0.2484x + 0.4029x^2 + 0.2662x^3$

error: $e_i = \sum (y_i - \theta_1 - \theta_2 x_i - \theta_3 x_i^2 - \theta_4 \ldots$

$= 5.0913 \times 10^{-6}$

Please look at the MATLAB file called HW3_Problem3and4.m for calculations and code explanations for this problem.

Output:

>>

degree1 =

-0.5125
1.6655

error =

0.3356

degree2 =

0.0851
-0.3114
1.1294

error =

0.0024

degree3 =

-0.0184
0.2484
0.4029
0.2662

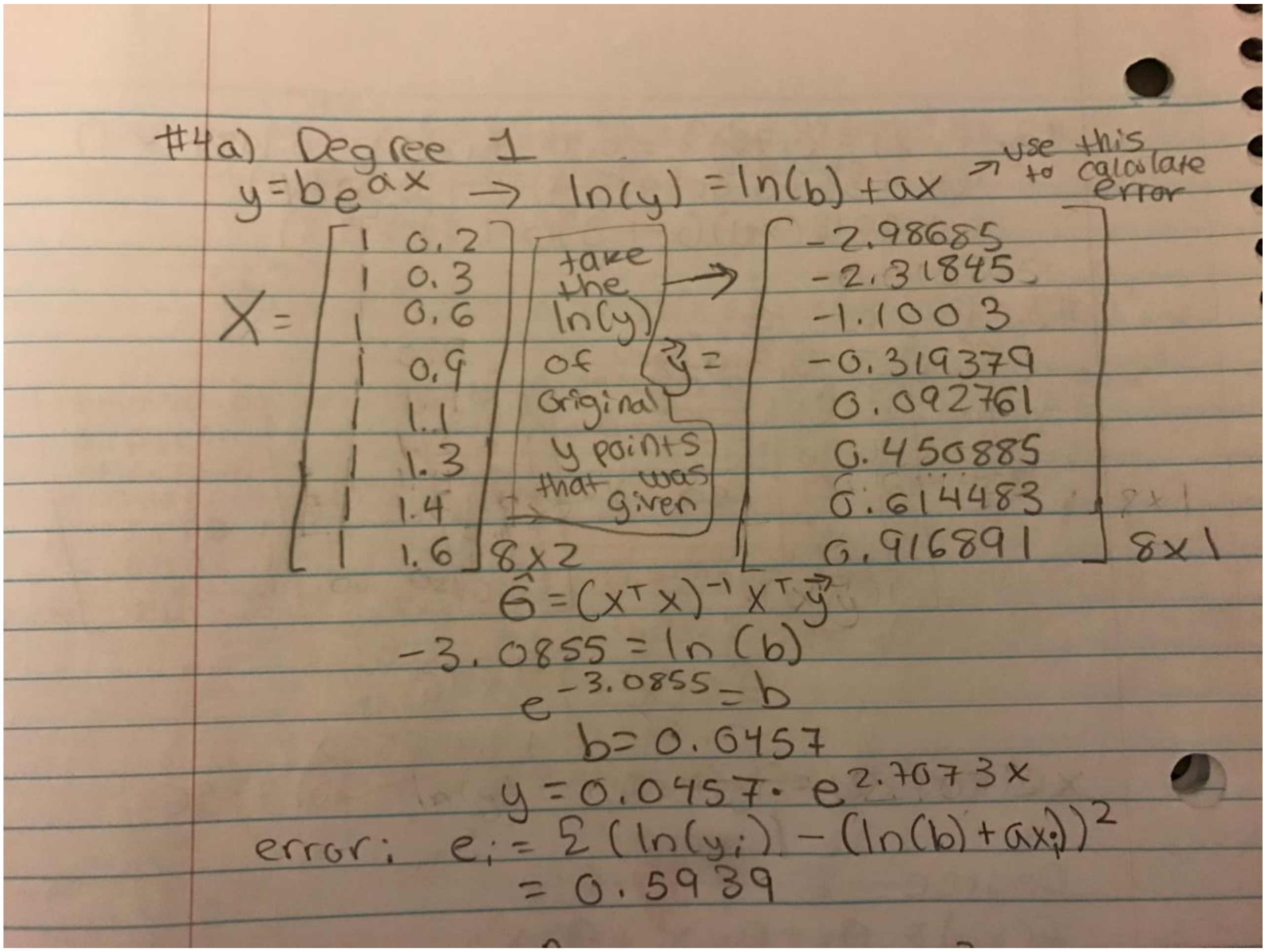
error =

5.0913e-06

**Part B**

Please look at the MATLAB file called HW3_Problem3and4.m for graph labeled figure 1.
Blue line – represents the linear degree one polynomial
Red line – represents the quadratic degree two polynomial
Green line – represents the cubic degree three polynomial

**4. Part A**



#4a) Degree 1

$y = be^{ax} \rightarrow \ln(y) = \ln(b) + ax$ use this to calculate error

$$X = \begin{bmatrix} 1 & 0.2 \\ 1 & 0.3 \\ 1 & 0.6 \\ 1 & 0.9 \\ 1 & 1.1 \\ 1 & 1.3 \\ 1 & 1.4 \\ 1 & 1.6 \end{bmatrix}_{8\times 2}$$

take the ln(y) of original y points that was given

$$\vec{y} = \begin{bmatrix} -2.98685 \\ -2.31845 \\ -1.1003 \\ -0.319379 \\ 0.092761 \\ 0.450885 \\ 0.614483 \\ 0.916891 \end{bmatrix}_{8\times 1}$$

$$\hat{\theta} = (X^T X)^{-1} X^T \vec{y}$$

$$-3.0855 = \ln(b)$$

$$e^{-3.0855} = b$$

$$b = 0.0457$$

$$y = 0.0457 \cdot e^{2.7073x}$$

error: $e_i = \sum (\ln(y_i) - (\ln(b) + ax_i))^2$

$= 0.5939$

Please look at the MATLAB file called HW3_Problem3and4.m for calculations and code explanations for this problem.

Output:

```
>>

logDegree1 =

  -3.0855
   2.7073


error4 =
```

0.5939

**BONUS: Part B**

#4b) $\min \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$

Output: $a_0, a_1$

Use fminsearch:

$\min_{x} f(x)$

Output $x$

In this case, $y = be^{ax}$

$\min \sum_{i=1}^{n} (y_i - b e^{ax})^2$

Output $(a, b)$

Output:

```
>>

ans =

   0.1326    1.8583
```

Please look at the MATLAB file called HW3_Problem3and4.m for graph labeled figure 2.
Blue line – represents the model solution from part A
Red line – represents the model solution in part B using the fminsearch function

My least squares estimates for a and b are smaller than my estimates in part A. The approximation using the fminsearch function gives me the better fit (red line). I can see this clearly when looking at the plot in figure 2.