Project: Gaussian Quadrature, Due: Tuesday, 12/11

This project (report and accompanying code) is due on **Tuesday, December 11**, by 11:59 PM. Your report should be well-organized, **typed** (using LaTeX or Microsoft Word) and saved as a .pdf for submission on Canvas. You must show all of your work within the report to receive full credit. For portions of the project requiring the use of MATLAB code, remember to also submit your .m-files on Canvas as a part of your completed project. Your code should be appropriately commented to receive full credit.

Project Description

Gaussian quadrature rules are defined to minimize the expected error obtained in the approximation formula

$$\int_a^b f(x)\ dx \approx \sum_{i=1}^n c_i f(x_i)$$

by choosing the nodes x_i and weights c_i that give the greatest degree of precision. In this project, you will explore the numerical implementation and application of Gaussian quadrature rules.

Problems

$\boxed{1}$ (10 points) In your own words, briefly describe the main ideas behind Gaussian quadrature rules. In particular, discuss various approaches for defining the weights and nodes of the quadrature formulas in order to obtain the desired precision, and describe how the Gaussian rules differ from the Newton-Cotes formulas. Your description should be clear and concise, including the key points and defining factors of the approaches, and around 1 page in length (no more than 2 pages).

As you research the methods, you should refer to the main text for our course as well as supplementary sources. For example, you may find the following references useful:

- G. H. Golub and G. Meurant (2010) *Matrices, Moments and Quadrature with Applications* – Ch. 6 (available online as an ebook through Gordon Library)
- N. Kovvali (2011) *Theory and Applications of Gaussian Quadrature Methods* – Ch. 3 (available online as an ebook through Gordon Library)

You may use additional references as you find helpful. Cite any references (including the course textbook) you use in a bibliography at the end of your description. For example, if you use [1] as a reference, then you should include the following bibliography entry at the end of your description:

[1] G. H. Golub and G. Meurant (2010) *Matrices, Moments and Quadrature with Applications.* Princeton University Press: Princeton, NJ.

You should **cite at least 2 references** (and Wikipedia does not count!).

$\boxed{2}$ In class we discussed how selecting the nodes x_i to be the roots of the nth Legendre polynomial $P_n(x)$ results in a Gaussian quadrature rule

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

of precision $2n - 1$. This specific rule is known as a **Gauss-Legendre** quadrature rule.

(a) (12 points) Verify the Gauss-Legendre quadrature formula when $n = 3$ using two different approaches:

- (a) Set up a system of equations to directly solve for the nodes x_i and weights c_i that achieve the desired precision. Note that this will result in a **nonlinear** system of equations – use MATLAB to show that the values of the nodes and weights given in Table 4.12 satisfy this system. You can solve this system with the help of MATLAB's built-in `fsolve` function by writing your system in the form $F(x) = 0$ and starting with initial guess `x0 = [1; 1; 1; 1; 0; -1]`.
- (b) Use the 3rd Legendre polynomial and Theorem 4.7 in the text to compute the nodes x_i and weights c_i.

Show that each approach results in the same nodes and weights for the quadrature formula!

(b) (6 points) Write a MATLAB **function** that outputs the nodes and weights of the Gauss-Legendre quadrature rules for $n = 2$, 3, 4, and 5. You may hard code the values as given in Table 4.12 in the text. Make a table displaying the nodes x_i and weights w_i that result from your code for each specified n.

(c) (10 points) Using your function from part (b), approximate the integral

$$\int_{3}^{3.5} \frac{x}{\sqrt{x^2 - 4}}\, dx$$

using Gauss-Legendre quadrature with $n = 2$, 3, 4, and 5. Compare your results for each n to the exact value of the integral, reporting the absolute error in each case.

Note: For any of the above problems for which you use MATLAB to help you solve, you must submit your code/.m-files as part of your work. Your code must run in order to receive full credit. If you include any plots, make sure that each has a title, axis labels, and readable font size, and include the final version of your plots as well as the code used to generate them.