Assignment 2

Andrew Van

1.) A.) We prove (2) by induction:

(i) $j=1$

$$m^{(1)} = V\Phi^{(1)}(\Sigma)\Sigma^{-1}U^Td$$

$$= Vw\Sigma^2\Sigma^{-1}U^Td$$

$$= wV\Sigma\underbrace{\Sigma\Sigma^{-1}}_{I}U^Td$$

$$= wV\Sigma U^Td = \boxed{wK^Td \ (a)}$$

$\phi^{(1)}(\sigma_i^2) = 1-(1-w\sigma_i^2)^1 = w\sigma_i^2$

$$\Rightarrow \Phi^{(1)}(\Sigma) = \begin{bmatrix} w\sigma_1^2 & & \\ & \ddots & \\ & & w\sigma_N^2 \end{bmatrix}$$

$$= w\Sigma^2$$

from $m^{(j+1)} = m^{(j)} + wK^T(d-Km^{(j)})$

$$m^{(1)} = m^{(0)} + wK^T(d-Km^{(0)}) \quad \text{since } m^{(0)}=0$$

$$= \boxed{wK^Td \ (b)}$$

$\boxed{a=b \text{ therefore (2) holds for } j=1}$

(ii) Assume (2) holds for j, then for $j+1$

$$m^{(j+1)} = V\Phi^{(j+1)}(\Sigma)\Sigma^{-1}U^Td$$

$\phi^{(j+1)}(\sigma_i^2) = 1-(1-w\sigma_i^2)^{j+1}$

$= 1-(1-w\sigma_i^2)^j(1-w\sigma_i)$

$$= V[I-(I-w\Sigma^2)^j(I-w\Sigma^2)]\Sigma^{-1}U^Td \quad \Rightarrow \Phi^{(j+1)} = I-(I-w\Sigma^2)^j(I-w\Sigma)$$

$$= VI\Sigma^{-1}U^Td - V(I-w\Sigma^2)^j(I-w\Sigma^2)\Sigma^{-1}U^Td$$

$$= VI\Sigma^{-1}U^Td - V(I-w\Sigma^2)^jI\Sigma^{-1}U^Td + V(I-w\Sigma^2)^jw\Sigma^2\Sigma^{-1}U^Td$$

$$= \underbrace{V[I-(I-w\Sigma^2)^j]\Sigma^{-1}U^Td}_{m^{(j)}} + V(I-w\Sigma^2)^jw\Sigma^2\Sigma^{-1}U^Td$$

$$= m^{(j)} + wV(I-w\Sigma^2)^j\Sigma U^Td$$

$$\Rightarrow \boxed{m^{(j+1)} - m^{(j)} = wV(I-w\Sigma^2)^j\Sigma U^Td} \quad \Phi^{(j)}(\Sigma) = [I-(I-w\Sigma^2)^j]$$

$$\boxed{\underset{(LHS)}{wK^T(d-Km^{(j)})} = \underset{(RHS)}{wV(I-w\Sigma^2)^j\Sigma U^Td}}$$

Expanding LHS

$$wK^T(d-Km^{(j)}) = wV\Sigma U^T(d-U\Sigma V^TV\Phi^{(j)}(\Sigma)\Sigma^{-1}U^Td) = wV\Sigma U^Td - wV\Sigma\underbrace{U^TU}_{I}\Sigma\underbrace{V^TV}_{I}[I-(I-w\Sigma^2)^j]\Sigma^{-1}U^Td$$

$$= wV\Sigma U^Td - wV\Sigma^2[I-(I-w\Sigma^2)^j]\Sigma^{-1}U^Td = wV\Sigma U^Td - wV\underbrace{\Sigma^2\Sigma^{-1}}_{\Sigma}U^Td + wV\Sigma^2(I-w\Sigma^2)^j\Sigma^{-1}U^Td$$

$$= wV\Sigma U^Td - wV\Sigma U^Td + wV\underbrace{\Sigma^2(I-w\Sigma^2)^j}\Sigma^{-1}U^Td = wV(I-w\Sigma^2)^j\Sigma^2\Sigma^{-1}U^Td$$

$$= \cancel{wV\Sigma U^Td} - \cancel{wV\Sigma U^Td} + wV\Sigma^2(I-w\Sigma^2)^j\Sigma^{-1}U^Td$$

This is diagonal so I can interchange

$= \boxed{wV(I-w\Sigma^2)^j\Sigma U^Td \text{ which equals RHS}}$

$\boxed{\text{From (i) and (ii), we show (2) is true by induction.}}$