BME 493/593, Spring 2019:
Computational Methods for Imaging Science
Assignment 1 (due Feb. 18, 2018)

Instructions: The problems below require a mix of paper-and-pencil work and MATLAB implementation. Remember that, as stated in the syllabus, you should both

1. Submit a hard copy of your solutions (which will be collected on Monday, Feb 18 **before** the class starts), **and**
2. Upload a digital copy of your solutions on Canvas by 9:59 AM on Monday, Feb 18.

Please include printouts of the MATLAB code together with the generated results and figures in the hard copy of your solutions.
The digital copy uploaded on Canvas must be a scan (or *readable* cellphone picture) of the hard copy that you handed in at the beginning of class. The instructors will grade only the hard copy of your solutions. The digital copy uploaded on Canvas only serves as proof that the assignment was turned in on time. If you are late or miss class on the day the assignment is due, uploading your solution on Canvas before the due date ensures that your assignment is considered on time.

For this assignment you will need the following files (that can be found on canvas):

- `deconv1D.m`: This script solves a 1D Gaussian deconvolution problem using Tikhonov regularization and illustrates the L-curve criterion.
- `deconv2D.m`: This script solves a 2D Gaussian deconvolution problem using Tikhonov regularization.
- `apply.m`: This function implements the *matrix-free* application of the normal equations left hand side matrix $K^T K + \alpha I$, where K is the 2D blurring operator.

- `circle.png`: The true image for Problem 3.

Problem 1: We consider a linear inverse problem in $\mathbb{R}^N$ ($N \in \mathbb{N}$):

$$\boldsymbol{d} = K\boldsymbol{m} + \boldsymbol{n}$$

with $\mathbb{R}^N$-vectors $\boldsymbol{d}$ (data), $\boldsymbol{m}$ (image/parameters) and $\boldsymbol{n}$ (noise). The matrix $K \in \mathbb{R}^{N\times N}$ is invertible. The singular value decomposition of K shows that there are orthonormal matrices $U, V \in \mathbb{R}^{N\times N}$ with columns $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ and singular values s_i, with $s_1 \geq s_2 \geq \ldots \geq s_N$, with $s_i \neq 0$, such that

$$K = U\text{diag}(s_i)V^T.$$

Note that the orthonormality of U and V implies that $U^T = U^{-1}$ and $V^T = V^{-1}$.

A) Show that, for $\alpha > 0$ the Tiknonov-filtered reconstruction of the parameter field

$$\boldsymbol{m}_\alpha = \sum_{i=1}^{N} \frac{s_i}{s_i^2 + \alpha}(\boldsymbol{u}_i^T \boldsymbol{d})\boldsymbol{v}_i$$

can also be found as

$$\boldsymbol{m}_\alpha = (K^T K + \alpha I)^{-1} K^T \boldsymbol{d}$$

Why is $K^T K + \alpha I$ invertible?

B) Show that $\boldsymbol{m}_\alpha$ is also the solution of the minimization problem

$$\min_{\boldsymbol{m}} \frac{1}{2}\|K\boldsymbol{m} - \boldsymbol{d}\|^2 + \frac{\alpha}{2}\|\boldsymbol{m}\|^2.$$

Hints: Use that $\|\boldsymbol{x}\|^2 = \boldsymbol{x}^T\boldsymbol{x}$ for vectors $\boldsymbol{x}$ and that at the minimum of a function its derivative is zero.

Problem 2: Discretize the blurring operator[1]

$$d(x) = \int_0^1 k(x - x')p(x')\,dx' \quad \text{for } 0 < x < 1.$$

[1] Use the MATLAB code `deconv1D.m` as an example, but remember to modify the convolution kernel (line 16), and the true image (line 21).

with 200 discretization points, with $k(x) = c^{-2}\max(0, c - |x|)$ with $c = 0.2$. Use the resulting matrix K to blur the function

$$\boldsymbol{m}_{true}(x) = \begin{cases} 0.75 & 0.1 < x < 0.25 \\ 0.25 & 0.3 < x < 0.32 \\ \sin^4(2\pi x) & 0.5 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Add normally distributed noise[2] $\boldsymbol{n}$ with zero mean and variance $\sigma^2 = 0.1$. The resulting blurred and noisy data is $\boldsymbol{d} = K\boldsymbol{m}_{true} + \boldsymbol{n}$.

A) Use the truncated singular value decomposition[3] filter (TSVD) with $\alpha = 0.0001, 0.001, 0.1, 1$ to compute the regularized reconstructions $\boldsymbol{m}_\alpha$.

B) Use the Tikhonov filter with the same values for α for the reconstruction.

C) Compute the (approximate) optimal value for the regularization parameter α in the Tikhonov regularization using the L-curve method.

D) Compute the (approximate) optimal value for the regularization parameter α in the Tikhonov regularization using Morozov's discrepancy criterion.

E) Compute the (approximate) optimal value for the regularization parameter α in the Tikhonov regularization using generalized cross validation.

F) Plot $\|\boldsymbol{m}_{true} - \boldsymbol{m}_\alpha\|$ as a function of α, where $\boldsymbol{m}_\alpha$ is the Tikhonov regularized solution. Which value of α (approximately) minimizes the norm of this difference?

Problem 3: We use Tikhonov regularization to reconstruct the blurred and noisy 2D image shown in Figure 1. A MATLAB code to create the data and to compute the Tikhonov reconstruction is provided (function `deconv2D.m`)[4]. Note that the linear system in the Tikhonov regularization is solved iteratively with the conjugate gradient (CG) method[5] (rather than with a direct solver)

[2] Use the MATLAB function randn.

[3] MATLAB provides the function `svd` to compute the singular value decomposition of a matrix.

[4] Make sure the image `circle.png` is in the same folder of `deconv2D.m`

[5] If you have not heard about this method yet, don't worry—we will cover it later in class. The CG method is simply an (approximate) way to solve linear systems without computing matrix factorizations.

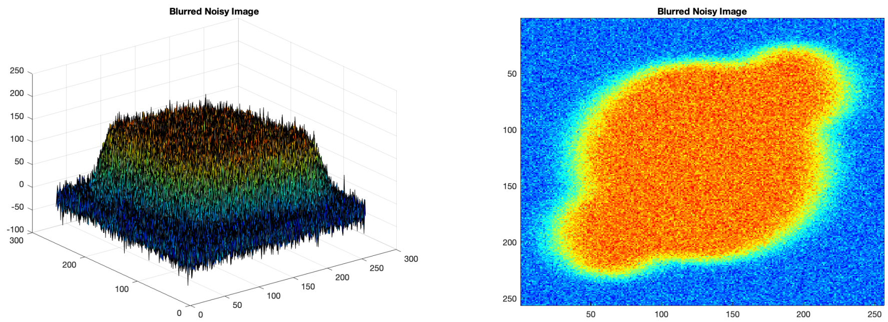

Figure 1: Noisy and blurred gray value image plotted as surface seen from side (left) and from top (right, in pseudo color). In MATLAB, images are represented as matrices, in which each entry corresponds to the gray value of a pixel.

since the 2D-blurring operator matrix K is not explicitly available (but we can apply it using 1D-blurring operators in x- and y-direction). Similar phenomena occur frequently not only in imaging science, but also in many other inverse problems related to various fields of science and engineering.

A) What would be the size of the matrix K if we followed the same approach as in the previous example?

B) Compute a (near-)optimal regularization parameter for Tikhonov regularization using the L-curve criterion.

C) Which value of α minimizes the norm of the difference between the real image and the Tikhonov reconstruction?