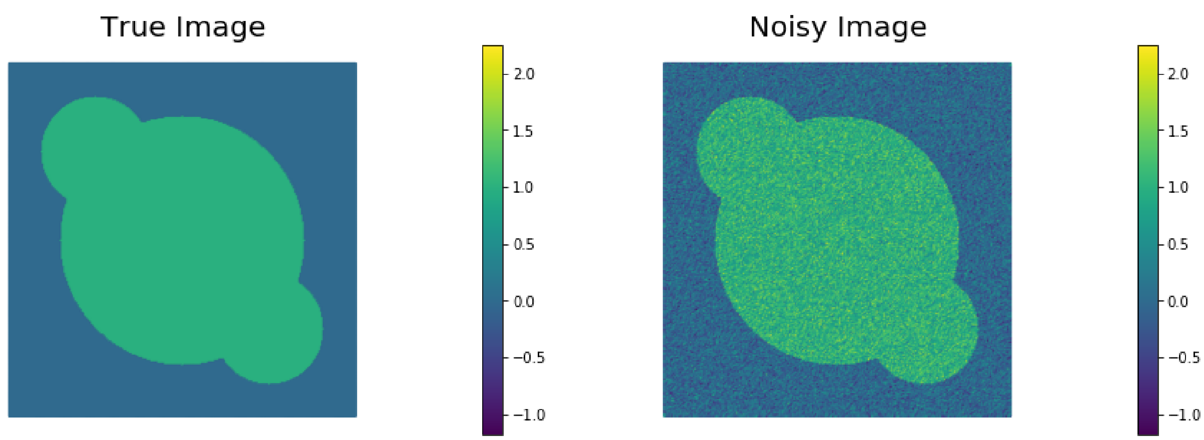
In [1]:
```
from __future__ import print_function, division, absolute_import

import math

import matplotlib.pyplot as plt
%matplotlib inline

import numpy as np
import scipy.io as sio

import dolfin as dl

from hippylib import nb

from unconstrainedMinimization import InexactNewtonCG


import logging

logging.getLogger('FFC').setLevel(logging.WARNING)
logging.getLogger('UFL').setLevel(logging.WARNING)
dl.set_log_active(False)
```

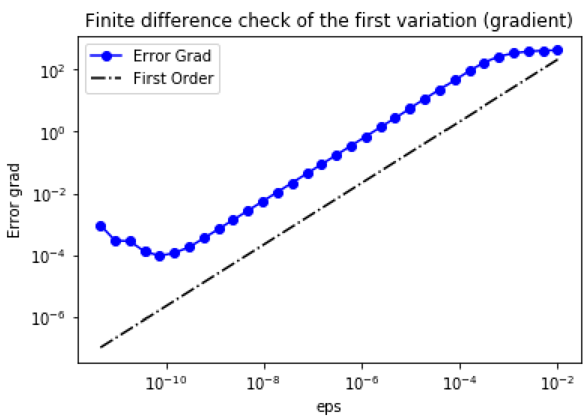
In [2]:
```
class Image(dl.Expression):
    def __init__(self, Lx, Ly, data, **kwargs):
        self.data = data
        self.hx = Lx/float(data.shape[1]-1)
        self.hy = Ly/float(data.shape[0]-1)

    def eval(self, values, x):
        j = int(math.floor(x[0]/self.hx))
        i = int(math.floor(x[1]/self.hy))
        values[0] = self.data[i,j]
```

In [3]:
```python
data = sio.loadmat('circles.mat')['im']

Lx = float(data.shape[1])/float(data.shape[0])
Ly = 1.

mesh = dl.RectangleMesh(dl.Point(0,0),dl.Point(Lx,Ly),data.shape[1], data.shape[0])
V = dl.FunctionSpace(mesh, "Lagrange",1)

trueImage = Image(Lx,Ly,data,degree = 1)
m_true  = dl.interpolate(trueImage, V)

np.random.seed(seed=1)
noise_std_dev = .3
noise = noise_std_dev*np.random.randn(data.shape[0], data.shape[1])
noisyImage = Image(Lx,Ly,data+noise, degree = 1)
d = dl.interpolate(noisyImage, V)

# Get min/max of noisy image, so that we can show all plots in the same scale
vmin = np.min(d.vector().get_local())
vmax = np.max(d.vector().get_local())

plt.figure(figsize=(15,5))
nb.plot(m_true, subplot_loc=121, mytitle="True Image", vmin=vmin, vmax = vmax)
nb.plot(d, subplot_loc=122, mytitle="Noisy Image", vmin=vmin, vmax = vmax)
plt.show()
```

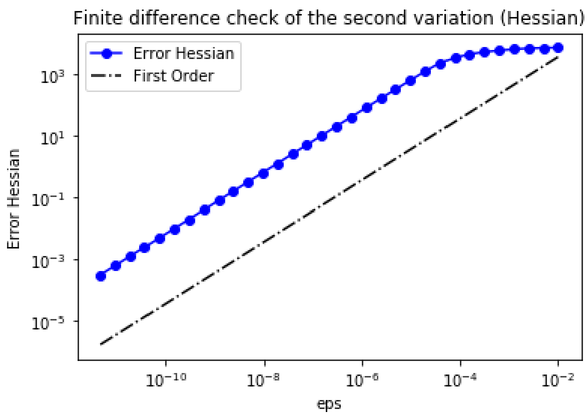


In [4]:
```
class TVDenoising:
    def __init__(self, V, d, alpha, beta):
        self.alpha   = dl.Constant(alpha)
        self.beta    = dl.Constant(beta)
        self.d       = d
        self.m_tilde  = dl.TestFunction(V)
        self.m_hat = dl.TrialFunction(V)

    def cost_reg(self, m):
        return dl.sqrt( dl.inner(dl.grad(m), dl.grad(m)) + self.beta)*dl.dx

    def cost_misfit(self, m):
        return dl.Constant(.5)*dl.inner(m-self.d, m - self.d)*dl.dx

    def cost(self, m):
        return self.cost_misfit(m) + self.alpha*self.cost_reg(m)

    def grad(self, m):
        grad_ls = dl.inner(self.m_tilde, m - self.d)*dl.dx

        TVm = dl.sqrt( dl.inner(dl.grad(m), dl.grad(m)) + self.beta)
        grad_tv = dl.Constant(1.)/TVm*dl.inner(dl.grad(m), dl.grad(self.m_tilde))*dl.dx

        grad = grad_ls + self.alpha*grad_tv

        return grad

    def Hessian(self,m):
        H_ls = dl.inner(self.m_tilde, self.m_hat)*dl.dx

        TVm = dl.sqrt( dl.inner(dl.grad(m), dl.grad(m)) + self.beta)
        A = dl.Constant(1.)/TVm * (dl.Identity(2) - dl.outer(dl.grad(m)/TVm, dl.grad(m)/TVm ) )
        H_tv = dl.inner(A*dl.grad(self.m_tilde), dl.grad(self.m_hat))*dl.dx

        H = H_ls + self.alpha*H_tv

        return H

    def LD_Hessian(self,m):
        H_ls = dl.inner(self.m_tilde, self.m_hat)*dl.dx

        TVm = dl.sqrt( dl.inner(dl.grad(m), dl.grad(m)) + self.beta)
        H_tv = dl.Constant(1.)/TVm *dl.inner(dl.grad(self.m_tilde), dl.grad(self.m_hat))*dl.dx

        H = H_ls + self.alpha*H_tv

        return H
```

In [5]:
```python
n_eps = 32
eps = 1e-2*np.power(2., -np.arange(n_eps))
err_grad = np.zeros(n_eps)

m0 = dl.interpolate(dl.Expression("x[0]*(x[0]-1)*x[1]*(x[1]-1)", degree=4), V)
alpha = 1.
beta   = 1e-4
problem = TVDenoising(V,d,alpha, beta)

J0 = dl.assemble( problem.cost(m0) )
grad0 = dl.assemble(problem.grad(m0) )

mtilde = dl.Function(V)
mtilde.vector().set_local(np.random.randn(V.dim()))
mtilde.vector().apply("")

mtilde_grad0 = grad0.inner(mtilde.vector())

for i in range(n_eps):
    Jplus = dl.assemble( problem.cost(m0 + dl.Constant(eps[i])*mtilde) )
    err_grad[i] = abs( (Jplus - J0)/eps[i] - mtilde_grad0 )

plt.figure()
plt.loglog(eps, err_grad, "-ob", label="Error Grad")
plt.loglog(eps, (.5*err_grad[0]/eps[0])*eps, "-.k", label="First Order")
plt.title("Finite difference check of the first variation (gradient)")
plt.xlabel("eps")
plt.ylabel("Error grad")
plt.legend(loc = "upper left")
plt.show()
```



In [6]:
```python
H_0 = dl.assemble( problem.Hessian(m0) )
H_0mtilde = H_0 * mtilde.vector()
err_H = np.zeros(n_eps)

for i in range(n_eps):
    grad_plus = dl.assemble( problem.grad(m0 + dl.Constant(eps[i])*mtilde) )
    diff_grad = (grad_plus - grad0)
    diff_grad *= 1/eps[i]
    err_H[i] = (diff_grad - H_0mtilde).norm("l2")

plt.figure()
plt.loglog(eps, err_H, "-ob", label="Error Hessian")
plt.loglog(eps, (.5*err_H[0]/eps[0])*eps, "-.k", label="First Order")
plt.title("Finite difference check of the second variation (Hessian)")
plt.xlabel("eps")
plt.ylabel("Error Hessian")
plt.legend(loc = "upper left")
plt.show()
```



In [7]:
```python
def TVsolution(alpha, beta):

    m = dl.Function(V)
    problem = TVDenoising(V, d, alpha, beta)

    solver = InexactNewtonCG()
    solver.parameters["rel_tolerance"] = 1e-5
    solver.parameters["abs_tolerance"] = 1e-9
    solver.parameters["gdm_tolerance"] = 1e-18
    solver.parameters["max_iter"] = 1000
    solver.parameters["c_armijo"] = 1e-5
    solver.parameters["print_level"] = -1
    solver.parameters["max_backtracking_iter"] = 10
    solver.solve(problem.cost, problem.grad, problem.Hessian, m)

    MSE  = dl.inner(m - m_true, m - m_true)*dl.dx
    J    = problem.cost(m)
    J_ls = problem.cost_misfit(m)
    R_tv = problem.cost_reg(m)

    print( "{0:15e} {1:15e} {2:4d} {3:15e} {4:15e} {5:15e} {6:15e}".format(
           alpha, beta, solver.it, dl.assemble(J), dl.assemble(J_ls), dl.assemble(R_tv), dl.assembl
e(MSE))
         )

    return m
```

Part e

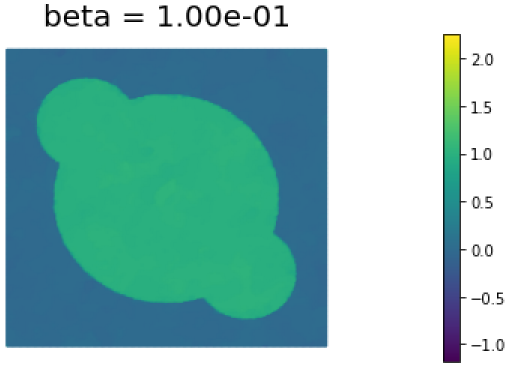
In [8]:
```
print ("{0:15} {1:15} {2:4} {3:15} {4:15} {5:15} {6:15}".format("alpha", "beta", "nit", "J", "J_ls"
, "R_tv", "MSE") )

alpha = 1e-3
betas = np.array((10.,1.,1e-1,1e-2,1e-3,1e-4))
for beta in betas:
    m = TVsolution(alpha, beta)
    plt.figure()
    nb.plot(m, vmin=vmin, vmax = vmax, mytitle="beta = {0:1.2e}".format(beta))
plt.show()
```

alpha	beta	nit	J	J_ls	R_tv	MSE
1.000000e-03	1.000000e+01	19	2.748766e-02	2.079902e-02	6.688631e+00	1.262773e-03
1.000000e-03	1.000000e+00	40	2.584654e-02	2.148072e-02	4.365817e+00	9.500933e-04
1.000000e-03	1.000000e-01	66	2.538788e-02	2.176078e-02	3.627106e+00	8.387293e-04
1.000000e-03	1.000000e-02	147	2.526200e-02	2.185685e-02	3.405153e+00	8.064175e-04
1.000000e-03	1.000000e-03	360	2.522709e-02	2.188693e-02	3.340160e+00	7.987682e-04
1.000000e-03	1.000000e-04	921	2.521711e-02	2.189623e-02	3.320886e+00	7.971665e-04

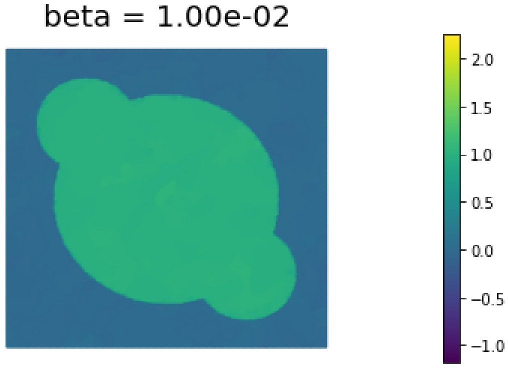
beta = 1.00e+01

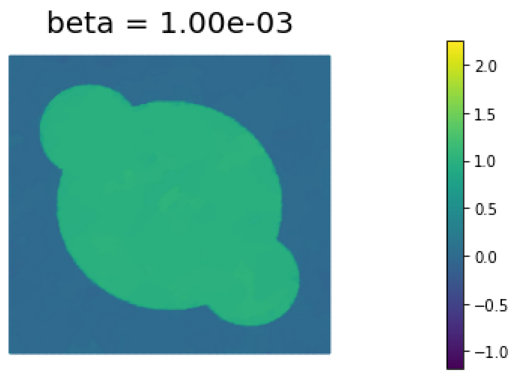
beta = 1.00e+00

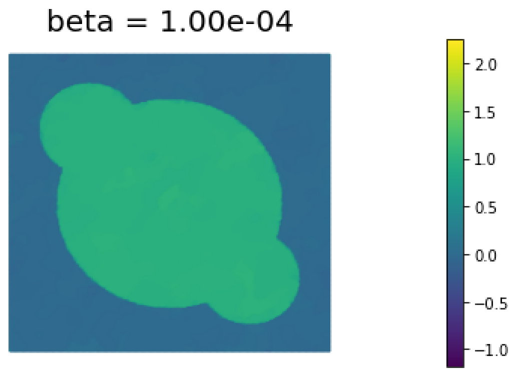
beta = 1.00e-01



beta = 1.00e-02







From part d, we know that for large β, the solution is more smoothed, but without preserving the edges, which corresponds to behavior similar to the Tikhinov regularization. With smaller values of β, the solution preserves the edges of image, since smoothing only occurs perpendicular to the gradient of the image. However, the cost to this is more ill-conditioning of the system.

We can see this as we go from high to low β. For high β, the number of Newton iteration is lower (less ill-conditioned), but the image edges are not as well. For low β, the number of Newton iterations is higher (more ill-conditioned), but the image edges are well preserved.

This file is part of **cmis_labs**, the teaching material for the S2019 BME 493/593 course on *Computational Methods in Imaging Science* at Washington University in St. Louis. Please see https://uvilla.github.io/cmis_labs (https://uvilla.github.io/cmis_labs) for more information and source code availability.

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