1.

Homework 4

1) Midpoint $f'(x_0) = \dfrac{f(x_0+h) - f(x_0-h)}{2h}$ and $h = 0.01$

Endpoint $f'(x_0) = \dfrac{[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)]}{2h}$

$\rightarrow f'(1.00) = \dfrac{[-3f(1.00) + 4f(1.00+0.01) - f(1.00+2(0.01))]}{2(0.01)}$

$= (-9.03 + 12.48 - 3.14)/0.02 = 15.5$

$\rightarrow f'(1.01) = \dfrac{f(1.01+0.01) - f(1.01-0.01)}{2(0.01)}$

$= (3.14 - 3.01)/0.02 = 6.5$

$\rightarrow f'(1.02) = \dfrac{f(1.02+0.01) - f(1.02-0.01)}{2(0.01)}$

$= (3.18 - 3.12)/0.02 = 3$

$\rightarrow f'(1.03) = \dfrac{f(1.03+0.01) - f(1.03-0.01)}{2(0.01)}$

$= (3.24 - 3.14)/0.02 = 5$

use $h = -0.01$

$\rightarrow f'(1.04) = \dfrac{[-3f(1.04) + 4f(1.04-0.01) - f(1.04-2(0.01))]}{-2(0.01)}$

$= (-9.72 + 12.72 - 3.14)/0.02 = 7$

$\mathcal{E}(1.00) = 0.98(15.5) + 0.142(3.01) = 15.19 + 0.42742 = 15.61742$

$\mathcal{E}(1.01) = 0.98(6.5) + 0.142(3.12) = 6.37 + 0.44304 = 6.81304$

$\mathcal{E}(1.02) = 0.98(3) + 0.142(3.14) = 2.94 + 0.44588 = 3.38588$

$\mathcal{E}(1.03) = 0.98(5) + 0.142(3.18) = 4.9 + 0.45156 = 5.35156$

$\mathcal{E}(1.04) = 0.98(7) + 0.142(3.24) = 6.86 + 0.46008 = 7.32008$

t	1.00	1.01	1.02	1.03	1.04
i	15.61742	6.81304	3.38588	5.35156	7.32008

2a.

2a) $f(x) = f(x_0) + (x-x_0) f'(x_0) + (x-x_0)^2 \frac{f''(x_0)}{2}$
$+ (x-x_0)^3 \frac{f^{(3)}(\xi)}{6}$

where ξ is b/w x_0 and x_0+2h

Let $x = x_0+2h$

$f(x_0+2h) = f(x_0) + (x_0+2h-x_0) f'(x_0)$
$+ (x_0+2h-x_0)^2 \frac{f''(x_0)}{2} + (x+2h-x_0)^3 \frac{f^{(3)}(\xi_1)}{6}$

$f(x_0+2h) = f(x_0) + 2h f'(x_0) + 4h^2 \frac{f''(x_0)}{2}$
$+ 8h^3 \frac{f^{(3)}(\xi_1)}{6}$ (equation (1))

AND $f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f'''(\xi_2)$
(equation (2))

Multiply equation (2) by 4:

$4f(x_0+h) = 4f(x_0) + 4hf'(x_0) + \frac{4h^2}{2} f''(x_0) + \frac{4h^3}{6} f'''(\xi_2)$
(equation (2.1))

Subtract eq. (2.1) from eq. (1) to get:

h^2 terms cancel!

$f(x_0+2h) - 4f(x_0+h) = -3f(x_0) - 2hf'(x_0) + O(h^3)$

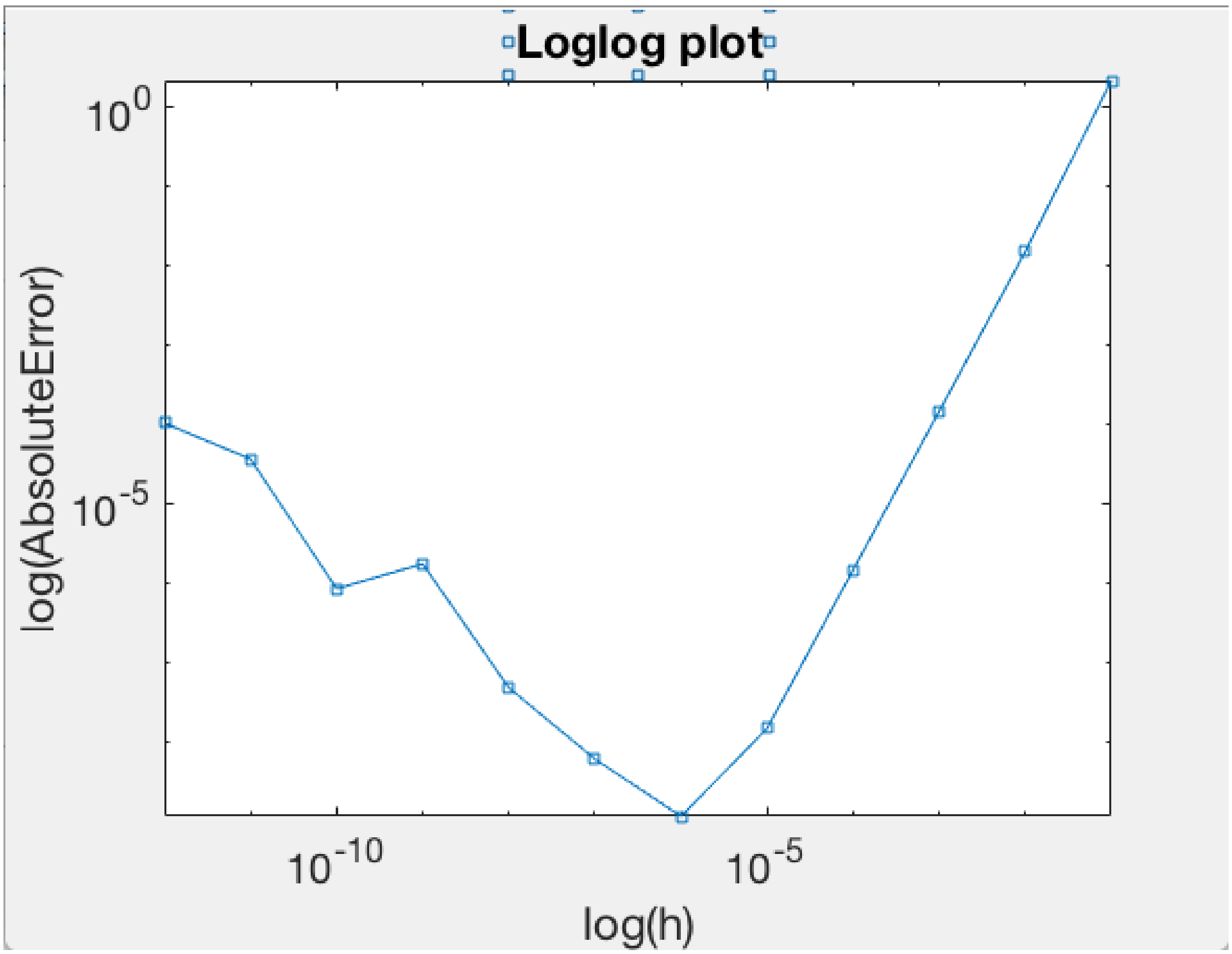
Rearrange $\boxed{\text{H.O.T}}$

✓ $f'(x_0) = \frac{-3f(x_0) + 4f(x_0+h) - f(x_0+2h)}{2h} + O(h^2)$

The 3-point endpoint formula has been derived!

2b.

h	0.1	0.01	0.001	0.0001	1e-05	1e-06	1e-07	1e-08	1e-09	1e-10	1e-11	1e-12
Abs. error	2.1535	0.015101	0.00014598	1.4549e-06	1.45e-08	1.1329e-09	5.9726e-09	4.7318e-08	1.729e-06	8.4086e-07	3.6368e-05	0.00010298



Decreasing h does not systematically decrease the error. I can see that the absolute error values start off big and decrease until it reaches 10^-9 after which the error begins to rise. This is because numerical differentiation is unstable. The round-off error accumulates since truncation error and round-off error are proportional, meaning if truncation error goes up, round-off error goes down and vice versa.

3.

3\. $N_1(h) = \frac{f(x_0+h) - f(x_0)}{h} \Rightarrow N_1(0.4) = \frac{f(0+0.4) - f(0)}{0.4}$

$= ([0.4 + e^{0.4}] - [0 + e^{0}]) / 0.4 = 2.229562$

$N_2(0.4) = 2N_1(\frac{0.4}{2}) - N_1(0.4) = 1.98446$

$N_3(0.4) = N_2(\frac{0.4}{2}) + \frac{N_2(\frac{0.4}{2}) - N_2(0.4)}{3}$

$\Rightarrow N_2(0.2) = 2N_1(0.1) - N_1(0.2)$

$= 2(2.051709) - 2.107014$

$= 1.996404$

$N_3(0.4) = 1.996404 + \frac{1.996404 - 1.98446}{3}$

$= 2.0004$

$f'(x) = 1 + e^{x}$ $\qquad N_1(0.2) = \frac{f(0+0.2) - f(0)}{0.2} = 2.107013$

$f'(0) = 1 + e^{0} = 2$