Incomplete Data Analysis

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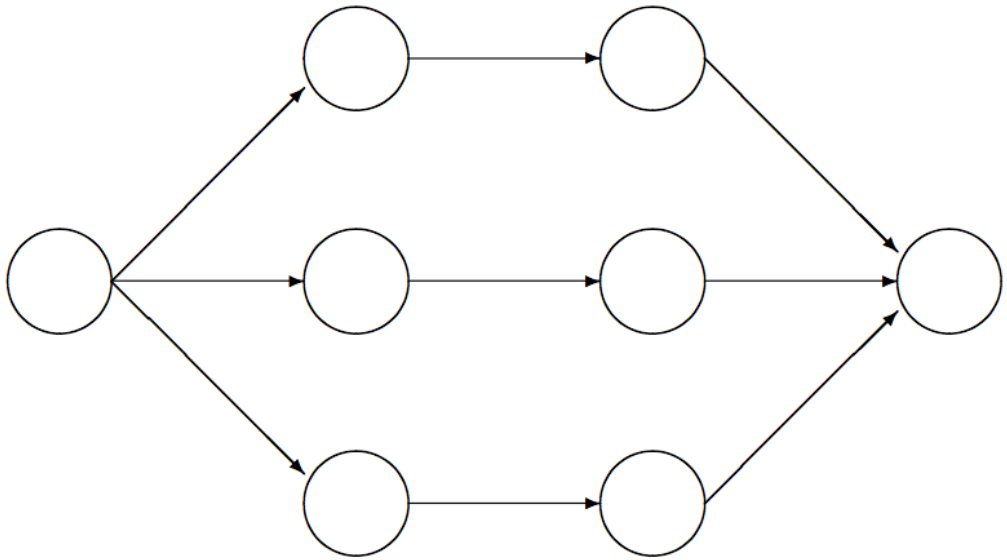
Multiple imputation



Incomplete data Imputed data Analysis results Pooled results

↪ In summary:

1. **Imputation**: impute multiple times.
2. **Analysis**: analyse each of the datasets.
3. **Pooling**: combine results, taking into account additional uncertainty.

Multiple imputation

Step 1

- ↪ Create a number ($M > 1$) of copies of the incomplete dataset, and use an appropriate procedure to impute the missing values in each of these copies.
- ↪ The imputed datasets are composed of a fixed portion – the observed data– and a varying portion – the imputed values. Since we do not know the true values that are missing it seems reasonable that the imputed values used in each copy should in general differ from each other.
- ↪ The choice of M is discussed later in the next set of slides.

Multiple imputation

Step 2

- We have created M imputed datasets that are now complete. How do we analyse them?
- For now, we will assume that our focus is on estimating a single (univariate) parameter, which we denote by θ.
- For instance, θ can be the mean or median of a variable, the proportion of individuals in a particular categorical (level) of a factor variable, a coefficient in a regression model, etc.
- For each imputed dataset, perform the analysis of interest (e.g., estimating the mean or fitting the regression model) that would have been performed in the absence of missing values. In the MI literature, the model of interest is sometimes referred to as the substantive model.
- Store the parameter estimate and its variance (the squared standard error). The estimate of θ obtained from the mth ($m = 1, \ldots, M$) complete dataset is denoted by $\widehat{\theta}^{(m)}$ and its (estimated) variance (squared standard error) by $\widehat{U}^{(m)}$.

Multiple imputation

Step 3

- After step 2, we have the results from M analyses, that is, we have $\widehat{\theta}^{(1)}, \ldots, \widehat{\theta}^{(M)}$, and $\widehat{U}^{(1)}, \ldots, \widehat{U}^{(M)}$.
- How do we now combine them to come up with a final estimate and how to measure the uncertainty about such estimate?
- According to the so-called **Rubin's rules**, the multiple imputation estimate of θ, $\widehat{\theta}^{\text{MI}}$, is the average of the M individual estimates, that is,

$$\widehat{\theta}^{\text{MI}} = \frac{1}{M} \sum_{m=1}^{M} \widehat{\theta}^{(m)}.$$

Multiple imputation

Step 3

- To estimate the variance of $\widehat{\theta}^{\text{MI}}$, we **do not** simply average the variances from each dataset. It is slightly more complicated, but not that much complicated!
- First, we calculate the **between-imputation** variance

$$B = \frac{1}{M-1} \sum_{m=1}^{M} \left(\widehat{\theta}^{(m)} - \widehat{\theta}^{\text{MI}} \right)^2,$$

- This is simply the usual unbiased sample variance formula applied to $\widehat{\theta}^{(1)}, \ldots, \widehat{\theta}^{(M)}$.
- B measures how much the estimates of θ vary across the imputed datasets.
- If there is very little missing data, the estimates from the different imputed datasets will be very similar and the between imputation variance will be small.
- The larger the amount of missing data, the larger the variability in the estimates between the imputed datasets, and the larger the between imputation variance will be.
- B thus captures uncertainty in $\widehat{\theta}^{\text{MI}}$ due to missing data.

Multiple imputation

Step 3

↪ Second, we calculate the **within-imputation** variance

$$\bar{U} = \frac{1}{M} \sum_{m=1}^{M} \widehat{U}^{(m)},$$

where $\widehat{U}^{(m)}$ is the estimated variance of $\widehat{\theta}^{(m)}$. This is simply the average of the individual variance estimates.

↪ The within imputation variance $\bar{U}$ is measuring the uncertainty due to the fact that the sample is of finite size (i.e., we are not using the entire population). This is the usual source of uncertainty in parameter estimates.

Multiple imputation

Step 3

- ↪ It is tempting to conclude that the **total variance** V^{MI} is equal to the sum of $\bar{U}$ and B, but that would be incorrect.
- ↪ We need to incorporate the fact that $\widehat{\theta}^{\text{MI}}$ itself is estimated using finite M, and thus only approximates $\widehat{\theta}^{\text{MI}}_{\infty}$, the estimator that would have been obtained for an infinitely large number of imputations $M = \infty$.
- ↪ Rubin (1987, eq. 3.3.5) shows that the contribution to the variance of this factor is systematic and equal to B_{∞}/M. Since B approximates B_{∞} (estimated between imputation variance for infinitely many imputations), we may write:

$$\begin{aligned} V^{\text{MI}} &= \bar{U} + B + \frac{B}{M} \\ &= \bar{U} + \left(1 + \frac{1}{M}\right) B, \end{aligned}$$

for the total variance of $\widehat{\theta}^{\text{MI}}$.

Multiple imputation

Step 3

- ↪ The inclusion of the term B/M is critical to make multiple imputation work at low values of M.
- ↪ Not including it would result in p-values that are too low or confidence intervals that are too short.

Multiple imputation

Step 3

↪ In summary, the total variance V^{MI} stems from three sources:

1. $\bar{U}$, the variance caused by the fact that we are taking a sample rather than observing the entire population. This is the conventional measure of variability.
2. B, the extra variance caused by the fact that there are missing values in the sample.
3. B/M, the extra simulation variance caused by the fact that $\widehat{\theta}^{\text{MI}}$ itself is estimated for finite M.

↪ Note that if there were no missing values then B would be equal to zero and the estimated total variance V^{MI} would be $\bar{U}$.

Multiple imputation

Rubin's rules–toy example

- A confidence interval for θ can be constructed based on V^{MI} and $\widehat{\theta}^{\text{MI}}$.
- Specifically, the $(1-\alpha)100\%$ confidence interval is then

$$\widehat{\theta}^{\text{MI}} \pm t_{\nu}\left(\frac{\alpha}{2}\right)\sqrt{V^{\text{MI}}},$$

  with $t_{\nu}\left(\frac{\alpha}{2}\right)$ is the $\alpha/2$ quantile of the t distribution with $\nu = (M-1)(1+1/r_M)^2$, where $r_M = (1+1/M)B/\bar{U}$ is the relative increase in variance due to missing values.

- Notice that r_M does not depend on the sample size of the observed data. This can lead to situations where the degrees of freedom are larger than those for the complete case analysis, which is inappropriate.
- To avoid this problem, Barnard and Rubin (1999) proposed an improvement to calculate the degrees of freedom. This improved version is implemented in the `mice` package.

Multiple imputation

Rubin's rules–toy example

↪ Suppose we take a survey of five people, measuring their height and weight. Only three of them disclosure their weight; the other two don't give it just because of random chance. The data are:

Height (inches)	Weight (pounds)
65	130
68	140
70	150
72	NA
75	NA

↪ The aim of the analysis (step 2) is to regress the weight on the height, that is, our statistical model of interest is

$$\text{weight} = \beta_0 + \beta_1 \text{height} + \varepsilon, \qquad \varepsilon \sim \text{N}(0, \sigma^2).$$

Multiple imputation

Rubin's rules–toy example

↪ Suppose that five plausible values for each missing weight have been generated (represented below in blue) to create five complete datasets.

Height	Weight–1	Weight–2	Weight–3	Weight–4	Weight–5
65	130	130	130	130	130
68	140	140	140	140	140
70	150	150	150	150	150
72	157	166	155	157	156
75	171	169	167	171	168
Estimated slope ($\widehat{\beta}_1$)	4.12	4.26	3.71	4.12	3.83
$\widehat{U} = \widehat{\text{var}}(\widehat{\beta}_1)$	(0.025)	(0.346)	(0.024)	(0.025)	(0.018)

Multiple imputation

Rubin's rules–toy example

- The final estimate for the slope is

$$\widehat{\beta}_1^{\text{MI}} = \frac{1}{5}(4.12 + 4.26 + 3.71 + 4.12 + 3.83) = 4.008$$

- The within imputation variance is

$$\bar{U} = \frac{1}{5}(0.025 + 0.346 + 0.024 + 0.025 + 0.018) = 0.0876$$

- The between imputation variance is

$$\begin{aligned} B &= \frac{1}{4}\{(4.12 - 4.008)^2 + (4.26 - 4.008)^2 + (3.71 - 4.008)^2 + (4.12 - 4.008)^2 + (3.83 - 4.008)^2\} \\ &= 0.05227 \end{aligned}$$

- Thus, the final estimate of the variance is

$$V^{\text{MI}} = 0.0876 + \left(1 + \frac{1}{5}\right) \times 0.05227 = 0.150324$$

Multiple imputation

Rubin's rules – multivariate case

- Extensions to the case where the parameter of interest is a p-component vector, say $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)'$, are straightforward.

- For the estimate of the parameter vector we have

$$\widehat{\boldsymbol{\theta}}^{\text{MI}} = \frac{1}{M} \sum_{m=1}^{M} \widehat{\boldsymbol{\theta}}^{(m)}.$$

- In the multivariate context, the within-imputation covariance matrix is the average of the M covariance matrices, namely

$$\bar{\mathbf{U}} = \frac{1}{M} \sum_{m=1}^{M} \widehat{\mathbf{U}}^{(m)},$$

where $\widehat{\mathbf{U}}^{(m)}$ is the covariance matrix from the completed dataset m.

Multiple imputation

Rubin's rules – multivariate case

- The between-imputation covariance matrix is as follows

$$\mathbf{B} = \frac{1}{M-1}\sum_{m=1}^{M}\left(\widehat{\boldsymbol{\theta}}^{(m)} - \widehat{\boldsymbol{\theta}}^{\text{MI}}\right)\left(\widehat{\boldsymbol{\theta}}^{(m)} - \widehat{\boldsymbol{\theta}}^{\text{MI}}\right)^{T},$$

  where $\widehat{\boldsymbol{\theta}}^{(m)}$ contains the parameter estimates from the mth imputed dataset, and $\widehat{\boldsymbol{\theta}}^{\text{MI}}$ is the vector of pooled point estimates (i.e., the arithmetic average of the $\widehat{\boldsymbol{\theta}}^{(m)}$ vectors).
- The diagonal elements of **B** contain the between imputation variance estimate for individual parameters, and the off-diagonal elements quantify the extent to which the between imputation fluctuation in one parameter is related to the between imputation fluctuation in another parameter.
- Considered as a whole, the between imputation covariance matrix represents the additional sampling fluctuation that results from the missing data.
- Finally, the total covariance matrix combined the within and between imputation covariance matrices as follows

$$\begin{aligned}\mathbf{V}^{\text{MI}} =& \bar{\mathbf{U}} + \mathbf{B} + \frac{1}{M}\mathbf{B},\\ =& \bar{\mathbf{U}} + \left(I + \frac{1}{M}\right)\mathbf{B},\end{aligned}$$

  where I is the identity matrix.

Multiple imputation

Some remarks on the 3 steps

- ↪ As we shall see, the only complex part of multiple imputation is step one: formulate a good imputation model.
- ↪ The specification of an appropriate imputation model is the key issue, since if this is misspecified, there is the potential for bias.
- ↪ The second step, producing the final estimate, is straightforward as it treats each imputed dataset as if it were a real dataset, we just have to do it M times.
- ↪ As Schafer (1997) says, multiple imputation works by "*solving an incomplete-data problem by repeatedly solving the complete-data version*".
- ↪ The third step involves simple arithmetic and typically we do not need to implement Rubin's rules manually as they are coded into most multiple imputation packages.