

Conformal Field Theory

English version

Aurélien Vandeweyer

This internship report consists of a presentation of the subject of conformal field theory, a subject that was studied during the first-year master's internship at the University of Mons, in the Department of Physics of the Universe, Fields and Gravitation, supervised by Evgeny Skvortsov. The subject is presented in what I hope is a pedagogical way that I would have enjoyed learning about it.

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1. Introduction and Acknowledgments

This internship report presents the knowledge acquired during my first-year Master's internship in Physics at UMONS. The objective was to discover a new field in physics, namely conformal field theory, with a particular focus on phase transitions in the finality. Since the subject was entirely new to me, this document was written in a pedagogical format in order to synthesize all the learning acquired during the internship. The main points covered are:

1. First approach to phase transitions;
2. Introduction to the renormalization group approach;
3. Study of conformal symmetries and their application to correlation functions;
4. The *Operator Product Expansion* and introduction to the bootstrap approach.

Many sources were used to study the subject, and they are listed at the end of the document. The document itself is mostly original in the sense that it aims to explain what was learned in a personal way.

I would like to thank Mr. Skvortsov for his guidance, his explanations and his availability!

1.1. General Facts About Phase Transitions

Phase transitions represent significant changes in the state of a system, resulting in a qualitative change in its macroscopic properties. These phenomena, which are encountered in various fields ranging from solid-state physics to fluid thermodynamics, offer a rather broad field of study and, even though they have been studied for a long time, still present a number of questions of great interest to science, particularly when we are interested in critical points and universality phenomena.

As a system approaches a critical point, it exhibits peculiar behavior: some quantities, such as the correlation length or the susceptibility, diverge or exhibit non-analytic variations. For example, in a continuous (or second-order) transition, the divergence of the correlation length implies that the system becomes scale-invariant: no characteristic length or order of magnitude dominates, and local fluctuations propagate to all scales. This “scalelessness” leads to power laws describing the behavior near the critical point, and whose coefficients—the critical exponents—turn out to be identical for very different systems.

This universality phenomenon means that physically distinct systems with identical symmetries and dimensions share the same critical exponents and correlation functions. For example, a liquid in equilibrium near its critical point and a magnet near its Curie temperature exhibit similar large-scale behaviors, despite their microscopic differences. This property has greatly contributed to the understanding of statistical physics (and (quantum) field theory!), by allowing systems to be grouped into universality classes that depend mainly on a few essential parameters (the spatial dimension, the symmetry of the order parameter, etc.) rather than on the microscopic details of the system.

The theoretical explanation of this universality is still a subject of research, traditionally through the renormalization group approach, which allows to study how interactions at different scales aggregate, more modern numerical or analytical methods have emerged to try

to learn more about critical exponents, such as the conformal field theory approach through the “bootstrap”, as will be discussed.

Finally, and as is often the case in science, and particularly in physics, the study of critical points has repercussions well beyond the physics of macroscopic systems. The ideas of scaling, invariance, and renormalization find applications in particle physics, cosmology, and even in some models of complex phenomena in biological and social systems. Thus, understanding phase transitions, and in particular critical points and universality, represents a bridge between seemingly disparate phenomena, illuminating fundamental regularities that prevail across a multitude of systems.

In conclusion to this first section, the subject of critical points and phase transitions is important and not yet fully understood. Its study is therefore of interest.

2. The renormalization group

Scale invariance is a very powerful concept in physics. It involves analyzing a system starting at a microscopic scale and then gradually “zooming out” to observe the system’s behavior at larger scales. The renormalization group (RG) is the mathematical procedure that allows this scaling to be carried out in a gradual and controlled manner, gradually eliminating details too fine to be relevant at larger scales.

Example: In a piece of metal, we are not concerned with the individual behavior of each electron (if that makes any sense), but rather with global properties such as its conductivity or magnetism.

Before going any further, let us clarify that the renormalization “group” has nothing to do with group theory. We speak of a group because it reflects the idea of “grouping” as we will see shortly. Let us also clarify that the renormalization group “method” is not a recipe to be applied generically (it must also be applied with caution so as not to change the nature or geometry of the system under consideration).

We will introduce the concept through various examples, both discrete and continuous.

2.1. 1D Ising spin chain

Consider a one-dimensional chain composed of N spins with a coupling constant $0 \leq J \leq 1$ between each neighbor.

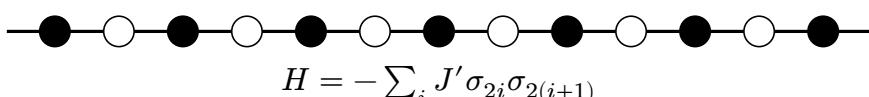


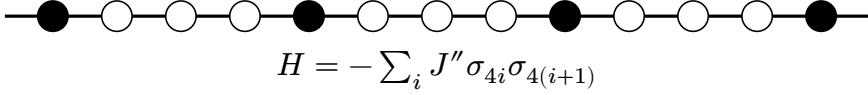
Figure 1: One-dimensional spin chain

The associated Hamiltonian is

$$H = - \sum_i J \sigma_i \sigma_{i+1}. \quad (1)$$

Rather than keeping all N spins, we could, for example, only consider half of these spins on the string and (for large N) still have a reasonably accurate description of the system by describing the remaining spins in terms of a new coupling constant, J' , which takes into account the fact that we have removed some of the spins. This is called “decimation”: in general we will sum (or integrate) a certain fraction of spins at each step, leaving behind a system with fewer spins than at the start but compensated with an “updated” coupling constant:





And so on...

Let's be more concrete and use a spin chain at a non-zero temperature, with the original Hamiltonian, the partition function therefore takes the following form:

$$z = \sum_{\{\sigma_i = \pm 1\}} e^{-\beta H} = \sum_{\{\sigma_i = \pm 1\}} \exp\left(\beta J \sum_i \sigma_i \sigma_{i+1}\right). \quad (2)$$

↑
 A microstate corresponds to a spin up or down

The goal is to take the sum over half of the spins in the system. To do this, we can simply consider skipping every other spin, as suggested above (which could be done mathematically by associating a number with each spin and summing over, for example, only odd-numbered spins). The *expected* partition function should therefore be of the following form:

$$z = \sum_{\{\sigma_i = \pm 1\}} \exp\left(\beta J' \sum_{2i} \sigma_{2i} \sigma_{2(i+1)}\right). \quad (3)$$

In fact, this is simply the original partition function, but where we have explicitly skipped every other step in the sum and modified the coupling constant accordingly. Let's see explicitly what this would give us for a chain of three spins if we carried out this decimation:

$$\begin{aligned} z &= \sum_{\{\sigma_i = \pm 1\}} \sum_{\{\sigma_1\}}^{i=0,2} \exp(\beta J(\sigma_0 \sigma_1 + \sigma_1 \sigma_2)) \\ &= \sum_{\{\sigma_i = \pm 1\}}^{i=0,2} [e^{\beta J(\sigma_0 + \sigma_2)} + e^{-\beta J(\sigma_0 + \sigma_2)}] \\ &= \sum_{\{\sigma_i = \pm 1\}}^{i=0,2} 2 \cosh[\beta J(\sigma_0 + \sigma_2)]. \end{aligned} \quad (4)$$

Comparing with eq. (3), we clearly haven't arrived at the desired result, but on the other hand, by comparison, it would give us a relationship between the coupling constant J and J' , and that's what we're looking for in the end! For N spins instead of three, eq. (4) is simply rewritten as

$$z = \sum_{\{\sigma_i = \pm 1\}} \prod_i 2 \cos[\beta J(\sigma_{2i} + \sigma_{2(i+1)})]. \quad (5)$$

Let's then compare this expression with eq. (3),

$$\begin{aligned}
& \sum_{\{\sigma_i\}} \prod_i 2 \cosh [\beta J (\sigma_{2i} + \sigma_{2(i+1)})] \\
& \stackrel{!}{=} \sum_{\{\sigma_i\}} e^{\beta J' \sum_i \sigma_{2i} \sigma_{2(i+1)}} \\
& \equiv \sum_{\{\sigma_i\}} \prod_i e^{\beta J' \sigma_{2i} \sigma_{2(i+1)}}.
\end{aligned} \tag{6}$$

To compare the left-hand side and the right-hand side, one way is to study the weights for each pair (σ_i, σ_{i+1}) . The spins σ_i and σ_{i+1} can take the values ± 1 . If $\sigma_i = \sigma_{i+1}$, then $\sigma_i + \sigma_{i+1} = \pm 2$ and the contribution of the cosh term becomes

$$2 \cosh(\beta J(\pm 2)) = 2 \cosh(2\beta J). \tag{7}$$

If $\sigma_i = -\sigma_{i+1}$, then $\sigma_i + \sigma_{i+1} = 0$ and the contribution of the term in cosh becomes

$$2 \cosh(\beta J \cdot 0) = 2 \cosh(0) = 2. \tag{8}$$

On the right side of the equality now, if $\sigma_i = \sigma_{i+1}$, then $\sigma_i \sigma_{i+1} = +1$ and the contribution in the exponential becomes

$$\exp(\beta J' \sigma_i \sigma_{i+1}) = \exp(\beta J'). \tag{9}$$

If $\sigma_i = -\sigma_{i+1}$, then $\sigma_i \sigma_{i+1} = -1$ and the contribution in the exponential becomes

$$\exp(\beta J' \sigma_i \sigma_{i+1}) = \exp(-\beta J'). \tag{10}$$

To make the two expressions identical configuration by configuration, we then compare the weight ratios between the parallel configuration and the antiparallel configuration:

- **LHS**

$$\frac{\text{weight}(+1)}{\text{weight}(-1)} = \frac{2 \cosh(2\beta J)}{2} = \cosh(2\beta J) \tag{11}$$

- **RHS**

$$\frac{\text{weight}(+1)}{\text{weight}(-1)} = \frac{\exp(\beta J')}{\exp(-\beta J')} = \exp(2\beta J') \tag{12}$$

we therefore deduce the following condition:

$$\cosh(2\beta J) = \exp(2\beta J') \iff 2\beta J' = \log[\cosh(2\beta J)] \tag{13}$$

in other words,

$$J' = \frac{1}{2\beta} \log [\cosh(2\beta J)] \tag{14}$$

So we have obtained an expression for the scaled coupling constant, J' , which physically describes the interactions between the remaining spins after a decimation step. Hence, for a given initial J (which corresponds to the still unrenormalized system), we have a recursive relation that we can apply a number of times and which will correspond, by construction, to this renormalization procedure where, at each step, we zoom out a little more and get rid of the microscopic details of the system!

What happens when we iterate enough times? Suppose the initial system is such that the temperature is high (i.e., β is very small). Then,

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \mathcal{O}(x^6) \quad (15)$$

which, in our case, gives us

$$\cosh(2\beta J) \approx 1 + 2(\beta J)^2. \quad (16)$$

Then,

$$\log[\cosh(2\beta J)] \approx \log(1 + 2(\beta J)^2) \approx 2(\beta J)^2 \quad (17)$$

where we used $\log(1 + x) \approx x$ for x to be small enough. We substitute this result into our formula for J' and then find the new relation

$$J' = \frac{1}{2\beta} \log[\cosh(2\beta J)] \approx \frac{1}{2\beta} [2(\beta J)^2] = \beta J^2. \quad (18)$$

So, when the temperature is high enough, and therefore β is small enough, we get a new coupling $J' \approx \beta J^2$ which is *smaller* than J (recall that $0 < J \leq 1$ and that $\beta \ll 1$). In other words, the more we “zoom out” the more the coupling constant tends towards zero, we can conclude that J is a useless variable for describing the large-scale properties of the system! We can also implement the previous recursive function numerically. For example, after a few iterations, we get the following table:

n	J_n
0	1
1	0.433781
2	0.0912725
3	0.00415956
4	8.65094e-06
5	3.74194e-11

It only takes a few iterations to realize that, indeed, the coupling constant becomes irrelevant at larger scales of the system. Physically, this means that thermal fluctuations dominate and spin-spin interactions do not contribute to the magnetic properties of the system. For a smaller temperature, the trend is the same but the convergence is slower. In this particular example, this means that the system is paramagnetic and there is no phase transition possible.

2.2. 1D spin chain with an external magnetic field

We will not go into as much detail here as before, but will illustrate a slightly different configuration that presents a phase transition that we will be able to identify thanks to the renormalization group. Let us consider the same spin chain as before, with the same coupling constant J between each spin, but this time we add an external transverse magnetic field h . The previous Hamiltonian is then expressed here as

$$H = - \sum_{\langle i,j \rangle} J \sigma_i^x \sigma_j^x - \sum_i h \sigma_i^z \quad (19)$$

We will apply the RG method, this time using a similar form of decimation, but one that better captures the physics of the system and is simpler to implement given the situation. Instead of counting the contribution of every other spin, we will count the contribution of pairs of spins, in other words, we form a “big spin” from two neighboring spins and choose to assign to the latter the lowest energy of the two composite spins in order to maintain the appearance of a low-energy system (if we did the opposite, at each step of the decimation, the system would potentially gain energy, which is not representative of a low energy system at the beginning). This decimation method is called “spin blocks”. We will not give the derivation of the recursive formula but only give the final result, which actually consists of a relation for the coupling constant and a relation for the external magnetic field:

$$\begin{aligned} J' &= \frac{J^2}{\sqrt{J^2 + h^2}}, \\ h' &= 2h \sqrt{\frac{1}{2} + \frac{J}{2\sqrt{J^2 + h^2}}} \sqrt{\frac{1}{2} - \frac{J}{2\sqrt{J^2 + h^2}}}. \end{aligned} \quad (20)$$

We can express the ratio h/J and study numerically what happens:

- For an initial ratio $h/J > 1$, after iterations, the ratio diverges and becomes infinite. This tells us that J is a less relevant variable than h . Physically, the material is in a paramagnetic phase;
- For an initial ratio $h/J < 1$, we find that the ratio converges to zero, which indicates that J is a more relevant variable than h . The material is in a ferromagnetic phase.
- For an initial ratio $h/J = 1$, we notice that the ratio does not change and remains at 1.

In other words, this means that the system is undergoing a phase transition! The fact that a parameter of the renormalization group (here, the ratio h/J) remains constant throughout the procedure reflects the scaling invariance of the system near a critical point, and we further note that there are two distinct trends of this parameter “before” and “after” this critical point, which illustrates well the existence of different phases for this particular system. In fact, a slightly more detailed analysis is needed to really conclude that a phase transition is possible, depending on the stability of the system (we’ll see an example later), but this initial approach using renormalization groups nevertheless points to a possible transient phenomenon.

2.3. 2D Ising model (idea)

How can we apply the renormalization group in a more realistic 2D model? The idea remains the same: we want to find a way to “zoom out.” We won’t go into any detail here, but only illustrate that an ill-advised choice of decimation leads to changing the geometry of the physical system, which is exactly what we want to avoid.

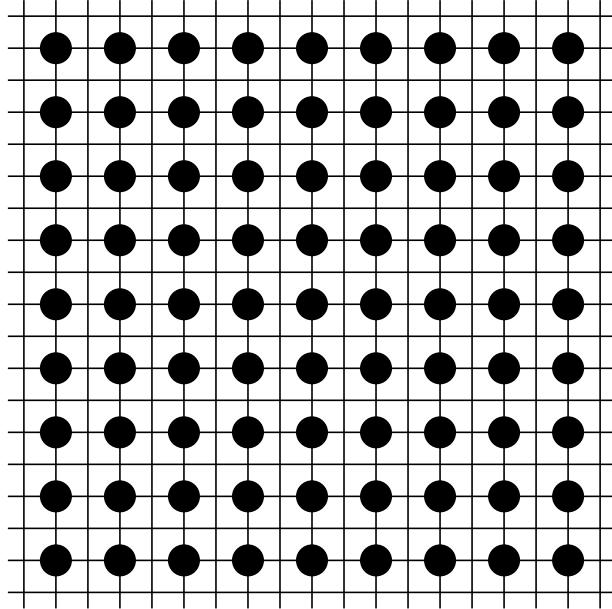


Figure 2: Two-dimensional configuration

If we apply the same decimation method as for the one-dimensional chain where we skip every other spin, then we end up with a configuration that no longer respects the geometry of the system, in fact we would end up with a collection of almost decoupled chains, as illustrated below:

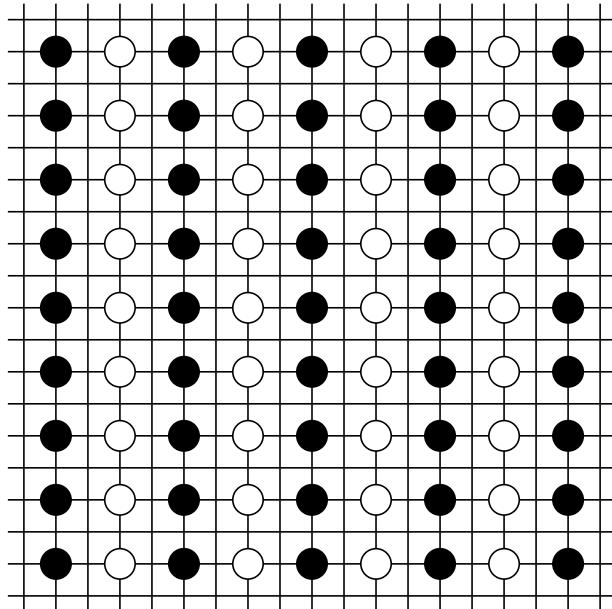


Figure 3: Configuration in which every other spin is removed: we end up with chains and the couplings between the spins are no longer very clear.

We could try a variation of this decimation method, where we remove every other spin on one row, and then the same on the next row but in a staggered manner:

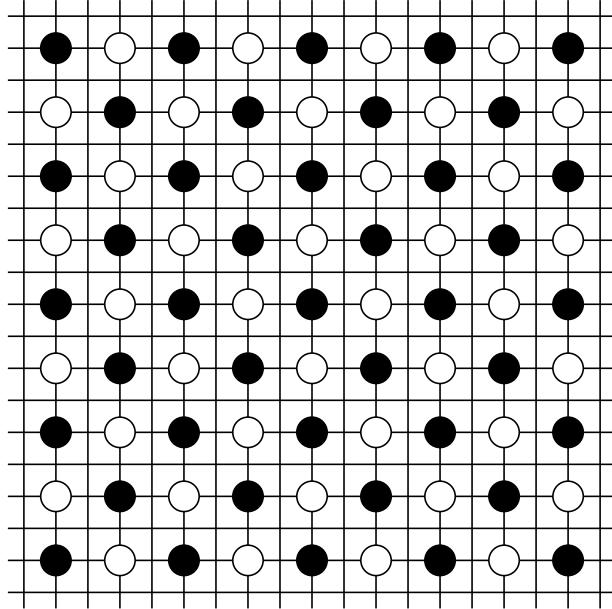


Figure 4: Configuration where the decimation is shifted for each row

This configuration seems to preserve the geometry of the system! However, a careful eye will realize that this configuration amounts to performing an initial rotation of the system where each spin is distant by a factor $r \rightarrow \sqrt{2}r$ where every second spin is removed; in other words, this decimation method once again changes the geometry of the spin network.

The conclusion to this is that choosing a decimation method in $d \geq 2$ must be done carefully so as not to alter the geometry of the network.

2.4. Renormalization group for a continuous system – Wilson’s method

So far, we have presented the ideas of the renormalization group (and have completely determined the equation for the simplest case) in discrete, lattice systems. We will now turn our attention to the continuous case. The method is essentially the same as before, but instead of starting in real space, we start with momentum space (this is the Wilson renormalization procedure). This approach requires a field description of the system, in particular the partition function is expressed as

$$z = \alpha \int [\mathcal{D}\varphi] e^{-S[\varphi]} \quad (21)$$

where α is a normalization factor.

Remark: The functional measure, often referred to as $[\mathcal{D}\varphi]$, basically means that

$$\int [\mathcal{D}\varphi] F[\varphi] \equiv \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \prod_x d\varphi(x) F[\varphi]. \quad (22)$$

The idea in Wilson’s approach is that we “integrate the high-energy physics” (i.e., “get rid” of the “fast modes,” i.e., short-wavelength functions) and keep only the long-wavelength

behavior (“slow modes”), which therefore describe the behavior of the system at larger scales. To do this, a standard approach is to express φ as a sum of slow and fast modes, or equivalently, as a sum of high- or low-energy modes. One will encounter the following notations:

$$\varphi = \varphi_{<} + \varphi_{>} \quad \text{or} \quad \varphi = \varphi_{\text{low}} + \varphi_{\text{high}}$$
(23)

which is often abbreviate as φ_l and φ_h . This will give us the following partition function:

$$z = \int [\mathcal{D}\varphi_{<}][\mathcal{D}\varphi_{>}] e^{-S[\varphi_{<}, \varphi_{>}]}.$$
(24)

The action S also takes an adapted form:

$$S[\varphi_{<}, \varphi_{>}] = S_0[\varphi_{<}] + S_1[\varphi_{>}] + \delta S[\varphi_{<}, \varphi_{>}]$$
(25)

where the last term corresponds to the mixed modes (which depend on both the weak and fast modes). Wilson’s procedure can be visualized as follows: we start with a system without any alteration, then we remove the high-frequency modes, and finally we “zoom out” by rescaling the pulses, as illustrated very schematically with the squirrel image below:

We consider the squirrel in all its aspects	We get rid of the high frequencies of the image (blurred image)	we zoom out to see the system on a larger scale

As this pictorial view suggests, this method will allow us to establish how the parameters evolve under rescaling after removing the microscopic information. Note that the squirrel maintains the same shape throughout the process.

2.4.1. Gaussian action

We will illustrate Wilson’s renormalization procedure using a simple example where we will consider a quadratic action (also sometimes called “Gaussian”) for a free scalar field $\varphi(k)$ in the space k . The action is expressed as

$$S[\varphi] = \frac{1}{2} \int_{|k| < \Lambda} d^d k (k^2 + r) |\varphi(k)|^2$$
(26)

where r is the mass term and Λ bounds the integrated k region of space. In physical terms, this action can be seen as the quadratic approximation of a Ginzburg-Landau model for a phase transition, but it is also found in other models. We start by separating $\varphi(k)$ into “slow modes” $\varphi_{<}$ and “fast modes” $\varphi_{>}$, more specifically,

- the **slow modes** $\varphi_{<}$ are such that $|k| < \Lambda/b$;

- the **fast modes** $\varphi_>$ are such that $\Lambda/b < |k| < \Lambda$

where b is a scaling factor (we will come back to this later). We can then express the φ field as a piecewise function,

$$\varphi(k) = \begin{cases} \varphi_<(k) & \text{if } |k| < \Lambda/b \\ \varphi_>(k) & \text{if } \Lambda/b < |k| < \Lambda. \end{cases} \quad (27)$$

In this case, the action takes the following form:

$$\begin{aligned} S[\varphi] &= \frac{1}{2} \left(\int_{|k| < \Lambda/b} d^d k (k^2 + r) |\varphi_<(k)|^2 + \int_{\Lambda/b < |k| < \Lambda} d^d k (k^2 + r) |\varphi_>(k)|^2 \right) \\ &\approx \frac{1}{2} \left(\int_{|k| < \Lambda/b} d^d k (k^2 + r) |\varphi_<(k)|^2 + \int_{\Lambda/b < |k| < \Lambda} \cancel{d^d k (k^2 + r)} \overline{|\varphi_>(k)|^2} \right) \quad (28) \\ &\approx \frac{1}{2} \left(\int_{|k| < \Lambda/b} d^d k (k^2 + r) |\varphi_<(k)|^2 \right. \\ &\quad \left. =: S_{\text{eff}}[\varphi_<] \right) \end{aligned}$$

where we have therefore “eliminated” the fast modes.

Remark: In reality, although the result is the same (up to a constant), simply “crossing out” the second term is not very rigorous. When considering the partition function,

$$z = \int [\mathcal{D}\varphi] e^{-S[\varphi]}, \quad (29)$$

the integral over $\varphi_>$ is a Gaussian independent of $\varphi_<$ and factors by simply producing a multiplicative constant (a factor $\exp(-1/2 \text{tr}(k^2 + r))$ to be precise), but in order not to go into computational details we take a shorter route. We therefore make the constant resulting from this process implicit.

Remembering the squirrel image, we want to return to an action that takes a similar form to the original. To get out of this, and this naturally introduces a *rescaling*, we set $\tilde{k} := bk$ and so the bound $|k| < \Lambda/b$ becomes $|\tilde{k}| < \Lambda$. The slow mode $\varphi_<$ must also be rescaled to “regain resolution” (like the third squirrel image), so we must set a $\tilde{\varphi}$ of the following form:

$$\tilde{\varphi}(\tilde{k}) := b^{-\Delta} \varphi_<(k = \tilde{k}/b) \quad (30)$$

where $\Delta := (d - 2)/2$ is a term introduced for dimensionality reasons (according to convention we take a different sign in front of Δ). Now, the effective action is rewritten

$$\begin{aligned}
S_{\text{eff}[\varphi_<]} &= \int_{|\tilde{k}|<\Lambda} (b^{-d} d^d \tilde{k}) \left(\frac{\tilde{k}^2}{b^2} + r \right) |\varphi_<(\tilde{k}/b)|^2 \\
&= \int_{|\tilde{k}|<\Lambda} d^d \tilde{k} b^{-d+2} (\tilde{k}^2 + b^2 r) |\varphi_<(\tilde{k}/b)|^2 \\
&= \int_{|\tilde{k}|<\Lambda} d^d \tilde{k} b^{-d+2} (\tilde{k}^2 + b^2 r) |b^{+\Delta} \varphi(\tilde{k})|^2 \\
&= \int_{|\tilde{k}|<\Lambda} d^d \tilde{k} b^{-d+2+2\Delta} (\tilde{k}^2 + b^2 r) |\varphi(\tilde{k})|^2 \\
&= \int_{|\tilde{k}|<\Lambda} d^d \tilde{k} (\tilde{k}^2 + r') |\varphi(\tilde{k})|^2
\end{aligned} \tag{31}$$

where we go from the first to the second line by factoring b , from the second to the third by introducing eq. (30) and where, on the last line,

$$-d + 2 + 2\Delta = -d + 2 + \mathcal{Z} \left(\frac{d-2}{\mathcal{Z}} \right) = 0, \tag{32}$$

and where we have set

$$r' := b^2 r. \tag{33}$$

So we have effectively rewritten S_{eff} in the same form as the original action. Let's recap what we have done:

1. We started from an action for the φ field;
2. We decomposed φ into “slow modes” and “fast modes”;
3. We got rid of fast modes;
4. We returned to an action having the same shape as originally by “zooming out”.

So the image we gave with the squirrel was really not misleading, we did exactly the same thing here. What do we have left now? The important point that emerged from this procedure is the redefinition (the *renormalization*) of the mass parameter eq. (33). As we saw previously with discrete Ising models, these parameters that appear during renormalization characterize how sensitive the system is to it at large scales. In the present case, since we are in a continuous case, we can go further: these recursive equations are similar to differential equations given an infinitesimal decimation procedure. Generally, we set

$$b = e^{dl} \approx 1 + dl \tag{34}$$

where dl is an “infinitesimal decimation step”. We then write

$$r' = b^2 r \iff r' = e^{2dl} r \approx (1 + 2dl)r \tag{35}$$

and, after some manipulations, we arrive at the *renormalization group flow equation*

$$\frac{dr}{dl} = 2r$$

(36)

This equation, as for the discrete cases, tells us how the mass term r changes as we “zoom out” to look at the system at larger scales. This equation presents the flux associated with figure Figure 5.

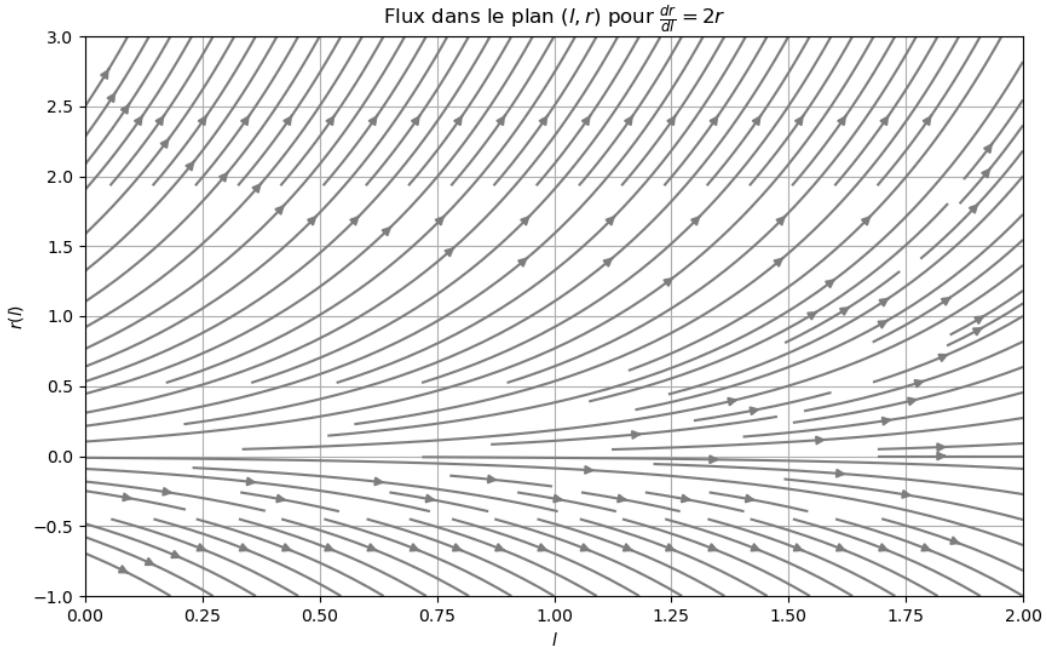


Figure 5: Flow graph of the equation derived by the Wilson renormalization procedure

Analytically (but we also see it on the graph Figure 5), if $r_{\text{initial}} > 0$ then $r(l) \rightarrow \infty$, if $r_{\text{initial}} < 0$ then $r(l) \rightarrow -\infty$ and if $r_{\text{initial}} = 0$ then it is a fixed point, but unstable (the slightest perturbation makes $r(l)$ diverge), which does not indicate the existence of a phase transition (the random fluctuations of the system would not allow the fixed point to remain fixed and would make it diverge). In a theory with interaction (typically φ^4) then the procedure would indeed bring us a stable fixed point, a sign of a phase transition.

To conclude the introduction given on the renormalization group method, we have seen that it is a powerful procedure based on the important idea that a system must be *invariant under rescaling*. By applying this principle through some discrete and continuous examples with adapted procedures, we have highlighted how the properties of systems evolve when viewed under larger scales, which has also allowed us to study the existence of phase transitions. There is of course much more to say, but it is beyond the scope of this document.

3. Conformal field theory

As mentioned and seen earlier, it seems that symmetries under dilation, or rescaling, play a rather important role. As always in physics, when discussing symmetries, it is appropriate to look at them more formally with the eye of group theory. The next section is thus devoted to the conformal group and its algebra, which provide the appropriate mathematical framework for understanding these symmetries. We will begin by clearly defining what is meant by “conformal transformation,” and from there we can develop the rest.

Remark: Conformal symmetries include symmetry under dilation: conformal invariance implies scale invariance, but the converse is not true in general.

Let us also note an important element: the study of a conformal theory is different depending on the dimension of the theory, in particular it is convenient to distinguish the case in dimension $d = 2$ from the more general case $d \geq 3$, in fact, as we will see, a conformal theory in two dimensions (like string theory) has an infinite-dimensional algebra, while, as we will see, a CFT in $d \geq 3$ has a finite-dimensional algebra. It follows that the study of conformal theories in $d \geq 3$ dimensions is less simpler and it is these that will interest us in the following. There are also one-dimensional CFTs, but we will only mention them.

In most of the cases that will interest us, we will also consider a flat spacetime, so $g_{\mu\nu} = \eta_{\mu\nu}$. The study of a CFT where the metric is not flat or is not sufficiently simple does not seem relevant: the physical cases whose conformal symmetries are of standard and principal interest do not involve gravity (an important counterexample is the AdS/CFT correspondence) and, if this were the case, analytical solutions would probably be too difficult to find, if any exist.

3.1. Conformal transformation

We will define a *conformal transformation* as a certain change of coordinates, a diffeomorphism, $x^\mu \rightarrow x'^\mu(x^\mu)$ leaving the metric invariant up to a function of the position that we will call *the scale factor*. In order for the new metric to be well-defined, positive and non-zero, the most natural way to formulate this is to write

$$\begin{cases} x^\mu \rightarrow x'^\mu(x^\mu) \\ g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \boxed{e^{\sigma(x)}} g_{\mu\nu}(x), \end{cases} \quad \text{scale factor} \quad (37)$$

but it is common to see other writing conventions, such as

$$\begin{cases} x^\mu \rightarrow x'^\mu(x^\mu) \\ g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \boxed{\Omega(x)^2} g_{\mu\nu}(x), \end{cases} \quad \text{scale factor} \quad (38)$$

or even

$$\begin{cases} x^\mu \rightarrow x'^\mu(x^\mu) \\ g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \boxed{\Lambda(x)} g_{\mu\nu}(x). \end{cases} \quad \text{scale factor} \quad (39)$$

Depending on what is more convenient, we will use one or the other of these conventions. Note that if the scale factor reduces to the constant 1, then this means that the transformation fully preserves the structure of the spacetime metric, except that we know that the set of transformations that leave the Minkowski metric invariant form the *Poincaré group*, which includes the Lorentz transformations. In other words, we can already see that the group of conformal transformations, the *conformal group*, contains the Poincaré group. We will use this fact in the following, as we expect to see the associated generators that we already know emerge again.

Remark: A conformal transformation and a Weyl transformation are two different transformations: a Weyl transformation is not a change of coordinates, but simply a rescaling of the metric.

It follows from the definition that a conformal transformation does not necessarily preserve distances, but will always locally preserve angles: if two curves intersect at an angle α then, after conformal transformation, they will always intersect at an angle α . The term “conformal” comes from the Latin “conformatilis” which means “of the same shape”, indeed a conformal transformation preserves the local “shape” of the figures, in the sense that it preserves the angles between the intersecting curves. Even if the distances and areas can be modified, the “conformity” of the angles is maintained.

Example: A simple example of a conformal transformation is a *dilation* (change of scale) in a flat 2-dimensional space. Consider the metric

$$ds^2 = dx^2 + dy^2 \quad (40)$$

and apply the transformation

$$x \rightarrow x' = \lambda x, \quad y \rightarrow y' = \lambda y \quad (41)$$

where $\lambda > 0$ is a constant. We then find that

$$\begin{aligned} ds'^2 &= (dx')^2 + (dy')^2 \\ &= \lambda^2 dx^2 + \lambda^2 dy^2 \\ &= \lambda^2 (dx^2 + dy^2) \end{aligned} \quad (42)$$

in other words,

$$g'_{\mu\nu}(x') = \lambda^2 g_{\mu\nu}(x). \quad (43)$$

This is indeed a conformal transformation, it is a simple but important example.

Finding an example that remains relatively simple is not easy, but that does not mean that they do not exist; in reality, it is mainly the results that will subsequently flow from the properties

of these transformations that will be of interest. In the meantime, we can still provide a less trivial example that illustrates the definition of conformal transformation.

Example: Let us still consider a Euclidean metric

$$ds^2 = dx^2 + dy^2 \quad (44)$$

and consider the following coordinate transformation:

$$x' = x^2 - y^2, \quad y' = 2xy. \quad (45)$$

To calculate $ds' = dx'^2 + dy'^2$, we calculate the following differentials:

$$dx' = 2xdx - 2ydy, \quad dy' = 2xdy + 2ydx \quad (46)$$

and so

$$\begin{aligned} (dx')^2 &= (2xdx - 2ydy)^2 = 4(xdx - ydy)^2 \\ (dy')^2 &= (2xdy + 2ydx)^2 = 4(xdy + ydx)^2 \end{aligned} \quad (47)$$

which gives, after calculation and rearrangement of the terms,

$$\begin{aligned} ds'^2 &= (dx')^2 + (dy')^2 \\ &= 4(x^2 + y^2)(dx^2 + dy^2). \end{aligned} \quad (48)$$

In other words, we have a transformation of the coordinates which is such that the new metric is proportional to the old one up to a function of the coordinates. More explicitly,

$$ds'^2 = \Omega(x, y)^2 ds^2 \quad \text{with} \quad \Omega(x, y) = 2\sqrt{x^2 + y^2}. \quad (49)$$

3.2. Derivation of the conformal group and algebra

First, we will work with an arbitrary infinitesimal transformation that we will then constrain to be conformal in order to obtain the useful relations for the following. Let us therefore consider a first-order infinitesimal transformation of a coordinate x^μ ,

$$x'^\mu = x^\mu + \varepsilon^\mu(x) + \mathcal{O}(\varepsilon^2) \quad (50)$$

and recall that for an arbitrary change of coordinate $x \rightarrow x'$, the metric transforms according to

$$\eta'_{\mu\nu}(x') = \eta_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu}. \quad (51)$$

By inserting eq. (50) into eq. (51) and keeping everything in first order, we find

$$\begin{aligned}
\eta_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} &= \eta_{\rho\sigma} \frac{\partial}{\partial x^\mu} (x^\rho + \varepsilon^\rho(x) + \mathcal{O}(\varepsilon^2)) \frac{\partial}{\partial x^\nu} (x^\sigma + \varepsilon^\sigma(x) + \mathcal{O}(\varepsilon^2)) \\
&= \eta_{\rho\sigma} \left(\delta_\mu^\rho \delta_\nu^\sigma + \delta_\mu^\rho \frac{\partial \varepsilon^\sigma}{\partial x^\nu} + \delta_\nu^\sigma \frac{\partial \varepsilon^\rho}{\partial x^\mu} \right) + \mathcal{O}(\varepsilon^2) \\
&= \eta_{\mu\nu} + \left(\frac{\partial \varepsilon_\mu}{\partial x^\nu} + \frac{\partial \varepsilon_\nu}{\partial x^\mu} \right) + \mathcal{O}(\varepsilon^2) \\
&= \eta_{\mu\nu} + (\partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu) + \mathcal{O}(\varepsilon^2)
\end{aligned} \tag{52}$$

where, in the last line, we simply used the notation $\partial/(\partial x^\mu) \equiv \partial_\mu$. For such a transformation to be conformal, we see that we must have $\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x) \eta_{\mu\nu}$ where $f(x)$ is any function. We can apply the inverse metric to both sides of this equality so as to determine an expression for $f(x)$:

$$\begin{aligned}
\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu &\stackrel{!}{=} f(x) \eta_{\mu\nu} \\
\iff \eta^{\mu\nu} \partial_\nu \varepsilon_\nu + \eta^{\mu\nu} \partial_\nu \varepsilon_\mu &= f(x) \eta^{\mu\nu} \eta_{\mu\nu} \equiv f(x) d \\
\iff f(x) &= \frac{2}{d} \partial^\mu \varepsilon_\mu.
\end{aligned} \tag{53}$$

We will sometimes denote the divergence $\partial^\mu \varepsilon_\mu =: (\partial \cdot \varepsilon)$ so as not to be burdened with silent indices. Substituting this expression for $f(x)$ in the original expression, we find a first intermediate expression:

$$\boxed{\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d} (\partial \cdot \varepsilon) \eta_{\mu\nu}} \tag{54}$$

This equation is the *Killing conformal equation*, which can also be written as

$$\boxed{\partial_{(\mu} \varepsilon_{\nu)} = \frac{1}{d} (\partial \cdot \varepsilon) \eta_{\mu\nu}} \tag{55}$$

To derive another useful relation, we will apply ∂^ν on eq. (54):

$$\begin{aligned}
\partial^\nu \partial_\mu \varepsilon_\nu + \partial^\nu \partial_\nu \varepsilon_\mu + \frac{2}{d} \partial^\nu [(\partial \cdot \varepsilon) \eta_{\mu\nu}] & \\
\iff \partial^\nu \partial_\mu \varepsilon_\nu + \square \varepsilon_\mu &= \frac{2}{d} \partial^\nu (\partial \cdot \varepsilon) \eta_{\mu\nu} \\
\iff \partial_\mu (\partial \cdot \varepsilon) + \square \varepsilon_\mu &= \frac{2}{d} \partial_\mu (\partial \cdot \varepsilon)
\end{aligned} \tag{56}$$

where we used the commutativity of partial derivatives and $\square := \partial_\mu \partial^\mu$. We apply ∂_ν to this result again to find

$$\partial_\mu \partial_\nu (\partial \cdot \varepsilon) + \square \partial_\nu \varepsilon_\mu = \frac{2}{d} \partial_\mu \partial_\nu (\partial \cdot \varepsilon) \tag{57}$$

we swap the indices:

$$\partial_\nu \partial_\mu (\partial \cdot \varepsilon) + \square \partial_\mu \varepsilon_\nu = \frac{2}{d} \partial_\nu \partial_\mu (\partial \cdot \varepsilon) \tag{58}$$

and we add eq. (57) with eq. (58), which gives us

$$\begin{aligned} \partial_\mu \partial_\nu (\partial \cdot \varepsilon) + \square \partial_\nu \varepsilon_\mu + \partial_\nu \partial_\mu (\partial \cdot \varepsilon) + \square \partial_\mu \varepsilon_\nu \\ = \frac{2}{d} \partial_\mu \partial_\nu (\partial \cdot \varepsilon) + \frac{2}{d} \partial_\nu \partial_\mu (\partial \cdot \varepsilon) \end{aligned} \quad (59)$$

After rearranging the terms and highlighting, we find

$$\partial_\mu \partial_\nu (\partial \cdot \varepsilon) \left[1 - \frac{2}{d} \right] + (\square \partial_\nu \varepsilon_\mu + \square \partial_\mu \varepsilon_\nu) = 0. \quad (60)$$

We can use the conformal Killing equation, eq. (54), to write

$$\partial_\mu \partial_\nu (\partial \cdot \varepsilon) \left[1 - \frac{2}{d} \right] + \frac{2}{d} \square (\partial \cdot \varepsilon) \eta_{\mu\nu} = 0. \quad (61)$$

We can highlight $(\partial \cdot \varepsilon)$,

$$[\eta_{\mu\nu} \square + (d-2) \partial_\mu \partial_\nu] (\partial \cdot \varepsilon) = 0 \quad (62)$$

then contract this equation with the metric $\eta^{\mu\nu}$ so as to find, finally and after rearrangement of the terms, our second intermediate result:

$$(d-1) \square (\partial \cdot \varepsilon) = 0 \quad (63)$$

where, as a reminder, $d > 0$ is the dimension of our spacetime obtained by contraction of the metric with itself. Note that if d were equal to 2 in eq. (62), we would not have obtained this result, while for any other $d > 0$ this result holds.

Remark: We also take advantage of talking about the dimensions of space-time to mention that we will call a conformal theory in dimension $d = 1$ a *conformal quantum mechanics*, indeed if we consider that the dimension is time, the one-dimensional quantum field theory describes the temporal evolution of a system living in zero spatial dimensions, that is to say at a single point, so that it is not really a field theory, but a quantum mechanics.

Now we can look at constructing the algebra of conformal transformations. As we already intuited earlier, some of the generators should be rather familiar given that the Poincaré group is a subgroup of the conformal group. To begin, note that the equation eq. (63) implies that $(\partial \cdot \varepsilon)$ must be at most linear in x^μ (indeed, recall that initially the parameter ε comes from the infinitesimal transformation $x'^\mu = x^\mu + \varepsilon^\mu(x) + \mathcal{O}(\varepsilon^2)$), which in turn implies that ε_μ must be at most quadratic in x^ν , in other words we must have “ $\varepsilon = a + bx + cx^2$ ”:

$$\varepsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho, \quad (64)$$

where a_μ , $b_{\mu\nu}$ and $c_{\mu\nu\rho}$ are parameters to be deduced. Note that $c_{\mu\nu\rho}$ is symmetric under the exchange of ν and ρ . Let's summarize what we have done so far:

1. We have considered a completely general infinitesimal change of coordinates;
2. We have forced this transformation to be compliant;

3. After some manipulation, we arrived at a quadratic expression for ε .

Since ε is constrained by the definition of the conformal transformation, and this is independent of position, we are able to study each term of eq. (64) individually. The first is the simplest to understand: a_μ corresponds to a *translation*, it follows that the associated generator is already known (we must remember the generators of the Poincaré group), it is $\hat{P}_\mu = -i\partial_\mu$, the momentum operator. The linear term in x^ν , $b_{\mu\nu}$, corresponds to a *rescaling*. To find (or identify!) the associated generator, we will use the conformal Killing equation eq. (54) and insert a linear term $\varepsilon_\mu = b_{\mu\nu}x^\nu$,

$$\begin{aligned}\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu &= \frac{2}{d}(\partial \cdot \varepsilon)\eta_{\mu\nu} \\ \iff \partial_\mu(b_{\nu\rho}x^\rho) + \partial_\nu(b_{\mu\rho}x^\rho) &= \frac{2}{d}(\partial^\rho(b_{\rho\sigma}x^\sigma))\eta_{\mu\nu} \\ \iff b_{\nu\rho}\delta_\mu^\rho + b_{\mu\rho}\delta_\nu^\rho &= b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d}(\eta^{\rho\sigma}b_{\rho\sigma})\eta_{\mu\nu} \\ \iff b_{(\mu\nu)} &= \frac{1}{d}(\eta^{\rho\sigma}b_{\rho\sigma})\eta_{\mu\nu} \\ \iff b_{(\mu\nu)} &\propto \eta_{\mu\nu},\end{aligned}\tag{65}$$

this teaches us that the symmetric part of $b_{\mu\nu}$ is proportional to the metric, in other words $b_{\mu\nu}$ must be written as follows:

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + \beta_{\mu\nu}\tag{66}$$

where α is a proportionality factor and $\beta_{\mu\nu}$ is the antisymmetric part of $b_{\mu\nu}$. Let's start with something we know: the antisymmetric object $\beta_{\mu\nu}$, in this context, must be identified with a *Lorentz rotation*:

$$\begin{aligned}x'^\mu &= \delta_\nu^\mu x^\nu + \beta_\nu^\mu x^\nu \\ &= (\delta_\nu^\mu + \beta_\nu^\mu)x^\nu,\end{aligned}\tag{67}$$

and we know that the generator of these Lorentz rotations is the angular momentum operator $\hat{L}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$! Then, concerning the symmetric part of $b_{\mu\nu}$, we see that it is nothing other than an infinitesimal *dilation*

$$\begin{aligned}x'^\mu &= x^\mu + \alpha x^\mu \\ &= (1 + \alpha)x^\mu\end{aligned}\tag{68}$$

whose associated generator is written $\hat{D} = -ix^\mu\partial_\mu$. Now, let's look at the quadratic term $c_{\mu\nu\rho}$, here the transformation and the associated generator are less obvious. We will start by developing an intermediate result that will allow us to study an expression with three indices (indeed, if we want to study $c_{\mu\nu\rho}$ which is of rank 3 then it's the least we can do). Let's consider the application of ∂_ρ on the conformal Killing equation eq. (54) and swap the indices cyclically:

$$\begin{aligned}\partial_\rho\partial_\mu\varepsilon_\nu + \partial_\rho\partial_\nu\varepsilon_\mu &= 2/d \eta_{\mu\nu}\partial_\rho(\partial \cdot \varepsilon) \quad (a) \\ \partial_\nu\partial_\rho\varepsilon_\mu + \partial_\mu\partial_\rho\varepsilon_\nu &= 2/d \eta_{\rho\mu}\partial_\nu(\partial \cdot \varepsilon) \quad (b) \\ \partial_\mu\partial_\nu\varepsilon_\rho + \partial_\nu\partial_\mu\varepsilon_\rho &= 2/d \eta_{\nu\rho}\partial_\mu(\partial \cdot \varepsilon) \quad (c)\end{aligned}\tag{69}$$

we then express, for example, -(a) + (b) + (c) in order to find a simplified expression (we could put the negation on any of the other equations, given that the derivatives commute), which gives us, after calculations, the following expression:

$$2\partial_\mu\partial_\nu\varepsilon_\rho = \frac{2}{d}(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)(\partial \cdot \varepsilon). \quad (70)$$

As before, to find $c_{\mu\nu\rho}$, we use the same trick: we insert the quadratic term $\varepsilon_\mu = c_{\mu\nu\rho}x^\nu x^\rho$ into the previous equation and expand. We first express the divergence explicitly,

$$\begin{aligned} (\partial \cdot \varepsilon) &\equiv \partial_\mu\varepsilon^\mu = \eta^{\mu\alpha}\partial_\mu\varepsilon_\alpha \\ &= \eta^{\mu\alpha}\partial_\mu(c_{\alpha\nu\rho}x^\nu x^\rho) \\ &= \eta^{\mu\alpha}c_{\alpha\nu\rho}\partial_\mu(x^\nu x^\rho) \\ &= \eta^{\mu\alpha}c_{\alpha\nu\rho}(\delta_\mu^\nu x^\rho + \delta_\mu^\rho x^\nu) \\ &= c_{\nu\rho}^\mu(\delta_\mu^\nu x^\rho + \delta_\mu^\rho x^\nu), \end{aligned} \quad (71)$$

which then gives us in the equation eq. (70) by passing a few simple details:

$$\begin{aligned} 2\partial_\mu\partial_\nu\varepsilon_\rho &= \frac{2}{d}(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)(\partial \cdot \varepsilon) \\ \iff 2c_{\mu\nu\rho}\delta_\mu^\rho &= \frac{2}{d}(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)c_{\nu\rho}^\mu(\delta_\mu^\nu x^\rho + \delta_\mu^\rho x^\nu) \\ \iff \dots \\ \iff c_{\mu\nu\rho} &= \eta_{\mu\nu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu \quad \text{where} \quad b_\alpha := \frac{1}{d}c_{\nu\alpha}^\nu \end{aligned} \quad (72)$$

where we have basically contracted, distributed and derived the remaining linear terms. So we do have an expression for the coefficient $c_{\mu\nu\rho}$, and we can then calculate the associated infinitesimal transformation $x'^\mu = x^\mu + \varepsilon^\mu$. We can start by writing

$$c_{\nu\rho}^\mu = \eta^{\mu\alpha}c_{\alpha\nu\rho} = \delta_\rho^\mu b_\nu + \delta_\nu^\mu b_\rho - \eta_{\nu\rho}b_\mu \quad (73)$$

to then have:

$$\begin{aligned} \varepsilon^\mu &= c_{\nu\rho}^\mu x^\nu x^\rho \\ &= \delta_\rho^\mu b_\nu x^\nu x^\rho + \delta_\nu^\mu b_\rho x^\nu x^\rho - \eta_{\nu\rho}b_\mu x^\nu x^\rho \\ &= b_\nu x^\nu x^\mu + b_\rho x^\mu x^\rho - b_\mu x^\mu x^\rho \\ &= (x \cdot b)x^\mu + (x \cdot b)x^\mu - b^\mu x^2 \\ &= 2(x \cdot b)x^\mu - b^\mu x^2, \end{aligned} \quad (74)$$

and so

$$\begin{aligned} x'^\mu &= x^\mu + \varepsilon^\mu \\ &= x^\mu + 2(x \cdot b)x^\mu - b^\mu x^2. \end{aligned} \quad (75)$$

This infinitesimal transformation is called “special conformal”. We deduce that the generator associated with it is

$$\hat{K}_\mu = -i(2x_\mu x^\nu \partial_\nu - (x^2) \partial_\mu). \quad (76)$$

To summarize, we have determined four infinitesimal generators, which we summarize here in the following table:

GENERATOR	INFINITESIMAL TRANSFORMATION
$\hat{P}_\mu = -i\partial_\mu$	$x'^\mu(x^\mu) = x^\mu + a^\mu$
$\hat{L}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$	$x'^\mu(x^\mu) = (\delta_\nu^\mu + \beta_\nu^\mu)x^\nu$
$\hat{D} = -ix^\mu \partial_\mu$	$x'^\mu(x^\mu) = (1 + \alpha)x^\mu$
$\hat{K}_\mu = -i(2x_\mu x^\nu \partial_\nu - (x^2) \partial_\mu)$	$x'^\mu(x^\mu) = x^\mu + 2(x \cdot b)x^\mu - b^\mu x^2$

The first two generators are therefore associated with the Poincaré group while the second two are induced by conformal symmetries (dilation and special conformal transformation respectively). We can explicitly rewrite all the terms of ε^μ expanded:

$$\varepsilon^\mu = a^\mu + b^\mu_\nu x^\nu + cx^\mu + d_\nu(\eta^{\mu\nu}x^2 - 2x^\mu x^\nu) \quad (77)$$

Translations, dilations, and Lorentz transformations are fairly intuitive to understand, but the special conformal transformation is a little less so. The associated general non-infinitesimal transformation can be written as

$$\begin{aligned} x'^\mu &= \frac{x^\mu - (x \cdot x)b^\mu}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)} \\ &= \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}. \end{aligned} \quad (78)$$

We can see that this is a transformation with singular points (but in infinitesimal form this is not the case, as can be seen in the table above and in any case in physics we are interested in infinitesimal transformations in the framework of Lie algebras). The denominator is indeed zero in $x^\mu = b^{-2}b^\mu$. This transformation can be understood as the composition of an inversion $x^\mu \rightarrow x^\mu/x^2 = y^\mu$, a translation $y^\mu \rightarrow y^\mu - b^\mu = z^\mu$ and another inversion $z^\mu \rightarrow z^\mu/z^2 = x'^\mu$. If we want to define a finite conformal special transformation that is globally defined then we must consider *compactification*, but this will not be discussed.

Remark: Groups are non-linear and complicated objects, which is why in physics we prefer to work with spaces that are tangent to them (we often choose tangent to the identity for convenience), which corresponds to Lie algebras, where any transformation is well defined everywhere.

Now that we have the generators, we can calculate the different commutators in order to deduce the Lie algebra of the conformal group. The commutation relations between the

generators of the Poincaré group are already known, and the others are therefore calculated more or less laboriously. We can therefore find the following relations:

$$\begin{aligned}
[\hat{D}, \hat{P}_\mu] &= i\hat{P}_\mu \\
[\hat{D}, \hat{K}_\mu] &= -i\hat{K}_\mu \\
[\hat{K}_\mu, \hat{P}_\nu] &= 2i(\eta_{\mu\nu}\hat{D} - \hat{L}_{\mu\nu}) \\
[\hat{K}_\rho, \hat{L}_{\mu\nu}] &= i(\eta_{\rho\mu}\hat{K}_\nu - \eta_{\rho\nu}\hat{K}_\mu) \\
[\hat{P}_\rho, \hat{L}_{\mu\nu}] &= i(\eta_{\rho\mu}\hat{P}_\nu - \eta_{\rho\nu}\hat{P}_\mu) \\
[\hat{L}_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}\hat{L}_{\mu\sigma} + \eta_{\mu\sigma}\hat{L}_{\nu\rho} - \eta_{\mu\rho}\hat{L}_{\nu\sigma} - \eta_{\nu\sigma}\hat{L}_{\mu\rho})
\end{aligned} \tag{79}$$

where all other commutators are zero. This therefore defines the Lie algebra of the conformal group.

3.2.1. Note on the case $d = 2$: Virasoro algebra

We will not go into the details of a conformal theory at $d = 2$ because that is a separate topic, but will simply make some remarks on why this case is distinguished from the $d \geq 3$ case, as we now have sufficient tools to understand how things differ. At $d = 2$, it is convenient to use a complex coordinate system, with $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$. By doing this, the conformal Killing equation eq. (54) takes the following form (after some manipulation):

$$\begin{cases} \partial_1 \varepsilon_1 = \partial_2 \varepsilon_1 \\ \partial_1 \varepsilon_2 = -\partial_2 \varepsilon_1 \end{cases} \tag{80}$$

in other words, we recognize the *Cauchy-Riemann equations* of complex analysis! Previously, we were looking for the ε satisfying the conformal Killing equation and found that there existed a finite number of them up to constants, but in the present case there are an *infinity* of solutions to these equations. More precisely, the solutions are any analytic function $z \mapsto f(z)$ and $\bar{z} \mapsto f(\bar{z})$. If we wanted to classify this set of transformations into generators, then, instead of having four, we would have an infinity of them: we must be able to generate any analytic function (in the complex sense), hence this necessary infinite number of generators. If we continue in this direction, we would arrive at a more “general” conformal theory, and in particular we would arrive at the Virasoro algebra, a complex Lie algebra of infinite dimension notably used in string theory. This is why we will not go into this further in this document.

3.2.2. Dimension of the conformal group and isomorphism with $\text{SO}(d + 1, 1)$

There is a connection between the conformal group, which we studied through its algebra, as is often the case, and the group $\text{SO}(d + 1, 1)$. Let us start by looking at the dimension of the algebra (i.e. the number of generators) for $d \geq 3$:

- **Translation:** there are d independent parameters for the translation, one for each direction of the d -dimensional spacetime;
- **Lorentz rotations:** we already know that the orthogonal group in dimension d (i.e. $\text{SO}(d - 1, 1)$ in relativity) has $d(d - 1)/2$ generators;

- **Expansions:** there is only one common expansion factor (a constant factor), so it contributes to only one generator;
- **Special conformal transformations:** knowing that the SCT boils down to an inversion, then a translation, then another inversion, then we deduce that there are as many generators as for the translation, in other words d .

All together we get the following sum:

$$\begin{aligned} d \text{ translations} + \frac{d(d-1)}{2} \text{ rotations} + 1 \text{ dilatation} + d \text{ SCT's} \\ = \frac{(d+2)(d+1)}{2} \text{ generators} \end{aligned} \quad (81)$$

and this is therefore the dimension of the conform algebra to $d \geq 3$. Note that this corresponds to the number of generators of the group $\text{SO}(d+1, 1)$ (depending on the signature adopted), in fact the group $\text{SO}(d+1, 1)$ is the set of orthogonal matrices in a space of dimension $d+2$ and, in general, the dimension of $\text{SO}(n)$ is $(n(n-1))/2$ or, in mixed signature $\text{SO}(p, q)$, the dimension is expressed analogously as $((p+q)(p+q-1))/2$. If we specify in the present case, i.e. $\text{SO}(d+1, 1)$, then we have

$$\dim(\text{SO}(d+1, 1)) = \frac{(d+2)(d+1)}{2} \quad (82)$$

and we therefore find the same dimension as for the conformal group.

Remark: In dimension d , rotations (or more generally orthogonal transformations) are made in two-dimensional planes. Such a “rotation plane” is defined by the choice of two axes among d possible, the number of ways to choose 2 axes among d is

$$\binom{d}{2} = \frac{d(d-1)}{2} \quad (83)$$

hence the result obtained for rotations if we no longer remember the number of generators of the orthogonal group.

Saying that two groups have the same dimension is necessary to say that they are isomorphic, but not sufficient. If we set

$$\begin{aligned} J_{\mu\nu} &:= L_{\mu\nu} \\ J_{-1,\mu} &:= \frac{1}{2}(P_\mu - K_\mu) \\ J_{0,\mu} &:= \frac{1}{2}(P_\mu + K_\mu) \\ J_{-1,0} &:= D, \end{aligned} \quad (84)$$

then it is possible to show that the generators $J_{\alpha\beta}$, with $a, b = -1, 0, \dots, n = p+q$ obey the Lorentz algebra with the metric $\tilde{\eta}_{\alpha\beta} = \text{diag}(-1, +1, -1, \dots, -1, +1, \dots, +1)$ but we are not going to show it here.

3.3. Note on the S matrix in conformal field theory

This subsection is not dedicated to establishing the S-matrix given conformal symmetries because the S-matrix *cannot* be constructed in a conformal field theory. This is what we will briefly discuss in this mini-section.

Remark: Talking about the S matrix in a context other than that of particles, as is the case in the study of phase transitions for example, is not very useful, but we will nevertheless take a small parenthesis to discuss this aspect which is slightly outside the scope of this document.

First of all, let us note that just because conformal field theory does not admit an S-matrix does not mean that the S-matrix, in another theory, does not admit conformal symmetries. Recall that the S-matrix is defined as the (unitary) matrix connecting *asymptotically* free particle states $|in\rangle$ and $|out\rangle$ in a Hilbert space. Now, let us recall that a conformal field theory is invariant under conformal symmetries, and more particularly under *dilation*. In other words, because of invariance under dilation, the very concept of being “asymptotically distant” no longer really makes sense: in CFT, there are no asymptotic states, and therefore *no* S-matrix, strictly speaking.

3.4. Correlation functions

In conformal field theory, *correlation functions* play the role of observables of the theory, a central object in CFT. Correlation functions in physics are quite similar to those found in statistics; they measure, as one might expect, the degree of correlation between two random variables. That is, the frequency with which two random variables have similar values. In a field theory context, the connection is less clear. In field theory, the n -point correlation function is defined as the average functional product of n fields at different positions,

$$\begin{aligned} G_n(x_1, x_2, \dots, x_n) &:= \langle \varphi(x_1)\varphi(x_2)\dots\varphi(x_n) \rangle \\ &:= \frac{\int [\mathcal{D}\varphi] e^{-S[\varphi]} \varphi(x_1)\dots\varphi(x_n)}{\int [\mathcal{D}\varphi] e^{-S[\varphi]}} \\ &=: \frac{1}{z_0} \int [\mathcal{D}\varphi] e^{-S[\varphi]} \varphi(x_1)\dots\varphi(x_n) \end{aligned} \tag{85}$$

and for time-dependent correlation functions, the time-order operator T must be included. The two-point correlation function $G_2(x, y)$ can be physically interpreted as the propagation amplitude of a particle between y and x . In a free theory, this is simply the Feynman propagator. The term “Green function” is sometimes used to refer to any correlation function, not just two-point correlation functions.

For the following, in the framework of conformal field theory, we will define a correlation function as a function of the form $\langle \varphi_1(x_1)\varphi_2(x_2)\dots\varphi_n(x_n) \rangle$ containing a finite number of objects (more specifically operators) and returning a real number. We will assume that the objects between $\langle \dots \rangle$ can commute without changing the value of the correlation function and that the correlation functions are linear, that is to say that

$$\langle A_1 \varphi_1(x_1) A_2 \varphi_2(x_2) \dots A_n \varphi_n(x_n) \rangle = (A_1 A_2 \dots A_n) \langle \varphi_1(x_1) \varphi_2(x_2) \dots \varphi_n(x_n) \rangle \quad (86)$$

where A_k is a constant and the φ 's are functions of the positions x^μ . The notion of correlation function in the framework of CFT should become clearer as it is used.

3.5. Scaling dimension of an operator

In the following, we will need an important concept that is directly related to dilations: the scaling dimension of an operator. The scaling dimension is a number associated with an operator that indicates how the latter behaves under dilation $x \rightarrow \lambda x$. This notion is related to dimensions in the sense of dimensional analysis. Given any operator $\mathcal{O}(x)$, the invariance under dilation $x \rightarrow \lambda x$ implies that

$$\mathcal{O}(x) \rightarrow \mathcal{O}(\lambda x) = \lambda^{-\Delta} \mathcal{O}(x) \quad (87)$$

where $\Delta \in \mathbb{R}$ is the *scale dimension* of the operator \mathcal{O} . The number Δ must exist so that the dimensions (hence to be understood as “the units”) are respected. In a theory that is not invariant under dilation, the Δ are no longer simple numbers but functions of the distance scales. This concept is not specific to conformal field theory.

Often, we will want to determine what the scaling dimension of an operator or field $\varphi(x)$ is, we can establish the general expression of this dimension for a field theory. In the natural unit system $\hbar = 1 = c$, the distance is the inverse of the mass, $[L] = [M]^{-1}$, and when we study the scaling units it is the exponents that interest us, so we will have that if m is a mass term and x is a position term, then we will take the convention that $\Delta_m = 1$ and $\Delta_x = -1$. To determine the scaling dimension of a field $\varphi(x)$, consider the action

$$S = \frac{1}{2} \int d^d x (\partial_\mu \varphi)^2. \quad (88)$$

In natural units, the action is dimensionless (since $\hbar = 1$, a simple scalar), so we deduce that $\Delta_S = 0$. We also deduce that the scale dimension of the measure is

$$\Delta_{d^d x} = -d \quad (89)$$

since $\Delta_x = -1$. We can then try to find the scaling dimension of φ :

$$\begin{aligned} \Delta_S &= \Delta_{d^d x} + 2(\Delta_{\partial_\mu} + \Delta_\varphi) \\ \iff 0 &= -d + 2\Delta_{\partial_\mu} + 2\Delta_\varphi \end{aligned} \quad (90)$$

where the partial derivative $\partial_\mu \equiv \partial/\partial x^\mu$, which implies that $\Delta_{\partial_\mu} = +1$, so in the end we have

$$\begin{aligned} 0 &= -d + 2 + 2\Delta_\varphi \\ \iff \boxed{\Delta_\varphi} &= \frac{d}{2} - 1 \end{aligned} \quad (91)$$

For example, in four dimensions $\Delta_\varphi = 1$ and in three dimensions $\Delta_\varphi = 1/2$.

Example: Now that we know what the general form of Δ_φ is, we can infer the scaling dimension of other parameters, for example consider the following interacting action:

$$S = \int d^d x \lambda \varphi^n \quad (92)$$

so,

$$\begin{aligned} \Delta_S &= \Delta_{d^d x} + \Delta_\lambda + n\Delta_\varphi \\ \iff 0 &= -d + \Delta_\lambda + n\left(\frac{d}{2} - 1\right) \\ \iff \Delta_\lambda &= d - n\left(\frac{d}{2} - 1\right) \end{aligned} \quad (93)$$

The scale dimension will be an important parameter, in fact it is part of the information needed to characterize an operator, as we will see later.

Example: We can take another more complicated example, it will actually be reduced to a very simple case (it is an action for the Ising model):

$$S = \frac{1}{2} \int d^3 x \left[-\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 - \frac{g}{4!} \varphi^4 \right]. \quad (94)$$

Here, the only unknown scaling dimension is Δ_g , so we can focus only on the third term of the sum (knowing that $d = 3$ in this example):

$$\begin{aligned} \Delta_S &= \Delta_{d^3 x} + (\Delta_g + 4\Delta_\varphi) \\ \iff 0 &= -3 + \Delta_g + 4\left(\frac{3}{2} - 1\right) \\ \iff \Delta_g &= 1. \end{aligned} \quad (95)$$

The relation eq. (87), $\mathcal{O}(x) \rightarrow \mathcal{O}(\lambda x) = \lambda^{-\Delta} \mathcal{O}(x)$, implies that the correlation functions, under scale invariance, must be such that

$$\langle \mathcal{O}_1(\lambda x_1) \mathcal{O}_2(\lambda x_2) \dots \rangle = \lambda^{-\Delta_1 - \Delta_2 - \dots} \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \dots \rangle. \quad (96)$$

Example: Let's take for example the 2-point correlation function $\langle \mathcal{O}(x) \mathcal{O}(0) \rangle$, then

$$\begin{aligned} \langle \mathcal{O}(x) \mathcal{O}(0) \rangle &\rightarrow \langle \mathcal{O}(\lambda x) \mathcal{O}(\lambda 0) \rangle \\ &= \lambda^{-(\Delta+\Delta)} \langle \mathcal{O}(x) \mathcal{O}(0) \rangle \\ &= \lambda^{-2\Delta} \langle \mathcal{O}(x) \mathcal{O}(0) \rangle. \end{aligned} \quad (97)$$

The scale invariance of correlation functions will be the starting point for developing, subsequently, an important equation (see the section on bootstrapping), we will return to this later.

3.6. Conserved currents and energy-momentum tensor

For simplicity, consider a scalar field $\varphi(x)$. We will assume that this field transforms (under a symmetry transformation) according to

$$\begin{cases} x^\mu \rightarrow x'^\mu(x) \\ \varphi'(x') = \mathcal{F}(\varphi(x)) \end{cases} \quad (98)$$

where \mathcal{F} is simply the transformation function of the scalar field. As a result, we have the following associated infinitesimal transformations:

$$\begin{aligned} x^\mu &\rightarrow x'^\mu(x) = x^\mu + \omega^A \frac{\delta}{\delta \omega^A} x^\mu, \\ \varphi(x) &\rightarrow \varphi'(x) = \varphi(x) + \omega^A \frac{\delta}{\delta \omega^A} \mathcal{F}(\varphi(x)) \end{aligned} \quad (99)$$

where $\omega^A \ll 1$ are infinitesimal (the capital subscript A indicates that several subscripts may be present, e.g. ω^{ijk} , but we denote ω^A for generality and convenience). We also introduce the following notation:

$$\omega^A \frac{\delta}{\delta \omega^A} =: \delta_\omega \quad (100)$$

where the ω should not be confused with an index but serves to identify the variation with respect to these infinitesimal parameters ω^A . If this infinitesimal transformation is a symmetry of the action then we must have $\delta_\omega S = S' - S \approx 0$ by Noether's theorem (here the symbol “ \approx ” indicates an equality modulo the Euler-Lagrange equations). The transformed action is therefore

$$\begin{aligned} S[\varphi, \partial_\mu \varphi] &= \int d^d x \mathcal{L}(\varphi, \partial_\mu \varphi) \\ \rightarrow S[\varphi', \partial'_\mu \varphi'] &= \int d^d x \left| \frac{\partial x'^\mu}{\partial x^\nu} \right| \mathcal{L}\left(\varphi + \delta_\omega \mathcal{F}, \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu [\varphi + \delta_\omega \mathcal{F}] \right) \end{aligned} \quad (101)$$

where, knowing that $\det(\mathbb{1} + \varepsilon) \approx 1 + \text{tr}(\varepsilon)$, we have

$$\begin{aligned} \left| \frac{\partial x'^\mu}{\partial x^\nu} \right| &= \left| \frac{\partial}{\partial x^\nu} (x^\mu + \delta_\omega x^\mu) \right| \\ &= |\delta_\nu^\mu + \partial_\nu \delta_\omega x^\mu| \\ &\approx 1 + \partial_\mu \delta_\omega x^\mu. \end{aligned} \quad (102)$$

Substituting this expression into the transformed action, expanding and integrating by parts and neglecting boundary terms as is customary, we arrive at the following expression:

$$\begin{aligned}
\delta_\omega S &= - \int d^d x j_A^\mu \partial_\mu \omega^A(x) \\
&\stackrel{\text{(IPP)}}{=} \dots \\
&= \int d^d x \partial_\mu j_A^\mu \omega^A(x)
\end{aligned} \tag{103}$$

where j_A^μ is the *Noether current*, defined by

$$j_A^\mu(x) := \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \eta_{\mu\nu} \mathcal{L} \right] \frac{\delta x^\nu}{\delta \omega^A} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\delta F}{\delta \omega^A}. \tag{104}$$

tenseur énergie-impulsion (canonique)

If the Euler-Lagrange equations are satisfied by the field $\varphi(x)$,

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \approx 0, \tag{105}$$

then the action is invariant under any arbitrary variation of the field: $\delta S = 0$ for all ω^A , which leads to the Noether current conservation law:

$$\partial_\mu j_A^\mu = 0 \tag{106}$$

and we will define the associated *Noether charge* as

$$Q := \int d^{d-1}x j_a^0. \tag{107}$$

In other words, a continuous symmetry of the action implies the existence of a conserved current. Note that adding the divergence of an antisymmetric tensor to the current j_A^μ does not affect its conservation, in other words it will always be possible to find an antisymmetric $A^{\mu\nu}$ such that

$$\partial_\mu j_A^\mu = 0, \tag{108}$$

Indeed,

$$\begin{aligned}
j_A^\mu &\rightarrow j_A^\mu + \partial_\nu A^{\mu\nu} \quad \text{where} \quad A^{\mu\nu} = -A^{\nu\mu} \\
\partial_\mu j_A^\mu &\rightarrow \underline{\partial_\mu j_A^\mu + \partial_\mu \partial_\nu A^{\mu\nu}} = 0
\end{aligned} \tag{109}$$

is 0 for current conservation

where the second term is zero by contraction of symmetric and antisymmetric objects. We therefore have the freedom to (re)define the current j_A^μ . We can now be interested in these currents given conformal symmetries.

Translation and zero trace

Consider an infinitesimal translation $x^\mu \rightarrow x'^\mu = x^\mu + \varepsilon^\mu$, from which we get that

$$\frac{\delta x^\mu}{\delta \varepsilon^\nu} = \delta_\nu^\mu \quad \text{and} \quad \frac{\delta \mathcal{F}}{\delta \varepsilon^\nu} = 0 \tag{110}$$

and it follows directly, by inserting this in eq. (104), that

$$\begin{aligned} j_A^\mu &= T_\nu^\mu \frac{\delta x^\nu}{\delta \varepsilon^A} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\delta \mathcal{F}}{\delta \varepsilon^A} \\ &= T_\nu^\mu \delta_A^\nu. \end{aligned} \quad (111)$$

In other words, given a translational symmetry, the conserved current *is* the (canonical) energy-momentum tensor:

$$T^{\mu\nu} = -\eta^{\mu\nu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu \varphi. \quad (112)$$

This *current* is not manifestly symmetric, but we can for example use the freedom on the addition of a $\partial_\mu A^{\mu\nu}$, where $A^{\mu\nu} = -A^{\nu\mu}$, in order to determine an expression that is manifestly symmetric and that remains conserved, but this is not very important for what follows.

A particular fact about the energy-momentum tensor can be deduced from the conservation of current associated with the translational symmetry that we have just developed:

$$\begin{aligned} 0 &= \partial_\mu j^\mu = \partial_\mu (T^{\mu\nu} \varepsilon_\nu) \\ &= T^{\mu\nu} \partial_\mu \varepsilon_\nu + \varepsilon_\nu \partial_\mu T^{\mu\nu} \\ &= \frac{1}{2} T^{\mu\nu} (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon) \end{aligned} \quad (113)$$

and, using the conformal Killing equation eq. (54), we find

$$\begin{aligned} 0 &= \frac{1}{d} (\partial \cdot \varepsilon) T_\mu^\mu \\ \iff & \boxed{T_\mu^\mu = 0}. \end{aligned} \quad (114)$$

In other words, under conformal symmetries, the energy-momentum tensor has zero trace. This is a fact specific to conformal field theories: every CFT has a zero trace energy-momentum tensor! In a conformal field theory, the fact that the energy-momentum tensor has zero trace reflects the invariance of the system under expansions. Concretely, this means that the dynamics of the system does not involve any dimensional parameter that would fix a particular scale, which is at the heart of conformal invariance.

The other symmetry transformations (rotation, dilation and special conformal) can also be associated with a current by a similar procedure, however we will not go into the details of the calculations because this is not very useful for the following and it is a work which is not specific to conformal field theory.

Rotation

Analogously to the study of Poincaré transformations, Lorentz rotations are associated with the angular momentum tensor, the variation $\delta x^\mu = \omega^\mu_\nu x^\nu$ gives rise to the current

$$M^\mu_{\rho\sigma} = x_\rho T_\sigma^\mu - x_\sigma T_\rho^\mu \quad (115)$$

where $T^{\mu\nu}$ is the energy-momentum tensor (the one that emerges from symmetries under translations).

Dilation

Consider a transformation $x^\mu \rightarrow x'^\mu = x^\mu + \varepsilon x^\mu$, in other words $\delta x^\mu = \varepsilon x^\mu$, we then show that the associated Noether current is

$$J_{\text{dilatation}}^\mu = x_\nu T^{\mu\nu}. \quad (116)$$

Special conformity

For the SCT, to the infinitesimal order, we have $\delta x^\mu = 2(b \cdot x)x^\mu - x^2 b^\mu$, from which we derive the following Noether current:

$$J_{\text{SCT}}^\mu = (2x^\rho x_\nu - x^2 \delta_\nu^\rho) T^{\mu\nu}. \quad (117)$$

For each of these currents, the form is not unique, in particular it is common to express everything in terms of the manifestly symmetric energy-momentum tensor $T^{\mu\nu}$ and, therefore, to add corrective terms.

3.6.1. The Ward-Takahashi identities

At the classical level, the invariance of the action under continuous symmetries implies the existence of a conserved current. However, at the quantum level, classical symmetries lead to constraints on the correlation function, known as the Ward-Takahashi identities. We will say that the symmetry is “anomalous” if the functional measure in the path integral does not exhibit the symmetry of the action, i.e. $[\mathcal{D}\varphi'] \neq [\mathcal{D}\varphi]$. In the following, we will always assume that this constraint holds. Assume that the classical action is invariant under the general transformation

$$\varphi'(x') = \mathcal{F}(\varphi(x)) \quad (118)$$

and that the symmetry is not anomalous, i.e. $[\mathcal{D}\varphi'] = [\mathcal{D}\varphi]$. Then

$$\begin{aligned} \langle \varphi(x'_1) \dots \varphi(x'_n) \rangle &\equiv \alpha \int [\mathcal{D}\varphi] \varphi(x'_1) \dots \varphi(x'_n) e^{-S[\varphi]} \\ &= \alpha \int [\mathcal{D}\varphi'] \varphi'(x'_1) \dots \varphi'(x'_n) e^{-S[\varphi']} \\ &= \alpha \int [\mathcal{D}\varphi] \mathcal{F}(\varphi(x'_1)) \dots \mathcal{F}(\varphi(x'_n)) e^{-S[\varphi]} \\ &\equiv \langle \mathcal{F}(\varphi(x'_1)) \dots \mathcal{F}(\varphi(x'_n)) \rangle \end{aligned} \quad (119)$$

where we moved from the first line to the second line by renaming $\varphi \rightarrow \varphi'$ and where α is simply a proportionality factor. We want to find an infinitesimal version of this result. Recall that an infinitesimal transformation can be written in terms of its generators G_A as

$$\varphi'_{\mathcal{A}}(x) = \varphi_{\mathcal{A}}(x) - i\omega^A(G_A)_{\mathcal{A}}^{\mathcal{B}} \varphi_{\mathcal{B}}(x), \quad (120)$$

where ω^A are, again, a set of infinitesimal parameters all grouped under a single large index A . Setting $\omega^A \rightarrow \omega^A(x)$, the variation of the action $\delta_\omega S[\varphi]$ is given by eq. (103),

$$\delta_\omega S[\varphi] = \int d^d x \partial_\mu j_A^\mu(x) \omega^A(x). \quad (121)$$

Let us set the product $X := \varphi(x_1) \dots \varphi(x_n)$ and note its variation $\delta_\omega X$, given explicitly by

$$\begin{aligned} \delta_\omega X &= \omega^A \frac{\delta}{\delta \omega^A} [\varphi(x_1) \varphi(x_2) \dots \varphi(x_n)] \\ &= \omega^A \left[\frac{\delta}{\delta \omega^A} \varphi(x_1) \varphi(x_2) \dots \varphi(x_n) + \varphi(x_1) \frac{\delta}{\delta \omega^A} \varphi(x_2) \dots \varphi(x_n) + \dots \right. \\ &\quad \left. + \varphi(x_1) \varphi(x_2) \dots \frac{\delta}{\delta \omega^A} \varphi(x_n) \right] \\ &= \omega^A \sum_{k=1}^n \left[\varphi(x_1) \varphi(x_2) \dots \frac{\delta}{\delta \omega^A} \varphi(x_k) \dots \varphi(x_n) \right] \end{aligned} \quad (122)$$

where we just apply the rule of the product of derivatives. With $\delta/\delta\omega^A = -iG_A$ we therefore obtain

$$\delta_\omega X = -i\omega^A \sum_{k=1}^n [\varphi(x_1) \varphi(x_2) \dots G_A \varphi(x_k) \dots \varphi(x_n)]. \quad (123)$$

Now, taking the integral of this expression multiplied by a Dirac d -delta (so that the expression obtained reduces identically to the previous expression), we have

$$\delta_\omega X = -i \int d^d x \sum_{k=1}^n \delta^{(d)}(x - x_k) [\varphi(x_1) \dots G_A \varphi(x_k) \dots \varphi(x_n)] \omega^A, \quad (124)$$

what we can insert into a $\langle \dots \rangle$ to find a first intermediate expression:

$$\langle \delta_\omega X \rangle = -i \int d^d x \sum_{k=1}^n \delta^{(d)}(x - x_k) \langle \varphi(x_1) \dots G_A \varphi(x_k) \dots \varphi(x_n) \rangle \omega^A. \quad (125)$$

Wanting to express our result in an integral will be useful for what follows. Let's now use another equivalent notation to express $\langle \delta_\omega X \rangle$ using the previous functional form. To do this, we consider the variation introduced previously on φ and explicitly express $\langle X \rangle$, which should naturally introduce a $\langle \delta_\omega X \rangle$:

$$\begin{aligned} \langle X \rangle &= \alpha \int [\mathcal{D}\varphi'] X' e^{-S[\varphi']} \\ &= \alpha \int [\mathcal{D}\varphi'] (X + \delta_\omega X) e^{-(S[\varphi] + \delta_\omega S[\varphi])} \\ &= \alpha \int [\mathcal{D}\varphi'] (X + \delta_\omega X) e^{-S[\varphi] - \int d^d x \partial_\mu j_A^\mu(x) \omega^A(x)} \end{aligned} \quad (126)$$

where, as before, we assume that the functional measure is invariant under the transformation. We can now expand the previous expression to first order in ω^A for the exponential:

$$\begin{aligned}
\langle X \rangle &= \alpha \int [\mathcal{D}\varphi'] (X + \delta_\omega X) e^{-S[\varphi] - \int d^d x \partial_\mu j_A^\mu(x) \omega^A(x)} \\
&= \alpha \int [\mathcal{D}\varphi'] (X + \delta_\omega X) \left[1 - \int d^d x \partial_\mu j_A^\mu(x) \omega^A(x) + \dots \right] e^{-S[\varphi]} \\
&= \langle X \rangle + \langle \delta_\omega X \rangle - \int d^d x \partial_\mu \left[\alpha \int [\mathcal{D}\varphi] j_A^\mu(x) X e^{-S[\varphi]} \right] \omega^A + \dots \quad (127) \\
&= \langle X \rangle + \langle \delta_\omega X \rangle - \int d^d x \partial_\mu \langle j_A^\mu(x) X \rangle \omega^A + \mathcal{O}(\omega^2) \\
\iff \langle \delta_\omega X \rangle &= \int d^d x \partial_\mu \langle j_A^\mu(x) X \rangle \omega^A
\end{aligned}$$

↑
on néglige ces termes d'ordre supérieur

Now that we have two expressions for $\langle \delta_\omega X \rangle$, all we need to do is equate them. So we use eq. (125) and eq. (127) to form the following equation:

$$\begin{aligned}
\int d^d x \partial_\mu \langle j_A^\mu(x) X \rangle \omega^A = \\
-i \int d^d x \sum_{k=1}^n \delta^{(d)}(x - x_k) \langle \varphi(x_1) \dots G_A \varphi(x_k) \dots \varphi(x_n) \rangle \omega^A,
\end{aligned} \quad (128)$$

and since the ω^A are arbitrary, we can rewrite eq. (128) as

$$\boxed{\partial_\mu \langle j_A^\mu(x) \varphi(x_1) \dots \varphi(x_n) \rangle + i \sum_{k=1}^n \delta^{(d)}(x - x_k) \langle \varphi(x_1) \dots G_A \varphi(x_k) \dots \varphi(x_n) \rangle = 0} \quad (129)$$

where we have re-expressed $X = \varphi(x_1) \varphi(x_2) \dots \varphi(x_n)$. The equation eq. (129) is the *Ward-Takahashi identity* for the current $j_A^\mu(x)$ and is an infinitesimal version of eq. (119) (this can be seen by integration of eq. (129) and some algebraic manipulations).

Remark: In the literature, the terms “Ward identity” and “Ward-Takahashi identity” are sometimes encountered; the two are related and sometimes interchanged. In fact, the distinction is often made when applying the Ward-Takahashi identity specifically to the elements of the matrix S in quantum field theory, where we will then simply speak of the Ward identity.

Very generally, Ward identities reflect the constraints imposed by a symmetry (internal or gauge for example) on the correlation functions.

3.6.2. Ward-Takahashi identity and conformal symmetries

Knowing the Noether currents associated with conformal symmetries, it is sufficient to use them in the Ward-Takahashi equation eq. (129) to deduce the conformal Ward identities.

Symmetry under translation

The generator of the translations is $P_\mu = -i\partial_\mu$ and the associated current is given by $T^{\mu\nu}$, so, by substituting in eq. (129), we obtain

$$\begin{aligned}
0 &= \partial_\mu \langle j_A^\mu(x) \varphi(x_1) \dots \varphi(x_n) \rangle + i \sum_{k=1}^n \delta^{(d)}(x - x_k) \langle \varphi(x_1) \dots G_A \varphi(x_k) \dots \varphi(x_n) \rangle \\
&= \partial_\mu \langle T^{\mu\nu} \varphi(x_1) \dots \varphi(x_n) \rangle + \sum_{k=1}^n \delta^{(d)}(x - x_k) \partial_k^\nu \langle \varphi(x_1) \dots \varphi(x_k) \dots \varphi(x_n) \rangle
\end{aligned} \tag{130}$$

where we used the “linearity” of the correlation functions. This is the Ward identity associated with translations. More compactly, we write

$$\boxed{\partial_\mu \langle T_\nu^\mu X \rangle = - \sum_k \delta^{(d)}(x - x_k) \frac{\partial}{\partial x_k^\nu} \langle X \rangle} \tag{131}$$

Similarly, but with a little more effort, we can obtain the other Ward identities associated with the currents eq. (115), eq. (116) and eq. (117), the results are less interesting, more complicated, and we will therefore not present them since they will not be of use to us later.

3.7. Primary and descendant operators

Conformal symmetries impose constraints on correlation functions. One way to classify the operators of a conformal field theory comes directly from the representation of the generators of the conformal group and is analogous to the creation and annihilation operators of the harmonic oscillator in quantum mechanics. To make this clear, it is useful to adopt a different convention than that used when deriving the generators associated with conformal transformations. We will use

GENERATOR	TRANSFORMATION
$\hat{P}_\mu = \partial_\mu$	Translation
$\hat{L}_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu$	Lorentz
$\hat{D} = x^\mu \partial_\mu$	Dilation
$\hat{K}_\mu = 2x_\mu x^\nu \partial_\nu - (x^2) \partial_\mu$	Special compliant

In this new convention, the $-i$ “disappears” compared to what we had derived earlier. It follows that the associated algebra changes, but we are not going to rewrite everything, only

$$\begin{aligned}
[\hat{D}, \hat{P}_\mu] &= \hat{P}_\mu \\
[\hat{D}, \hat{K}_\mu] &= -\hat{K}_\mu.
\end{aligned} \tag{132}$$

With this convention, it is easier to realize the analogy with the creation and destruction operators in quantum mechanics:

$$\begin{aligned}
[\hat{D}, \hat{P}_\mu] &= \hat{P}_\mu &\leftrightarrow [\hat{N}, \hat{a}^\dagger] &= \hat{a}^\dagger \\
[\hat{D}, \hat{K}_\mu] &= -\hat{K}_\mu &\leftrightarrow [\hat{N}, \hat{a}] &= -\hat{a}
\end{aligned} \tag{133}$$

where \hat{a}^\dagger and \hat{a} are the up and down operators for the harmonic oscillator, respectively, and \hat{N} is the number operator. This similarity suggests that \hat{P}_μ and \hat{K}_μ can also be understood as up and down operators for \hat{D} , which is indeed the case. We will choose to work in a basis

where our states have a well-defined eigenvalue, Δ , under dilation (this will come back a little later!) We note

$$|\Delta\rangle = \text{state with dimension } \Delta. \quad (134)$$

These states will be created by acting with an operator $\hat{O}_\Delta(0)$ at the origin, on the vacuum

$$\hat{O}_\Delta(0)|0\rangle \equiv |\Delta\rangle. \quad (135)$$

Remark: In our discussions we can restrict ourselves to the operators inserted at $x = 0$, at the origin, since the transformation properties at any other point can be obtained by applying a translation,

$$\hat{O}(x) = e^{x^\mu \hat{P}_\mu} \hat{O}(0) e^{-x^\mu \hat{P}_\mu}, \quad (136)$$

and using the Baker–Campbell–Hausdorff formula as well as the commutation relations of conformal algebra. It is just more convenient to use $\hat{O}(0)$.

By state-operator correspondence, it is possible to generate all the states of our Hilbert space by acting with a local operator at the origin. These operators, \hat{O}_Δ , will satisfy the following commutation relation:

$$\hat{D}\hat{O}_\Delta(0) - \hat{O}_\Delta(0)\hat{D} \equiv [\hat{D}, \hat{O}_\Delta(0)] = \Delta\hat{O}_\Delta(0). \quad (137)$$

We can see that this is all consistent by acting on our state with \hat{D} :

$$\begin{aligned} \hat{D}|\Delta\rangle &= \hat{D}\hat{O}_\Delta(0)|0\rangle \\ &= ([\hat{D}, \hat{O}_\Delta(0)] + \hat{O}_\Delta(0)\hat{D})|0\rangle \\ &= \Delta\hat{O}_\Delta(0)|0\rangle + 0 \\ &= \Delta|\Delta\rangle \end{aligned} \quad (138)$$

where we used the fact that \hat{D} is associated with a symmetry of the system, and therefore it cancels the vacuum state (because $|0\rangle$ is invariant under the symmetries of any physical system). This shows that $|\Delta\rangle$ is an eigenstate of \hat{D} with eigenvalue Δ .

To return to the initial discussion, in the same way that the descent operator lowers the eigenvalue of the counting operator \hat{N} for the states of the harmonic oscillator, the operator \hat{K}_μ will act as a lowering operator for the dilation operator \hat{D} . As in the harmonic oscillator, we will therefore have a state of lowest weight for our states of given dimension. We will call these states the “primary states” and the operators that will act on the vacuum to generate them are called *primary operators* and we will use the calligraphic notation $\hat{\mathcal{O}}_\Delta$ to refer to them specifically. Unlike the harmonic oscillator, now, where there is only one lowering operator, in conformal field theories we will generally have several (sometimes even infinitely many) primary states (remember that the harmonic oscillator can indeed only have one ground state). We want these states to be annihilated by \hat{K}_μ . To this end, we impose that the primary operators commute with \hat{K}_μ ,

$$[\hat{K}_\mu, \hat{\mathcal{O}}_\Delta(0)] = 0, \quad (139)$$

it is often only this relation that is given to define the *primary operators*. We can see explicitly what this implies that the primary states are annihilated by \hat{K}_μ :

$$\begin{aligned} \hat{K}_\mu |\Delta\rangle &= \hat{K}_\mu \hat{\mathcal{O}}_\Delta(0) |0\rangle \\ &= \hat{\mathcal{O}}_\Delta(0) \hat{K}_\mu |0\rangle \quad (\text{commute}) \\ &= 0 \end{aligned} \quad (140)$$

where, in the last line, we used the fact that \hat{K}_μ is associated with a symmetry of the system, and therefore that it annihilates the vacuum, as before. In other words, this commutation relation accounts for the desired annihilation.

What about \hat{P}_μ now? The states we discussed earlier are the “lowest” ones (if we keep the similarity with the harmonic oscillator), so we would like to see what happens if we act on them with \hat{P}_μ , which is supposed to behave like a rise operator. Consider a new state $|\psi\rangle$ that is created by acting \hat{P}_μ on an eigenstate of the dilation operator,

$$\hat{P}_\mu |\Delta\rangle = |\psi\rangle \quad (141)$$

Is this still an eigenstate of \hat{D} ? We can check this by explicitly calculating:

$$\begin{aligned} \hat{D}|\psi\rangle &= \hat{D}\hat{P}_\mu |\Delta\rangle \\ &= ([\hat{D}, \hat{P}_\mu] + \hat{P}_\mu \hat{D}) |\Delta\rangle \\ &= (\hat{P}_\mu + \hat{P}_\mu \hat{D}) |\Delta\rangle \\ &= \hat{P}_\mu |\Delta\rangle + \hat{P}_\mu \hat{D} |\Delta\rangle \\ &= \hat{P}_\mu |\Delta\rangle + \hat{P}_\mu \Delta |\Delta\rangle \\ &= (1 + \Delta) \hat{P}_\mu |\Delta\rangle \\ &= (1 + \Delta) |\psi\rangle. \end{aligned} \quad (142)$$

We thus see that $|\psi\rangle$ is indeed an eigenstate of \hat{D} , with a value $\Delta + 1$. Concretely, if we start from a state created by the action of a primary operator $|\Delta\rangle$, then we apply \hat{K}_μ , we obtain a new state, called “descendant”. Generally speaking, from a primary operator, the operators \hat{K}_μ generate a set of descendant operators, each corresponding to a new state whose dimension increases by 1 each time:

$$\begin{aligned} \hat{\mathcal{O}}_\Delta &= \text{primary dimension operator } \Delta \\ \hat{P}_\mu \hat{\mathcal{O}}_\Delta &= \text{descendant dimension operator } \Delta + 1 \\ &\vdots \\ &\vdots \\ \hat{P}_{\mu_1} \dots \hat{P}_{\mu_n} \hat{\mathcal{O}}_\Delta &= \text{descendant dimension operator } \Delta + n \end{aligned} \quad (143)$$

Therefore, given a primary operator, it will always be associated with a set (often infinite) of descendant operators, we will say that they form a “family”. In other words, given that the

primary operators generate the descendants by the action of \hat{P}_μ , it will never be necessary to specify operators other than the primaries, which will greatly simplify discussions in the future.

It turns out, moreover, that the transformation properties of primary operators are simple, it is possible to show that they transform like tensor densities (this is not *as good* as transforming like a tensor, but it is still a good thing):

$$\hat{\mathcal{O}}_\Delta^A(\tilde{x}^\mu) = \left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right|^{\Delta/D} \xrightarrow[\text{Jacobian}]{} L_A^B \hat{\mathcal{O}}_\Delta^B(x^\mu) \quad (144)$$

Lorentz representation

Descending operators do not transform as well, and we will therefore only invoke them implicitly, through the primary operators from which they originate by the action of \hat{P}_μ .

3.8. The operator product expansion (OPE)

We now come to an important concept (yet another one!), the *Operator Product Expansion*. We'll introduce the concept using states to get a hook on something already known, but we'll quickly get rid of them since they're not necessary and will be more of an aid than anything else. The goal is to write the operator product $\varphi(x)\varphi(0)$ as a sum of operators inserted at a single point (we'll come back to the *why* right after). Consider the state $|\psi\rangle$ given by

$$|\psi\rangle = \varphi_1(x)\varphi_2(0)|0\rangle \quad (145)$$

where φ_1 and φ_2 are two arbitrary primary operators (for simplicity, we consider scalars, but the discussion remains valid for indexed operators). In the framework of a conformal theory, we have at our disposal the dilation operator \hat{D} and, in the same way that in quantum mechanics we can diagonalize the Hamiltonian operator and obtain a basis of eigenstates, we will diagonalize this dilation operator: we rewrite $|\psi\rangle$ as a sum in an eigenstate basis of \hat{D} ,

$$|\psi\rangle = \sum_k C_k |\Delta_k\rangle \quad (146)$$

where the state $|\Delta_k\rangle$ includes a primary operator and all its descendants (in fact, to have a complete basis, we must sum over all the operators). Let's rearrange this expression:

$$\begin{aligned} |\psi\rangle &= \varphi_1(x)\varphi_2(0)|0\rangle \\ &= \sum_k C_k |\Delta_k\rangle \\ &= \sum_{\varphi_I} C_{\Delta,I}(x, \partial) \varphi_I(0)|0\rangle \end{aligned} \quad (147)$$

where this time we make explicit the fact that we are summing over the primary operators, the descendant operators of φ having been taken into account in the coefficient C by the action of the partial derivative (recall that a descendant operator is defined up to a constant as an n -partial derivative of a primary operator). Each term of the sum therefore includes a primary operator and all its descendants. We have also included an index Δ for the scaling dimension of the primary operator and an index I for the Lorentz representation associated with this

operator. We have expressed everything in terms of states, but this was only to start from an analogy of something known and we can forget the fact that we were talking about states,

$$\begin{aligned} |\psi\rangle &= \varphi_1(x)\varphi_2(0)|0\rangle \\ &= \sum_{\varphi_I} C_{\Delta,I}(x, \partial)\varphi_I(0)|0\rangle \end{aligned} \quad (148)$$

and “promote” the discussion in terms of operators only, which then simply gives us equality between operators,

$$\boxed{\varphi_1(x)\varphi_2(0) = \sum_{\varphi_I} C_{\Delta,I}(x, \partial)\varphi_I(0)} \quad (149)$$

This is called *Operator Product Expansion* (OPE for short); and it doesn’t just work by magic, there’s also context to consider:

1. the OPE is *true* only in a correlation function;
2. the other operators that are in the correlation function must be “sufficiently far away” from the product considered.

Remark: If we try to be formal, the second point can be interpreted as follows: the other operators that are in the correlation function must be located outside a sphere of radius $|x|$, otherwise there are convergence problems and the OPE is not guaranteed to converge.

In fact, depending on the point of view adopted, the preceding discussion is either postulated for conformal field theory or demonstrable. In pure CFT, without any real other prerequisite that has not been presented in this document, we cannot do better than to take the preceding points as axiom and will, moreover, assume that the OPE converges in the framework of CFT.

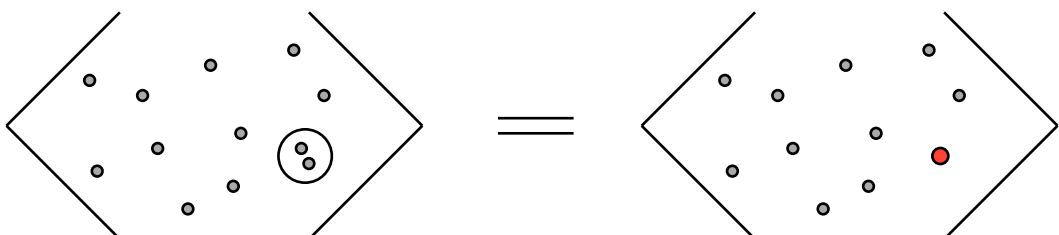


Figure 6: Visualization of an OPE

Figure Figure 6 schematically illustrates this new notion of *Operator Product Expansion*, where we see that two operators “sufficiently close” to each other in a correlation function are replaced by a new and unique operator by means of the OPE. From then on, any correlation function with $n > 2$ points can be reduced to a correlation function with $(n - 1)$ points by means of the OPE.

Remark: It turns out that, numerically, the OPE is a rapidly converging operation. This will be particularly interesting for bootstrapping.

One might ask the following question: given that we have said several times that a dilation-invariant theory no longer really has a notion of being “close” or “far” (remember for example the argument given for the S matrix), why do we say that the OPE between two operators only occurs if the latter are “sufficiently close”? The answer is simple: here, we are comparing *relative* distances, which therefore eliminates dilation-invariance from the discussion, and which therefore allows us to properly establish the OPE.

As has been noted, the OPE is defined with respect to the relative distance, hence the arbitrary choice of the product $\varphi(x)\varphi(0)$ in the definition eq. (149), we could imagine a completely equivalent definition where we take the product $\varphi(x_1)\varphi(x_2)$:

$$\varphi(x_1)\varphi(x_2) = \sum_{\varphi_I} C_{\Delta,I}(x_1 - x_2, \partial) \varphi_I\left(\frac{x_1 + x_2}{2}\right) \quad (150)$$

and if we consider a product $\varphi_i(x_1)\varphi_j(x_2)$, then we adapt the definition to write

$$\varphi_i(x_1)\varphi_j(x_2) = \sum_k C_{ijk}^{\Delta,I} \varphi_k\left(\frac{x_1 + x_2}{2}\right). \quad (151)$$

↑
Simple summation, no Einstein notation

The notations may vary depending on convention or usage, but the idea remains the same. In these notations, it may be easier to express the second condition mentioned earlier: we will have a valid OPE iff $|x_1 - x_2| \ll |x_1 - x_l|$ for all $l \neq 2$.

Remark: Knowing that there are an infinite number of descendant operators given a primary operator, this sum must be understood as a series expansion.

It is possible, only by dimensional analysis, to determine the first contribution of the OPE between two scalars. For simplicity, let us take $\varphi_1(x)\varphi_2(0)$ where φ_1 has a scaling dimension Δ_1 and φ_2 has a scaling dimension Δ_2 . Then, it follows that

$$\varphi_1(x)\varphi_2(0) \sim \frac{1}{|x|^{\Delta_1 + \Delta_2}} \quad (152)$$

The expression eq. (152) follows from the principle of conformal invariance: for the product of operators $\varphi_1(x)\varphi_2(0)$ to respect scale transformations, its dependence on $|x|$ must compensate for the total dimension $\Delta_1 + \Delta_2$ of the two operators, hence the form $1/|x|^{\Delta_1 + \Delta_2}$. We will see a little later in detail, in Section 3.9, how conformal symmetries influence the form of correlation functions.

3.8.1. Example – free and massless boson

Consider the action of a free boson, given by

$$S = \frac{1}{2}g \int d^2x (\partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^2) \quad (153)$$

where $g \in \mathbb{R}$ is simply a normalization parameter. This action gives rise to the Klein-Gordon equation, in fact with $\mathcal{L}(\varphi, \partial_\mu \varphi) = g/2(\partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^2)$ we have

$$\begin{aligned} & \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \approx 0 \\ \iff & \frac{g}{2} \left[2m^2 \varphi - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} (\partial_\mu \varphi \partial^\mu \varphi) \right] \approx 0 \\ \iff & \frac{g}{2} \left[2m^2 \varphi - \eta^{\mu\nu} \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} (\partial_\mu \varphi \partial_\nu \varphi) \right] \approx 0 \quad (154) \\ \iff & \frac{g}{2} [2m^2 \varphi - \eta^{\mu\nu} \partial_\mu (\partial_\nu \varphi + \delta^{\mu\nu} \partial_\mu)] \approx 0 \\ \iff & \frac{g}{2} [2m^2 \varphi - 2\partial_\mu \partial^\mu \varphi] \approx 0 \\ \iff & g(-\square + m^2) \varphi(x) \approx 0 \quad (\text{Klein Gordon}). \end{aligned}$$

where $\square \equiv \partial_\mu \partial^\mu$ is the d'Alembertian. We can define the following correlation function:

$$K(x, y) := \langle \varphi(x) \varphi(y) \rangle \quad (155)$$

Well, in the context of a quantum field theory, we will prefer to speak of a *propagator*, in what interests us these terms are interchangeable. Since the action leads to the Klein-Gordon equation, we know that this propagator must satisfy

$$g(-\square + m^2) K(x, y) = \delta^{(2)}(x - y) \quad (156)$$

In other words, the propagator for φ is a Green's function of the Klein-Gordon equation. Using this fact, we are able to find the expression for $K(x, y)$. To do this, let's make the coordinate change $r := |x - y|$ so that

$$\square = \partial_x^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right). \quad (157)$$

Not considering mass interaction, $m = 0$, eq. (156) is then written

$$\begin{aligned} & g(-\square + m^2) K(x, y) = \delta^{(2)}(x - y) \\ \iff & g \left[-\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + 0 \right] K(r) = \delta^{(2)}(r) \\ \iff & -g \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial K}{\partial r} \right) = \delta^{(2)}(r). \quad (158) \end{aligned}$$

In polar coordinates, we can show that the Dirac delta is expressed as

$$\delta^{(2)}(r) = \frac{\delta(r)}{2\pi r}, \quad (159)$$

we then end up with

$$-\frac{g}{r} \frac{\partial}{\partial r} \left(r \frac{\partial K}{\partial r} \right) = \frac{\delta(r)}{2\pi r} \quad (160)$$

and all that remains is for us to solve this differential equation:

$$\begin{aligned} & -\frac{g}{r} \frac{\partial}{\partial r} \left(r \frac{\partial K}{\partial r} \right) = \frac{\delta(r)}{2\pi r} \\ \iff & -2\pi g \frac{d}{dr} \left(r \frac{dK}{dr} \right) = \delta(r) \\ \iff & -2\pi g \int \frac{d}{dr} \left(r \frac{dK}{dr} \right) dr = \int \delta(r) dr \\ \iff & -2\pi g \left(r \frac{dK}{dr} \right) = 1 \\ \iff & \frac{dK}{dr} = -\frac{1}{2\pi gr} \\ \iff & \boxed{K(r) = -\frac{1}{2\pi g} \log(r) + C} \end{aligned} \quad (161)$$

So, we have explicitly calculated the correlation function of the real free scalar field. According to the authors (see for example *Di Francesco*), this solution also takes the form

$$K(r) = -\frac{1}{4\pi g} \log(r^2) + C \quad (162)$$

but eq. (161) and eq. (162) are of course completely equivalent since $\log(a^b) \equiv b \log(a)$, this last notation is in fact more useful when we want to move to complex coordinates. By re-expressing r in Cartesian coordinates, we finally found

$$\begin{aligned} K(\mathbf{x}, \mathbf{y}) := \langle \varphi(\mathbf{x}) \varphi(\mathbf{y}) \rangle &= -\frac{1}{2\pi g} \log(|\mathbf{x} - \mathbf{y}|) + C \\ &\equiv -\frac{1}{4\pi g} \log(|\mathbf{x} - \mathbf{y}|^2) + C. \end{aligned} \quad (163)$$

The OPE of $\partial\varphi$ and $\partial\varphi$

By moving to complex coordinates, precisely, with $\mathbf{x} = (x, y)$ which we re-express by defining $z := x + iy$ and $\mathbf{y} = (u, v)$ which we re-express with $\omega := y + iv$ and knowing the identity $|z - \omega|^2 \equiv (z - \omega)(\bar{z} - \bar{\omega})$, the previous result can be rewritten as

$$\langle \varphi(z, \bar{z}) \varphi(\omega, \bar{\omega}) \rangle = -\frac{1}{4\pi g} [\log(z - \omega) + \log(\bar{z} - \bar{\omega})] + C. \quad (164)$$

Now let's take the derivative of both sides of eq. (164),

$$\begin{aligned} \langle \partial_z \varphi(z, \bar{z}) \partial_\omega \varphi(\omega, \bar{\omega}) \rangle &= \frac{1}{4\pi g} \frac{1}{(z - \omega)^2} \\ \langle \partial_{\bar{z}} \varphi(z, \bar{z}) \partial_{\bar{\omega}} \varphi(\omega, \bar{\omega}) \rangle &= \frac{1}{4\pi g} \frac{1}{(\bar{z} - \bar{\omega})^2} \end{aligned} \quad (165)$$

and from there it is now clear that we have the following OPEs:

$$\begin{aligned} \partial_z \varphi(z, \bar{z}) \partial_\omega \varphi(\omega, \bar{\omega}) &\sim \frac{1}{4\pi g} \frac{1}{(z - \omega)^2} + \dots \\ \partial_{\bar{z}} \varphi(z, \bar{z}) \partial_{\bar{\omega}} \varphi(\omega, \bar{\omega}) &\sim \frac{1}{4\pi g} \frac{1}{(\bar{z} - \bar{\omega})^2} + \dots \end{aligned} \tag{166}$$

The OPE of T and $\partial\varphi$

The classical energy-momentum tensor for the free, massless boson is given by

$$T_{\mu\nu} = g \left(\partial_\mu \varphi \partial^\nu \varphi - \frac{1}{2} \eta_{\mu\nu} \partial_\rho \varphi \partial^\rho \varphi \right) \tag{167}$$

and in complex coordinates,

$$\begin{aligned} T(z, \bar{z}) &\equiv -2\pi T_{zz} = -2\pi g \left[(\partial\varphi)^2 - \frac{1}{2} \eta_{zz} \partial_\rho \varphi \partial^\rho \varphi \right] = -2\pi g (\partial\varphi)^2 \\ \bar{T}(z, \bar{z}) &\equiv -2\pi \bar{T}_{zz} = -2\pi g \left[(\bar{\partial}\varphi)^2 - \frac{1}{2} \eta_{\bar{z}\bar{z}} \partial_\rho \varphi \partial^\rho \varphi \right] = -2\pi g (\bar{\partial}\varphi)^2 \end{aligned} \tag{168}$$

Remark: The “quantum version” of this energy-momentum tensor is usually presented with normal order and Wick’s theorem, but this is beyond the scope of this paper and we will just stick to the results without going too deep, the goal is to see what the OPE looks like on concrete cases after all.

We can therefore show that after quantization,

$$\begin{aligned} T(z, \bar{z}) &= -2\pi g : \partial\varphi \partial\varphi : \\ &= -2\pi g \lim_{\omega \rightarrow z} \lim_{\bar{\omega} \rightarrow \bar{z}} [\partial\varphi(z, \bar{z}) \partial\varphi(\omega, \bar{\omega}) - \langle \partial\varphi(z, \bar{z}) \partial\varphi(\omega, \bar{\omega}) \rangle] \end{aligned} \tag{169}$$

The OPE of T and $\partial\varphi$ can be calculated using Wick’s theorem:

$$\begin{aligned}
T(z, \bar{z})\partial\varphi(w, \bar{w}) &= -2\pi g \lim_{z_1 \rightarrow z} \lim_{\bar{z}_1 \rightarrow \bar{z}} (\partial\varphi(z, \bar{z})\partial\varphi(z_1, \bar{z}_1) - \langle \partial\varphi(z, \bar{z})\partial\varphi(z_1, \bar{z}_1) \rangle)\partial\varphi(w, \bar{w}) \\
&\sim -2\pi g \lim_{z_1 \rightarrow z} \lim_{\bar{z}_1 \rightarrow \bar{z}} [(\langle \partial\varphi(z, \bar{z})\partial\varphi(w, \bar{w}) \rangle \partial\varphi(z_1, \bar{z}_1)) \\
&\quad + \langle \partial\varphi(z, \bar{z})\partial\varphi(z_1, \bar{z}_1) \rangle \partial\varphi(w, \bar{w}) \\
&\quad + \langle \partial\varphi(z_1, \bar{z}_1)\partial\varphi(w, \bar{w}) \rangle \partial\varphi(z, \bar{z}) \\
&\quad - \langle \partial\varphi(z, \bar{z})\partial\varphi(z_1, \bar{z}_1) \rangle \partial\varphi(w, \bar{w})] \\
&\sim -2\pi g \lim_{z_1 \rightarrow z} \lim_{\bar{z}_1 \rightarrow \bar{z}} (\langle \partial\varphi(z, \bar{z})\partial\varphi(w, \bar{w}) \rangle \partial\varphi(z_1, \bar{z}_1) \\
&\quad + \langle \partial\varphi(z_1, \bar{z}_1)\partial\varphi(w, \bar{w}) \rangle \partial\varphi(z, \bar{z})) \\
&\sim -4\pi g \langle \partial\varphi(z, \bar{z})\partial\varphi(w, \bar{w}) \rangle \partial\varphi(z, \bar{z}) \\
&\sim \frac{\partial\varphi(z, \bar{z})}{(z - w)^2}
\end{aligned} \tag{170}$$

where, in the last line, we used the result eq. (166). We can then expand $\partial\varphi$ around $(\omega, \bar{\omega})$,

$$\partial\varphi(z, \bar{z}) \approx \partial\varphi(\omega, \bar{\omega}) + \partial^2\varphi(\omega, \bar{\omega})(z - \omega) + \frac{1}{2!}\partial^3\varphi(\omega, \bar{\omega})(z - \omega)^2 + \dots \tag{171}$$

which ultimately gives us the following OPE:

$$T(z)\partial\varphi(\omega, \bar{\omega}) \sim \frac{\partial\varphi(\omega, \bar{\omega})}{(z - \omega)^2} + \frac{\partial^2\varphi(\omega, \bar{\omega})}{(z - \omega)} + \dots$$

(172)

The OPE of T and T

A similar derivation to the previous one can be done to calculate $T(z)T(\omega)$, but the calculation is much heavier, so we will skip the big calculation step for convenience. We have

$$T(z)T(\omega) \sim \frac{1/2}{(z - \omega)^4} + \frac{2T(\omega)}{(z - \omega)^2} + \frac{\partial T(\omega)}{(z - \omega)} + \dots$$

(173)

where, roughly, there are two ways to perform a Wick contraction $\partial\varphi\partial\varphi$ and there are four ways to perform a single Wick contraction, and after expanding we find eq. (173).

Remark: In the context of a CFT in $d = 2$, it will often be indicated that the *central charge* associated with this OPE is here equal to 1. More generally, the OPE of the energy-momentum tensor is written

$$T(z)T(\omega) \sim \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{(z-\omega)} \quad (174)$$

where $c \in \mathbb{R}$ is a constant that is sometimes called the “central charge”. In the context of a CFT in $d \geq 3$, this notion is quite arbitrary and is actually not useful (indeed, we are just giving a name to a constant...). In fact, the central charge is useful in a two-dimensional CFT, where it corresponds to an eigenvalue of a Casimir (hence the name).

In the present case, it is clear that $c = 1$.

Remark: If we want to go a little further and understand *on the surface* how we calculated the previous results, we present here a very brief description of the concepts invoked above. For two operators \hat{A} and \hat{B} we define their *contraction* as being

$$\hat{A}^\bullet \hat{B}^\bullet \equiv \hat{A}\hat{B} - : \hat{A}\hat{B} : \quad (175)$$

where $: \hat{A}\hat{B} :$ indicates that the operators \hat{A} and \hat{B} are placed in *normal order*: all creation (or related) operators are placed to the left of all destruction (or related) operators. Contraction is therefore defined as the difference between their ordinary product and their normal-ordered product, which “measures” the extent to which the ordinary product is not already normally ordered.

Example: For example, let's take

$$: \hat{b}^\dagger \hat{b} : = \hat{b}^\dagger \hat{b}. \quad (176)$$

Since the two operators are already in the desired order, Wick's notation doesn't give anything special. A more interesting example is

$$: \hat{b} \hat{b}^\dagger : = \hat{b}^\dagger \hat{b} \quad (177)$$

where here we see that the operators have been explicitly ordered. An example with more than two operators could be

$$: \hat{b}^\dagger \hat{b} \hat{b} \hat{b}^\dagger \hat{b} \hat{b}^\dagger \hat{b} : = \hat{b}^\dagger \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} \hat{b} = (\hat{b}^\dagger)^3 \hat{b}^4. \quad (178)$$

Wick's theorem states that the product $\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots$ of creation operators and destruction operators can be expressed as the sum

$$\begin{aligned} \hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots &= : \hat{A}\hat{B}\hat{C}\hat{D}\hat{E}\hat{F}\dots : + \sum_{\text{simple}} : \hat{A}^\bullet \hat{B}^\bullet \hat{C}\hat{D}\hat{E}\hat{F}\dots : \\ &\quad + \sum_{\text{double}} : \hat{A}^\bullet \hat{B}^{\bullet\bullet} \hat{C}^{\bullet\bullet} \hat{D}^\bullet \hat{E}\hat{F}\dots : \quad (179) \\ &\quad + \dots \end{aligned}$$

We will not go into detail now or into the calculations using Wick contractions, but will simply mention them when it is useful to know where a result comes from.

3.9. Conformal symmetries and correlation functions

We have already seen a few times that conformal symmetries constrain correlation functions, this section is dedicated to a more in-depth study of this fact. We will focus the discussion on scalar operators since it is simpler to reason about them, but of course this generalizes to any kind of operator, with the only difference being that other objects will have to appear to handle indices. To keep it simple, we do not want to bother with such objects (which are really

nothing more than constants). We will denote these scalar operators by \mathcal{O}_Δ , where Δ is their dimension, and consider them to be primary operators.

Recall that correlation functions play the role of observables in a conformal field theory, i.e. correlation functions must be independent of the choice of configuration. Therefore, in CFT, any correlation function must be invariant under conformal transformations. We therefore impose that

$$\langle \tilde{\mathcal{O}}_1(x_1^\mu) \tilde{\mathcal{O}}_2(x_2^\mu) \dots \tilde{\mathcal{O}}_n(x_n^\mu) \rangle \stackrel{!}{=} \langle \mathcal{O}_1(x_1^\mu) \mathcal{O}_2(x_2^\mu) \dots \mathcal{O}_n(x_n^\mu) \rangle \quad \forall x_i^\mu \quad (180)$$

It is by applying this rule (which *must* exist for the theory to be physical) that we will be able to deduce the form of the scalar correlation functions at one, two and three points, as we will see just after.

Remark: In eq. (180), we changed the operators “ $\tilde{\mathcal{O}}_i \rightarrow \mathcal{O}_i$ ” but kept them evaluated at the same points, as it wouldn’t make much sense to compare correlation functions evaluated at different points! In the following, it will actually be more convenient to evaluate correlation functions at the transformed points, \tilde{x}^μ rather than at the original points, x^μ .

Let us also recall that primary operators transform like densities (see eq. (144)), in other words, for scalar operators,

$$\begin{aligned} \tilde{\mathcal{O}}_\Delta^A(\tilde{x}^\mu) &= \left| \frac{\partial x^\mu}{\partial x^\nu} \right|^{\Delta/D} L_A^B \mathcal{O}_\Delta^B(x^\mu) \\ \Leftrightarrow \tilde{\mathcal{O}}_\Delta(\tilde{x}^\mu) &= \left| \frac{\partial x^\mu}{\partial x^\nu} \right|^{\Delta/D} \mathcal{O}_\Delta(x^\mu) \end{aligned} \quad (181)$$

where, as announced, we thus get rid of the L_A^B which adds nothing to the discussion.

3.9.1. One-point scalar correlation functions

Let’s apply the eq. (180) constraint to the one-point scalar correlation function,

$$\langle \mathcal{O}_\Delta(\tilde{x}^\mu) \rangle \stackrel{!}{=} \langle \tilde{\mathcal{O}}_\Delta(\tilde{x}^\mu) \rangle \quad (182)$$

where, as noted just before, we evaluate it at the point \tilde{x}^μ rather than at x^μ for convenience. This constraint must be verified for all conformal transformations.

Translation

For translations, $\tilde{x}^\mu = x^\mu + a^\mu$, the Jacobian is written

$$\left| \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \right| = \left| \frac{\partial x^\mu}{\partial x^\nu} + \frac{\partial a^\mu}{\partial x^\nu} \right| = |\delta^\mu_\nu| = 1. \quad (183)$$

So our operator does not change under translation, $\tilde{\mathcal{O}}_\Delta(\tilde{x}^\mu) = \mathcal{O}_\Delta(x^\mu)$, hence

$$\langle \mathcal{O}_\Delta(\tilde{x}^\mu) \rangle = \langle \mathcal{O}_\Delta(x^\mu) \rangle. \quad (184)$$

Knowing that this must be true for all x^μ , we can conclude that the correlation function at a point is in fact a constant under translation. Let us check that the other conformal symmetry transformations agree with this statement.

Dilation

For dilations, $\tilde{x}^\mu = \lambda x^\mu$ or $x^\mu = \tilde{x}^\mu/\lambda$. The Jacobian is then written

$$\left| \frac{\partial x^\mu}{\partial \tilde{x}^\mu} \right| = \left| \frac{\delta_\nu^\mu}{\lambda} \right| = \left| \frac{1}{\lambda} \right| = \lambda^{-D}, \quad (185)$$

a scalar operator is therefore transformed as

$$\tilde{\mathcal{O}}_\Delta(\tilde{x}^\mu) = \left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right|^{\Delta/D} \mathcal{O}_\Delta(x^\mu) = \lambda^{-\Delta} \mathcal{O}_\Delta(x^\mu). \quad (186)$$

Note that we had already intuited this result earlier. In a one-point correlation function, we then have

$$\begin{aligned} \langle \mathcal{O}_\Delta(\tilde{x}^\mu) \rangle &= \langle \tilde{\mathcal{O}}_\Delta(\tilde{x}^\mu) \rangle \\ &= \langle \lambda^{-\Delta} \mathcal{O}_\Delta(x^\mu) \rangle \\ &= \lambda^{-\Delta} \langle \mathcal{O}_\Delta(x^\mu) \rangle \quad \forall x^\mu \end{aligned} \quad (187)$$

but, previously for translations, we had found that

$$\langle \mathcal{O}_\Delta(\tilde{x}^\mu) \rangle = \text{const.} =: C \quad \forall x^\mu, \quad (188)$$

which indicates that

$$C = \lambda^{-\Delta} C. \quad (189)$$

This equation *must* hold for any dilation factor λ , which is true if $\Delta = 0$ or if $C = 0$. Since the case $\Delta = 0$ corresponds to a zero-dimensional operator, it can only be a constant and is therefore uninteresting. From this, we can conclude that all point correlation functions must vanish:

$$\langle \mathcal{O}_\Delta(x^\mu) \rangle = 0 \quad \forall x^\mu \text{ and for } \Delta \neq 0$$

(190)

The other conformal transformations follow trivially (this will give 0 each time).

3.9.2. Two-point scalar correlation functions

We do the same for the two-point scalar function, so we impose that

$$\langle \mathcal{O}_{\Delta_1}(\tilde{x}_1^\mu) \mathcal{O}_{\Delta_2}(\tilde{x}_2^\mu) \rangle \stackrel{!}{=} \langle \tilde{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \tilde{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle \quad (191)$$

and we study conformal symmetries according to this constraint.

Translation

Let us note, to begin with, that $\langle \mathcal{O}_{\Delta_1}(\tilde{x}_1^\mu) \mathcal{O}_{\Delta_2}(\tilde{x}_2^\mu) \rangle$ is a function of positions returning a number, it will be easier, in this case, to write that

$$\langle \mathcal{O}_{\Delta_1}(\tilde{x}_1^\mu) \mathcal{O}_{\Delta_2}(\tilde{x}_2^\mu) \rangle =: f(\tilde{x}_1^\mu, \tilde{x}_2^\mu). \quad (192)$$

We also know from the previous point that, under translations, the primary scalar operators transform according to $\tilde{\mathcal{O}}_\Delta(\tilde{x}^\mu) = \mathcal{O}_\Delta(x^\mu)$, so by inserting this result in the right-hand side of eq. (191) we find

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1}(\tilde{x}_1^\mu) \mathcal{O}_{\Delta_2}(\tilde{x}_2^\mu) \rangle &= \langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \rangle \\ \iff f(\tilde{x}_1^\mu, \tilde{x}_2^\mu) &= f(x_1^\mu, x_2^\mu). \end{aligned} \quad (193)$$

Furthermore, we have that $\tilde{x}^\mu = x^\mu + a^\mu$, which allows us to write that

$$\begin{aligned} f(\tilde{x}_1^\mu, \tilde{x}_2^\mu) &= f(x_1^\mu, x_2^\mu) \\ \iff f(x_1^\mu + a^\mu, x_2^\mu + a^\mu) &= f(x_1^\mu, x_2^\mu) \quad \forall a^\mu. \end{aligned} \quad (194)$$

This result must be true regardless of the value of a^μ , which suggests that the a^μ term must vanish at some point (otherwise the equality would never hold). The only way for a^μ to vanish is by having a function dependent only on $x_1^\mu - x_2^\mu$,

$$f(x_1^\mu, x_2^\mu) = f(x_1^\mu - x_2^\mu), \quad (195)$$

in other words, the two-point correlation function must be a function not of two positions, but rather of the displacement between these two positions:

$$\langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \rangle = f(x_1^\mu - x_2^\mu). \quad (196)$$

We can't say much more at the moment.

Rotation

For the Lorentz transformations, we have $\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu$, the Jacobian is then expressed

$$\left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right| = \left| (\Lambda^{-1})^\mu_\nu \delta^\nu_\mu \right| = 1 \quad (197)$$

since the determinant of the Lorentz transformation matrix is equal to 1. The primary scalar operators are therefore transformed in the same way under translations and under rotations:

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1}(\tilde{x}_1^\mu) \mathcal{O}_{\Delta_2}(\tilde{x}_2^\mu) \rangle &= \langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \rangle \\ \iff f(\tilde{x}_1^\mu, \tilde{x}_2^\mu) &= f(x_1^\mu, x_2^\mu), \end{aligned} \quad (198)$$

and using our result found for translations, we find that

$$f(\tilde{x}_1^\mu - \tilde{x}_2^\mu) = f(x_1^\mu - x_2^\mu), \quad (199)$$

or even

$$f(\Lambda^\mu_\nu(x_1^\nu - x_2^\nu)) = f(x_1^\mu - x_2^\mu). \quad (200)$$

In other words, applying a rotation has no effect on the result. Since rotations have no effect on $x_1^\mu - x_2^\mu$, this tells us that the function f must depend not on the displacement but only on the magnitude of the latter:

$$\langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \rangle = f(|x_1^\mu - x_2^\mu|). \quad (201)$$

Dilation

Under dilation, $\tilde{\mathcal{O}}_\Delta(\tilde{x}^\mu) = \lambda^{-\Delta} \mathcal{O}_\Delta(x^\mu)$. We substitute this into eq. (191):

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1}(\tilde{x}_1^\mu) \mathcal{O}_{\Delta_2}(\tilde{x}_2^\mu) \rangle &= \langle \tilde{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \tilde{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle \\ &= \langle \lambda^{-\Delta_1} \mathcal{O}_{\Delta_1}(x_1^\mu) \lambda^{-\Delta_2} \mathcal{O}_{\Delta_2}(x_2^\mu) \rangle \\ &= \lambda^{-\Delta_1} \lambda^{-\Delta_2} \langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \rangle \\ &= \lambda^{-(\Delta_1 + \Delta_2)} f(|x_1^\mu - x_2^\mu|) \end{aligned} \quad (202)$$

where we have combined all the results obtained so far for the two-point correlation function. Using the function notation f , we have therefore currently found that

$$f(|\tilde{x}_1^\mu - \tilde{x}_2^\mu|) = \lambda^{-(\Delta_1 + \Delta_2)} f(|x_1^\mu - x_2^\mu|), \quad (203)$$

or even

$$f(\lambda |x_1^\mu - x_2^\mu|) = \lambda^{-(\Delta_1 + \Delta_2)} f(|x_1^\mu - x_2^\mu|). \quad (204)$$

Under a set of conditions that are always encountered in physics (💡), we can always expand f in a power series:

$$f(|x_1^\mu - x_2^\mu|) = \sum_k^\infty c_k |x_1^\mu - x_2^\mu|^k, \quad (205)$$

we can then substitute this series into the previous result to find

$$\sum_k^\infty c_k \lambda^k |x_1^\mu - x_2^\mu|^k = \lambda^{-(\Delta_1 + \Delta_2)} \sum_k^\infty c_k |x_1^\mu - x_2^\mu|^k, \quad (206)$$

which allows us to better compare the expansion factors: for equality to hold for all values of λ , each power of λ on the left side must correspond to the same power on the right side; this is only possible if all terms are zero except the one whose exponent is $k = -(\Delta_1 + \Delta_2)$, which then simply brings us to

$$\langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \rangle = C |x_1^\mu - x_2^\mu|^{-(\Delta_1 + \Delta_2)} \quad C \in \mathbb{R} \quad (207)$$

where C is the constant of the power development associated with $k = -(\Delta_1 + \Delta_2)$.

Special conformal transformation

We come to the last symmetry transformation with which the two-point scalar correlation function will be constrained. Recall that $\tilde{x}^\mu = x^\mu + 2(x \cdot b)x^\mu - b^\mu x^2$, we could calculate the Jacobian and try to get by as best we can, but in fact it is simpler to remember that the special conformal transformation is composed of an inversion, then a translation, then an inversion

and since we already know the result of the symmetry under translation, it is enough to tackle the “subsymmetry” under inversion. As was presented earlier, the inversion considered is given by

$$x^\mu = \frac{\tilde{x}^\mu}{\tilde{x}^2} \quad (208)$$

from where

$$\left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right| = \left| \frac{\delta_\nu^\mu}{\tilde{x}^2} \right| = \frac{1}{\tilde{x}^{2D}} \quad (209)$$

because $\tilde{x}^2 \equiv \tilde{x}^\alpha \tilde{x}_\alpha \in \mathbb{R}$ is a scalar. Under inversion, a scalar primary operator therefore transforms as

$$\tilde{\mathcal{O}}_\Delta(\tilde{x}^\mu) = \left| \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \right|^{\Delta/D} \mathcal{O}_\Delta(x^\mu) = \left(\frac{1}{\tilde{x}^{2D}} \right)^{\Delta/D} \mathcal{O}_\Delta(x^\mu) = \frac{1}{\tilde{x}^{2\Delta}} \mathcal{O}_\Delta(x^\mu) \quad (210)$$

what we replace in eq. (191),

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1}(\tilde{x}_1^\mu) \mathcal{O}_{\Delta_2}(\tilde{x}_2^\mu) \rangle &= \langle \tilde{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \tilde{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \rangle \\ &= \left\langle \frac{1}{\tilde{x}_1^{2\Delta_1}} \mathcal{O}_{\Delta_1}(x_1^\mu) \frac{1}{\tilde{x}_2^{2\Delta_2}} \mathcal{O}_{\Delta_2}(x_2^\mu) \right\rangle \\ &= \frac{1}{\tilde{x}_1^{2\Delta_1}} \frac{1}{\tilde{x}_2^{2\Delta_2}} \langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \rangle. \end{aligned} \quad (211)$$

We now use eq. (207) on the left and right of this equality, in order to obtain

$$\begin{aligned} C |\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{-(\Delta_1 + \Delta_2)} &= \frac{1}{\tilde{x}_1^{2\Delta_1}} \frac{1}{\tilde{x}_2^{2\Delta_2}} C |x_1^\mu - x_2^\mu|^{-(\Delta_1 + \Delta_2)} \\ \iff \frac{\tilde{x}_1^{2\Delta_1} \tilde{x}_2^{2\Delta_2}}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1 + \Delta_2}} &= \frac{1}{|x_1^\mu - x_2^\mu|^{\Delta_1 + \Delta_2}} \end{aligned} \quad (212)$$

which can also be put in the following form:

$$\frac{\tilde{x}_1^{2\Delta_1} \tilde{x}_2^{2\Delta_2}}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta_1 + \Delta_2}} = \left(\frac{\tilde{x}_1^2 \tilde{x}_2^2}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^2} \right)^{\frac{\Delta_1 + \Delta_2}{2}}. \quad (213)$$

We can notice that, if $\Delta_1 = \Delta_2 =: \Delta$, then the previous equality takes the following form:

$$\begin{aligned} \frac{\tilde{x}_1^{2\Delta} \tilde{x}_2^{2\Delta}}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^{\Delta + \Delta}} &= \left(\frac{\tilde{x}_1^2 \tilde{x}_2^2}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^2} \right)^\Delta \\ \iff \left(\frac{\tilde{x}_1^2 \tilde{x}_2^2}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^2} \right)^\Delta &= \left(\frac{\tilde{x}_1^2 \tilde{x}_2^2}{|\tilde{x}_1^\mu - \tilde{x}_2^\mu|^2} \right)^\Delta \\ \iff 0 &= 0, \end{aligned} \quad (214)$$

in other words, eq. (213) is only satisfied when $\Delta_1 = \Delta_2$. Symmetry under the special conformal transformation therefore tells us that the two-point scalar correlation function is in fact zero if $\Delta_1 \neq \Delta_2$ (otherwise eq. (213) has no solution). Therefore, eq. (207) can in all generality be written as

$$\langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \rangle = \frac{C \delta_{\Delta_1 \Delta_2}}{|x_1^\mu - x_2^\mu|^{\Delta_1 + \Delta_2}} \quad (215)$$

where $\delta_{\Delta_1 \Delta_2}$ is a Kronecker delta,

$$\delta_{\Delta_1 \Delta_2} = \begin{cases} 1 & \text{if } \Delta_1 = \Delta_2, \\ 0 & \text{otherwise} \end{cases}, \quad (216)$$

which concludes our analysis of conformal symmetries for the two-point scalar correlation function. To summarize what we did to get here,

1. Under translations, we arrive at the conclusion that the correlation function must be a function of the displacement $x_1^\mu - x_2^\mu$;
2. Under rotations, we actually show that the correlation function is rather a function of the magnitude of the displacement, $|x_1^\mu - x_2^\mu|$;
3. Under dilations, we find the general form of this correlation function;
4. Under special conformal transformation, we conclude that the general form is stricter and exhibits a Kronecker delta.

3.9.3. Three-point scalar correlation functions

Let's continue with the three-point scalar correlation function. As before, we should have

$$\langle \mathcal{O}_1(x_1^\mu) \mathcal{O}_2(x_2^\mu) \mathcal{O}_3(x_3^\mu) \rangle \stackrel{!}{=} \langle \tilde{\mathcal{O}}_1(x_1^\mu) \tilde{\mathcal{O}}_2(x_2^\mu) \tilde{\mathcal{O}}_3(x_3^\mu) \rangle. \quad (217)$$

The calculation steps will be quite similar to the previous cases, but more tedious if you want to repeat the same calculation steps.

Translation & Rotation

We have seen above that translations and rotations have a similar Jacobian and, therefore, can be grouped in the same discussion (this should not be too surprising given that these are precisely the components of a Poincaré transformation). Following the same result that we managed to derive for the two-point scalar correlation function, we deduce that here too it will be a function dependent on the magnitude of the displacements $x_i^\mu - x_j^\mu$, that is to say

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \mathcal{O}_{\Delta_3}(x_3^\mu) \rangle &= f(|x_1^\mu - x_2^\mu|, |x_2^\mu - x_3^\mu|, |x_3^\mu - x_1^\mu|) \\ &\equiv f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|). \end{aligned} \quad (218)$$

Dilation

For the dilation, we will have as before

$$\begin{aligned}
\langle \mathcal{O}_{\Delta_1}(\tilde{x}_1^\mu) \mathcal{O}_{\Delta_2}(\tilde{x}_2^\mu) \mathcal{O}_{\Delta_3}(\tilde{x}_3^\mu) \rangle &= \langle \tilde{\mathcal{O}}_{\Delta_1}(\tilde{x}_1^\mu) \tilde{\mathcal{O}}_{\Delta_2}(\tilde{x}_2^\mu) \tilde{\mathcal{O}}_{\Delta_3}(\tilde{x}_3^\mu) \rangle \\
&= \langle \lambda^{-\Delta_1} \mathcal{O}_{\Delta_1}(x_1^\mu) \lambda^{-\Delta_2} \mathcal{O}_{\Delta_2}(x_2^\mu) \lambda^{-\Delta_3} \mathcal{O}_{\Delta_3}(x_3^\mu) \rangle \\
&= \lambda^{-\Delta_1} \lambda^{-\Delta_2} \lambda^{-\Delta_3} \langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \mathcal{O}_{\Delta_3}(x_3^\mu) \rangle \\
&= \lambda^{-(\Delta_1 + \Delta_2 + \Delta_3)} \langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \mathcal{O}_{\Delta_3}(x_3^\mu) \rangle,
\end{aligned} \tag{219}$$

which, combined with the result for the Poincaré transformations, gives us

$$\begin{aligned}
f(|\tilde{x}_{12}^\mu|, |\tilde{x}_{23}^\mu|, |\tilde{x}_{31}^\mu|) &= \lambda^{-(\Delta_1 + \Delta_2 + \Delta_3)} f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) \\
\Leftrightarrow f(\lambda|x_{12}^\mu|, \lambda|x_{23}^\mu|, \lambda|x_{31}^\mu|) &= \lambda^{-(\Delta_1 + \Delta_2 + \Delta_3)} f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|).
\end{aligned} \tag{220}$$

In order to compare the expansion factors, we expand the three-point scalar correlation function as a power series to obtain

$$f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{31}^\mu|) = \sum_i^{\infty} \sum_j^{\infty} \sum_k^{\infty} c_{ijk} |x_{12}^\mu|^i |x_{23}^\mu|^j |x_{31}^\mu|^k. \tag{221}$$

We substitute this into the previous result,

$$\begin{aligned}
&\sum_i^{\infty} \sum_j^{\infty} \sum_k^{\infty} c_{ijk} \lambda^i |x_{12}^\mu|^i \lambda^j |x_{23}^\mu|^j \lambda^k |x_{31}^\mu|^k \\
&= \lambda^{-(\Delta_1 + \Delta_2 + \Delta_3)} \sum_i^{\infty} \sum_j^{\infty} \sum_k^{\infty} c_{ijk} |x_{12}^\mu|^i |x_{23}^\mu|^j |x_{31}^\mu|^k
\end{aligned} \tag{222}$$

and, with an analysis similar to what was done in the previous case, we deduce that this equality is true only if all the terms are zero, except those for which

$$i + j + k = -(\Delta_1 + \Delta_2 + \Delta_3), \tag{223}$$

which leads to

$$\langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \mathcal{O}_{\Delta_3}(x_3^\mu) \rangle = \sum_{i+j+k=-(\Delta_1+\Delta_2+\Delta_3)} c_{ijk} |x_{12}^\mu|^i |x_{23}^\mu|^j |x_{31}^\mu|^k \tag{224}$$

Triple sum over i, j, k such that their sum gives $-(\Delta_1 + \Delta_2 + \Delta_3)$

because, in fact, the condition eq. (223) can be verified for several terms a priori.

Special conformal transformation

As before, we only need to impose the transformation under inversion. In the case of the two-point scalar correlation function, we arrived at the constraint that “ $\Delta_1 = \Delta_2$ ”. Here, since there are three points, we will have three constraints on the i, j, k whose result we give directly and without development (it is a long calculation and without much interest):

$$\begin{cases} i = \Delta_1 + \Delta_2 - \Delta_3 \\ j = \Delta_1 + \Delta_3 - \Delta_2 \\ k = \Delta_2 + \Delta_3 - \Delta_1. \end{cases} \tag{225}$$

Combining everything together, we finally arrive at a result similar to what we found previously (but constrained differently):

$$\langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \mathcal{O}_{\Delta_3}(x_3^\mu) \rangle = \frac{C_{123}}{|x_{12}^\mu|^i |x_{23}^\mu|^j |x_{31}^\mu|^k} \quad (226)$$

where the powers $i, j, k \in \{(i, j, k) \mid \text{eq. (225)} \text{ is respected}\}$ and where the constants C_{123} come from the power expansion associated with the constraint eq. (225).

3.9.4. Summary

We have thus shown that the one-, two- and three-point scalar correlation functions are completely determined up to a set of constants,

$\langle \mathcal{O}_\Delta(x^\mu) \rangle = 0$	(1 point)
$\langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \rangle = \frac{C \delta_{\Delta_1 \Delta_2}}{ x_1^\mu - x_2^\mu ^{\Delta_1 + \Delta_2}}$	(2 points)
$\langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \mathcal{O}_{\Delta_3}(x_3^\mu) \rangle = \frac{C_{123}}{ x_{12}^\mu ^i x_{23}^\mu ^j x_{31}^\mu ^k}$	(3 points)

This set of constants is important for defining a specific conformal field theory, sometimes said to “determine the degree of interaction of operators” and will also be known as the “CFT data”, to which we add the dimensions.

3.10. The conformal bootstrap

In the previous subsection, we did not go beyond three-point scalar correlation functions. We will now discuss four-point scalar correlation functions and see that this discussion will lead us to the state of the art of CFT research (or at least a large counterpart of CFT research).

Remark: As already announced earlier, we stick with scalar operators because they are the simplest to consider: no transformation matrix needs to intervene to know how the indices are transformed, and we do not want to complicate things in the name of generality and completeness.

3.10.1. Four-point scalar correlation function

For a four-point scalar correlation function, we would want

$$\langle \mathcal{O}_1(x_1^\mu) \mathcal{O}_2(x_2^\mu) \mathcal{O}_3(x_3^\mu) \mathcal{O}_4(x_4^\mu) \rangle \stackrel{!}{=} \langle \tilde{\mathcal{O}}_1(x_1^\mu) \tilde{\mathcal{O}}_2(x_2^\mu) \tilde{\mathcal{O}}_3(x_3^\mu) \tilde{\mathcal{O}}_4(x_4^\mu) \rangle. \quad (227)$$

If we follow the same steps as before for the Poincaré transformations (translations + rotations), then we arrive at

$$\langle \mathcal{O}_1(x_1^\mu) \mathcal{O}_2(x_2^\mu) \mathcal{O}_3(x_3^\mu) \mathcal{O}_4(x_4^\mu) \rangle = f(|x_{12}^\mu|, |x_{23}^\mu|, |x_{34}^\mu|, |x_{41}^\mu|), \quad (228)$$

in other words, there are four quantities $|x_{ij}^\mu|$. It turns out that some combinations of these four quantities form functions which are in fact invariant under all conformal transformations, we can form two of them and we will then speak of *cross sections*:

$$u := \frac{|x_{12}^\mu| |x_{34}^\mu|}{|x_{13}^\mu| |x_{24}^\mu|} \quad \text{and} \quad v := \frac{|x_{12}^\mu| |x_{34}^\mu|}{|x_{23}^\mu| |x_{14}^\mu|} \quad (229)$$

Invariance under Poincaré

Since u and v are functions only of the magnitudes of the separation between the points, invariance under translation and rotation is automatically satisfied.

Invariance under dilation

Invariance under dilation is also trivially satisfied:

$$\tilde{u} = \frac{|\tilde{x}_{12}^\mu| |\tilde{x}_{34}^\mu|}{|\tilde{x}_{13}^\mu| |\tilde{x}_{24}^\mu|} = \frac{\lambda |x_{12}^\mu| \lambda |x_{34}^\mu|}{\lambda |x_{13}^\mu| \lambda |x_{24}^\mu|} = u. \quad (230)$$

Invariance under special conformal transformation

As before, we don't need to look at the SCT completely but only at the inversion $\tilde{x}^\mu = x^\mu / x^2$. To simplify the calculations, we can look at the squares x_{ij}^2 rather than the norms $|x_{ij}|$, which makes the discussion easier without changing the result since $|x| \equiv \sqrt{x^2}$. Let's then apply the inversion to the points x_i and x_j (we will denote I for the inversion):

$$I(x_i) = \frac{x_i}{x_i^2}, \quad I(x_j) = \frac{x_j}{x_j^2}, \quad (231)$$

so

$$I(x_i) - I(x_j) = \frac{x_i}{x_i^2} - \frac{x_j}{x_j^2} = \frac{x_j^2 x_i - x_i^2 x_j}{x_i^2 x_j^2}, \quad (232)$$

and we can take the square:

$$[I(x_i) - I(x_j)]^2 = \frac{(x_j^2 x_i - x_i^2 x_j)^2}{x_i^4 x_j^4}. \quad (233)$$

Hence,

$$(x_j^2 x_i - x_i^2 x_j)^2 = x_i^2 x_j^2 (x_i - x_j)^2 \equiv x_i^2 x_j^2 x_{ij}^2, \quad (234)$$

from where

$$[I(x_i) - I(x_j)]^2 = \frac{x_i^2 x_j^2 x_{ij}^2}{x_i^4 x_j^4} = \frac{x_{ij}^2}{x_i^2 x_j^2}. \quad (235)$$

In other words, under inversion,

$$x_{ij}^2 \mapsto \frac{x_{ij}^2}{x_i^2 x_j^2}. \quad (236)$$

Applying this to u , we will have

$$\begin{aligned} x_{12}^2 &\mapsto \frac{x_{12}^2}{x_1^2 x_2^2}, & x_{34}^2 &\mapsto \frac{x_{34}^2}{x_3^2 x_4^2}, \\ x_{13}^2 &\mapsto \frac{x_{13}^2}{x_1^2 x_3^2}, & x_{24}^2 &\mapsto \frac{x_{24}^2}{x_2^2 x_4^2} \end{aligned} \tag{237}$$

which gives rise to

$$\tilde{u} = \frac{\tilde{x}_{12}^2 \tilde{x}_{34}^2}{\tilde{x}_{13}^2 \tilde{x}_{24}^2} = \frac{\frac{x_{12}^2}{x_1^2 x_2^2} \frac{x_{34}^2}{x_3^2 x_4^2}}{\frac{x_{13}^2}{x_1^2 x_3^2} \frac{x_{24}^2}{x_2^2 x_4^2}} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = u. \tag{238}$$

Which clearly shows that u is invariant under all conformal transformations. The same reasoning also applies to v .

The invariance of the functions u and v under conformal transformations implies that the four-point scalar correlation function is no longer completely determined up to a set of constants, and this conclusion generalizes to correlation functions with $n \geq 4$ points, as well as to non-scalar operators.

3.10.2. Crossing symmetry

Since we are somewhat “stuck” with scalar correlation functions at $n \geq 4$ points, we will have to resort to an approximation to move forward, but it turns out that we already know it: the Operator Product Expansion (OPE). The OPE allows us to obtain an $n - 1$ point correlation function from an n point correlation function, and since we have seen that the one-, two-, and three-point correlation functions are entirely determined up to a set of constants, it will be enough to perform the OPE repeatedly until we reach one of these fixed forms, which, as a reminder, will also produce another set of constants, and it is the set of these constants together that is best called “CFT data”. It is all these constraints brought by the CFT that will give rise to the bootstrap algorithm, as we will see a little later.

Consider a four-point scalar correlation function that is developed by two successive OPEs, which can be written in all generality as

$$\begin{aligned} &\langle \overline{\mathcal{O}_1(x_1)} \overline{\mathcal{O}_2(x_2)} \overline{\mathcal{O}_3(x_4)} \overline{\mathcal{O}_4(x_4)} \rangle \\ &= \sum_{m,m'} \lambda_{12m} \lambda_{34m'} C_a(x_{12}, \partial_2) C_b(x_{34}, \partial_4) \langle \mathcal{O}_m^a(x_2) \mathcal{O}_{m'}^b(x_4) \rangle \end{aligned} \tag{239}$$

where we have indicated with small bars on which operators the OPE considered is carried out (we sometimes speak of *contraction channels*). The right side of this equality is completely determined since we know the form of the two-point correlation function. By properties of correlation functions, it is also possible to contract different operators by OPE without the result being different:

$$\begin{aligned} &\langle \overline{\mathcal{O}_1(x_1)} \overline{\mathcal{O}_2(x_2)} \overline{\mathcal{O}_3(x_4)} \overline{\mathcal{O}_4(x_4)} \rangle \\ &= \sum_{m,m'} \lambda_{14m} \lambda_{23m'} C_a(x_{14}, \partial_4) C_b(x_{23}, \partial_3) \langle \mathcal{O}_m^a(x_4) \mathcal{O}_{m'}^b(x_3) \rangle \end{aligned} \tag{240}$$

In other words, we “change the contraction channel”. The two correlation functions must be the same, eq. (239) and eq. (240) must be equal regardless of the contraction channel. We will speak of “crossing symmetry” or *crossing symmetry* in English.

Remark: We’re slightly modifying the notations we used to introduce Operator Product Expansion to make them more relevant to the current context. However, this is only a notational change; the idea remains unchanged.

3.10.3. Bootstrap equation

In order to arrive at the equation we want to present for the bootstrap, we will consider the simplest possible case and will, to do so, consider a four-point scalar correlation function for which the operators are all similar (and of dimension Δ , let’s say) and rewrite eq. (239) and eq. (240) in a much simpler form where the coefficients will all be absorbed into some functions:

$$\begin{aligned} \langle \overline{\mathcal{O}(x_1)} \mathcal{O}(x_2) \overline{\mathcal{O}(x_3)} \mathcal{O}(x_4) \rangle &= \frac{G(u, v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}}, \\ \langle \overline{\mathcal{O}(x_1)} \overline{\mathcal{O}(x_2)} \overline{\mathcal{O}(x_3)} \mathcal{O}(x_4) \rangle &= \frac{G(v, u)}{x_{14}^{2\Delta} x_{23}^{2\Delta}} \end{aligned} \quad (241)$$

where G is a function of the cross-sections u and v introduced above and also containing the constants that appear in eq. (239) and eq. (240). The OPE is “hidden” in this function G , but note that the form of this four-point scalar correlation function is also expected from the developments we made above to determine the form of the one-, two-, and three-point scalar correlation functions: we expect the four-point scalar correlation function to take a similar form but where “the set of constants determining it” is replaced by “a function of the cross-sections u and v ” (and some constants). With a little abuse of notation (because it is not entirely the case but it will be enough for us), we will speak of *conformal blocks* for these G functions. Now, as was noted in the previous subsection, these two correlation functions must be equal, so we must have that

$$\frac{G(u, v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}} \stackrel{!}{=} \frac{G(v, u)}{x_{14}^{2\Delta} x_{23}^{2\Delta}}. \quad (242)$$

We can expand the calculations to find a more appreciable expression:

$$\begin{aligned}
& \frac{G(u, v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}} = \frac{G(v, u)}{x_{14}^{2\Delta} x_{23}^{2\Delta}} \\
\iff & \frac{x_{14}^{2\Delta} x_{23}^{2\Delta}}{x_{12}^{2\Delta} x_{34}^{2\Delta}} G(u, v) = G(v, u) \\
\iff & \left(\frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} \right)^\Delta G(u, v) = G(v, u) \\
\iff & \left(\frac{v}{u} \right)^\Delta G(u, v) = G(v, u)
\end{aligned} \tag{243}$$

or even

$$\boxed{\left(\frac{v}{u} \right)^\Delta G(u, v) - G(v, u) = 0} \tag{244}$$

This little equation is the fundamental equation of the bootstrap algorithm! It has various names, sometimes we talk about the *conformal bootstrap equation* or the *crossing equation* since it comes directly from the contraction symmetries of the OPE.

Remark: We have indeed

$$\frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} \equiv \frac{\left(\frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{24}^2} \right)}{\left(\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \right)} = \frac{v}{u} \tag{245}$$

where we have, as has already been done once, used the square convention rather than the magnitude convention (so as not to drag a root around all the time...)

If we want to be a little more explicit, eq. (244) would take the form of a sum over the primary operators and would explicitly show the constant indices and coefficients. By making the contents of G explicit in eq. (241), we have an expression of the following form:

$$\langle \overbrace{\mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4)}^{\text{Sum over primary operators}} \rangle = \frac{G(v, u)}{x_{14}^{2\Delta} x_{23}^{2\Delta}} = \frac{1}{x_{14}^{2\Delta} x_{23}^{2\Delta}} \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 g_{\Delta, l}(v, u) \tag{246}$$

↑
OPE coefficients
↓
 $\lambda_{\mathcal{O}}^2$

and, applying the crossing symmetry in a very similar way to what we did for another OPE contraction channel, we finally arrive at the following equation:

$$\boxed{\sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 \left[\left(\frac{v}{u} \right)^\Delta g_{\Delta, l}(u, v) - g_{\Delta, l}(v, u) \right] = 0} \tag{247}$$

where the $g_{\Delta,l}$ are, themselves, what we call “conformal blocks”. This form is in fact eq. (244) put under a slightly more explicit form, showing the constant coefficients, the indices as well as the sum over the primary operators.

3.10.4. Idea and illustration of the bootstrap algorithm

We will not go into detail on the whole topic, as this would probably require an entire document. We will therefore limit ourselves to offering an overview of the bootstrap algorithm, but let us begin by explaining what “bootstrap” means and what is meant by it. The English expression “*to bootstrap [...]*” is always used to mean “*self - something*”, which implies an approach that is self-initiated and “self-sustaining”. In the case of conformal bootstrapping, this expression emphasizes the idea of solving a theory by imposing, through its own constraints (symmetries, observables, ...), an internal consistency, that is, without needing to use anything other than what has been presented so far: we solve a physical problem *without* Lagrangian or Hamiltonian, the theory is sufficient in itself.

The “CFT data”, which are present in the coefficients of eq. (247), are the parameters that we are trying to determine when we talk about the bootstrap procedure. These parameters, as mentioned earlier, are related to the physics that we are trying to describe (of course, we must first give ourselves a model). The algorithm then proceeds as follows in broad outline:

1. We start with an initial set of parameters (Δ, l) (if we use our previous notations);
2. We apply the equation eq. (247) on these parameters and the constraints of the physical system studied also determine an altered form of eq. (247) (for example we know that the Ising model has a symmetry \mathbb{Z}_2 and this must be taken into account);
3. The result of this iteration will provide us with the following information: a region of the parameter space is eliminated: “it does not contain a solution to the problem”;
4. We then choose a new set of parameters (Δ, l) from the region(s) not eliminated by the last iteration;
5. We start again at step 2 and stop at the desired iteration;
6. At the end of the procedure, we find ourselves with a solution space small enough to draw conclusions.

This algorithm therefore works by eliminating regions at each step and it has been shown in the founding articles that convergence is rapid. In very few iterations, it is possible to arrive at a region of the parameter space which is smaller than the uncertainty errors and which is more precise than the numerical Monte Carlo methods (which were until then the best available with regard to the Ising model for example).

This is the idea behind the bootstrap algorithm; going into detail and explicitly showing how it works is beyond the scope of this document, but we refer, for example, to Joshua D. Qualls’s *Lectures on Conformal Field Theory* for a more detailed presentation. Figures Figure 7, Figure 8 and Figure 9 graphically illustrate the method we have explained in the case of the three-dimensional Ising model.

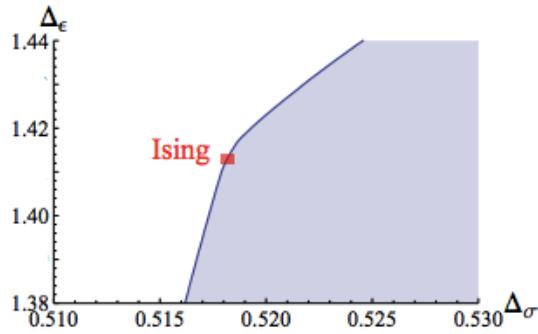


Figure 7: First iteration of the bootstrap algorithm (source: [1])

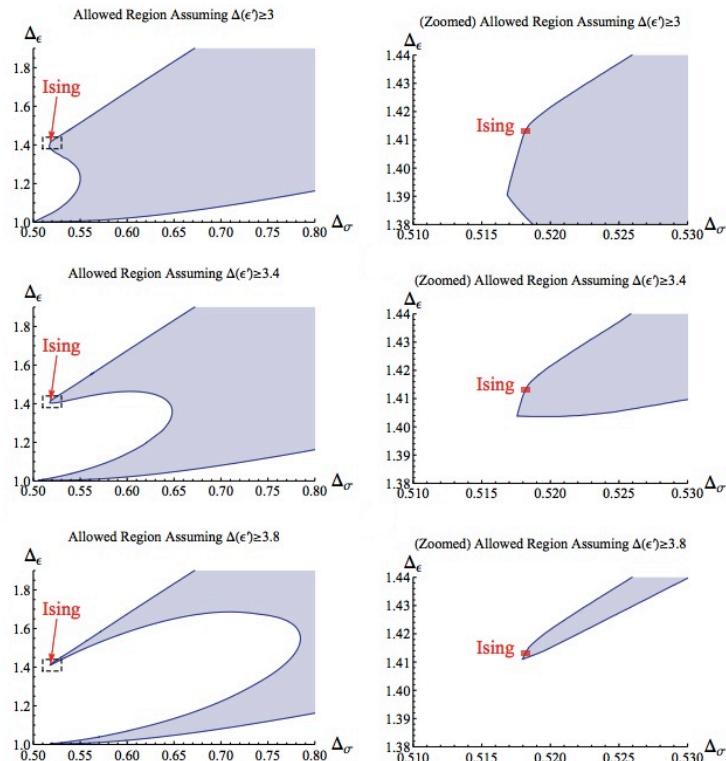


Figure 8: Several iterations of the bootstrap algorithm (source: [1]).
Reading is done column by column.

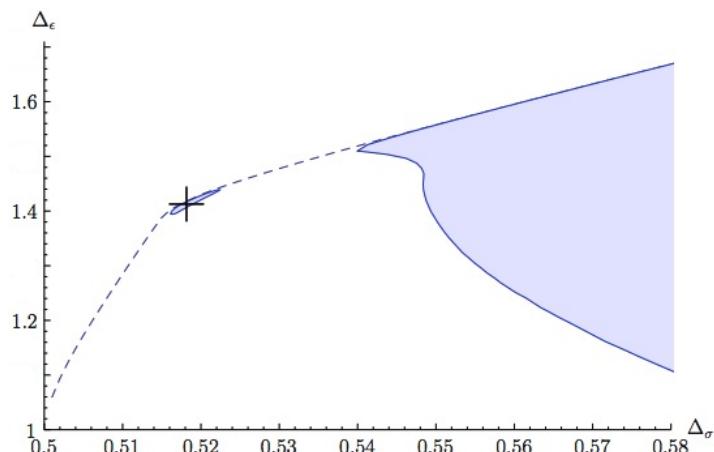


Figure 9: Isolation of a region after multiple iterations (source: [1])

After iterating a number of times, figure Figure 9 clearly illustrates that there is a region of the parameter space (here “ Δ_ϵ ” and “ Δ_σ ”) that holds up and is not eliminated from the possibility space. Error bars have been added to the graph to show that this region seems to contain the solution to the problem and that the number of iterations has been able to isolate them precisely enough. Figure Figure 9 is itself taken from the original article [2], where the authors found $\Delta_\sigma = 0.51820(14)$ and $\Delta_\epsilon = 1.4127(11)$, which are critical exponents of the three-dimensional Ising model (which are commonly expressed in terms of *scaling dimension* in field theory). The same authors gave a comparison of the bootstrap method with Monte Carlo simulations, their result is presented in figure Figure 10.

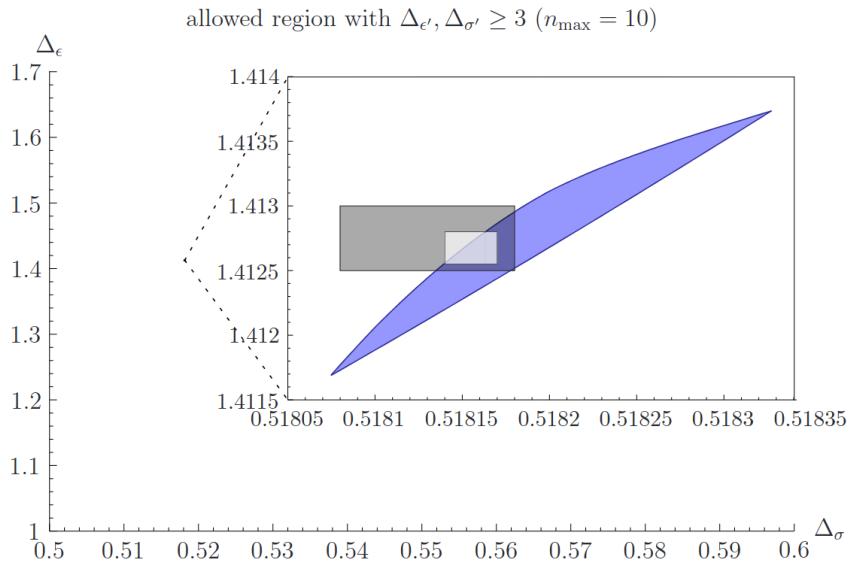


Figure 10: Comparison of numerical simulation methods (source: [2])

In figure Figure 10, the dark gray rectangle is the Monte Carlo prediction while the small light gray triangle is the bootstrap prediction. The blue region is simply a bound that the authors previously established.

To summarize, the bootstrap procedure offers the following advantages:

- There is no need to directly use Lagrangian, Hamiltonian, partition function, action, ... the CFT is sufficient to solve the problem, which is not the case for the renormalization group approach: the conformal bootstrap is simpler;
- The method is rapidly convergent;
- The method is more accurate than alternative numerical simulations.

The idea of the method is not new and dates back to the 1970s, but it experienced a resurgence of interest around 2008 when the method was successfully applied to the concrete physical case of the three-dimensional Ising model, for which we presented more recent results. However, there is still much to be done, and few other models have been studied so far.

4. Conclusion

To conclude, during the four-week internship of the 1^{er} year of the master's degree in theoretical physics at UMons, I was introduced to conformal field theory with the main guideline being the study of phase transitions via the bootstrap approach and this document, this internship report, reflects my understanding of the subject.

5. References

The work presented in this document is based on the knowledge acquired during the M1 internship in physics at the University of Mons, the references which were used during the latter are listed below.

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