1. Barrier

$S_0=100,\ \delta=0,\ \sigma=0.2,\ r=0.03$

1-yr 120-up & out put barrier 130, B-S price $=2$

• B-S price 1year 120-up and in put barrier 130

up and in put + up and out put = put

up and in put + 2 = put

$$d_1 = \frac{\ln\left(\frac{100}{120}\right)+\left(0.03+\frac{1}{2}(0.2)^2\right)(1)}{0.2\sqrt{1}} = -0.6616$$

$$d_2 = d_1 - 0.2\sqrt{1} = -0.8616$$

$$\text{put B-S} = -S\Phi(d_1) + Ke^{-rt}\Phi(-d_2)$$
$$= -100\Phi(0.6616) + 120e^{-0.03}\Phi(0.8616)$$
$$\approx -100\Phi(0.66) + 120e^{-0.03}\Phi(0.86)$$
$$\approx -100(0.7454) + 120e^{-0.03}(0.8051)$$
$$= 19.21668 \approx 19.22$$

$$19.22 - 2 = \boxed{17.22}$$

2. Barrier

$S_0=100,\ u=1.2,\ d=0.8,\ T=1,\ n=3,\ r=0.03,\ \delta=0,\ h=\frac{1}{3}$

1-yr 50-call up and out barrier 110

$(S_T-100)^+ \mathbb{1}_{\{S_0<110\}} \mathbb{1}_{\{\max_{0\le t\le T} S_t<110\}}$

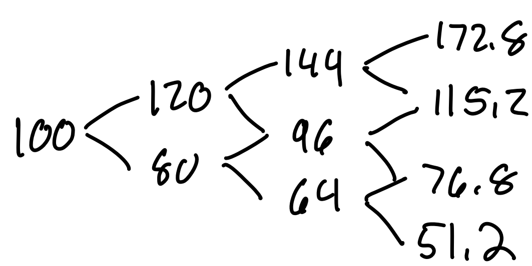
$uS_0 = 1.2(100) = 120$

$dS_0 = 0.8(100) = 80$

$u^2S_0 = 1.2^2(100) = 144$

$udS_0 = 1.2(0.8)(100) = 96$

$d^2S_0 = (0.8)^2(100) = 64$



$u^3S_0 = (1.2)^3(100) = 172.8$

$u^2dS_0 = (1.2)^2(0.8)(100) = 115.2$

$ud^2S_0 = (1.2)(0.8)^2(100) = 76.8$

$d^3S_0 = (0.8)^3(100) = 51.2$

$uuu = 0, \quad 120, 144, 172 > 110$

$uud = 0, \quad 120, 144, 115.2 > 110$

$udu = 0, \quad 120, 115 > 110$

$udd = 0, \quad 120 > 110$

$duu = 0, \quad 115.2 > 110$

$dud = (76.8 - 50) = 26.8$

$ddu = (76.8 - 50) = 26.8$

$ddd = (51.2 - 50) = 1.2$

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = \frac{e^{0.03(\frac{1}{3})} - 0.8}{1.2 - 0.8} = 0.52$$

$q^* = 1 - p^* = 1 - 0.52 = 0.48$

$$\text{price} = e^{-0.03}[26.8(q^{*2})p^* + 26.8(q^{*2})p^* + 1.2(q^{*3})]$$
$$= e^{-0.03}[26.8(0.48)^2(0.52) + 26.8(0.48)^2(0.52) + 1.2(0.48)^3]$$
$$= \boxed{6.36}$$

3. <u>Asian</u>

$S_0 = 100, \ u = 1.1, \ d = 0.9, \ T = 1, \ n = 3, \ r = 0.03, \ \delta = 0, \ k = 0$

call: $[[S(\tfrac{1}{3})S(\tfrac{2}{3})S(1)]^{\frac{1}{3}} - 100]^+$

$h = \frac{T}{n} = \frac{1}{3}$

$uS_0 = 110$

$dS_0 = 90$

$u^2S_0 = 121$

$udS_0 = 99$

$d^2S_0 = 81$

$u^3S_0 = 133.1$

$u^2dS_0 = 108.9$

$ud^2S_0 = 89.1$

$d^3S_0 = 72.9$

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & 1/3 & 2/3 & 3/3 \\
 & & & 133.1 \\
 & & 121 \arrow[ur] \arrow[dr] & \\
 & 110 \arrow[ur] \arrow[dr] & & 108.9 \\
100 \arrow[ur] \arrow[dr] & & 99 \arrow[ur] \arrow[dr] & \\
 & 90 \arrow[ur] \arrow[dr] & & 89.1 \\
 & & 81 \arrow[ur] \arrow[dr] & \\
 & & & 72.9
\end{tikzcd}$$

$C_{uuu} = ((110 \cdot 121 \cdot 133.1)^{\frac{1}{3}} - 100) = (120.97 - 100) = 20.97 \quad p^{*3}$

$C_{uud} = ((110 \cdot 121 \cdot 108.9)^{\frac{1}{3}} - 100) = (113.17 - 100) = 13.17 \quad p^{*2}q$

$C_{udu} = ((110 \cdot 99 \cdot 108.9)^{\frac{1}{3}} - 100) = (105.85 - 100) = 5.85 \quad p^{*}q^{*}p^{*}$

$C_{udd} = ((110 \cdot 99 \cdot 89.1)^{1/3} - 100) = (99 - 100) = 0 \quad q^{*}p^{*2}$

$C_{duu} = ((90 \cdot 99 \cdot 108.9)^{1/3} - 100) = (99 - 100) = 0 \quad q^{*2}p^{*}$

$C_{dud} = ((90 \cdot 99 \cdot 89.1)^{1/3} - 100) = (92.59 - 100) = 0 \quad q^{*}p^{*}q^{*}$

$C_{ddu} = ((90 \cdot 81 \cdot 89.1)^{1/3} - 100) = (86.60 - 100) = 0 \quad q^{*2}p^{*}$

$C_{ddd} \quad ((90 + 81 + 72.9)^{1/3} - 100) = (81 - 100) = 0 \quad q^{*3}$

$p^{*} = \dfrac{e^{0.03(\frac{1}{3})} - 0.9}{1.1 - 0.9} = 0.56 \qquad q^{*} = 1 - p^{*} = 1 - 0.55 = 0.45$

price is $e^{-0.03(1)}[20.97(p^{*3}) + 13.17(p^{*2})(q^{*}) + 5.85p^{*}q^{*}p^{*}]$

$e^{-0.03(1)}[20.97(0.55)^3 + 13.17(0.55)^2(0.45) + 5.85(0.55)(0.45)(0.55)]$

$= 5.89834 \approx \boxed{5.90}$

4. <u>Chooser</u>

$T=2,\ t_0=1,\ S_0=10,\ \sigma=0.2,\ r=0.05,\ K=9$

$C(0,S,K,T)=S\Phi(d_1)-Ke^{-rT}\Phi(d_2)$

$P(0,S,K,T)=-S\Phi(-d_1)+Ke^{-rT}\Phi(-d_2)$

for call:

$C(0,10,9,2)\ ,\ d_1=\dfrac{\ln(\frac{10}{9})+(0.05+\frac{1}{2}(0.2)^2)(2)}{0.2\sqrt{2}}=0.87$

$d_2=0.87-0.2\sqrt{2}=0.58$

$C(0,10,9,2)=10\Phi(0.87)-9e^{-0.1}\Phi(0.58)$ ← using table

$=10(0.8078)-9e^{-0.1}(0.7190)$

$=2.22279\approx 2.22$

for put:

$p(0,10,9e^{-0.05},1)\ ,\ d_1=\dfrac{\ln(\frac{10}{9e^{-0.05}})+[0.05+\frac{1}{2}(0.2)^2](1)}{0.2\sqrt{1}}=1.12$

$d_2=1.12-0.2\sqrt{1}=0.92$

$p(0,10,9e^{-0.05},1)=-10\Phi(-1.12)+9e^{-0.05}e^{-0.05(2)}\Phi(-0.92)$

$=-10(0.1335)+9e^{-0.05}e^{-0.05}(0.1814)$

$=0.1422\approx 0.14$

Value of Chooser:

$2.22+0.14=\boxed{2.36}$

5. $dS_t = rS_t dt + \sigma S_t dB_t$, $S_0 = S$ where r and σ are constants

Let $n \geq 2$ be any positive integer, K be any positive number, we want to find $E[e^{-rT}[(S_T)^n - K]^+]$

Since σ and r are constants, we can apply ito's formula and get that the Geometric Brownian motion

$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma B_t} \Rightarrow (S_t)^n = (S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma B_t})^n$

so $E[e^{-rT}[(S_0 e^{(r-\frac{1}{2}\sigma^2)t + \sigma B_t})^n - K]^+]$

now, $e^{-rT} E[(S_t^n - K)^+] = e^{-rt} E[(S_t^n - K)1_{\{S_t > K\}}]$

$= e^{-rT} P(S_t^n > K) E[S_T^n]$

$= e^{-rT} E[S_T^n] \cdot P(S_t^n > K)$

$P(S_t^n > K) = P((Se^{(r-\frac{1}{2}\sigma^2)T + \sigma B_t})^n > K)$

$= P((e^{(r-\frac{1}{2}\sigma^2)T + \sigma B_t})^n > \frac{K}{S})$

$= P([(r-\frac{1}{2}\sigma^2)T + \sigma B_t]n > \ln(\frac{K}{S^n}))$

$= P(\sigma B_t n > \ln(\frac{K}{S^n}) - [(r-\frac{1}{2}\sigma^2)T])$

$= P(Z > \frac{\ln(\frac{K}{S^n}) - (r-\frac{1}{2}\sigma^2)Tn}{\sigma\sqrt{T}})$

$= P(Z < \frac{\ln(\frac{S^n}{K}) - (r-\frac{1}{2}\sigma^2)Tn}{\sigma\sqrt{T} \cdot n})$

$= \Phi(\frac{\ln(\frac{S^n}{K}) - (r-\frac{1}{2}\sigma^2)Tn}{\sigma\sqrt{T} \cdot n}) = \Phi(d_2)$

$\Rightarrow$ price is $e^{-rT} E[(S_t)^n] \cdot P(S_t^n > K)$

$= e^{-rT} S_t^n e^{(r-\frac{1}{2}\sigma^2)Tn} \Phi(d_2)$

6. • Δ call: <u>positive</u>

Since $\Delta = V_x(t,x) = \Phi(d_1)$ and we know $\Delta = \frac{\partial v}{\partial x}$. For a call $V(t,x) = (x-k)^+$, it's partial derivative with respect to x $\frac{\partial v}{\partial x} = \Phi(d_1)$. Since $\Phi(d_1)$ is CDF for the normal dist. any cumulative probability must be greater than zero.

• Δ put: <u>negative</u>

When $V(x,t) = -x\Phi(-d_1) + ke^{-rt}\Phi(-d_2)$ for a put $V(x,t) = (x-s)^+$
$\Delta = V_x(t,x) = \frac{\partial v}{\partial x} = -\Phi(-d_1) = -1 \cdot \Phi(-d_1)$ so a negative value is always returned because any positive cdf multiplied by -1 is always negative. $V'(t,x) = \Phi(d_1) + 0$

• Γ call: <u>positive</u>

$\Gamma = \frac{\partial^2 v}{\partial x^2} = \frac{\partial \Delta}{\partial x}$ since $\Delta = \frac{\partial v}{\partial x}$, so $\frac{\partial \Delta}{\partial x} = \frac{\partial}{\partial x}\Phi(d_1)$.
Since $\Phi(d_1)$ must be positive bc normal CDF, the partial differentiation is a positive pdf. Thus, Γ call is non-negative.

• Γ put: <u>negative</u>

$\Gamma = \frac{\partial^2 v}{\partial x^2} = \frac{\partial \Delta}{\partial x}$ since $\Delta = \frac{\partial v}{\partial x}$, so we have $\frac{\partial \Delta}{\partial x} = \frac{\partial}{\partial x}(-1)(-d_1)$
Since $(-1)\Phi(-d_1)$ is always negative $(-1 \cdot 1 = -1)$, the partial differentiation is a positive pdf multiplied by -1, therefore Γ put is negative

7.

$$\Omega := V_x \cdot \frac{x}{V} = \frac{\Delta x}{V}$$

$$\Omega = \frac{\frac{\partial V}{V}}{\frac{\partial x}{x}} = \frac{\partial V}{\partial x} \frac{x}{V}$$

- for a no dividend call, we know that Δ is positive. from the black scholes equation we have

$$V = x\Phi(d_1) - Ke^{-rt}\Phi(d_2) \le x\Phi(d_1) = \Delta x \Rightarrow V < x\Phi(d_1) = \Delta x$$

$$\Rightarrow 1 \le \Delta = \frac{x}{s\Phi(d_1) - Ke^{-rt}\Phi(d_2)} = \frac{\Delta x}{V} \qquad \hookrightarrow V \le \Delta x \Rightarrow 1 \le \frac{\Delta x}{V}$$

$$\Rightarrow 1 \le \frac{\Delta x}{V} = \Omega$$

$$\Rightarrow 1 \le \Omega$$

thus, since delta is positive and less than Δx, our numerator is greater than denom and is always positive

- for a no dividend put, we know that Δ is negative from proof in problem 6, so $\Omega = \frac{\Delta x}{V} < 0$. From black scholes equation,

$V = -x\Phi(-d_1) + Ke^{-rt}\Phi(-d_1) \ge -x\Phi(-d_1) = \Delta x$ so, we can divide V on both sides to get $1 \ge \frac{\Delta x}{V}$. So we know that Ω must be less than or equal to 1 (meaning numerator $\Delta x \le V$. Since Δ is negative for a no-dividend put, $\frac{\Delta x}{V}$ must be negative. Thuss less than zero. Therefore, $\Omega \le 0$

8. $dX_t = [\sqrt{1+X_t^2} + \frac{1}{2} X_t]dt + \sqrt{1+X_t^2}\, dB_t = 0$

has the form $dX_t = b(X_t)\,dt + \sigma\, dB_t$, $X_0 = 0$

Let's find an equation to use for Ito's formula

$dG_t = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2} f_{xx}(t, X_t)(dX_t)^2$

let $G_t = f(t, X_t) = \sinh^{-1}(X_t) - t \Rightarrow \frac{d}{dx}\sinh^{-1}(X_t) = \frac{1}{\sqrt{1+X_t^2}}$.

$$dG_t = \frac{1}{\sqrt{1+X_t^2}}dX_t - \frac{1}{2}\left(\frac{X_t}{\sqrt{(1+X_t^2)^3}}\right)(X_t^2+1) - dt$$

$$= \frac{[(\sqrt{1+X_t^2} + \frac{1}{2}X_t)dt + \sqrt{1+X_t^2}\, dB_t]}{\sqrt{1+X_t^2}} - \frac{X_t}{2\sqrt{1+X_t^2}}dt$$

$$= \frac{\cancel{\sqrt{1+X_t^2}}dt}{\cancel{\sqrt{1+X_t^2}}} + \frac{X_t\, dt}{2\sqrt{1+X_t^2}} + \frac{\cancel{\sqrt{1+X_t^2}}}{\cancel{\sqrt{1+X_t^2}}}dB_t - \frac{X_t}{2\sqrt{1+X_t^2}}dt$$

$$= \cancel{dt} + \cancel{\frac{X}{2\sqrt{1+X_t^2}}dt} + dB_t - \cancel{\frac{X_t}{2\sqrt{1+X_t^2}}dt}$$

$$= dB_t$$

$G_t = G_0 + B_t = \int_0^t dB_s + \int_0^t ds$

when $G_t = G_0 + B_t = f(t, X_t) = \sinh^{-1}(X_t + t)$ we want $X_t = \sinh(X_0 + B_t) - t$. So we are left with $f(t, X_t) = \sin^{-1}(\sin(X_0) + B_t) + t = X_0 + B_t$. Thus, $X_t = \sinh(\sinh^{-1}(X_0) + T + B_t)$