Homework 4, Due: Thursday, 2/7

This assignment is due on **Thursday, February 7**, by 11:59 PM. Your assignment should be well-organized, typed (or neatly written and scanned) and saved as a .pdf for submission on Canvas. You must show all of your work to receive full credit. For problems requiring the use of MATLAB code, remember to include your output in the main body of your solution .pdf and to also submit your .m-files on Canvas as a part of your completed assignment. Your code should be appropriately commented to receive full credit.

Problems

$\boxed{1}$ Consider the Gauss-Seidel iterative method for solving linear systems of the form $\mathsf{A}\mathbf{x} = \mathbf{b}$.

(a) (6 points) Write a MATLAB **function** implementing the Gauss-Seidel method. Name your function `gauss_seidel.m`. List the input and output variables of your function, and insert comments to describe what each line of the code does. You may use `jacobi.m` as a starting point for modification.

Demonstrate that your function works by using it to approximate the solution to

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 6x_2 - 2x_3 &= 1 \\ 3x_1 + 3x_2 + 7x_3 &= 3 \end{aligned}$$

starting with $\mathbf{x}^{(0)} = \mathbf{0}$ and stopping when

$$||\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}||_\infty < 10^{-6}.$$

Report the solution and the number of iterations the algorithm took to converge.

(b) (4 points) Explore the effects of using different stopping criteria in your code for Gauss-Seidel from part (a). In particular, modify your `gauss_seidel.m` function to stop when

$$||\mathsf{A}\mathbf{x}^{(k)} - \mathbf{b}||_\infty < 10^{-6}$$

and

$$\frac{||\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}||_\infty}{||\mathbf{x}^{(k)}||_\infty} < 10^{-6}.$$

Discuss your findings, noting any differences in solving the linear system from part (a) starting with $\mathbf{x}^{(0)} = \mathbf{0}$, including the number of iterations it takes the algorithm to converge.

(c) (4 points) Explore the effects of using different stopping tolerances in your code for Gauss-Seidel from part (a). In particular, using the stopping criterion

$$||\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}||_\infty < \varepsilon$$

for tolerances $\varepsilon = 10^{-2}$, 10^{-6}, and 10^{-10}, note any differences in solving the linear system from part (a) starting with $\mathbf{x}^{(0)} = \mathbf{0}$, including the number of iterations it takes the algorithm to converge.

(d) (4 points) Explore the effects of using different initial guesses $\mathbf{x}^{(0)}$ in your code for Gauss-Seidel from part (a). In particular, choose at least three different initial vectors and note any differences in solving the linear system from part (a), stopping when

$$||\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}||_\infty < 10^{-6}.$$

Discuss your findings, highlighting any noticeable differences, including the number of iterations it takes the algorithm to converge.

$\boxed{2}$ The linear system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 7 \\ x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

has the solution $\mathbf{x} = (1, 2, -1)^\mathsf{T}$.

(a) (4 points) Show that $\rho(\mathsf{T}_\mathrm{J}) = 0$, where T_J is the iteration matrix associated with Jacobi's method.

(b) (4 points) Use Jacobi's method with $\mathbf{x}^{(0)} = \mathbf{0}$ to approximate the solution to the linear system to within 10^{-5} in the ℓ_∞ norm. You may do this numerically using `jacobi.m`. Report your solution and the number of iterations it took the algorithm to converge.

(c) (4 points) Show that $\rho(\mathsf{T}_\mathrm{GS}) = 2$, where T_GS is the iteration matrix associated with the Gauss-Seidel method.

(d) (4 points) Show that the Gauss-Seidel method applied as in part (b) fails to give a good approximation in 25 iterations. You may do this numerically using your function `gauss_seidel.m` from Problem 1. Report your approximation after 25 iterations.

$\boxed{3}$ (4 points) Prove that

$$||\mathbf{x}^{(k)} - \mathbf{x}|| \le ||\mathsf{T}||^k ||\mathbf{x}^{(0)} - \mathbf{x}||$$

where T is an $n \times n$ iteration matrix with $||\mathsf{T}|| < 1$ and

$$\mathbf{x}^{(k)} = \mathsf{T}\mathbf{x}^{(k-1)} + \mathbf{c}, \quad k = 1, 2, \ldots,$$

with $\mathbf{x}^{(0)}$ arbitrary, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{x} = \mathsf{T}\mathbf{x} + \mathbf{c}$.

Note: For any of the above problems for which you use MATLAB to help you solve, you must submit your code/.m-files as part of your work. Your code must run in order to receive full credit. If you include any plots, make sure that each has a title, axis labels, and readable font size, and include the final version of your plots as well as the code used to generate them.