BME 493/593, Spring 2019: Computational Methods for Imaging Science Assignment 2 (due March 4, 2018)

Instructions: The problems below require a mix of paper-and-pencil work and MATLAB implementation. Remember that, as stated in the syllabus, you should both

1. Submit a hard copy of your solutions (which will be collected on Monday, March 4 **before** the class starts), **and**
2. Upload a digital copy of your solutions on Canvas by 9:59 AM on Monday, March 4.

Please include printouts of the MATLAB code together with the generated results and figures in the hard copy of your solutions.
The digital copy uploaded on Canvas must be a scan (or *readable* cellphone picture) of the hard copy that you handed in at the beginning of class. The instructors will grade only the hard copy of your solutions. The digital copy uploaded on Canvas only serves as proof that the assignment was turned in on time. If you are late or miss class on the day the assignment is due, uploading your solution on Canvas before the due date ensures that your assignment is considered on time.

For this assignment you will need the following files (that can be found on canvas):

- `problem1.m, problem2.m, problem3.m`: This scripts contain the definition of the forward problems (i.e. the imagining operator $\boldsymbol{K}$, the true image $\boldsymbol{m}_{\text{true}}$, and the measured data $\boldsymbol{d}$), which you will need to answer the question below.
- `paralleltomo.m`: This function creates the matrix $\boldsymbol{K}$ for a 2D parallel-beam tomography test problem. This function and the functions below are part of the *AIR Tools II package* by P. C. Hansen and J. S. Jorgensen (`http://www2.compute.dtu.dk/~pcha/AIRtoolsII/`).

- `phantomgalley.m`: This file contains a collection of 2D phantoms to test reconstruction algorithms.
- `purge_rows.m`: This functions removes zero rows of $\boldsymbol{K}$ and corresponding entries in $\boldsymbol{d}$.

Problem 1: Consider the Landweber's method to solve the linear inverse problem

$$\boldsymbol{d} = K\boldsymbol{m} + \boldsymbol{n}$$

in a least squares setting. Here $\boldsymbol{d}$ (data), $\boldsymbol{m}$ (image/parameters) and $\boldsymbol{n}$ (noise) are $\mathbb{R}^N$-vectors. The matrix $\boldsymbol{K} \in \mathbb{R}^{N\times N}$ is invertible, and admits the singular value decomposition

$$\boldsymbol{K} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T,$$

where $\boldsymbol{U}, \boldsymbol{V} \in \mathbb{R}^{N\times N}$ are orthonormal matrices whose columns represent the left and right singular vectors (respectively), and $\boldsymbol{\Sigma} = \text{diag}(\sigma_i)$ ($\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N > 0$) is a diagonal matrix collecting the singular values of $\mathbf{K}$.

Using the above notation, the update step for the Landweber's method reads

$$\boldsymbol{m}^{(j+1)} = \boldsymbol{m}^{(j)} + \omega \boldsymbol{K}^T(\boldsymbol{d} - \boldsymbol{K}\boldsymbol{m}^{(j)}), \tag{1}$$

where j is the iteration number, and the step length ω satisfies $0 < \omega < 2\|\boldsymbol{K}^T\boldsymbol{K}\|_2^{-1} = 2/\sigma_1^2$.

A) Show that, for $\boldsymbol{m}_0 = \mathbf{0}$, the iterate $\boldsymbol{m}^{(j)}$ at the j-th iteration can be expressed as the filtered SVD solution

$$\boldsymbol{m}^{(j)} = \boldsymbol{V}\boldsymbol{\Phi}^{(j)}(\boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1}\boldsymbol{U}^T\boldsymbol{d}, \tag{2}$$

where $\boldsymbol{\Phi}^{(j)}(\boldsymbol{\Sigma}) = \text{diag}(\phi^{(j)}(\sigma_1^2), \phi^{(j)}(\sigma_2^2), \ldots, \phi^{(j)}(\sigma_N^2))$ is a diagonal matrix whose entries are the filter functions

$$\phi^{(j)}(\sigma_i^2) = 1 - (1 - \omega\sigma_i^2)^j. \tag{3}$$

Hint: You can prove (**??**) by induction. That is, first show that (**??**) holds for $j = 1$, and then show that (**??**) holds for $j + 1$ assuming that it is valid for j.

B) Generate *loglog* plots of the Landweber filter function $\phi^{(j)}(\sigma_i^2)$ in (**??**) with σ_i^2 ranging in $(0, 1]$. How does the behavior of $\phi^{(j)}(\sigma_i^2)$ change as j and ω vary?

C) Consider the discrete 1D image deblurring problem $\boldsymbol{K}\boldsymbol{m} = \boldsymbol{d}$ as implemented `problem1.m`. Here, $\boldsymbol{K} \in \mathbb{R}^{N\times N}$ (N=128) is the matrix stemming from discretization of a Gaussian blurring operator with kernel $k(x,y) = 1/(\sqrt{2\pi}\gamma)\exp\left\{-\|x-y\|^2/(2\gamma^2)\right\}$ and $\gamma = 0.03$. The vector $\boldsymbol{d} = \boldsymbol{K}\boldsymbol{m}_{true}$ represents the noiseless data corresponding to the true image

$$\boldsymbol{m}_{true}(x) = \begin{cases} 1 & 0.2 < x < 0.3 \\ \sin(4\pi x) & 0.5 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Implement the Landweber method and solve the inverse problem $\boldsymbol{d} = \boldsymbol{K}\boldsymbol{m}$ using 1000 iterations with step length $\omega = 2$. Use $\boldsymbol{m}^{(0)} = \boldsymbol{0}$ as initial guess. Report in a *semilogy* scale the Euclidean norm of the residual $\boldsymbol{d} - \boldsymbol{K}\boldsymbol{m}^{(j)}$ and of the error $\boldsymbol{m}_{true} - \boldsymbol{m}^{(j)}$ as a function of the iteration number j.

D) Add normally distributed noise $\boldsymbol{n}$ with zero mean and variance $\sigma^2 = 10^{-4}$ to $\boldsymbol{d}$. The resulting blurred and noisy data is then $\hat{\boldsymbol{d}} = \boldsymbol{K}\boldsymbol{m}_{true} + \boldsymbol{n}$. Solve again the inverse problem for the noisy data $\hat{\boldsymbol{d}}$ using the Landweber method (1000 iterations, step length $\omega = 2$, and $\boldsymbol{m}^{(0)} = \boldsymbol{0}$). Report the Euclidean norm of the residual and of the error as in the previous point. What do you observe?

E) Implement the L-curve to find out when to stop the Landweber iterations, so that the solution is robust with respect to noise in the data. Use $\omega = 1$ as step length.

Problem 2: Implement the Algebraic Reconstruction Technique (ART), Cimmino (SIRT), and the Simultaneous Algebraic Reconstruction Techinque (SART) algorithms to solve the system of linear equations

$$\boldsymbol{d} = \boldsymbol{K}\boldsymbol{m}, \tag{4}$$

where the sparse system matrix $\boldsymbol{K} \in \mathbb{R}^{q\times n}$ and the data $\boldsymbol{d} \in \mathbb{R}^q$ are constructed in the MATLAB script `problem2.m`. Specifically, the system matrix $\boldsymbol{K}$ describes the data acquisition process of an X-ray CT imaging system, see e.g. Section 7.4.1 of A. C. Kak and M. Slaney, *Principles of Computerized Tomographic Imaging*, SIAM, 2001. The object $\boldsymbol{m}$ is discretized on a Cartesian grid with 64×64 pixels, thus $n = 4096$. The measured data $\boldsymbol{d}$ correspond to 90 projections, taken in increments of 2^o from 0^o to 178^o inclusive. Each projection consists of 90 rays over a distance equal to the diagonal of the image (see `paralleltomo.m`). For some projections certain rays do not intersect with the image, therefore the corresponding rows in $\boldsymbol{K}$ contain all zeros and are eliminated from the matrix (see `purge_rows.m`). After elimination of such rows, the matrix $\boldsymbol{K}$ has $q = 7328$ rows and $n = 4096$ columns.

A) Reconstruct the image $\boldsymbol{m}$ from the noiseless data $\boldsymbol{d} = \boldsymbol{K}\boldsymbol{m}_{\text{true}}$ using $1000 \times q$ ART updates. That is, perform 1000 swipes over the q rows of $\boldsymbol{K}$. Report in a *semilogy* scale the Euclidean norm of the residual $\boldsymbol{d} - \boldsymbol{K}\boldsymbol{m}^{(j)}$ and of the error $\boldsymbol{m}_{true} - \boldsymbol{m}^{(j)}$ as a function of the iteration number j.

B) Reconstruct the image $\boldsymbol{m}$ from the noiseless data $\boldsymbol{d} = \boldsymbol{K}\boldsymbol{m}_{\text{true}}$ using 1000 SART updates. Compare the convergence history (norm of the residual and of the error) of SART with the one of ART.

C) Reconstruct the image $\boldsymbol{m}$ from the noiseless data $\boldsymbol{d} = \boldsymbol{K}\boldsymbol{m}_{\text{true}}$ using 1000 SIRT updates. Compare the convergence history (norm of the residual and of the error) of SIRT with those of ART and SART.

D) Consider the noisy data $\hat{\boldsymbol{d}} = \boldsymbol{K}\boldsymbol{m}_{\text{true}} + \boldsymbol{n}$, where $\boldsymbol{n}$ is a normally distributed random vector with zero mean an variance $\sigma^2 = 10^{-4}\|\boldsymbol{K}\boldsymbol{m}_{\text{true}}\|_\infty^2$, where $\|\boldsymbol{K}\boldsymbol{m}_{\text{true}}\|_\infty$ denotes the maximum of $\boldsymbol{K}\boldsymbol{m}_{\text{true}}$. Solve the inverse problem $\hat{\boldsymbol{d}} = \boldsymbol{K}\boldsymbol{m}$ using ART, SIRT, SART. What do you observe?

E) Implement a stopping criterion for ART that is robust with respect to noise in the data using Morozov discrepancy principle.

Problem 3: Implement the Expectation Maximization (EM) algorithm to solve the system of linear equations

$$\boldsymbol{d} = \boldsymbol{K}\boldsymbol{m}, \tag{5}$$

where the sparse system matrix $\boldsymbol{K} \in \mathbb{R}^{q \times n}$ and the data $\boldsymbol{d} \in \mathbb{R}^q$ have non-negative entries. Use the true image $\boldsymbol{m}_{\text{true}}$, the imaging operator $\boldsymbol{K}$, and the data $\boldsymbol{d} = \boldsymbol{K}\boldsymbol{m}_{\text{true}}$ that are constructed in the MATLAB script `problem3.m`.

A) Reconstruct the image $\boldsymbol{m}$ from the noiseless data $\boldsymbol{d} = \boldsymbol{K}\boldsymbol{m}_{\text{true}}$ using 100 EM iterations. Report in a *semilogy* scale the Kullback–Leibler distance $D_{\text{KL}}(\boldsymbol{d}||\boldsymbol{K}\boldsymbol{m}^{(j)})$ and the Euclidean error norm $\|\boldsymbol{m}_{true} - \boldsymbol{m}^{(j)}\|$ as a function of the iteration number j.

B) Design an appropriate stopping criterion and use the EM algorithm to reconstruct the image $\boldsymbol{m}$ from noisy data $\hat{\boldsymbol{d}}$. Specifically, let $\hat{\boldsymbol{d}}$ be a realization of a random vector distributed according to a Poisson distribution[1] with parameter $\boldsymbol{\lambda} = \boldsymbol{K}\boldsymbol{m}_{\text{true}}$.

[1] Use the MATLAB function `poissrnd`