Lectures 14 and 15

Summary

These lectures were on the method of induction.

We note how implications are proved versus used:

To *prove* an implication, you may assume the antecedent, and under that assumption, prove the consequence.

To *use* an implication you need additional information. If you know both p and $p \Rightarrow q$, you can conclude q (*modus ponens*, if you like Latin). If you know both $\neg q$ and $p \Rightarrow q$, you can conclude $\neg p$ (*modus tollens*).

If $\{p_n \mid n \in \mathbb{Z}\}$ is is a set of statements, the *method of induction* is a way to show

$$\bigwedge_{n=0}^{\infty} p_n$$

by showing

$$p_0 \wedge \left[\bigwedge_{n=0}^{\infty} (p_n \Rightarrow p_{n+1}) \right]$$

p_0 is called the *Base Case*.

The *Induction Step* is proving for any particular n that $(p_n \Rightarrow p_{n+1})$. In proving implication $(p_n \Rightarrow p_{n+1})$ we assume the antecedent, p_n, and under that assumption prove the consequence p_{n+1}. The assumption of p_n is called the *Induction Hypothesis*.

We did several examples.

In particular, we showed that if $|X| = n$, them

$$|\mathcal{P}_k(X)| = \frac{n!}{k!(n-k)!}$$

Exercises on Lectures 14 and 15

1. Show that $1 + 2 + 3 + 4 + \cdots + n = n(n+1)/2$ is true for all $n \geq 0$ by induction.

2. Show that $1 + 3 + 5 + 7 + \cdots + (2n+1) = (n+1)^2$ is true for all $n \geq 0$ by induction.

3. Show that $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = [n(n+1)/2]^2$ is true for all $n \geq 0$ by induction.

4. Show that $n^2 - n$ is even for all $n \in \mathbb{Z}$ by induction on $|n|$.

5. Show that $4^n + 15n - 1$ is evenly divisible by 9 for all $n \geq 0$.

6. Show that if $n > 0$ and n is odd, then $n^2 - 1$ is divisible by 8. [Note the the next odd number after n is $n + 2$].

7. Show that if $2^n \leq n!$ for all but finitely many values of n.

   [Hint: Find a value n_0 and show $2^n \leq n!$ for all $n \geq n_0$, with n_0 as the "base case".]

8. Let $\epsilon > 0$ be given. Show that $(1+\epsilon)^n \geq 1 + n\epsilon$ for all $n \geq 0$.

9. Show that for all $n \geq 1$ that the polynomial $p_n = x^n - 3^n$ can always be written as $(x-3)q_n(x)$, where $q_n(x)$ is a polynomial with only integer coefficients.