Lectures 16 and 17

We discussed strong induction, in which the induction hypothesis is strengthened to

$$\wedge_{n=0}^{\infty} p_n.$$

We actually did discuss conjunctive normal form, and disjunctive normal form, which I had thought to have done in lecture 13.

We showed that every Boolean expression can be written at least one way in CNF and DNF. So, in particular, $\neg$, $\wedge$, and $\vee$ are sufficient to express any Boolean function.

We noted, but did not prove, some properties of Pascal's Triangle.

These exercises continue that. The earlier exercises posted on CNF and DNF should now be done.

Exercises for Lectures 16 and 17

1. Let X be a set with an odd number of elements: $|X| = 2n+1$, $n \geq 0$. Show that the elements in the even positions of the $2n+1$'st row of Pascal's Triangle have the same sum as those in the odd positions:

$$\sum_{k=0}^{n} |\mathcal{P}_{2k}(X)| = \sum_{k=0}^{n} |\mathcal{P}_{2k+1}(X)|$$

   [You may do this by induction, but a direct proof is better.]

2. Let $|X| = 2n$, with $n \geq 0$. Prove by induction (a golden argument) that

$$\sum_{k=0}^{n} |\mathcal{P}_{2k}(X)| = \sum_{k=1}^{n} |\mathcal{P}_{2k-1}(X)|$$

   [Note: The sum on the right has one fewer terms than that on the left.]

3. Show that each of the sums in Exercises 1 and 2 is a power of 2.

4. Let the elements element in the i'th position of the j'th row of Pascal's Triangle be denoted by $\binom{j}{i}$.

   Show that for all $n \in \mathbb{N}$ and all $k \leq n$, that $\sum_{i=n}^{k} \binom{i}{k} = \binom{i+1}{k+1}$

   [Hint: Organize your induction by letting k be fixed, and prove the statement by induction on n, $n \geq k$, with base case $n = k$.]