1

Given the equation:

$$F[k_1,k_2] = \sum_{n_2=0}^{N_2-1}\sum_{n_1=0}^{N_1-1} f[n_1,n_2]e^{-j2\pi\left(\frac{n_1k_1}{N_1}+\frac{n_2k_2}{N_2}\right)}$$

We will show that $F[k_1+pN_1, k_2+rN_2] = F[k_1,k_2]$, where $\{p,r\}\in\mathbb{Z}$. To do this, we will set $k_1 = k_1+pN_1$ and $k_2 = k_2+rN_2$ and simplify the right-hand side to obtain the original equation:

$$\begin{aligned}
F[k_1+pN_1, k_2+rN_2] &= \sum_{n_2=0}^{N_2-1}\sum_{n_1=0}^{N_1-1} f[n_1,n_2]e^{-j2\pi\left(\frac{n_1(k_1+pN_1)}{N_1}+\frac{n_2(k_2+rN_2)}{N_2}\right)} \\
&= \sum_{n_2=0}^{N_2-1}\sum_{n_1=0}^{N_1-1} f[n_1,n_2]e^{-j2\pi\left(\frac{n_1k_1}{N_1}+\frac{n_2k_2}{N_2}+\frac{n_1pN_1}{N_1}+\frac{n_2rN_2}{N_2}\right)} \\
&= \sum_{n_2=0}^{N_2-1}\sum_{n_1=0}^{N_1-1} f[n_1,n_2]e^{-j2\pi\left(\frac{n_1k_1}{N_1}+\frac{n_2k_2}{N_2}\right)}e^{-j2\pi(pn_1+rn_2)}
\end{aligned}$$

Since $\{p,r\}\in\mathbb{Z}$ (are integers) and $\{n_1,n_2\}\in\mathbb{Z}^+$ (are positive integers), then $(pn_1+rn_2)\in\mathbb{Z}$ (will also be some integer).

We'll define $q = (pn_1+rn_2)$ to represent some integer. Then:

$$= \sum_{n_2=0}^{N_2-1}\sum_{n_1=0}^{N_1-1} f[n_1,n_2]e^{-j2\pi\left(\frac{n_1k_1}{N_1}+\frac{n_2k_2}{N_2}\right)}e^{-j2\pi q}$$

We can see that $e^{-j2\pi q} = 1$, for all integers $q\in\mathbb{Z}$.

This is because $e^{-j2\pi} = 1$, and multiplying the phase term by an integer, q, keeps the phase in multiples of 2π, which means it is always 1. So:

$$= \sum_{n_2=0}^{N_2-1}\sum_{n_1=0}^{N_1-1} f[n_1,n_2]e^{-j2\pi\left(\frac{n_1k_1}{N_1}+\frac{n_2k_2}{N_2}\right)} = F[k_1,k_2]$$

Q.E.D.

2

Given the inverse Fourier Transform formula:

$$f[n_1,n_2] = \sum_{k_2=0}^{N_2-1}\sum_{k_1=0}^{N_1-1} F[k_1,k_2]e^{j2\pi\left(\frac{n_1k_1}{N_1}+\frac{n_2k_2}{N_2}\right)}$$

We will show that $f[n_1+pN_1, n_2+rN_2] = f[n_1,n_2]$, where $\{p,r\}\in\mathbb{Z}$. To show this, we will use the same approach as **1**:

$$f[n_1+pN_1, n_2+rN_2] = \sum_{k_2=0}^{N_2-1}\sum_{k_1=0}^{N_1-1} F[k_1,k_2]e^{j2\pi\left(\frac{(n_1+pN_1)k_1}{N_1}+\frac{(n_2+rN_2)k_2}{N_2}\right)}$$

$$= \sum_{k_2=0}^{N_2-1}\sum_{k_1=0}^{N_1-1} F[k_1,k_2]e^{j2\pi\left(\frac{n_1k_1}{N_1}+\frac{n_2k_2}{N_2}+\frac{pN_1k_1}{N_1}+\frac{rN_2k_2}{N_2})\right)}$$

$$= \sum_{k_2=0}^{N_2-1}\sum_{k_1=0}^{N_1-1} F[k_1,k_2]e^{j2\pi\left(\frac{n_1k_1}{N_1}+\frac{n_2k_2}{N_2}\right)}e^{j2\pi(pk_1+rk_2)}$$

Since $\{p,r\} \in \mathbb{Z}$ (are integers) and $\{k_1,k_2\} \in \mathbb{Z}^+$ (are positive integers), then $(pk_1+rk_2) \in \mathbb{Z}$ (will also be some integer).

We'll define $s = (pk_1+rk_2)$ to represent some integer. Then:

$$= \sum_{k_2=0}^{N_2-1}\sum_{k_1=0}^{N_1-1} F[k_1,k_2]e^{j2\pi\left(\frac{n_1k_1}{N_1}+\frac{n_2k_2}{N_2}\right)}e^{j2\pi s}$$

We can see that $e^{j2\pi s}=1$, for all integers $s \in \mathbb{Z}$.

This is because $e^{j2\pi}=1$, and multiplying the phase term by an integer, s, keeps the phase in multiples of 2π, which means it is always 1. So:

$$= \sum_{k_2=0}^{N_2-1}\sum_{k_1=0}^{N_1-1} F[k_1,k_2]e^{j2\pi\left(\frac{n_1k_1}{N_1}+\frac{n_2k_2}{N_2}\right)} = f[n_1,n_2]$$

Q.E.D.

3

First, we will consider the an arbitrary measurement $F[k_1,k_2]$. To achieve this, k_1,k_2 must satisfy the following relation:

$$k_x = k_1\Delta k_x$$
$$k_y = k_2\Delta k_y$$

We also have:

$$\Delta k_x = \bar{\gamma}G_x\Delta t_x$$
$$\Delta k_y = \bar{\gamma}G_y\Delta t_y$$

then combining these with the previous set of equations, we have:

$$k_x = \bar{k}_1\gamma G_x\Delta t_x$$
$$k_y = \bar{k}_2\gamma G_y\Delta t_y$$

Thus, k_1,k_2 scale the values of k_x,k_y by acting as integer weights.

For $F[k_1=3,k_2=5]$, a measurement must be made at $k_x=3\Delta k_x$ and $k_y=5\Delta k_y$. For $F[k_1=1,k_2=1]$, a measurment must by made at $k_x=\Delta k_x$ and $k_y=\Delta k_y$. If we examine the scanning parameter equation:

$$k_x = \bar{\gamma}G_x t_x$$
$$k_y = \bar{\gamma}G_y t_y$$

We can see that there are two free parameters to manipulate, the gradient strength (G_x, G_y) and the time the gradient is on (t_x, t_y). To achieve $3k_x$, this means either setting the gradient to $3G_x$ (3 times the strength relative to the gradient strength at $k_1 = 1$ or $k_x = \Delta x$) or setting the gradient time length to $3\Delta t_x$. Similarly, to achieve $5k_x$ to $5G_y$ (5 times the stength relative to the gradient strength at $k_2 = 1$ or $k_y = \Delta y$) or setting the gradient time length to $5\Delta t_y$.

However, we are not limited to changing each parameter individually, we can scale both parameters in proportion to each other as long as they are equal to k_1, k_2. For example, we can also achieve $3k_x$ by setting the gradient strength to $\sqrt{3}G_x$ and the gradient time to $\sqrt{3}\Delta t_x$.