Conformal Field Theory

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Introduction

General facts about phase transitions

A **phase transition** is a transformation of a system from one phase to another.

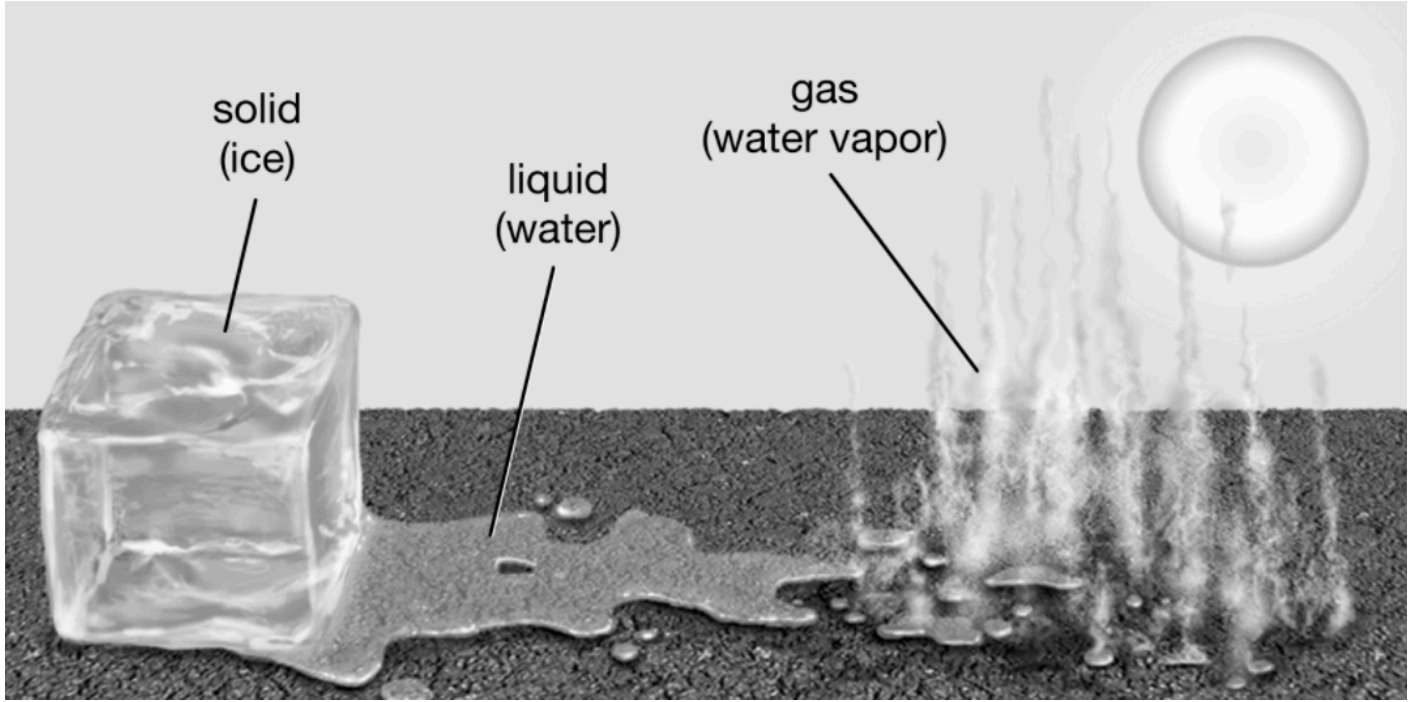


Figure 1: Well know example of transition

General facts about phase transitions

Critical point are the end points of a phase equilibrium curve.

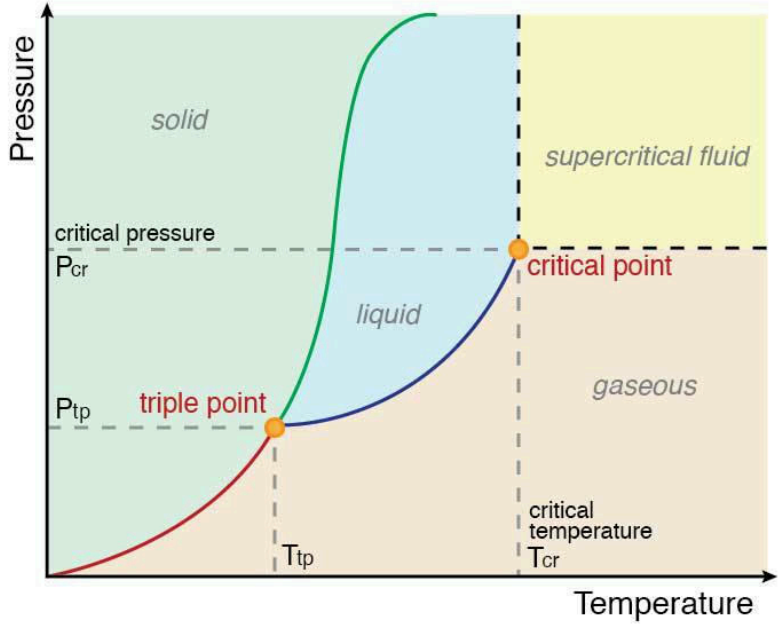


Figure 2: Example of critical point

Critical exponents describe the behavior of physical quantities near continuous phase transitions. Typical examples: $C \sim |T - T_c|^{-\alpha}$ and $M \sim |T - T_c|^{\beta}$.

	d=2	d=3	d=4	general expression
α	0	0.11008708(35)	0	$2 - d/(d - \Delta_\epsilon)$
β	1/8	0.32641871(75)	1/2	$\Delta_\sigma/(d - \Delta_\epsilon)$
γ	7/4	1.23707551(26)	1	$(d - 2\Delta_\sigma)/(d - \Delta_\epsilon)$
δ	15	4.78984254(27)	3	$(d - \Delta_\sigma)/\Delta_\sigma$
η	1/4	0.036297612(48)	0	$2\Delta_\sigma - d + 2$
ν	1	0.62997097(12)	1/2	$1/(d - \Delta_\epsilon)$
ω	2	0.82966(9)	0	$\Delta_{\epsilon'} - d$

Figure 3: Some critical exponents of the Ising model

General facts about phase transitions

It is believed (but not proven) that they are **universal**: they do not depend on the details of the physical system, but only on some of its <u>general features</u>.

General facts about phase transitions

The physics of phase transitions is well understood, but **important questions** remain about critical exponents. How to study them?

Main approaches:

- phenomenological (e.g. Landau) ;
- renormalization group (e.g. Wilson RG) ;
- numerical simulations (e.g. Monte Carlo, Metropolis).

Using the idea of **scale invariance** we can formally go further by studying **conformal field theory**. That was the goal of the internship:

- Get an idea of what is scale invariance using the **renormalization group** ;
- Study of **conformal symmetries** ;
- Apply these symmetries to fields and **correlation functions** ;
- Introduction to the **conformal bootstrap** procedure.

(*critical exponents were just a pretext for studying conformal field theory*)

The renormalization group and the idea of scale invariance

- The **renormalisation group** approach makes it possible to study the behaviour of a system at different **scales**.
- **Wilson**'s idea: remove the **high-energy** degrees of freedom and **zoom out** to see the system on larger scales. The **coupling constants** would **change** accordingly.

Look at the squirrel in all its aspects

Get rid of all the high frequencies in the image

Zoom out to see the system on a larger scale and “gain resolution”

Summary of the RG method

- **Scale invariance** definitely leads to some interesting result ;
- The renormalization group approach is a **good first approach** ;
- **In practice** this is getting very <u>complicated</u>.

$\longrightarrow$ we want to get rid of all the unnecessary details of calculation

The Conformal Group and Algebra

Conformal transformation

- Since **scale symmetry** seems important, let's study it with a more formal approach ;
- Introducing the **conformal transformation**,

Conformal transformation

This is a diffeomorphism $x^\mu \to \tilde{x}^\mu(x^\mu)$ leaving the metric invariant up to a function of the position:

$$\begin{cases} x^\mu \to \tilde{x}^\mu(x^\mu) \\ g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = e^{\sigma(x)}\, g_{\mu\nu}(x) \end{cases}$$

where $e^{\sigma(x)}$ is the function of the position. Its form is somewhat arbitrary.

- We will often restrict ourselves to flat metrics, $g^{\mu\nu} = \eta^{\mu\nu}$.

- A conformal transformation leaves the metric invariant up to a function of the position, say $\Lambda(x)$. What if $\Lambda(x) = 1$?

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(x)$$

- We know that the set of transformations that leave the metric unchanged form the **Poincaré group** $\longrightarrow$ we expect it to be a **subgroup** of the **conformal group**.

The conformal group and algebra

generator	infinitesimal transformation	name
$\hat{P}_\mu = -i\partial_\mu$	$\tilde{x}^\mu(x^\mu) = x^\mu + a^\mu$	translation
$\hat{L}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$	$\tilde{x}^\mu(x^\mu) = (\delta^\mu_\nu + \beta^\mu_\nu)x^\nu$	rotation
$\hat{D} = -ix^\mu\partial_\mu$	$\tilde{x}^\mu(x^\mu) = (1+\alpha)x^\mu$	dilatation
$\hat{K}_\mu = -i(2x_\mu x^\nu\partial_\nu - (x^2)\partial_\mu)$	$\tilde{x}^\mu(x^\mu) = x^\mu + 2(x\cdot b)x^\mu - b^\mu x^2$	S.C.T

$$\varepsilon^\mu = \overbrace{a^\mu}^{\text{translation}} + \underbrace{b^\mu{}_\nu x^\nu}_{\text{lorentz}} + \overbrace{cx^\mu}^{\text{dilation}} + \underbrace{d_\nu(\eta^{\mu\nu}x^2 - 2x^\mu x^\nu)}_{\text{special conform}}$$

The non-zero commutators are as follows:

$$\left[\hat{D}, \hat{P}_\mu\right] = i\hat{P}_\mu$$

$$\left[\hat{D}, \hat{K}_\mu\right] = -i\hat{K}_\mu$$

$$\left[\hat{K}_\mu, \hat{P}_\nu\right] = 2i\left(\eta_{\mu\nu}\hat{D} - \hat{L}_{\mu\nu}\right)$$

$$\left[\hat{K}_\rho, \hat{L}_{\mu\nu}\right] = i\left(\eta_{\rho\mu}\hat{K}_\nu - \eta_{\rho\nu}\hat{K}_\mu\right)$$

$$\left[\hat{P}_\rho, \hat{L}_{\mu\nu}\right] = i\left(\eta_{\rho\mu}\hat{P}_\nu - \eta_{\rho\nu}\hat{P}_\mu\right)$$

$$\left[\hat{L}_{\mu\nu}, L_{\rho\sigma}\right] = i\left(\eta_{\nu\rho}\hat{L}_{\mu\sigma} + \eta_{\mu\sigma}\hat{L}_{\nu\rho} - \eta_{\mu\rho}\hat{L}_{\nu\sigma} - \eta_{\nu\sigma}\hat{L}_{\mu\rho}\right)$$

Operators, Correlation Functions, and Operator Product Expansion

Primary and descendants operators

- In a conformal theory, we can *classify* operators as **primaries** and **descendants**.
- Basically, a **primary operator** transforms as

$$\hat{\tilde{\mathcal{O}}}^A_\Delta(\tilde{x}^\mu) = \underbrace{\boxed{\left|\frac{\partial x^\mu}{\partial \tilde{x}^\nu}\right|}}_{\text{Jacobian}}{}^{\Delta/D} \, \overbrace{\boxed{L^A{}_B}}^{\text{Lorentz representation}} \hat{\mathcal{O}}^B_\Delta(x^\mu)$$

where Δ is the **operator's dimension** and D is the space dimension,

- and any **descendant operator** is obtained from a given primary operator by applying the $\hat{P}_\mu$ generator to it, n times.

$\longrightarrow$ Only **primary operators** matter

Correlation functions

In broad terms, a **correlation function**, in CFT, is a statistical measure that quantifies the interactions between different local operators at different points in space-time.

- We denote them by

$$\boxed{G_n(x_1, x_2, ..., x_n) \equiv \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ ...\ \mathcal{O}_n(x_n)\rangle}$$

- In QFT, they are related to path integrals and functional derivatives.
- Correlation functions homogeneous function.

Correlation functions are the **observables** of the theory, so they must be invariant under *any* coordinate transformation:

$$\boxed{\langle\tilde{\mathcal{O}}_1(x_1^\mu)\tilde{\mathcal{O}}_2(x_2^\mu)\,\dots\,\tilde{\mathcal{O}}_n(x_n^\mu)\rangle \stackrel{!}{=} \langle\mathcal{O}_1(x_1^\mu)\mathcal{O}_2(x_2^\mu)\,\dots\,\mathcal{O}_n(x_n^\mu)\rangle \quad \forall x_i^\mu}$$

$\longrightarrow$ let's apply **conformal transformations** to them under this constraint!

Note: we will only use *scalar operators* to get ride of indices. This means that

$$\hat{\tilde{\mathcal{O}}}_\Delta(\tilde{x}^\mu) = \left|\frac{\partial x^\mu}{\partial\tilde{x}^\nu}\right|^{\Delta/D}\hat{\mathcal{O}}_\Delta(x^\mu)$$

1, 2, 3-point correlation functions under conformal symmetries

By applying the conformal symmetries for 1, 2 and 3-point functions, we obtain

$$\langle \mathcal{O}_\Delta(x^\mu) \rangle = 0 \qquad \text{(1 point)}$$

$$\left\langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \right\rangle = \frac{C \delta_{\Delta_1 \Delta_2}}{\left| x_1^\mu - x_2^\mu \right|^{\Delta_1 + \Delta_2}} \qquad \text{(2 points)}$$

$$\left\langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \mathcal{O}_{\Delta_3}(x_3^\mu) \right\rangle = \frac{C_{123}}{\left| x_{12}^\mu \right|^i \left| x_{23}^\mu \right|^j \left| x_{31}^\mu \right|^k} \qquad \text{(3 points)}$$

where $x_{ij}^\mu := x_i^\mu - x_j^\mu$ and where δ_{kl} is the Kronecker delta.

$\longrightarrow$ These correlation functions are **fully determined**.

Cross-sections

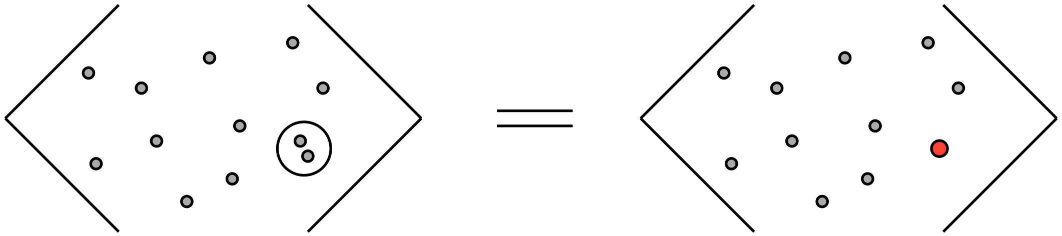
- For the **4-point** correlation function, and above, this will not work anymore.
- At this point, we can form some quantities called **cross-sections** that are invariant under all conformal transformations:

$$u := \frac{\left|x_{12}^{\mu}\right|\left|x_{34}^{\mu}\right|}{\left|x_{13}^{\mu}\right|\left|x_{24}^{\mu}\right|} \qquad \text{and} \qquad v := \frac{\left|x_{12}^{\mu}\right|\left|x_{34}^{\mu}\right|}{\left|x_{23}^{\mu}\right|\left|x_{14}^{\mu}\right|}$$

$\longrightarrow$ The $n \geq 4$-points correlation functions are therefore no longer fully determined, we should make **approximations** to go further...

Operator Product Expansion

In a **correlation function**, when two operators are "close", we can combine them into a single operator using a series expansion. The big picture is the following:



and mathematically (simplified), this means...

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_{\mathcal{O}_I} C_{\Delta,I}(x,\partial)\mathcal{O}_I(0)$$

This is called **Operator Product Expansion** (OPE). Some warnings:

- the OPE is *valid* only in a correlation function;
- the other operators that are in the correlation function must be "sufficiently far away" from the product considered.

$\longrightarrow$ **OPE** gives us a $(n-1)$-point correlation function from a n-point one!

The bootstrap equation

The bootstrap equation

Using together the **crossing symmetries**, the **OPE**, and the associativity of the **correlation functions**, this gives, after calculations, the **bootstrap equation**:

$$\boxed{\sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 \left[\left(\frac{v}{u} \right)^{\Delta} g_{\Delta}(u, v) - g_{\Delta}(v, u) \right] = 0}$$

where

- u and v are the cross sections ;
- g_{Δ} is a **conformal block** ;

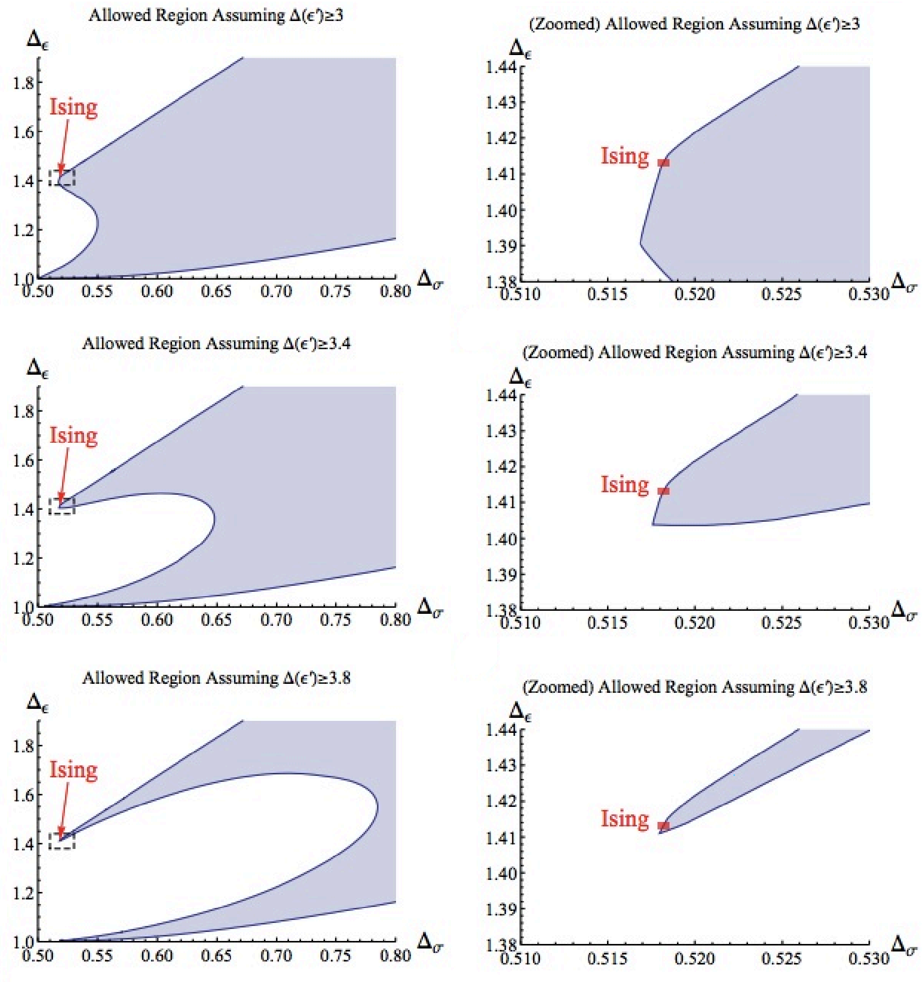
The bootstrap algorithm

Using the previous result, we can establish an algorithm that gives us precise values of the parameters of interest (particularly **critical exponents**):

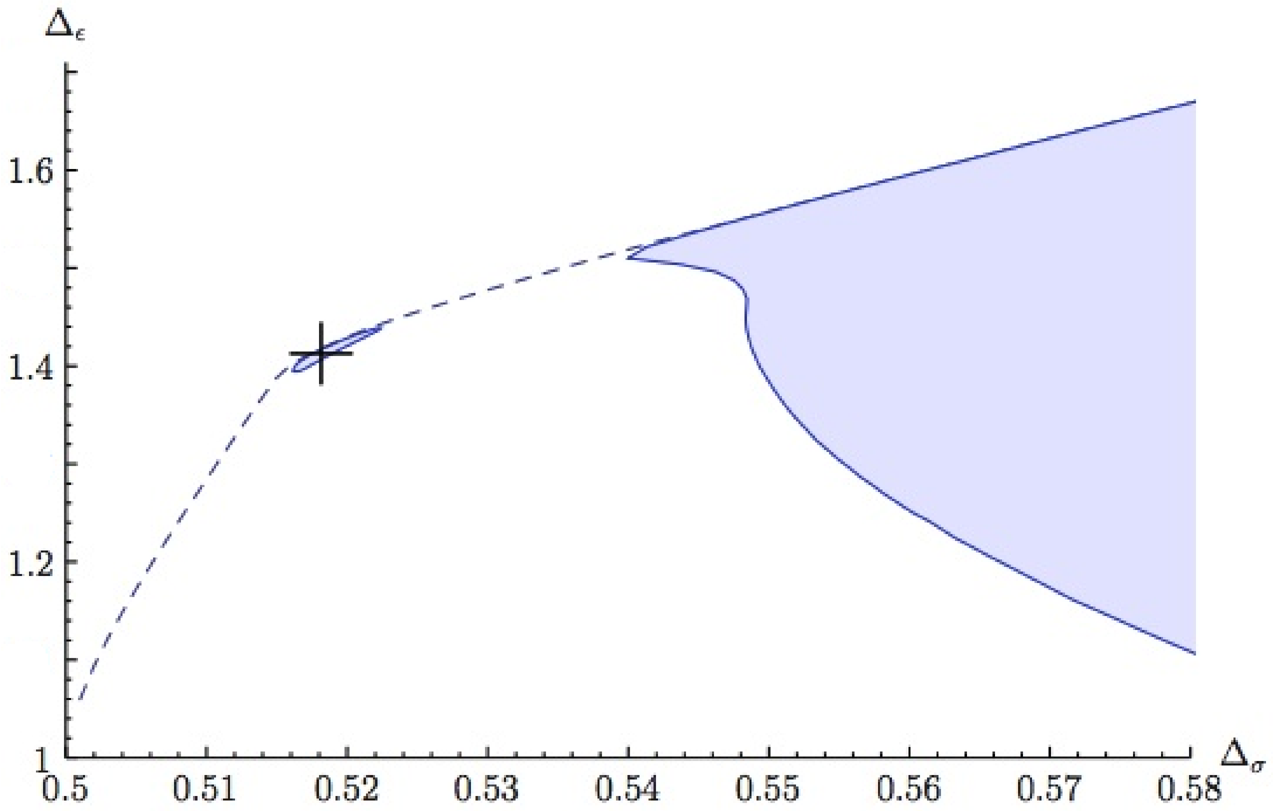
1. Start with **initial parameters** ;
2. Apply the equation adapted to the **physical constraints** of the system ;
3. **Eliminate** the parameter region **incompatible** with the gotten solution ;
4. Select **new parameters** in non-eliminated regions ;
5. **Repeat** steps 2-4 as many times as required ;
6. At the end, we find have a **solution space** small enough to draw conclusions.

This procedure is known as the **bootstrap algorithm**.

Results — back to the Ising model

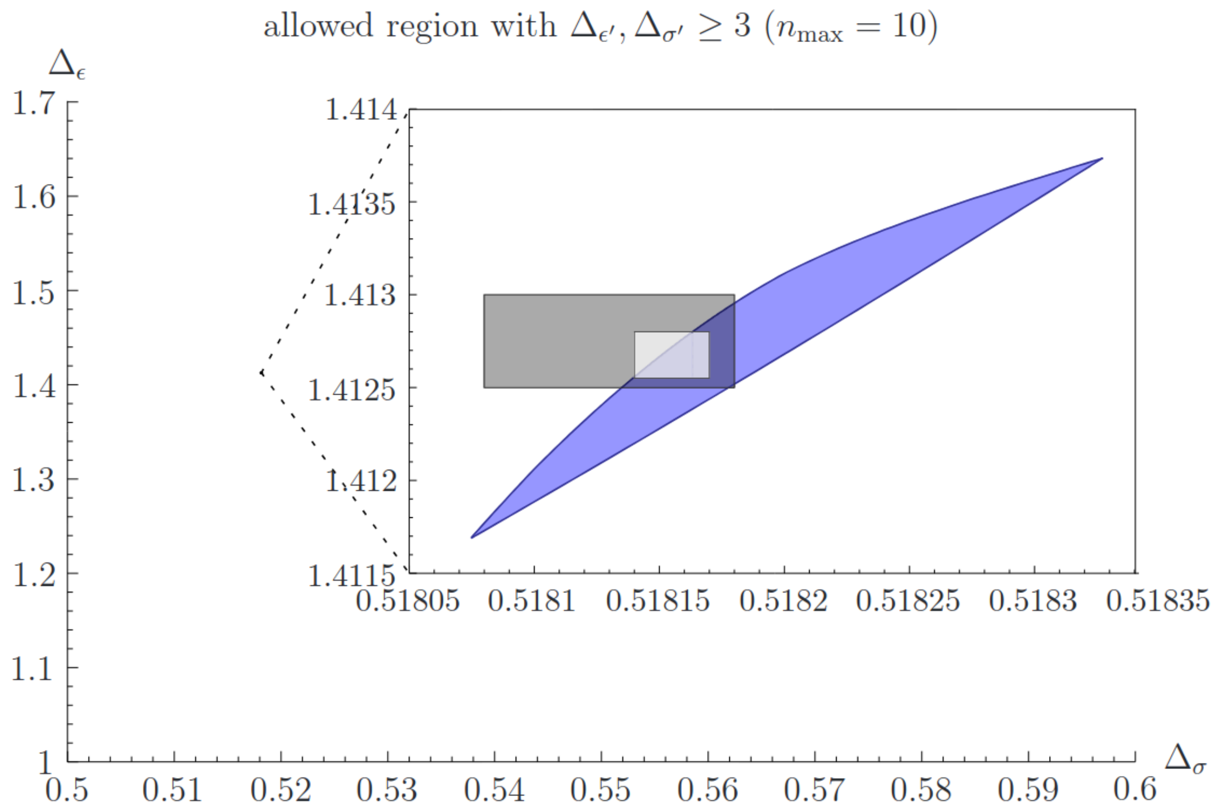


The space of solutions is reduced to a region that is smaller than the error bars:



Results — back to the Ising model

This method far **outperforms** all previous simulations (e.g. with Monte Carlo).



Summary

To summarize, the **bootstrap procedure** offers the following advantages:

- There is **no need** to use Lagrangians, Hamiltonians, partition function, action, ...
- The method is **rapidly convergent** ;
- The method is **more accurate** than alternative numerical simulations.

Conclusion

Conclusion

- Conformal field theory is a **powerful** theory for studying phase transitions ;
- It is **naturally adapted** to the problem ;
- An **efficient algorithm** has been established to compute critical exponents ;
- Conformal field theory is **simpler** than traditional QFT.

Note: The bootstrap approach is still relatively new, and few results have been calculated so far. The Ising model is one of the few.

— *Thank you for your attention.*

References

Slide figures

- Figure 1: *Unacademy – phase transition* ;
- Figure 2: *labxchange – Phase Diagram for Water* ;
- Figure 3: *Wikipedia – Ising Model Critical Exponents* ;
- Conformal bootstrap figures: [1]

Internship

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