Homework 3: CS4032

1. $LUx=b$

$$\begin{bmatrix}1&0&0\\2&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}2&3&-1\\0&-2&1\\0&0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

$Ly=b$

$$\begin{bmatrix}1&0&0\\2&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}2\\-1\\1\end{bmatrix} \qquad \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}2\\-5\\3\end{bmatrix}$$

$Ux=y$

$$\begin{bmatrix}2&3&-1\\0&-2&1\\0&0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-5\\3\end{bmatrix} \qquad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-3\\3\\1\end{bmatrix}$$

2. a) $$A=\begin{bmatrix}1&2&-1\\1&2&3\\2&-1&4\end{bmatrix}\begin{matrix}R_2\to R_2-R_1\\R_3\to R_3-2R_1\end{matrix}\begin{bmatrix}1&2&-1\\0&0&4\\0&-5&6\end{bmatrix}\xleftrightarrow{R_2 \leftrightarrow R_3}\begin{bmatrix}1&2&-1\\0&-5&6\\0&0&4\end{bmatrix}=U$$

$$L=\begin{bmatrix}1&0&0\\2&1&0\\1&0&1\end{bmatrix} \qquad P=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$ Since only one switch

$$\underset{A}{\begin{bmatrix}1&2&-1\\1&2&3\\2&-1&4\end{bmatrix}}=\underset{P^T}{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}}\cdot\underset{L}{\begin{bmatrix}1&0&0\\2&1&0\\1&0&1\end{bmatrix}}\cdot\underset{U}{\begin{bmatrix}1&2&-1\\0&-5&6\\0&0&4\end{bmatrix}} \checkmark$$

b) The LU decomposition and permutation matrix are different on MATLAB because when calculating it by hand we are using Doolittle's method, but MATLAB may be using a different method such as partial pivoting, so the answers vary.

By running these lines of code from the file LUprobHW3.m :

```
A=[1 2 -1;1 2 3;2 -1 4];
[L,U,P] = lu(A)
```

The following was outputted:

```
>> LUprobHW3

L =

    1.0000         0         0
    0.5000    1.0000         0
    0.5000    1.0000    1.0000


U =

    2.0000   -1.0000    4.0000
         0    2.5000    1.0000
         0         0   -4.0000


P =

     0     0     1
     0     1     0
     1     0     0
```

Continued on next page...

3. Must prove all properties of matrix norm to be true.

(1.) Verify property 1, $||A|| \geq 0$

Let's say matrix $A = a_{ij}$ and is $n \times n$. $x \in \mathbb{R}^n$

So $||Ax||_\infty = \max_{1 \leq i \leq n} \{ \sum_{j=1}^{n} |a_{ij} x_j| \}$. We can see that

$||A||_\infty = \max_{||x||=1} ||Ax||_\infty$ so $||A|| \geq 0$ ✓

(2.) Verify property 2, $||A|| = 0$ iff $A = 0$

Same as above, let's say matrix $A = a_{ij}$ and is $n \times n$.

x is equal to identity matrix, $x = (1, 0, 0 \ldots 0)^T$

So $||Ax|| = 0$ which means $||A|| = 0$ ✓

(3.) Verify property 3, $||\alpha A|| = |\alpha| \, ||A||$

Let's replace A by αA in the original def. $||A|| = \max_{||x||=1} ||Ax||$

$||\alpha A|| = \max_{||x||=1} ||(\alpha A)x||$

$= \max_{||x||=1} \{ |\alpha| \, ||Ax|| \}$ Vector norm property that we can use since $||\cdot||$ is a vector norm

$= |\alpha| \max_{||x||=1} ||Ax||$ Can factor out scalar #

So, $|\alpha| \, ||A|| = ||\alpha A||$ ✓

(4.) Verify property 4, $||A+B|| \leq ||A|| + ||B||$

Consider the separate definitions for A and B:

$||A|| = \max_{||x||=1} ||Ax||$ and $||B|| = \max_{||x||=1} ||Bx||$

When adding A and B: $||A+B|| = \max_{||x||=1} ||(A+B)x||$

And we can distribute x: $||A+B|| = \max_{||x||=1} ||(Ax+Bx)||$

$\leq \{ ||Ax|| + ||Bx|| \}$

$\leq \max_{||x||=1} ||Ax|| + \max_{||x||=1} ||Bx||$

$\leq ||A|| + ||B||$ ✓

(5.) Verify property 5, $||AB|| \leq ||A|| \cdot ||B||$

Like above, we write def. for A and B

$||A|| = \max_{||x||=1} ||Ax||$ and $||B|| = \max_{||x||=1} ||Bx||$

When multiplying A and B: $||AB|| = \max_{||x||=1} ||(AB)x||$

Can group differently! $= ||A(Bx)||$

Vector norm property since $||\cdot||$ is a vector norm $\leq \{ ||A|| \, ||Bx|| \}$

Can take out A and apply def. $\leq ||A|| \cdot \max_{||x||=1} ||Bx||$

→ We have proved the proof! $\leq ||A|| \, ||B||$ ✓

4. a) $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0 \cdot \begin{vmatrix} 1 & 0 \\ 2-\lambda & 0 \end{vmatrix} - 0 \cdot \cancel{\begin{vmatrix} 2-\lambda & 0 \\ 1 & 0 \end{vmatrix}} + (4-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$

$= (4-\lambda)\cdot[(2-\lambda)(2-\lambda) - 1]$

$= (4-\lambda)\cdot[3 - 4\lambda + \lambda^2]$

$= 12 - 3\lambda - 16\lambda + 4\lambda^2 + 4\lambda^2 - \lambda^3$

$= -\lambda^3 + 8\lambda^2 - 19\lambda + 12$

$= -(\lambda^3 - 8\lambda^2 + 19\lambda - 12)$

$= -(\lambda - 4)(\lambda - 3)(\lambda - 1)$

So $\lambda = 4, \lambda = 3, \lambda = 1$ ← eigenvalues

For $\lambda = 4$: $(A - 4I : 0)$

$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ eigenvector is $x = \begin{bmatrix} x_1 = 0 \\ x_2 = 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 3$: $(A - 3I : 0)$

$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to -R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + (-1)R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 - x_2 = 0, \ x_3 = 0$ $x = \begin{bmatrix} x_1 = x_2 \\ x_2 \\ x_3 = 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ← eigenvector

For $\lambda = 1$: $(A - 1I : 0)$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \xrightarrow[\frac{1}{3}R_3 \to R_3]{R_2 \to R_2 + -1R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ eigenvector is $x = \begin{bmatrix} x_1 = -x_2 \\ x_2 \\ x_3 = 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

4b) $\rho(A) = \max|\lambda|$ Biggest eigenvalue $= 4$

So, $\rho(A) = 4$

c) l_2 norm of $A = \sqrt{\rho(AA^T)} = \sqrt{\rho(A^2)} = \sqrt{4^2} = 4$

$= \sqrt{(\rho(A))^2}$

5. Let's say that A is an $n \times n$ matrix.

Then, $A^T = A \Rightarrow AA^T = AA = A^2$

Let's also say that x is the eigenvector of A corresponding to the eigenvalue λ

So this rule applies: $Ax = \lambda x$

$A^2x = A(Ax)$ factor and group

$= A(\lambda x)$ apply above rule

$= \lambda(Ax)$ regroup

$= \lambda(\lambda x)$ apply above rule again

$= \lambda^2 x$ Simplify

So λ^2 is the eigenvalue of A^2

$\rho(A^2) = (\rho(A))^2$ and this rule: $||A_2|| = \sqrt{\rho(A^TA)}$

$||A_2|| = \sqrt{\rho(A^TA)} = \sqrt{\rho(A^2)} = \sqrt{(\rho(A))^2} = \rho(A)$

So, $||A_2|| = \rho(A)$ ✓