1

Using the forward Fourier equation:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

and the equation for $x[n] = \sin(\frac{\pi}{4}n)$ and $N = 8$, we solve for $k = 1, 3, 7$.

Note that:

$$x[0] = \sin(\frac{\pi}{4}0) = 0$$
$$x[1] = \sin(\frac{\pi}{4}1) = \frac{\sqrt{2}}{2}$$
$$x[2] = \sin(\frac{\pi}{4}2) = 1$$
$$x[3] = \sin(\frac{\pi}{4}3) = \frac{\sqrt{2}}{2}$$
$$x[4] = \sin(\frac{\pi}{4}4) = 0$$
$$x[5] = \sin(\frac{\pi}{4}5) = -\frac{\sqrt{2}}{2}$$
$$x[6] = \sin(\frac{\pi}{4}6) = -1$$
$$x[7] = \sin(\frac{\pi}{4}7) = -\frac{\sqrt{2}}{2}$$

1.1 $k = 1$

$$
\begin{aligned}
X[1] &= \sum_{n=0}^{7} \sin(\frac{\pi}{4}n)e^{-j\frac{2\pi}{8}n} \\
&= \sin(\frac{\pi}{4}0)e^{-j\frac{2\pi}{8}0} + \sin(\frac{\pi}{4}1)e^{-j\frac{2\pi}{8}1} + \sin(\frac{\pi}{4}2)e^{-j\frac{2\pi}{8}2} + \sin(\frac{\pi}{4}3)e^{-j\frac{2\pi}{8}3} \\
&+ \sin(\frac{\pi}{4}4)e^{-j\frac{2\pi}{8}4} + \sin(\frac{\pi}{4}5)e^{-j\frac{2\pi}{8}5} + \sin(\frac{\pi}{4}6)e^{-j\frac{2\pi}{8}6} + \sin(\frac{\pi}{4}7)e^{-j\frac{2\pi}{8}7} \\
&= 0 + \frac{\sqrt{2}}{2}e^{-j\frac{\pi}{4}} + e^{-j\frac{\pi}{2}} + \frac{\sqrt{2}}{2}e^{-j\frac{3\pi}{4}} + 0 - \frac{\sqrt{2}}{2}e^{-j\frac{5\pi}{4}} - e^{-j\frac{3\pi}{2}} - \frac{\sqrt{2}}{2}e^{-j\frac{7\pi}{4}}
\end{aligned}
$$

Using Euler's formula (note this is the negative variant): $e^{-jx} = \cos(x) - j\sin(x)$

$$
\begin{aligned}
&= \frac{\sqrt{2}}{2}(\cos(\frac{\pi}{4}) - j\sin(\frac{\pi}{4})) + (\cos(\frac{\pi}{2}) - j\sin(\frac{\pi}{2})) + \frac{\sqrt{2}}{2}(\cos(\frac{3\pi}{4}) - j\sin(\frac{3\pi}{4})) \\
&- \frac{\sqrt{2}}{2}(\cos(5\frac{\pi}{4}) - j\sin(\frac{5\pi}{4})) - (\cos(\frac{3\pi}{2}) - j\sin(\frac{3\pi}{2})) - \frac{\sqrt{2}}{2}(\cos(\frac{7\pi}{4}) - j\sin(\frac{7\pi}{4}))
\end{aligned}
$$

Evaluating all the cos and sin terms, we get:

$$
\begin{aligned}
&= \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}) + (0 - j) + \frac{\sqrt{2}}{2}(-\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}) - \frac{\sqrt{2}}{2}(-\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}) - (0 + j) - \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}) \\
&= (\frac{1}{2} - j\frac{1}{2}) + (0 - j) + (-\frac{1}{2} - j\frac{1}{2}) + (\frac{1}{2} - j\frac{1}{2}) + (0 - j) + (-\frac{1}{2} - j\frac{1}{2}) \\
&= (1 - j) + (0 - 2j) + (-1 - j) = -4j
\end{aligned}
$$

Which matches $X[1]$ in the plot.

1.2 $k = 3$

Since $x[0]$ and $x[4]$ are 0, we can skip the terms that depend on them.

For $k = 3$, the exponential component of the e term will be $-j\frac{2\pi}{8}3n$. So each term after the application of Euler's formula will be $\cos(\frac{3n\pi}{4}) - j\sin(\frac{3n\pi}{4})$. I'll start my computations from there:

$$
\begin{aligned}
X[3] &= \frac{\sqrt{2}}{2}(\cos(\frac{3\pi}{4}) - j\sin(\frac{3\pi}{4})) + (\cos(\frac{3\pi}{2}) - j\sin(\frac{3\pi}{2})) + \frac{\sqrt{2}}{2}(\cos(\frac{9\pi}{4}) - j\sin(\frac{9\pi}{4})) \\
&- \frac{\sqrt{2}}{2}(\cos(15\frac{\pi}{4}) - j\sin(\frac{15\pi}{4})) - (\cos(\frac{9\pi}{2}) - j\sin(\frac{9\pi}{2})) - \frac{\sqrt{2}}{2}(\cos(\frac{21\pi}{4}) - j\sin(\frac{21\pi}{4})) \\
&= \frac{\sqrt{2}}{2}(-\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}) + (0 + j) + \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}) - \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}) - (0 - j) - \frac{\sqrt{2}}{2}(-\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}) \\
&= (-\frac{1}{2} - j\frac{1}{2}) + (0 + j) + (\frac{1}{2} - j\frac{1}{2}) + (-\frac{1}{2} - j\frac{1}{2}) + (0 + j) + (\frac{1}{2} - j\frac{1}{2}) \\
&= (-1 - j) + (0 + 2j) + (1 - j) = 0
\end{aligned}
$$

Which matches $X[3]$ in the plot.

1.3 $k = 7$

For $k = 7$, the exponential component of the e term will be $-j\frac{2\pi}{8}7n$. So each term after the application of Euler's formula will be $\cos(\frac{7n\pi}{4}) - j\sin(\frac{7n\pi}{4})$. Then:

$$
\begin{aligned}
X[7] = &\frac{\sqrt{2}}{2}(\cos(\frac{7\pi}{4}) - j\sin(\frac{7\pi}{4})) + (\cos(\frac{7\pi}{2}) - j\sin(\frac{7\pi}{2})) + \frac{\sqrt{2}}{2}(\cos(\frac{21\pi}{4}) - j\sin(\frac{21\pi}{4})) \\
&- \frac{\sqrt{2}}{2}(\cos(35\frac{\pi}{4}) - j\sin(\frac{35\pi}{4})) - (\cos(\frac{21\pi}{2}) - j\sin(\frac{21\pi}{2})) - \frac{\sqrt{2}}{2}(\cos(\frac{49\pi}{4}) - j\sin(\frac{49\pi}{4})) \\
&= \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}) + (0 + j) + \frac{\sqrt{2}}{2}(-\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}) - \frac{\sqrt{2}}{2}(-\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}) - (0 - j) - \frac{\sqrt{2}}{2}(\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}) \\
&= (\frac{1}{2} + j\frac{1}{2}) + (0 + j) + (-\frac{1}{2} + j\frac{1}{2}) + (\frac{1}{2} + j\frac{1}{2}) + (0 + j) + (-\frac{1}{2} + j\frac{1}{2}) \\
&= (1 + j) + (0 + 2j) + (-1 + j) = 4j
\end{aligned}
$$

Which matches $X[7]$ in the plot.

2

Using the inverse Fourier equation:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$$

and the values of $X[k]$ obtained from the plot (and $N = 8$), we solve for $n = 5$.

We note that $X[k]$ is 0, except when $k = 1$ or $k = 7$. Therefore, we only need to worry about those terms. Then:

$$x[5] = \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{\pi}{4}k5}$$

$$= \frac{1}{8}(X[1]e^{j\frac{5\pi}{4}} + X[7]e^{j\frac{35\pi}{4}})$$

$$= \frac{1}{8}(-4je^{j\frac{5\pi}{4}} + 4je^{j\frac{35\pi}{4}})$$

Using Euler's formula (note this is the positive variant): $e^{jx} = \cos(x) + j\sin(x)$

$$= \frac{1}{8}(-4j(\cos(\frac{5\pi}{4}) + j\sin(\frac{5\pi}{4})) + 4j(\cos(\frac{35\pi}{4}) + j\sin(\frac{35\pi}{4})))$$

$$= \frac{1}{8}(-4j(-\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}) + 4j(-\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}))$$

$$= \frac{1}{8}(-4j(-j\frac{\sqrt{2}}{2}) + 4j(j\frac{\sqrt{2}}{2}))$$

$$= \frac{1}{8}(-4\frac{\sqrt{2}}{2} - 4\frac{\sqrt{2}}{2}) = \frac{1}{8}(\frac{-8\sqrt{2}}{2}) = \frac{-\sqrt{2}}{2}$$

Which matches $x[5]$ in the plot.

3

We start from the equation we have from 2, only we don't input a value of n:

$$x[n] = \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{\pi}{4}kn}$$
$$= \frac{1}{8}(X[1]e^{j\frac{n\pi}{4}} + X[7]e^{j\frac{7n\pi}{4}})$$
$$= \frac{1}{8}(-4je^{j\frac{n\pi}{4}} + 4je^{j\frac{7n\pi}{4}})$$

Euler's formula again...

$$= \frac{1}{8}((-4j(\cos(\frac{n\pi}{4}) + j\sin(\frac{n\pi}{4})) + 4j(\cos(\frac{7n\pi}{4}) + j\sin(\frac{7n\pi}{4})))$$
$$= \frac{-j}{2}(\cos(\frac{n\pi}{4}) + j\sin(\frac{n\pi}{4})) + \frac{j}{2}(\cos(\frac{7n\pi}{4}) + j\sin(\frac{7n\pi}{4}))$$
$$= \frac{-j}{2}\cos(\frac{n\pi}{4}) + \frac{1}{2}\sin(\frac{n\pi}{4}) + \frac{j}{2}\cos(\frac{7n\pi}{4}) + \frac{-1}{2}\sin(\frac{7n\pi}{4})$$
$$= \frac{1}{2}\sin(\frac{n\pi}{4}) + \frac{-1}{2}\sin(\frac{7n\pi}{4})$$

We will take a short detour to prove: $\frac{-1}{2}\sin(\frac{7n\pi}{4}) = \frac{1}{2}\sin(\frac{n\pi}{4})$

Cancelling the $\frac{1}{2}$ on both sides and using $\sin(-x) = -\sin(x)$, we get:

$$\sin(\frac{-7n\pi}{4}) = \sin(\frac{n\pi}{4})$$

It's easy to see that $\frac{-7\pi}{4}$ is the same angle as $\frac{\pi}{4}$. It's just defined from the other direction!

So they are the same.

Back to solving $x[n]$! Substitute: $\frac{-1}{2}\sin(\frac{7n\pi}{4})$ with $\frac{1}{2}\sin(\frac{n\pi}{4})$

$$x[n] = \frac{1}{2}\sin(\frac{n\pi}{4}) + \frac{1}{2}\sin(\frac{n\pi}{4}) = \sin(\frac{n\pi}{4})$$

Which is $x[n]$!

4

Starting with the forward Fourier equation and $x[n] = \sin(\frac{n\pi}{4})$ for $N = 8$, we have:

$$X[k] = \sum_{n=0}^{7} \sin(\frac{n\pi}{4}) e^{-j\frac{\pi}{4}kn}$$

Using the relation: $\sin(x) = \dfrac{e^{jx} - e^{-jx}}{2j}$

$$= \frac{1}{2j}\sum_{n=0}^{7}(e^{j\frac{n\pi}{4}} - e^{-j\frac{n\pi}{4}})e^{-j\frac{n\pi}{4}k}$$

We see that: $e^{j\frac{-\pi}{4}} = e^{j\frac{7\pi}{4}}$

$$= \frac{1}{2j}\sum_{n=0}^{7}(e^{j\frac{n\pi}{4}} - e^{j\frac{7n\pi}{4}})e^{-j\frac{n\pi}{4}k}$$

$$= \frac{1}{2j}\sum_{n=0}^{7}(e^{j\frac{n\pi}{4}(1-k)} - e^{j\frac{n\pi}{4}(7-k)})$$

$$= \frac{1}{2j}\sum_{n=0}^{7}e^{j\frac{n\pi}{4}(1-k)} - \frac{1}{2j}\sum_{n=0}^{7}e^{j\frac{n\pi}{4}(7-k)}$$

Each sum is almost in the form of the kronecker delta function: $\delta_{ab} = \dfrac{1}{N}\displaystyle\sum_{c=1}^{N} e^{2\pi j\frac{c}{N}(a-b)}$

Which is alternatively defined as: $\delta_{ab} = \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases}$

We multiply and divide by N for each term, adjust the exponent, and reduce the fraction coefficients:

$$= -4j\frac{1}{8}\sum_{n=0}^{7}e^{j\frac{2n\pi}{8}(1-k)} + 4j\frac{1}{8}\sum_{n=0}^{7}e^{j\frac{2n\pi}{8}(7-k)}$$

Let $n = n' - 1$, then:

$$= -4j\frac{1}{8}\sum_{n'=1}^{8}e^{2\pi j\frac{n'-1}{8}(1-k)} + 4j\frac{1}{8}\sum_{n'=1}^{8}e^{2\pi j\frac{n'-1}{8}(7-k)}$$

$$= -4je^{2\pi j\frac{-1}{8}(1-k)}\frac{1}{8}\sum_{n'=1}^{8}e^{2\pi j\frac{n'}{8}(1-k)} + 4je^{2\pi j\frac{-1}{8}(7-k)}\frac{1}{8}\sum_{n'=1}^{8}e^{2\pi j\frac{n'}{8}(7-k)}$$

$$= -4je^{2\pi j\frac{-1}{8}(1-k)}\delta_{1k} + 4je^{2\pi j\frac{-1}{8}(7-k)}\delta_{7k}$$

Because of the delta functions, we see that this function only has values at $k = 1$ and $k = 7$.

And when k is either of those values, the exponential term reduces to 1, since $e^0 = 1$. So:

$$X[k] = \begin{cases} -4j & \text{if } k = 1 \\ 4j & \text{if } k = 7 \\ 0 & \text{otherwise} \end{cases}$$

Which is $X[k]$!