Bayesian Data Analysis

Vanda Inácio & Ken Newman

University of Edinburgh

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General information

- **Lecturers**: Vanda Inácio & Ken Newman.
- **Email**: `vanda.inacio@ed.ac.uk` and `ken.newman@bioss.ac.uk`.
- **Office**: 4601 (Vanda) and 3617 (Ken), JCMB.
- **Lectures and location**: Friday, 13:00-15:00, Swann 7.15, odd weeks.
- **Computer labs**: Friday, 13:00-15:00, KB Centre, level 3, even weeks.

Assessment

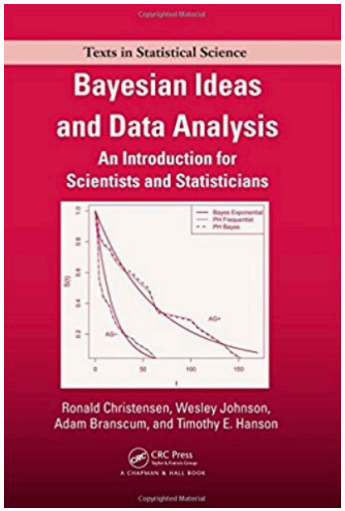
- ↪ 100% coursework: two homework assignments.
- ↪ Homework assignment 1 (50%) will be delivered in week 5. Solutions should be handed in week 7.
- ↪ Homework assignment 2 (50%) will be delivered in week 10. Solutions should be handed in week 12.
- ↪ Both homework assignments are to be done individually and consist of a sample of applied problems requiring the use of statistical software. A report containing all the analyses and conclusions should be delivered.

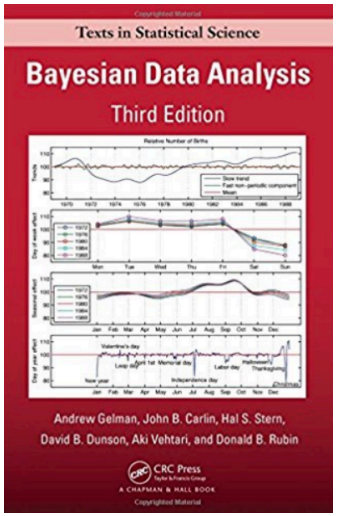
Scope

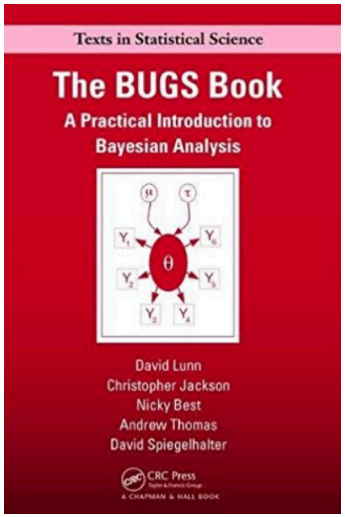
- The goal of this course is to provide practical experience of applying Bayesian analyses to a range of statistical models.
- The statistical analyses will be conducted using the widely used computer package WinBUGS.
- Topics:
  - Brief overview of main Bayesian ideas.
  - Introduction to WinBUGS.
  - Linear and generalised linear models (fixed effects).
  - Hierarchical Bayesian models: linear and generalised linear models with random effects.
  - Further topics (which might include: models with missing data, measurement error, mixture models).

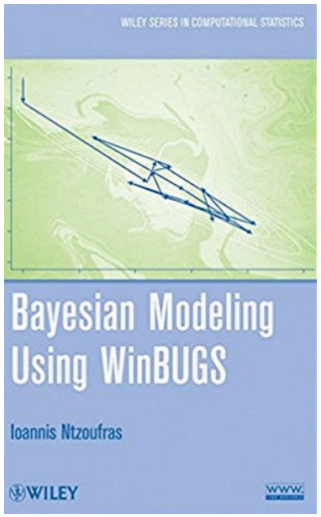
Recommended textbooks

- The course material (slides, worksheets, and other support material) contain all the information needed for the course.
- However, there is a wide variety of books, at a wide range of levels, that cover all or part of the material within this course that may be of interest to supplement the provided material and/or provide additional examples.
- Personally, we like



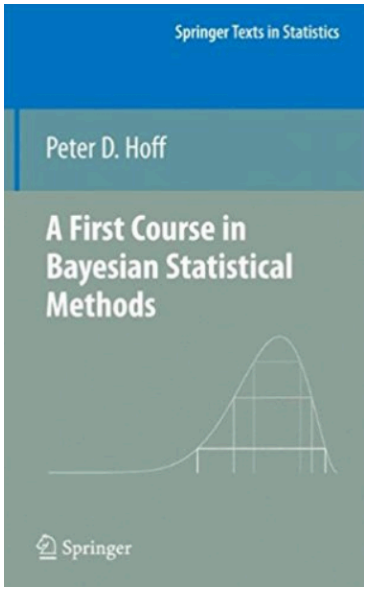


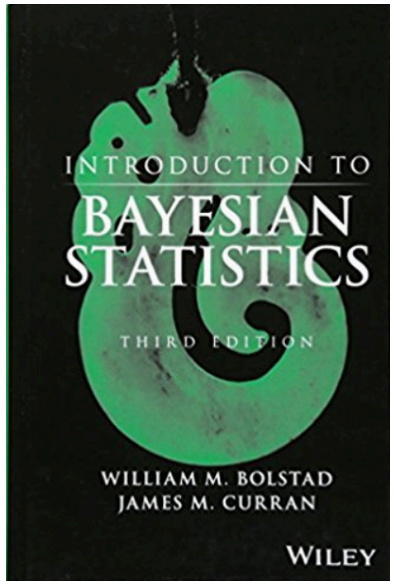


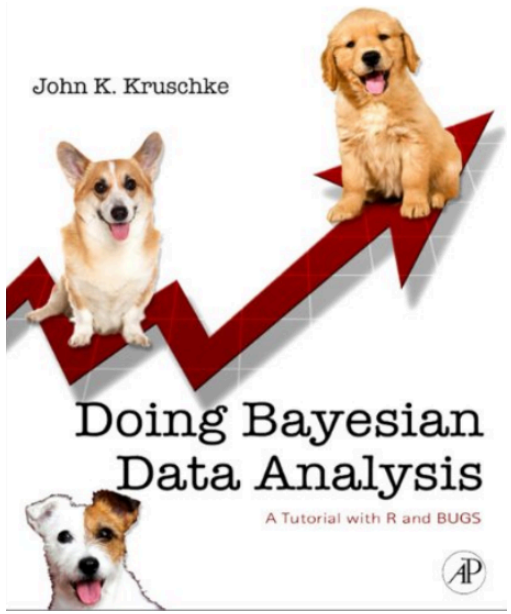


Additional textbooks

Popular introductory texts

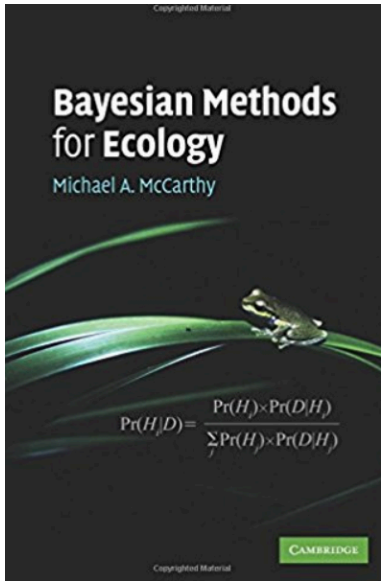


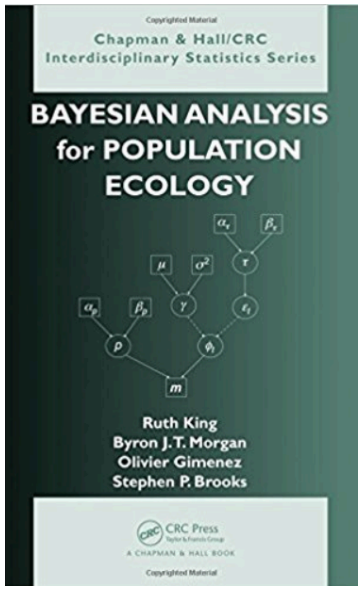


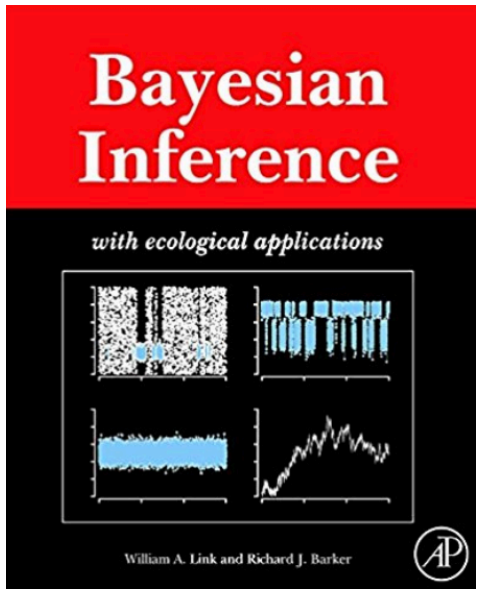


Additional textbooks

Ecological applications (covers from `amazon.co.uk`)

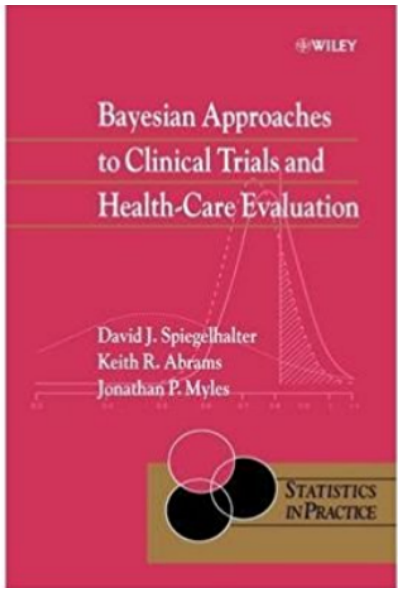


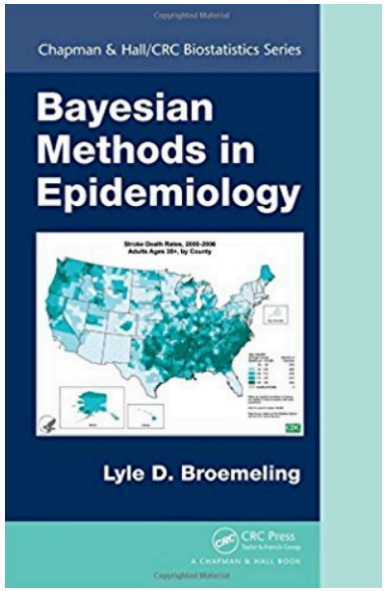


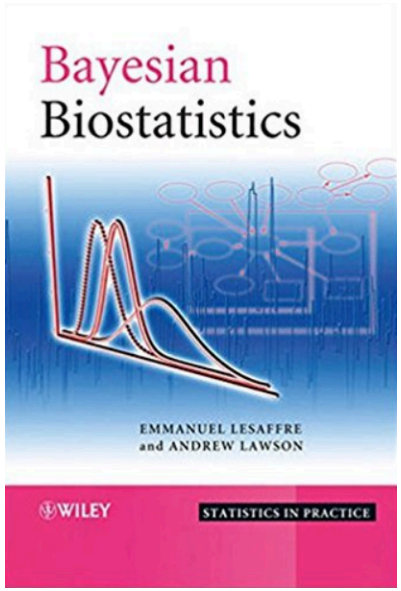


Additional textbooks

Medical/epidemiological applications (covers from amazon.co.uk)

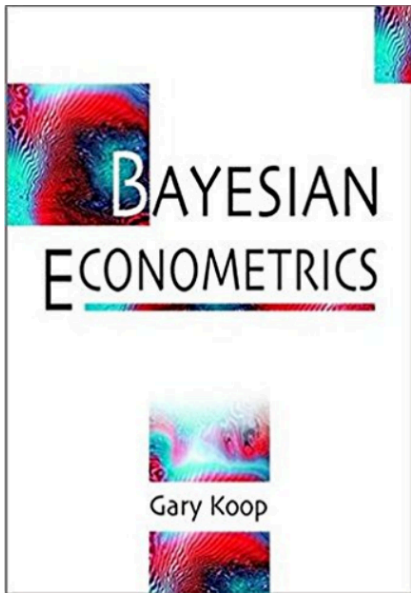


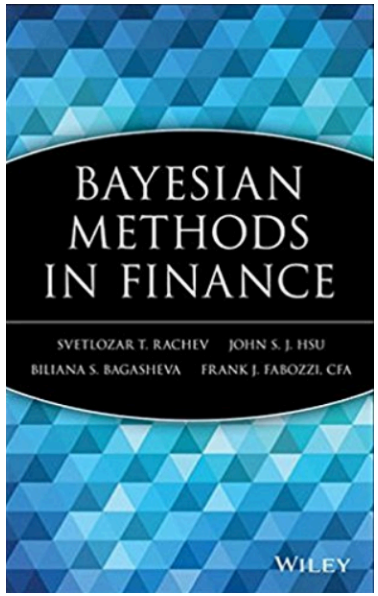


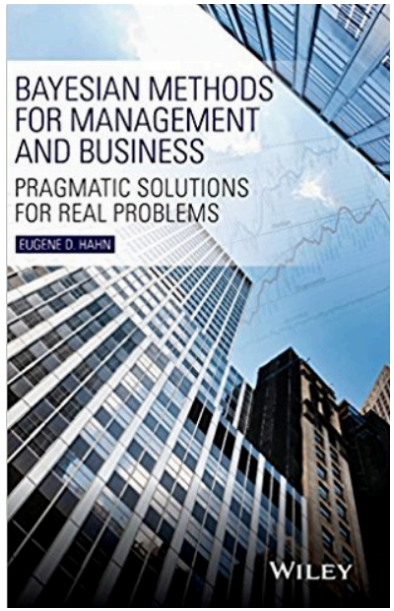


Additional textbooks

Finance/business/econometric applications (covers from amazon.co.uk)

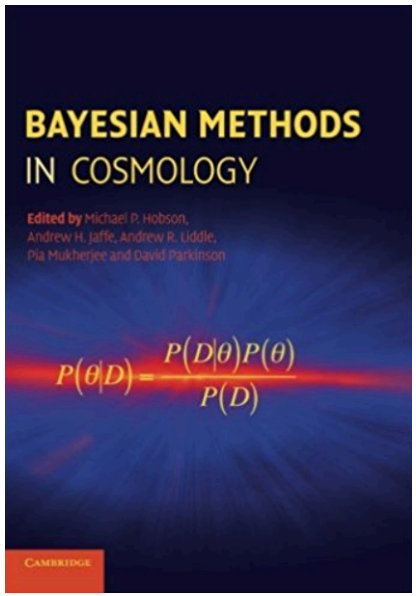


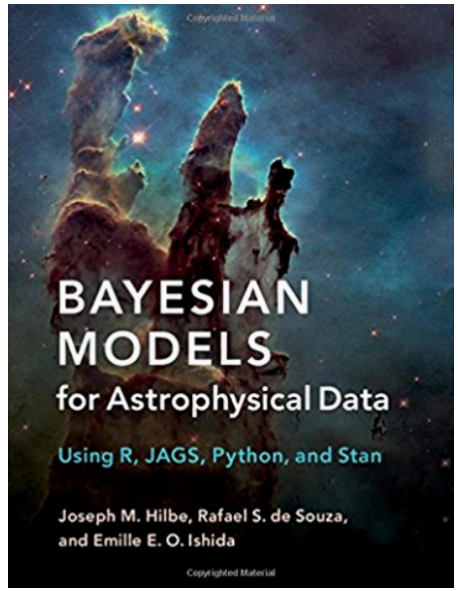




Additional textbooks

Astrophysics/cosmology applications (covers from `amazon.co.uk`)





Review of Bayesian inference

- Bayesian methods have been widely applied in many areas of science (e.g., medicine, finance, ecology, physics, psychology, etc).
- Motivations for adopting Bayesian approach vary:
  - natural and coherent way of thinking about science and learning,
  - pragmatic choice that is suitable for the problem in hand.
- Spiegehalter et al. (2004) define a Bayesian approach as

  'the explicit use of external information in the design, monitoring, analysis, interpretation, and reporting of a [scientific investigation].'

- These authors argue that a Bayesian approach is
  - more flexible in adapting to each unique situation,
  - more efficient in using all available evidence,
  - more useful in providing relevant quantitative summaries,

  than traditional methods.

Review of Bayesian inference

Bayesian vs frequentist statistics

- The **frequentist** approach can be regarded as a procedure that quantifies uncertainty (p-value, confidence interval, etc) in terms of repeating the process that generated the data many times.
- Parameters are fixed and unknown, on the data is random.
- Aims to be objective.
- **Bayesian** represent their uncertainty about parameters with probability distributions and treat them as random variables.
- We can thus, under a Bayesian framework, make probability statements about model parameters.
- This is in contrast with the frequentist framework where probability statements only concern the data.

- Both approaches have pros and cons. When both are applicable they are unlikely to give different answers.

Review of Bayesian inference

Main components

- Suppose we have a parameter $\theta \in \Theta$ on which we wish to make inference.
- The main 'ingredients' of Bayesian inference are:
  - The prior distribution, $p(\theta)$, which represents the initial beliefs concerning the parameter prior to any data being observed.
  - The likelihood $f(\mathbf{y} \mid \theta)$, which plays a key role in statistical inference, Bayesian or not. Represents the information contained in the data $\mathbf{y}$ about the parameter θ.
  - The posterior distribution, $p(\theta \mid \mathbf{y})$, which updates the prior beliefs with respect to θ, following the data $\mathbf{y}$ being observed.
- Bayesian learning combines past experience (prior) with new data (likelihood) in a mathematically coherent way (Bayes' Theorem) to form the current state of knowledge (posterior).

Review of Bayesian inference

Bayes' theorem

↪ Remember that Bayes' theorem (continuous version) tells us

$$p(\theta \mid \mathbf{y}) = \frac{f(\mathbf{y} \mid \theta)p(\theta)}{f(\mathbf{y})},$$

where $f(\mathbf{y}) = \int_{\Theta} f(\mathbf{y} \mid \theta)p(\theta)\mathrm{d}\theta$ is the marginal distribution data of the data.

↪ Since the marginal distribution does not depend on θ (we are integrating it out), we can write

$$p(\theta \mid \mathbf{y}) \propto f(\mathbf{y} \mid \theta)p(\theta),$$

i.e., the posterior distribution is proportional to the likelihood times the prior distribution.

Review of Bayesian inference

Summarising posterior distributions

- All Inference about the parameter of interest θ is based on the posterior distribution.
- The information contained in the posterior distribution can be summarised in different ways as appropriate to the inference goal, e.g.
  - Means, standard deviations, medians.
  - Probability of exceeding a certain threshold, say θ_0, $\Pr(\theta > \theta_0 \mid \mathbf{y})$.
  - Credibility intervals.

Review of Bayesian inference

Decision theory

- ↪ What is the 'best' one-number summary of the posterior to be used as an estimator?
- ↪ This depends on the situation, and in particular, on the penalty associated with different types of errors (e.g., maybe overestimation is way worse than underestimation).
- ↪ Decision theory is used to form estimators with good properties.
- ↪ Let $L(\theta, a)$ be the loss associated with using a as the estimate, when the true value is θ. Note that for simplicity we only write a but, in fact, it is $a(\mathbf{y})$.
- ↪ The corresponding Bayes estimator is then chosen to minimise the expectation of the loss function with respect to the posterior distribution, i.e. the posterior expected loss.
- ↪ Mathematically, the Bayes estimate, $\widehat{\theta}$, is defined such that

$$\begin{aligned}\widehat{\theta} &= \min_{a \in \Theta} \mathbb{E}_{\text{post.}} \left[L(\theta, a) \right] \\ &= \min_{a \in \Theta} \left[\int_{\theta \in \Theta} L(\theta, a) p(\theta \mid \mathbf{y}) \mathrm{d}\theta \right].\end{aligned}$$

Review of Bayesian inference

Decision theory

↪ Three commonly used loss functions and corresponding Bayes estimates are:

Loss	Bayes estimate
$L(\theta, a) = (\theta - a)^2$ (quadratic loss)	$\widehat{\theta} = \mathbf{E}_{\text{post.}}(\theta) = \int_{\theta \in \Theta} \theta p(\theta \mid \mathbf{y}) \mathrm{d}\theta$
$L(\theta, a) = \lvert\theta - a\rvert$ (absolute error loss)	$\widehat{\theta} = \text{median}_{\text{post}}(\theta)$
$L(\theta, a) = I(\theta \neq a)$ (zero/one loss)	$\widehat{\theta} = \arg\max_{\theta} p(\theta \mid \mathbf{y})$

↪ Appropriate loss functions for hypothesis testing and interval estimation do also exist.

Choice of prior distribution

- Picking the prior is obviously important and uniquely Bayesian.
- Some types of priors include (these categories are not mutually exclusive):
  - Informative/expert priors
  - Non-informative/vague priors
  - Conjugate priors
- It is also important to try several priors in a sensitivity analysis.

Choice of prior distribution

Informative/expert priors

- A major advantage of the Bayesian approach is the ability to include expert prior information.
- This can come from either an expert in the field or from past data (literature, pilot study, etc).
- We call elicitation to the process of extracting prior knowledge in a suitable manner to permit the formulation of a suitable prior distribution that represents the expert/historical information as accurately as possible.
- Spiegelhalter et al. (2004), sections 5.2, 5.3, and 5.4, contain a good discussion on how priors might be elicited from experts or historical data.

Choice of prior distribution

Informative/expert priors

- What about if we ask more than one expert and they don't agree?
- Suppose we are interested in a quantity θ and that expert j recommends prior $\theta \sim \mathrm{N}(\mu_j, \sigma_j)$.
- We could weight the experts using a mixture model

$$p(\theta) = \sum_{j=1}^{J} \omega_j \mathrm{N}(\theta \mid \mu_j, \sigma_j^2),$$

where ω_j is the weight given to expert j ($\sum_{j=1}^{J} \omega_j = 1$).

Choice of prior distribution

Non-informative/vague priors

- What should we do if we do not have any prior information concerning the parameter of interest?
- Bayes himself suggested that when this is the case, the Uniform prior should be used, so that $p(\theta) = c$, for all θ.
- In this case we clearly have

$$p(\theta \mid \mathbf{y}) \propto f(\mathbf{y} \mid \theta),$$

  i.e., the posterior distribution has the same shape as the likelihood function.
- In this case the mode of the posterior distribution coincides with the MLE.
- If the parameter space Θ is unbounded, this prior will be improper (regardless the value of c), that is, $\int_\Theta p(\theta) \mathrm{d}\theta = \infty$. Improper priors can be used but we must check if the resulting posterior is proper (i.e., if it integrates to one).

Choice of prior distribution

Non-informative/vague priors

- As argued by Raiffa and Schlaiffer (1961), if one is ignorant about θ, one should also be ignorant about θ^2, and one cannot find a distribution that is uniform on both θ and θ^2.
- To see this, suppose that we place a Uniform prior on $\theta \in [0, 1]$, so that $p(\theta) = 1$. The corresponding prior on $\psi = \theta^2$ is $p(\psi) = \frac{1}{2\sqrt{\psi}}$, which is obviously non-uniform on ψ.
- In practice, priors should be specified on the parameter that the statistician is interested in within the analysis.

Choice of prior distribution

Non-informative/vague priors: Jeffreys' prior

- Jeffreys (1961) proposed a class of priors that are invariant to transformations.
- The Jeffrey's prior is $p(\theta) \propto \sqrt{I(\theta)}$, where $I(\theta)$ is the expected Fisher information.
- Remember that

$$I(\theta) = \mathbf{E}\left(\frac{\mathrm{d}}{\mathrm{d}\theta} \log f(\mathbf{Y} \mid \theta)\right)^2,$$

  which in regular cases equals to $I(\theta) = -\mathbf{E}\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \log f(\mathbf{Y} \mid \theta)\right)$.

- For the case of a multivariate parameter vector, say $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)^T$, Jeffreys' prior is given by

$$p(\boldsymbol{\theta}) \propto \sqrt{\det I(\boldsymbol{\theta})},$$

  with $I(\boldsymbol{\theta}) = -\mathbb{E}\left(\frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T} \log f(\mathbf{Y} \mid \boldsymbol{\theta})\right)$.

Choice of prior distribution

Non-informative/vague priors: Jeffreys' prior

- Let us exemplify the calculation of Jeffreys' prior on binomial data.
- To do that, let us suppose that $Y \sim \text{Bin}(n, \theta)$, where Y denotes the number of 'successes' out of n trials, and where the probability of success is θ.
- The likelihood is

$$f(y \mid \theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y},$$

and the log likelihood is

$$\log f(y \mid \theta) = C + y \log \theta + (n-y) \log(1-\theta).$$

- The first derivative is

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log f(y \mid \theta) = \frac{y}{\theta} - \frac{n-y}{1-\theta},$$

and the second derivative is

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \log f(y \mid \theta) = -\frac{y}{\theta^2} - \frac{n-y}{(1-\theta)^2}.$$

Choice of prior distribution

Non-informative/vague priors: Jeffreys' prior

- Thus,

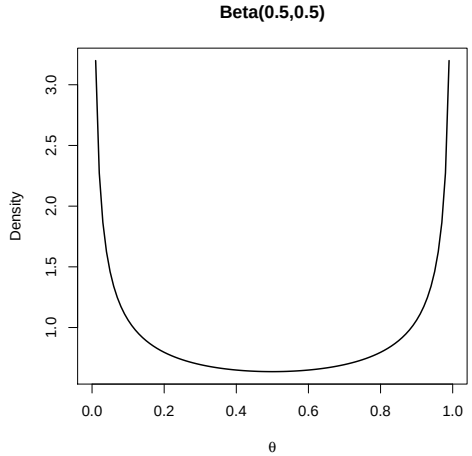
$$I(\theta) = -\mathbb{E}\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\log f(Y \mid \theta)\right) = \frac{1}{\theta^2}\mathbb{E}(Y) + \frac{1}{(1-\theta)^2}(n - \mathbb{E}(Y)).$$

- Remember that $Y \sim \text{Bin}(n, \theta)$, implies $E(Y) = n\theta$.

- Therefore,

$$I(\theta) = \frac{n}{\theta(1-\theta)}.$$

- We can then conclude that Jefrreys's prior is $p(\theta) \propto \theta^{-1/2}(1-\theta)^{-1/2}$, which is a Beta$(1/2, 1/2)$ distributions and gives greater plausibility to values near 0 and 1 than to values in between (see figure below).



Choice of prior distribution

Non-informative/vague priors: Jeffreys' prior

↪ Let us now compute Jeffreys' prior for a normal mean θ (assuming that the variance σ^2 is known).

↪ With $Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} \text{N}(\theta, \sigma^2)$, the likelihood is

$$f(\mathbf{y} \mid \theta) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{\sigma^2}(y_i - \theta)^2 \right\} \right]$$
$$= (2\pi\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \theta)^2 \right\},$$

and the corresponding log likelihood is

$$\log f(\mathbf{y} \mid \theta) = C - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \theta)^2.$$

Choice of prior distribution

Non-informative/vague priors: Jeffreys' prior

↪ The first and second derivatives are, respectively

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log f(\mathbf{y} \mid \theta) = \frac{n}{\sigma^2}(\bar{y} - \theta),$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \log f(\mathbf{y} \mid \theta) = -\frac{n}{\sigma^2}.$$

↪ The expected Fisher information is then $I(\theta) = \frac{n}{\sigma^2}$, which does not depend on θ, thus implying that $p(\theta) \propto 1, \forall\theta$ ($\Rightarrow$ improper prior).

Choice of prior distribution

Non-informative/vague priors: Jeffreys' prior

- We will now consider the Poisson case. Let $Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} \text{Poisson}(\theta)$, $\theta > 0$.
- The likelihood and log likelihood are, respectively, given by

$$f(\mathbf{y} \mid \theta) = \prod_{i=1}^{n} \left\{ \frac{e^{-\theta} \theta^{y_i}}{y_i!} \right\}$$
$$= \frac{e^{-n\theta} \theta^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!},$$

and

$$\log f(\mathbf{y} \mid \theta) = -n\theta + \sum_{i=1}^{n} y_i \log \theta + C.$$

- The first derivative is

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log f(\mathbf{y} \mid \theta) = -n + \frac{1}{\theta} \sum_{i=1}^{n} y_i,$$

while the second derivative is

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \log f(\mathbf{y} \mid \theta) = -\frac{\sum_{i=1}^{n} y_i}{\theta^2}.$$

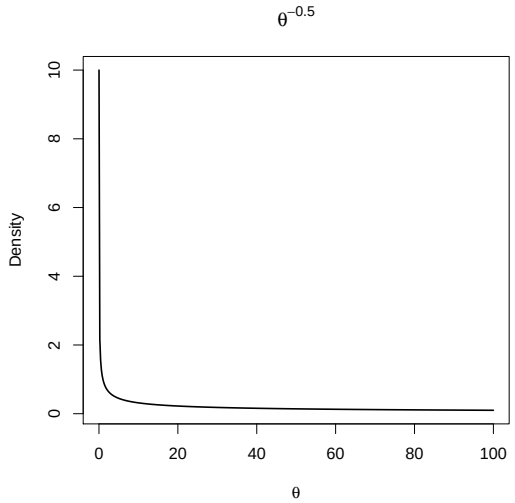
Choice of prior distribution

Non-informative/vague priors: Jeffreys' prior

↪ Thus, the expected Fisher information is

$$I(\theta) = \frac{1}{\theta^2} n\mathbb{E}[Y] = \frac{n}{\theta},$$

implying that $p(\theta) \propto \theta^{-1/2}$.



↪ Jeffreys' prior is improper in this case, but can be approximated by a Gamma distribution with parameters $\alpha = 1/2$ and $\beta \to 0$.

Choice of prior distribution

Non-informative/vague priors: Jeffreys' prior

- ↪ Jeffreys' prior is objective in that there is no prior tuning. It is a means of constructing a prior in the absence of prior information.
- ↪ Although Jeffreys' prior has the desirable property of being invariant to reparameterisations it can lead to improper priors.
- ↪ Alternative vague or non-informative prior distributions often have a reasonable mean for the distribution, but with a large variance parameter.
- ↪ Several different priors may be considered, each of which may be described to be vague or non-informative, and the sensitivity of the posterior on these priors investigated.

Choice of prior distribution

Conjugate priors

- A conjugate prior leads to a posterior from the same parametric family as the prior.
- There are long lists of conjugacies that we should be aware of

  `https://en.wikipedia.org/wiki/Conjugate_prior`

- Conjugate priors are used often for computational convenience because the posterior has a closed form.
- In fancier models, conjugate priors facilitate Gibbs sampling which is the easiest Bayesian computational algorithm.

Choice of prior distribution

Conjugate priors: beta-binomial model

↪ Let us suppose $Y \sim \text{Bin}(n, \theta)$, whose likelihood is

$$f(y \mid \theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}.$$

↪ It is mathematically convenient to use a Beta(a, b) prior distribution for θ because it has a similar form to the binomial likelihood. Its density function is

$$p(\theta) = \frac{1}{B(a,b)} \theta^{a-1}(1-\theta)^{b-1} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1}.$$

↪ If $\theta \sim \text{Beta}(a, b)$, then

$$\mathbb{E} = \frac{a}{a+b},$$

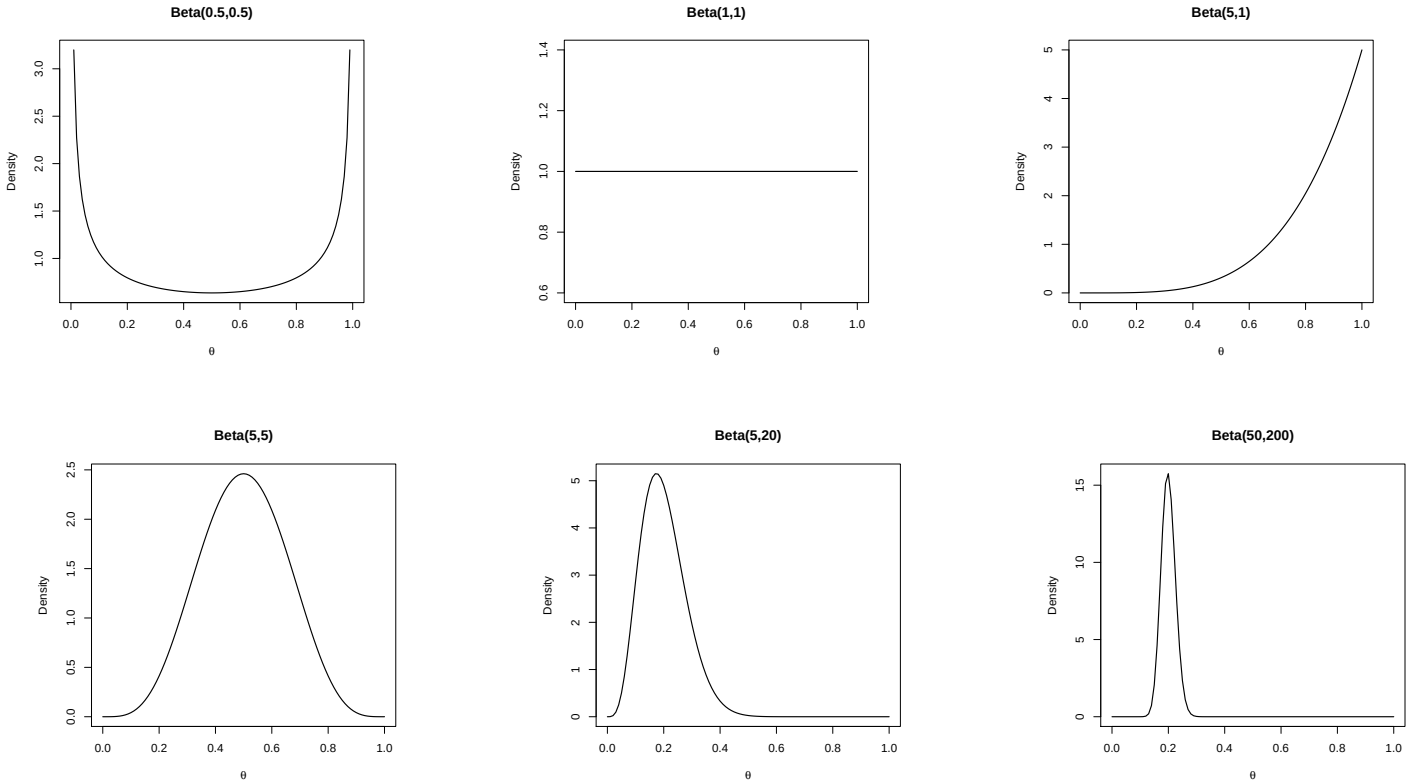
and variance

$$\text{var}(\theta) = \frac{ab}{(a+b)^2(a+b+1)}.$$

Choice of prior distribution

Conjugate priors: beta-binomial model

↪ The beta distribution can take several different shapes.



Choice of prior distribution

Conjugate priors: beta-binomial model

↪ Combining the beta prior distribution for θ with the binomial likelihood results in the following posterior distribution

$$\begin{aligned} p(\theta \mid y) &\propto f(y \mid \theta) p(\theta) \\ &= \binom{n}{y} \theta^y (1-\theta)^{n-y} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1-\theta)^{b-1} \\ &\propto \theta^{a+y-1} (1-\theta)^{b+n-y-1}. \end{aligned}$$

↪ That is, the posterior distribution turns out to be another Beta distribution

$$\theta \mid y \sim \text{Beta}(a+y, b+n-y)$$

Choice of prior distribution

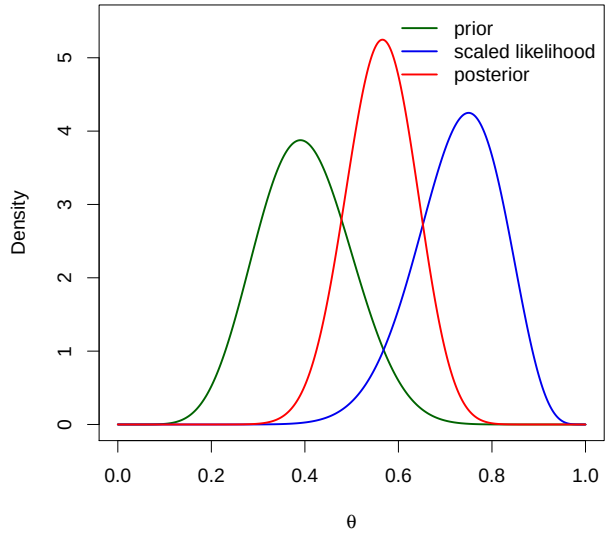
Conjugate priors: beta-binomial model

- Some comments are in order.
- The prior mean is $E(\theta) = \frac{a}{a+b}$.
- The data mean (mle) is y/n.
- The posterior mean $\mathbb{E}[\theta \mid y] = \frac{a+y}{a+b+n}$.
- Can interpret prior information as being equivalent to having observed $a-1$ successes in $a+b-2$ prior trials.
- With fixed a and b, as y and n increase, $\mathbb{E}[\theta \mid y] \to \frac{y}{n}$ (the mle), and the variance tends to zero.
- This is a general phenomenon: as n increases, posterior distribution gets more concentrated and the likelihood dominates the prior.

Choice of prior distribution

Conjugate priors: beta-binomial model

- Consider a drug to be given for relief of chronic pain.
- Experience with similar compounds has suggest that response rates, say θ, between 0.2 and 0.6 could be feasible.
- Interpret this as a distribution with mean 0.4 and standard deviation 0.1.
- A Beta(9.2, 13.8) distribution has these properties.
- Suppose we treat $n = 20$ volunteers with the compound and observe $y = 15$ positive responses.
- The parameters of the Beta distribution are updated to $9.2 + 15 = 24.2$ and $13.8 + 20 - 15 = 18.8$.



Choice of prior distribution

Conjugate priors: beta-binomial model

- Note that the likelihood, although a function of the parameter, it is not a density and so, in particular, does not integrate to one.
- In order to plot the likelihood along with the prior and posterior distributions, it is convenient that the three are in the same scale
- Therefore, we have rescaled the likelihood function so that it integrates to one.
- See `R` script `rescaled_likelihood`.

Choice of prior distribution

Conjugate priors: poisson-gamma model

- Suppose we have $Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} \text{Poisson}(\theta)$.
- We have already seen that the likelihood is

$$f(\mathbf{y} \mid \theta) = \prod_{i=1}^{n} \left\{ \frac{e^{-\theta} \theta^{y_i}}{y_i!} \right\}$$

- The kernel of the Poisson likelihood (as a function of θ) has the same form as that of a Gamma(a, b) prior for θ

$$p(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}.$$

- This parameterisation of the Gamma distributions in terms of the shape parameter a and rate parameter b, has mean a/b and variance a/b^2.

Choice of prior distribution

Conjugate priors: poisson-gamma model

↪ This implies the following posterior

$$\begin{aligned} p(\theta \mid \mathbf{y}) &\propto f(\theta \mid \mathbf{y}) p(\theta) \\ &= \left\{ \prod_{i=1}^{n} \frac{e^{-\theta} \theta^{y_i}}{y_i!} \right\} \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} \\ &\propto e^{-n\theta} \theta^{\sum_{i=1}^{n} y_i} \theta^{a-1} e^{-b\theta} \\ &= \theta^{a+\sum_{i=1}^{n} y_i - 1} e^{-\theta(b+n)} \end{aligned}$$

↪ We recognise this as the kernel of a gamma distribution with parameters $a + n\bar{y}$ and $b + n$, that is

$$\theta \mid \mathbf{y} \sim \text{Gamma}(a + n\bar{y}, b + n).$$

Choice of prior distribution

Conjugate priors: poisson-gamma model

- Note that

$$\mathbb{E}(\theta \mid \mathbf{y}) = \frac{a + n\bar{y}}{b + n} = \bar{y}\left(\frac{n}{n+b}\right) + \frac{a}{b}\left(1 - \frac{n}{n+b}\right).$$

- The posterior mean is then a compromise between prior mean a/b and the mle $\bar{y}$.

Choice of prior distribution

Conjugate priors: normal-normal model

- Let us now consider $Y_1, \ldots, Y_n \overset{\text{iid}}{\sim} \text{N}(\theta, \sigma^2)$, with σ^2 known. Further, let $\theta \sim \text{N}(\mu_0, \sigma_0^2)$.
- We've already seen the 'simplified' form of the likelihood

$$f(\mathbf{y} \mid \theta) = (2\pi\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \theta)^2 \right\}.$$

- The posterior is

$$\begin{aligned} p(\theta \mid \mathbf{y}, \sigma^2) &\propto f(\mathbf{y} \mid \theta) p(\theta) \\ &\propto \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i} (y_i - \theta)^2 \right\} \exp\left\{ -\frac{1}{2\sigma_0^2} (\theta - \mu_0)^2 \right\} \\ &\propto \exp\left\{ -\frac{1}{2\sigma^2\sigma_0^2} \left[\theta^2 (n\sigma_0^2 + \sigma^2) - 2\theta (n\bar{y}\sigma_0^2 + \mu_0\sigma^2) \right] \right\} \end{aligned}$$

Choice of prior distribution

Conjugate priors: normal-normal model

↪ This can be recognised as the kernel of a normal distribution with mean

$$\mu_n = \frac{n\bar{y}\sigma_0^2 + \mu_0\sigma^2}{n\sigma_0^2 + \sigma^2} = \frac{\frac{\mu_0}{\sigma_0^2} + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}},$$

and variance

$$\sigma_n^2 = \frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2} = \frac{1}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}}.$$

↪ The posterior parameters μ_n and σ_n^2 combine the prior parameters μ_0 and σ_0^2 with terms from the data.

↪ For instance, notice that

$$\mu_n = \frac{\tau_0}{\tau_0 + n\tau}\mu_0 + \frac{n\tau}{\tau_0 + n\tau}\bar{y},$$

where $\tau_0 = 1/\sigma_0^2$ (prior precision) and $\tau = 1/\sigma^2$ (sampling precision), and so the posterior mean is a weighted average of the prior mean and the sample mean.

Choice of prior distribution

Conjugate priors: normal-gamma model

- Suppose now– and this is unrealistic, but just to make a point– that θ is known, but the variance σ^2 is unknown.
- It is often convenient in Bayesian statistics to work with the precision, $\tau = 1/\sigma^2$.
- Let us take $\tau \sim \text{Gamma}(a, b)$.
- Then the posterior of τ is

$$\begin{aligned} p(\tau \mid \mathbf{y}, \theta) &\propto f(\mathbf{y} \mid \tau) p(\tau) \\ &\propto \tau^{n/2} \exp\left\{ -\frac{\tau}{2} \sum_{i=1}^{n} (y_i - \theta)^2 \right\} \tau^{a-1} e^{-b\tau} \\ &= \tau^{a+n/2-1} \exp\left\{ -\tau \left(b + \frac{1}{2} \sum_{i} (y_i - \theta)^2 \right) \right\}. \end{aligned}$$

- That is,

$$\tau \mid \mathbf{y}, \theta \sim \text{Gamma}\left(a + \frac{n}{2}, b + \frac{1}{2} \sum_{i} (y_i - \theta)^2 \right).$$

Choice of prior distribution

Conjugate priors: normal-gamma model

↪ A common choice is to take both a and b very small, then the posterior is approximately

$$\tau \mid \mathbf{y}, \theta \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\sum_i (y_i - \theta)^2\right),$$

and so

$$\mathbb{E}[\tau \mid \mathbf{y}, \theta] = \left(\frac{1}{n}\sum_i (y_i - \theta)^2\right)^{-1},$$

so that the posterior expectation of the precision is (approximately) the sample precision (but with a divisor of n and not $n - 1$).

Choice of prior distribution

Conjugate priors: mixture priors

- Conjugate priors are convenient, but sometimes they might be not flexible enough.
- Mixtures of conjugate priors are a good alternative and they are actually quite flexible.
- And the good new is that mixtures of conjugate priors are also conjugate. A mixture prior is

$$p(\theta) = \sum_{j=1}^{J} \pi_j p_j(\theta),$$

where p_j are conjugate priors with different hyperparameters and the mixture weights $\pi_j \in [0, 1]$ sum to one.

- **Question**: If $Y \sim \text{Bin}(n, \theta)$ and $p(\theta) = \pi\text{Beta}(\theta \mid a_1, b_1) + (1 - \pi)\text{Beta}(\theta \mid a_2, b_2)$, then $\theta \mid y$ is ... (see first practical lab.).

Predictive inference

- We now consider prediction.
- The prior predictive distribution of $\mathbf{y}$ is

$$f(\mathbf{y}) = \int_{\Theta} f(\mathbf{y} \mid \theta) p(\theta) \mathrm{d}\theta,$$

  also called marginal distribution of the data, usually in the frequentist approach.
- It is useful to check whether the model (likelihood+prior) gives (un)reasonable predictions.
- The posterior predictive distribution of a future observation z, given the data $\mathbf{y}$ is

$$\begin{aligned} f(z \mid \mathbf{y}) &= \int_{\Theta} f(z \mid \theta, \mathbf{y}) p(\theta \mid \mathbf{y}) \mathrm{d}\theta \\ &= \int_{\Theta} f(z \mid \theta) p(\theta \mid \mathbf{y}) \mathrm{d}\theta, \end{aligned}$$

  where the second equality is due to the conditional independence of Y and Z.
- Notice that the predictive distribution only depends on z and $\mathbf{y}$.

Predictive inference

- Let us consider a beta binomial experiment, where $Y \sim \text{Bin}(n, \theta)$ and $\theta \sim \text{Beta}(a, b)$.
- We have seen that the posterior distribution $\theta \mid y \sim \text{Beta}(a + y, b + n - y)$.
- For ease of notation, let $c = a + y$ and $d = b + n - y$.
- Now let us consider a further random quantity Z for which we judge $Z \sim \text{Bin}(m, \theta)$ and Z and Y are conditionally independent given θ.
- So, having conducted n trials, we consider conducting further m trials.
- We seek the predictive distribution of Z (the number of successes in m new trials) given the observed y.

Predictive inference

↪ We have

$$
\begin{aligned}
f(z \mid y) &= \int_{\Theta} f(z \mid \theta) p(\theta \mid y) \mathrm{d}\theta \\
&= \int_0^1 \binom{m}{z} \theta^z (1-\theta)^{m-z} \frac{1}{B(c,d)} \theta^{c-1} (1-\theta)^{d-1} \mathrm{d}\theta \\
&= \binom{m}{z} \frac{1}{B(c,d)} \int_0^1 \theta^{c+z-1} (1-\theta)^{d+m-z-1} \mathrm{d}\theta \\
&= \binom{m}{z} \frac{B(c+z, d+m-z)}{B(c,d)} \\
&= \binom{m}{z} \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)} \frac{\Gamma(c+z)\Gamma(d+m-z)}{\Gamma(c+d+m)}.
\end{aligned}
$$

↪ We say that $Z \mid y$ is the Binomial-Beta distribution with parameters c, d, and m.