MODULE 2

ELASTIC PROPERTIES OF THE MATERIALS

Introduction

The study of strength of materials is to provide the means of analysing and designing various machines and load bearing structures. Elasticity is an elegant and fascinating subject that deals with determination of the stress, strain and displacement distribution in an elastic solid under the influence of external forces. Following the assumptions of linear, small deformation theory, the formulation establishes a mathematical model providing solutions to problems that have applications in many engineering and scientific fields. Civil engineering applications include stress and defection analysis of structures like rods, beams, plates, shells soil, rock concrete etc. Mechanical engineering uses elasticity in numerous problems of thermal stress analysis, fracture mechanics, fatigue, and design of machine elements. Material engineering uses elasticity to determine the stress fields of crystalline solids, dislocations, microstructures etc. Applications in aeronautical engineering include stress fluctuations, fracture, fatigue analysis in aero structures. The subject also provides the basis for study of materials behaviour in plasticity and viscoelasticity.

Elasticity:

The property of material body to regain its original shape and size on removal of the deforming forces is called ***elasticity***.

Within certain elastic limit steel and quarts show elastic properties. The elastic property is desirable for materials used in tools and machines.

Importance of Elasticity in Engineering Applications:

A sound knowledge of elastic properties of materials is very essential in the field of engineering. Engineers can make better choice of the materials for their use by knowing the nature of its stress-strain curve. Pure metals are soft by property. They are ductile and have low tensile strength. Hence they are rarely sued in engineering applications. Alloys are generally harder than pure metals. They are produced by blending (mixing) different metals after which, they exhibit unique properties that are different from metals mixed. These alloys offer better elastic properties useful for engineering applications.

Plasticity:

Bodies which does not show any tendency to recover their original condition are said to be Plastic and the property is called plasticity.

E.g. – polyethylene, Polystyrene etc.

This property of the material is necessary for forging, stamping images on coins and ornamental works.

Load: The term load implies the combination of external forces acting on a body and its effect is to change the form or the dimensions of the body. It is essentially a deforming force.

Stress: When the deforming forces acting on a body, the restoring or recovering force per unit area of cross section set up inside the body is called stress.

Stress = $\frac{Restoring\ Force}{cross\ sectiponal\ area} = \frac{F}{A}$, S I unit of stress is N/m^2.

Types of stress:

Tensile Stress:

When a body or section is subjected to two equal and opposite pulls and if it tends to pull apart the particles of the material causing extension in the direction of application of load, then the load is called **tensile load** and the corresponding stress induced is known as **tensile stress**. **Longitudinal stress or tensile stress is applied along the length and hence causes change in length**.

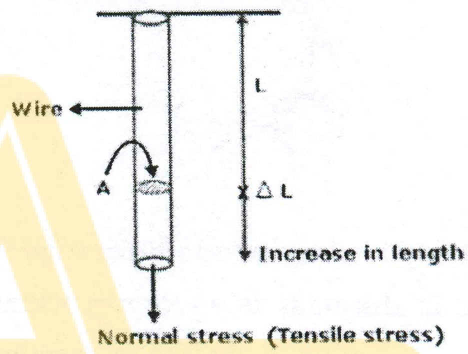
Compressive stress (Pressure or volume stress):

When a body or section is subjected to two equal and opposite pushes and it tends to push the particles of the material nearer causing shortening in the direction of application of load ,then the load is called **compressive load** and the corresponding stress induced is known as **compressive stress**.

Tangential stress or shearing stress:

When a body is subjected to a force acting along the tangential direction, the body experiences a turning or twisting effect resulting in the change in the shape of the body without any change in its volume, the stress induced is called **shearing stress** or **tangential stress**.

Strain: When a body is subjected to external force, there will be change in dimensions of the body. The change in dimension is called deformation. *The ratio of change in dimension of body or deformation to the original dimension of the body is called known as* ***strain.***



If a bar is subjected to a direct load and hence a stress, the bar will change in length. If the bar has an original length L and changes by an amount ΔL, the strain produce is defined as follows:

$$\text{Strain} = \frac{\textit{change in dimension of the body}}{\textit{original dimension}} = \frac{\Delta L}{L}$$

Strain is thus, a measure of the deformation of the material and is a non-dimensional Quantity i.e. it has no units.

Since in practice, the extensions of materials under load are very very small, it is often convenient to measure the strain in the form of strain x 10^{-6} i.e., micro strain.

Types of stress:

Linear or Longitudinal Strain:

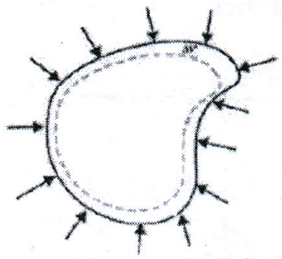
When a force is applied, the ratio of change in the length to the original length of a body is known as longitudinal strain.

$$\text{Longitudinal Strain} = \frac{\textit{change in length of the body}}{\textit{original length}} = \frac{\Delta L}{L}$$

Volume strain:

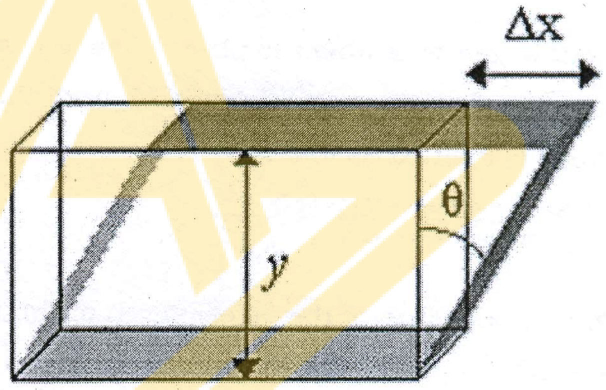
When uniform pressure is applied normally on all over the surface of a body, the body undergoes a change in its volume. The ratio of change in the volume to the original volume is called as volume strain.

Volume strain= $\frac{\Delta V}{V}$



Shear Strain:

When a body is subjected to tangential force, the angular displacement of a reference line in the body is as known shear strain. The shearing angle itself is the measure of the ratio of change in dimension to the original dimension.



$$Shearing\ strain\ = tan\theta\ = \ \Delta x\ /\ y$$

When the angle of shear is small,

$$Shearing\ strain\ = \theta\ = \ \Delta x\ /\ y$$

Hooke's law

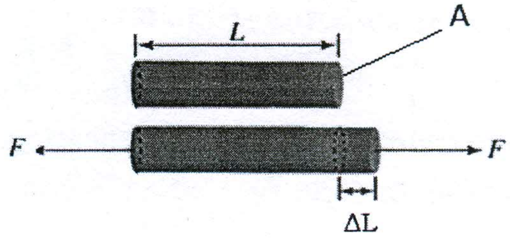
Hooke's law states that when a material is loaded within its elastic limit, **stress is directly proportional strain.**

It means that the ratio of stress to strain is constant within the elastic limit. This constant is known as Modulus of elasticity.

$$\frac{stress}{strain} = \text{constant or modulus of elasticity}$$

Types of Elastic Moduli:

1. Young's Modulus of Elasticity (Y)



When a wire is acted upon by two equal and opposite forces in the direction of its length, the length of the body is changed. The change in length per unit length ($\Delta L/L$) is called the longitudinal strain and the restoring force (which is equal to the applied force in equilibrium) per unit area of cross-section of wire is called the longitudinal stress.

For small change in the length of the wire, the ratio of the longitudinal stress to the corresponding strain is called the **Young's modulus of elasticity (Y)** of the wire.

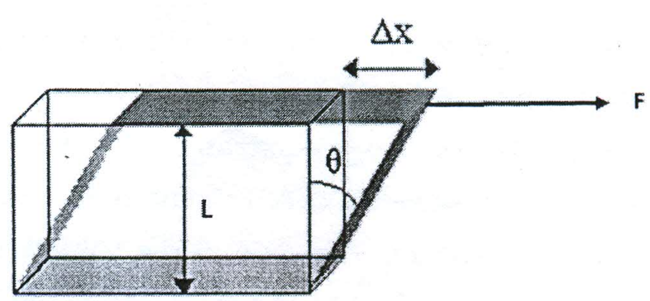
Thus, Y = Longitudinal stress / Linear Strain = $\dfrac{(F|A)}{(\Delta L|L)} = \dfrac{FL}{A\Delta L}$

Let there be a wire of length 'L' and radius 'r'. Its one end is clamped to a rigid support and a mass M is attached at the other end. Then

$$F = Mg \text{ and } A = \pi r^2$$

Substituting in above equation, we have, $Y = \dfrac{MgL}{(\pi r^2)\Delta L}$.

2. Modulus of Rigidity (n or η)



When a body is acted upon by an external force tangential to a surface of the body, the opposite surfaces being kept fixed, it suffers a change in shape of the body, and its volume remains unchanged. Then the body is said to be sheared. The tangential force acting per unit area of the surface is called the **'shearing stress' (F/A).**

The ratio of displacement to perpendicular distance between the two surfaces is known as **shearing strain (θ).**

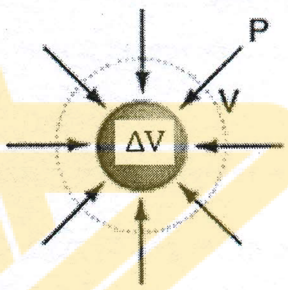
Shearing strain $\theta = \frac{\Delta x}{L}$ when θ is small.

For small strain, the ratio of the shearing stress to the shearing strain is called the **'modulus of rigidity'** of the material of the body. It is denoted by 'n or η'.

Rigidity modulus (n or η) = Tangential stress / shear Strain

$$\mathbf{n} = \frac{F/A}{\Delta x/L} = \frac{F/A}{\theta} = \frac{F}{A\theta}$$

3. Bulk Modulus of Elasticity (K)



Describes volumetric elasticity or the tendency of an object to deform in all directions when uniformly loaded in all directions. It is defined as volumetric stress over volumetric strain, and is the inverse of compressibility.

$$K = \frac{\text{Volumetric stress}}{\text{Volume strain}} = \frac{FV}{A\Delta V}$$

When a uniform pressure (normal force) is applied all over the surface of a body, the volume of the body changes. The change in volume per unit volume of the body is called the **'volume strain'** and the normal force acting per unit area of the surface (**pressure**) is called the normal stress or **volume stress.**

For small strains, the ratio of the volume stress to the volume strain is called the **'Bulk modulus' of the material of the body**. It is denoted by K.

Then, $$K = \frac{-P}{\frac{\Delta V}{V}}$$

Negative sign in formula implies that when the pressure increases volume decreases and vice-versa.

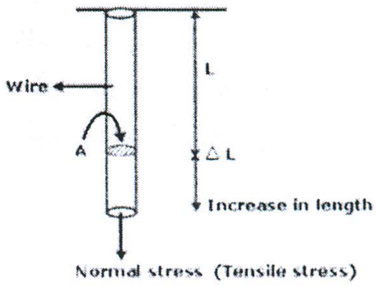
The reciprocal of the Bulk modulus of the material of a body is called the "**compressibility**' of that material. Thus, $Compressibility = 1/K$

Longitudinal Strain Coefficient (α)

The longitudinal strain produced per unit stress is called longitudinal strain coefficient.

$$\alpha = \frac{Longitudinal\ Strain}{Applied\ Stress}$$

$$= \frac{\frac{\Delta L}{L}}{T} = \frac{\Delta L}{TL}$$



The extension produced due to the applied stress 'T' is $\Delta L = TL\alpha$

Lateral Deformation:

When there is a Longitudinal Strain in a material due to the deforming forces acting along the length, there is always a change in the thickness or diameter of the material. This change occurs in a direction perpendicular to the direction of the deforming force and is called lateral change.

Lateral stain:

If a deforming force acting on a wire of circular cross section with the original diameter 'D', produces a change 'd' in its diameter then,

Lateral Strain $= \frac{d}{D}$

Lateral Strain Coefficient (β)

The lateral strain produced per unit stress is called lateral strain coefficient.

$$\beta = \frac{Lateral\ Strain}{Applied\ Stress}$$

$$\beta = \frac{\frac{d}{D}}{T} = \frac{d}{TD}$$

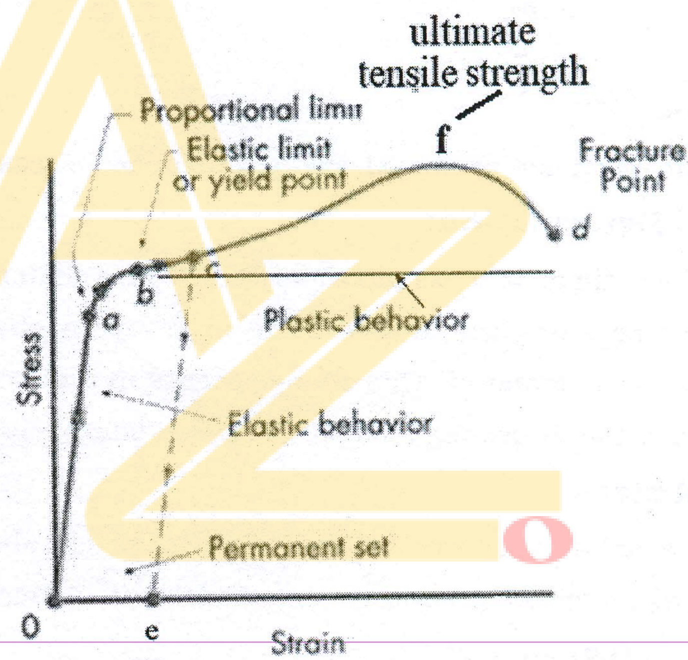
Poisson's ratio (σ):

When a material is stretched, the increase in its length (Linear strain α) is accompanied by decrease in cross section (lateral strain β). Within the elastic limit, the lateral strain is proportional to longitudinal strain and the ratio between them is a constant for a material known as **Poisson ratio (σ)**.

$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} \quad \text{i.e,} \quad \sigma = \frac{\beta}{\alpha}$$

Stress-Strain Diagram

The relationship between stress and strain is studied by plotting a graph for various values of stress and the accompanying strain and is known as stress-strain diagram.



The stress strain graph for a material is as shown in the fig.

1. The linear part **oa** of the curve shows that the strain produced is directly proportional to the stress or the Hook's law is obeyed perfectly up to a. The stress corresponding to the point **'a'** is known as proportional limit beyond which Hooke's law is obeyed.
2. With an increase in the stress beyond **'a'** the strain increases more rapidly and the curve has smaller and smaller slope until the point 'b'. Up to 'b' material exhibits perfect elasticity but does not Hooke's law. The stress at

which the linear relationship between stress and strain ceases to hold good is referred to as the **elastic limit or yield strength** of the material.

3. As soon as the elastic limit is crossed, the strain increases more rapidly for a small change in stress. In this region the material does not regain original dimension when the stress is removed. Thus the material said to have a permanent set and the deformation is plastic deformation. This region is called as plastic region where Hooke's law is not valid.
4. If the material is unloaded beyond **b** say at Point **c**, the curve will proceed from Point **c** to Point **e**. If the material is loaded again from Point **e**, the curve will follow back to Point **c** with the same Elastic Modulus (slope) and thus shows perfect elastic behavior.
5. The material now has a higher yield strength of Point **c**. Increasing the yield strength of a material by permanently straining (deforming) it is called **Strain Hardening**. Strain Hardening takes place up to a maximum stress indicated by **f.** This largest value of stress which a material withstand without breaking is known **ultimate tensile strength**.
6. After **f**, material exhibits drastic increase in the strain for small or no increase in stress. The micro crack generates and propagates in the material due to continuous concentration of stress which results in fracture of material (**failure**) at **d**.

Elastic fatigue:

The state of temporary loss in elasticity due to repeated or continuous strain is called **Elastic fatigue**. It is a form of localized failure that occurs in structures or materials which are subjected to dynamic or repeated stresses for a long period of time. Fatigue cracks form wherever there are stress concentrations. Stress gets concentrated not only due to applied stress but also on geometry of components, other variables such as corrosion, temperature, metallurgical structure etc. Fatigue is a process where a material fails elastically below the ultimate strength or within in the elastic limit due to its repeated use under stresses.

For example substances like quartz, phosphor, bronze etc. may be employed in manufacturing of galvanometers, electrometers etc, after knowing their elastic properties.

FACTORS AFFECTING ELASTICITY

Some material will have change in their elastic property because of the following factors.

a. Effect of stress
b. Effect of annealing
c. Change in temperature
d. Presence of impurities
e. Due to the nature of crystals

a) Effect of stress:

When a material is subjected to large number of cycles of stresses, it loses its elastic property even within the elastic limit. Therefore the working stress on the material should be kept lower than the ultimate tensile strength and the safety factor.

b) Effect of Annealing:

Annealing is a process by which the material is heated to a very high temperature and then it is slowly cooled. Usually this process is adopted for the material to increase the softness and ductility in the material. But annealing a material results in the formation of large crystal grains, which ultimately reduces the elastic property of the material.

c) Effect of Temperature:

The elastic property of the materials decreases with increase in the temperature due to decease in the strength of inter molecular forces in materials with increase in temperature. But elasticity of invar steel (alloy) does not change with change of temperature.

Examples:

- The elastic property of lead increases when the temperature is decreased.
- The carbon filament becomes plastic at higher temperatures.

d) Effect of impurities:

The addition of impurities produces variation in the elastic property of the materials. The increase and decrease of elasticity depends upon the type of impurity added to it. If impurity added is more elastic than the material, then elasticity of the material increases and vice versa.

Suitable impurities can alter the elastic properties of metals as they settle between the grains and brings connectivity between two grains.

Examples:

- When potassium is added to gold, the elastic property of gold increases.
- When carbon is added to molten iron, the elastic property of iron decreases provided the carbon content should be more than 1% in iron.
- If the carbon added is less 1% or in minute quantity, the elastic property of iron increases.

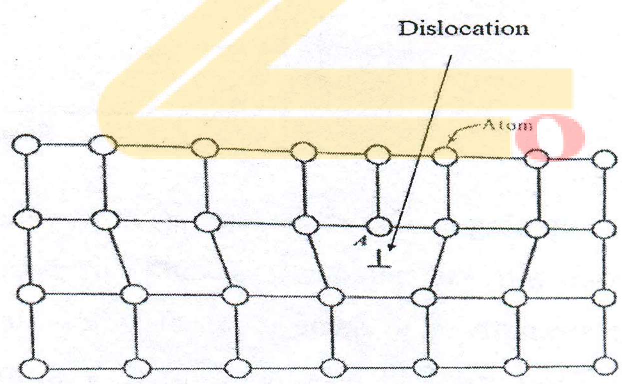
e) Nature of crystals

Elasticity of materials depends on crystalline nature of materials such as single crystals or poly crystals. Single crystals are more elastic than poly crystals due to the presence of grains in pol crystals.

Strain hardening:

Strain hardening, also known as **work hardening**, is the strengthening of a metal or polymer by plastic deformation. This strengthening occurs because of dislocation movements and dislocation generation within the crystal structure of the material.

Reason:

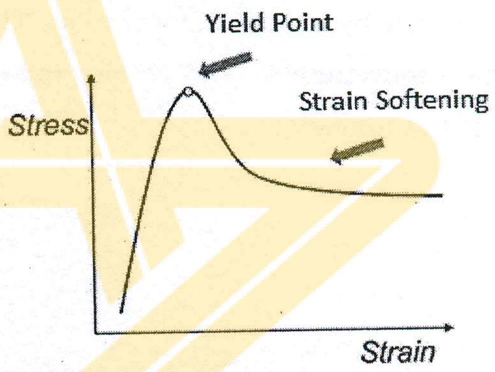


Strain hardening is due to the dislocations in the material. Dislocation is an irregularity caused within the crystal structure.

- As the deformation of the material occur in the plastic region under stress, the dislocations in the material increases.
- The dislocation interaction is repulsive in nature. There will be large number of dislocations in metals.

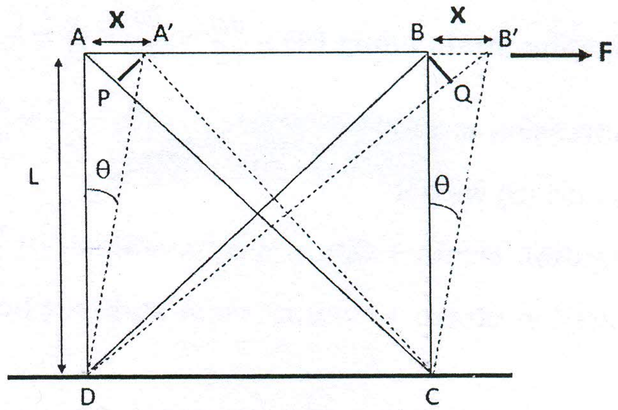
- There will be compressive stresses in the region of increased atom density (Above A) and tensile stresses in region of lesser atom density (Below A).
- Similar types of dislocations will repel due to similar types of stresses around them.
- When external stress applied, these dislocations move in the direction of stress.
- An obstacle for one dislocation, will halt the entire queue behind it.
- When the applied stress is sufficiently large, the train type of dislocations will overcome the obstacles by bypassing it.
- As the dislocation density increases the further deformation of the material become difficult, this is called **Work Hardening** or **Strain Hardening**.

Strain Softening:



Strain softening is defined as the region in which the stress in the material decreases with an increase in strain. This is observed beyond the ultimate tensile strength or in some materials after the yield point as show in the graph. The curve will have a negative slope in these region. The negative slope indicates the softening of the material over this range.

Relation between Shearing Strain, Longitudinal (Elongation) strain compression strain:



Consider a cube of side 'L', whose lower surface CD is fixed to a rigid support. Let a tangential force 'F' is applied at the upper surface along AB of the cube in a direction as shown in figure. The applied tangential force causes the relative displacements at different parts of the cube, so that, A moves to A' and B moves to B' through a small angle 'θ'. Due to this, the diagonal AC will be shortened to A'C and diagonal DB will be increased to a length DB'. We have an extension Strain along DB and compression strain along AC. The angle 'θ' is the angle of shear (shearing strain) which is very small in magnitude.

From figure, $tan\theta = \frac{BB'}{AD} = \frac{x}{L}$

Since 'θ' is very small in magnitude, shearing strain, $\theta = \frac{x}{L}$ ---------- (1)

A'P and BQ are the perpendiculars drawn as shown in the figure.

$$\left.\begin{aligned} &\text{Elongation strain along BD} = \frac{QB'}{BD} \\ &\text{Compression strain along AC} = \frac{AP}{AC} \end{aligned}\right\} \text{------------------ (2)}$$

From figure, $AC = BD = \sqrt{2}L$,

In the isosceles right angled triangle ABD, $\angle ABD = 45°$

Since 'θ' is very small, $\angle ABD = \angle QB'B = 45°$

From right angled triangle QBB',

$$QB' = BB'cos45°$$

$$QB' = \frac{x}{\sqrt{2}} \text{ -------- (3)}$$

From right angled triangle APA',

$$AP = AAcos45°$$

$$AP = \frac{x}{\sqrt{2}} \text{ -------- (4)}$$

Therefore, from (2)

Elongation strain along BD = $\frac{QB'}{BD} = \frac{\frac{x}{\sqrt{2}}}{\sqrt{2}L} = \frac{x}{2L} = \frac{\theta}{2}$ ------- (5)

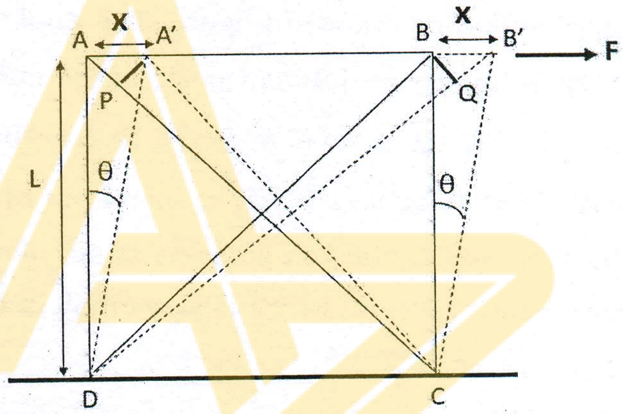
Compression strain along AC= $\frac{AP}{AC} = \frac{\frac{x}{\sqrt{2}}}{\sqrt{2}L} = \frac{x}{2L} = \frac{\theta}{2}$ ----- (6)

Adding (5) and (6) we get

Elongation strain + Compression stain = $\frac{\theta}{2} + \frac{\theta}{2} = \theta$

i.e. Elongation strain + Compression stain = Shearing strain

Expression for Rigidity Modulus (η), linear strain (α) & lateral strain (β):



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The shearing stress acting on the body, $T = \frac{F}{A} = \frac{F}{L^2}$

From figure, shearing strain, $\theta = \frac{x}{L}$ ---------- (1)

Shearing stress along AB is equivalent to elongation stress along BD and compressive stress along AC. Let α be the longitudinal strain coefficient and β be the lateral strain coefficient respectively. Then

Elongation along BD due to the tensile stress along BD $= BD.T.\alpha$

Elongation along BD due to the compressive stress along AC $= AC.T.\beta$

$= BD.T.\beta$ (Since AC =BD)

Total extension along BD is

QB' $= BD.T\,(\alpha+\beta) = \sqrt{2}LT\,(\alpha+\beta)$ ---- (2) [since $BD = \sqrt{2}L$]

Also, from right angled triangle QBB',

$$QB' = BB'cos45°$$

$$QB' = \frac{x}{\sqrt{2}} \text{ -------- (3)}$$

Eqn (2) becomes,

$$\frac{x}{\sqrt{2}} = \sqrt{2}LT\,(\alpha+\beta$$

$$\frac{x}{LT} = 2\,(\alpha+\beta$$

$$\frac{\theta}{T} = 2\,(\alpha+\beta \text{ (since from (1))}$$

Taking the reciprocal,

$$\frac{T}{\theta} = \frac{1}{2(\alpha+\beta)}$$

But $\frac{T}{\theta} = \eta$, the rigidity modulus

Therefore, $\eta = \frac{1}{2(\alpha+\beta)}$ ---------- (4)

Young's modulus, Y:

Young's modulus, Y $= \frac{Longitudinal\ Stress}{Linear\ strain}$

$$= \frac{1}{\frac{Linear\ Strain}{Longitudinal\ Stress}} = \frac{1}{\alpha}$$

Therefore, $Y = \frac{1}{\alpha}$ ----- (5)

This is the relation between young's modulus (Y) and linear strain (α).

<u>Relation between $Y - \eta - \sigma$</u>

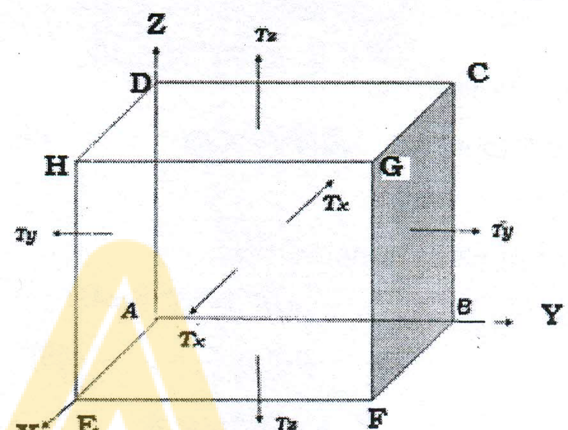
We have from (4); $2(\alpha+\beta) = \frac{1}{\eta}$

$$2\alpha\left(1+\frac{\beta}{\alpha}\right) = \frac{1}{\eta}$$

$$2\eta(1+\sigma) = \frac{1}{\alpha}\left(\because \frac{\beta}{\alpha} = \sigma\right)$$

$$2\eta(1+\sigma) = Y \left(\because \frac{1}{\alpha} = Y\right)$$

Relation between Bulk Modulus (K), Linear strain (α) and lateral strain (β)



unit cube

Consider a cube ABCDEFGH of unit sides. The Force F be acting on each face of the cube. Let stresses T_X, T_Y and T_Z act perpendicular to faces of the cube as shown in the figure.

Let α be the longitudinal strain coefficient and β be the lateral strain coefficient respectively.

Elongation produced along X axis = $T_X .\alpha.1$

Compression produced along X axis $=(T_y.\beta.1+T_Z.\beta.1)$

After the deformation, change in the length of the sides of the cube

$$Along\ X-axis\ = 1+\alpha T_x - \beta(T_y+T_z)$$

$$Along\ Y-axis\ = 1+\alpha T_y - \beta(T_x+T_z)$$

$$Along\ Z-axis\ = 1+\alpha T_z - \beta(T_y+T_x)$$

Volume of the cube after deformation

$$= \left(1+\alpha T_x - \beta(T_y+T_z)\right)\left(1+\alpha T_y - \beta(T_x+T_z)\right)\left(1+\alpha T_z - \beta(T_y+T_x)\right)$$

Since α and β are very small we can Neglect terms containing α.β, α^2, β^2 in the Product.

Volume of the cube after deformation

$$= 1+\alpha(T_x+T_y+T_z) - 2\beta(T_x+T_y+T_z)$$

$$=1+(\alpha-2\beta)(T_X+T_Y+T_Z)$$

If uniform stresses are applied in all directions then, $T_x = T_y = T_z = T$

Volume of the cube after deformation $= 1 + 3T(\alpha - 2\beta)$

Increase in volume of the cube $= = 1 + (\alpha - 2\beta)(3T) - 1 = (\alpha - 2\beta)(3T)$

Instead of stress acting, If the inward pressure is applied,

The reduction in volume of the cube $= (\alpha - 2\beta)(3P)$

$$\boxed{\therefore \text{ Bulk Modulus (K)} = \frac{P}{3P(\alpha - 2\beta)} = \frac{1}{3(\alpha - 2\beta)}}$$

Rewriting the above the equation,

$$K = \frac{1}{3\alpha\left(1 - \frac{2\beta}{\alpha}\right)} = \frac{1}{3\alpha(1 - 2\sigma)} = \frac{Y}{3(1 - 2\sigma)}$$

This is the **relation between Y, K and σ**

RELATION BETWEEN ELASTIC CONSTANTS, Y, K and η:

We have $\alpha - 2\beta = \frac{1}{3K}$ ......(1)

And $2(\alpha + \beta) = \frac{1}{\eta}$ --------- (2)

Adding (1) and (2)

$$\alpha - 2\beta + 2\alpha + 2\beta = \frac{1}{\eta} + \frac{1}{3K}$$

$$3\alpha = \frac{3K + \eta}{3K\eta}$$

$$\alpha = \frac{3K + \eta}{9K\eta}$$

Taking reciprocal, $\frac{1}{\alpha} = \frac{9K\eta}{3K + \eta}$

$$Y = \frac{9K\eta}{3K + \eta} \quad \left(\because \frac{1}{\alpha} = Y\right)$$

Relation between $K - \eta - \sigma$

We have, $K = \frac{1}{3(\alpha - 2\beta)} = \frac{1}{3\alpha\left(1 - \frac{2\beta}{\alpha}\right)}$

$$\Rightarrow K = \frac{1}{3\alpha(1 - 2\sigma)} = \frac{Y}{3(1 - 2\sigma)} \Rightarrow \boxed{Y = 3K(1 - 2\sigma)} \text{ ------ (i)}$$

Also, $\alpha + \beta = \frac{1}{2\eta}$. $\Rightarrow \eta = \frac{1}{2(\alpha + \beta)}$

$$\eta = \frac{1}{2\alpha\left(1+\frac{\beta}{\alpha}\right)} = \frac{1}{2\alpha(1+\sigma)} = \frac{Y}{2(1+\sigma)}$$

$$\Rightarrow Y = 2\eta(1+\sigma) \quad \text{-------} \quad \text{(ii)}$$

From relations (i) & (ii), we have, $3K(1-2\sigma) = 2\eta(1+\sigma)$

$$3K - 2\eta = \sigma(2\eta + 6K)$$

$$\sigma = \frac{3K - 2\eta}{6K + 2\eta}$$

LIMITS of σ:

We have

$$Y = 3K(1-2\sigma) \text{ and } Y = 2\eta(1+\sigma)$$

Therefore, $3K(1-2\sigma) = 2n(1+\sigma)$

1. If σ be a positive quantity, (1-2σ) should be positive

$2\sigma<1 \quad = \sigma < 0.5,$

A perfect incompressible material deformed elastically at small strains would have a Poissons ratio exactly 0.5.

If σ be a negative quantity, (1 + σ) should be positive. This implies that $\sigma < -1$

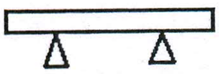
Thus the limiting values of σ lies between -1 and 0.5. Negative value of σ would mean that on being extended, a body should also expand laterally. This hardly happens ordinarily. Similarly a value of σ = 0.5 would mean that substance is perfectly incompressible. Generally the limiting values of Poisson's ratio of different materials varies from -1 to 0.5

BENDING OF BEAM:

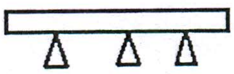
A homogenous body of uniform cross section whose length is large compared to its other dimensions is called a beam.

Types of Beam:

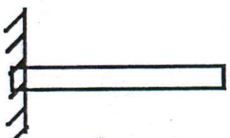
Depending on the support, beams are classified as following four types



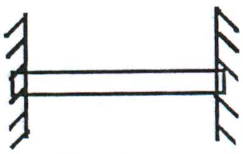
Simple Beam



Continuous Beam



Cantilever beam



Fixed beam

1. **Simple beam**: It is bar resting upon supports at its ends and is the most commonly used.
2. **Continuous beam**: It is a bar resting upon more than two supports.
3. **Cantilever beam**: It is a beam whose one end is fixed and the other end is free.
4. **Fixed beam**: A beam fixed at its both ends is called a fixed beam.

Neutral Surface and Neutral Axis:

Consider a uniform beam MN whose one end is fixed at M (Fig a). The beam can be thought of as made up of a number of parallel layers and each layer in turn as made up of a number of thin parallel longitudinal filaments or fibers in the plane of the layer. If a cross section of the beam along its length and perpendicular to these layers is taken the filaments of different layers appear like straight lines piled one above the other along the length of the beam (Fig b).

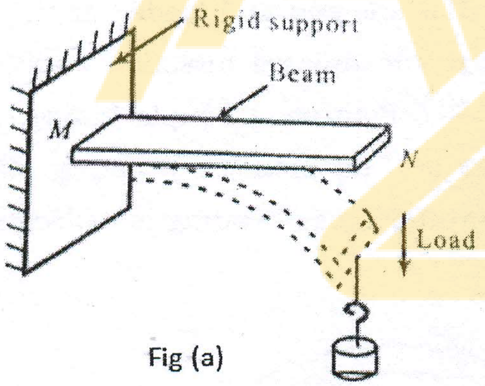


Fig (a)

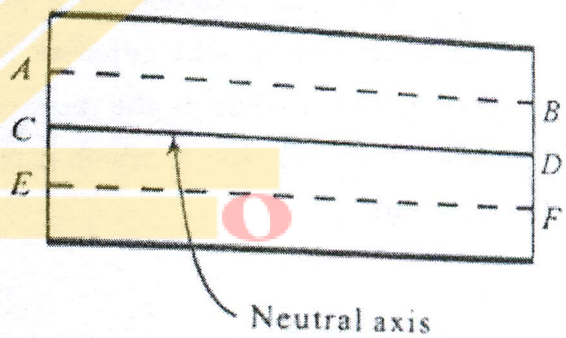
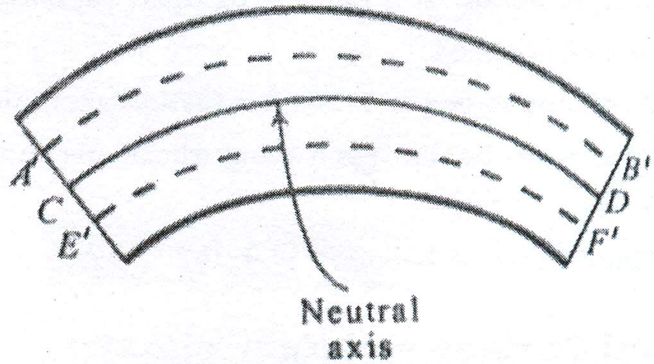


Fig (b)

If a load is attached to the free end of the beam, the beam bends. The successive layers along with constituent filaments are strained. A filament like AB of an upper layer will be elongated to A^1B^1 and the one like EF of a lower layer will be compressed to E^1F^1. The layer (CD) that does not undergo any change in the dimension is called as the **neutral layer** or **neutral surface**. A filament in the neutral layer whose length always remains the same is called the **neural axis.**

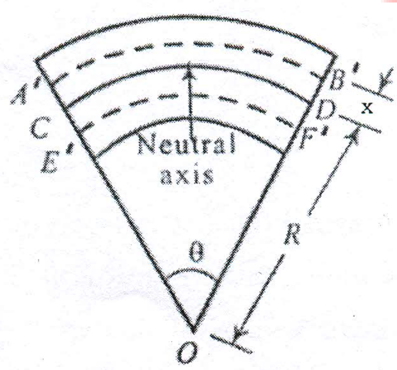


Neutral surface: is that layer of a uniform beam which does not undergo any change in its dimension when the beam is subjected to bending within the elastic limit.

Neutral Axis is an axis in the cross section of a beam along which there are no longitudinal stresses or stain. The length of this axis remains the same when the bean is subjected to bending.

Expression for bending moment of a beam:

Consider a uniform beam fixed at one end and loaded at the other. As a result an equal reaction force acts in the upward direction at the fixed end. These two equal and opposite forces constitute a couple known as bending couple due to which the beam bends. The moment (rotating effect) of the bending couple due to which a beam undergoes bending is called the **bending moment**.



PART OF BENT BEAM

Due to the bending, a layer like AB above the neutral layer will be elongated to A^1B^1 and the one like EF below the neutral layer will be compressed to E^1F^1. CD is the neutral surface which does not undergo any change in its length. The layers of the bent beam forms the part of concentric circles with centre at O. Let 'R' be the radius of the circle to which the neutral surface forms a part. Let 'θ' be the angle subtended by the layers at the common centre 'O' and 'x' be the separation between the successive layers.

The change in length of layer AB is $= A'B' - AB$

$$= A'B' - CD \ (\because AB = CD)$$

But $\quad CD = AB = R\theta$ (From fig.)

If 'x' is the separation between the successive layers then from figure

$$A'B' = (R + x)\theta$$

Change in length $= A'B' - CD$

$$= (R + x)\theta - R\theta$$

$$= x\theta$$

$$Linear\ strain = \frac{change\ in\ length}{oiginal\ length} = \frac{x\theta}{R\theta} = \frac{x}{R} \text{ ---------- (1)}$$

Young's modulus, $Y = \frac{Longitudinal\ stress}{Linear\ strain}$

$$Longitudinal\ stress = Y \times Linear\ Strain$$

$$= Y \times \frac{x}{R} \text{ -------------------- (2)}$$

But $\quad Longitudinal\ stress = \frac{F}{a}$

Where F is the force acting on the beam and 'a' is the area of the layer 'AB'.

Therefore $\frac{F}{a} = Y \times \frac{x}{R}$

Or $\quad F = \frac{Yxa}{R}$

The moment of this force about the neutral axis

=Force $\times$ $distance\ of\ AB\ from\ neutralais$

$$= F \times x = \frac{Yax^2}{R} \text{ --------- (3)}$$

Moment of the force acting on the entire layer

$$= \sum \frac{Yax^2}{R}$$

$$= \frac{Y}{R} \sum ax^2$$

Here $\sum ax^2$ is called the **geometric moment of Inertia** I_g

i.e. $I_g = \sum ax^2 = AK^2$; where K is called the radius of gyration about the neutral axis.

Therefore Moment of the force or bending moment $= \frac{Y}{R} I_g$

Expression for bending moment for a beam of

Circular cross sections; $I_g = \frac{\pi r^2}{4}$; r is the radius of the beam

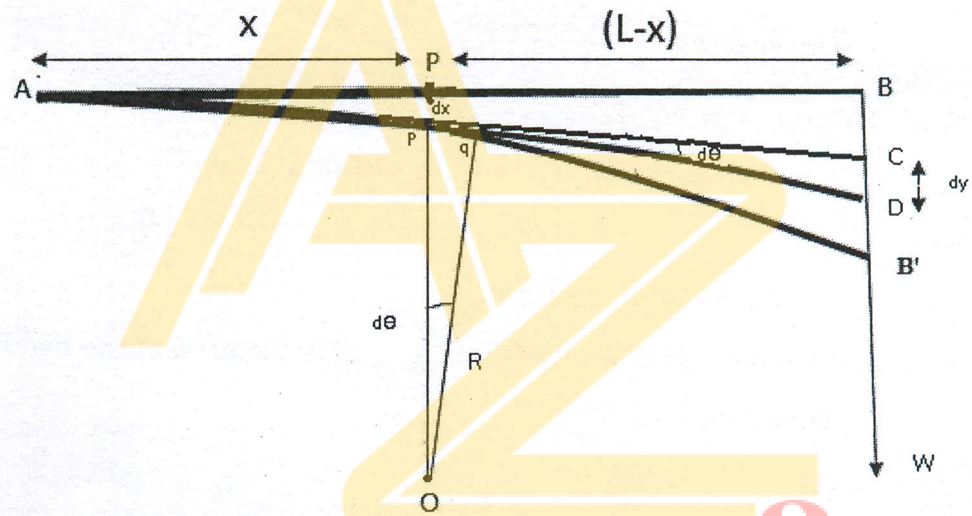
$\therefore$ **Bending moment** = $\frac{Y}{R}\left(\frac{\pi r^2}{4}\right)$

Rectangular cross sections; $I_g = \frac{bd^3}{12}$; b is the breadth and d is the thickness of the beam

$\therefore$ **Bending moment** = $\frac{Y}{R}\left(\frac{bd^3}{12}\right)$

<u>SINGLE CANTILEVER:</u>

A beam fixed horizontally at one end and loaded at the other is called a **Cantilever**.



Consider a cantilever of length 'L' fixed at one end and loaded at the other end with a load 'W'. Let AB be the neutral axis of the cantilever. Consider a section P of the beam at a distance **'x'** from A as shown in the figure.

Bending moment about P = Force× perpendicular distance

$= W\,(L - X)$ --- (1)

Due to this moment the beam bends such that the end B moves to B'.

But bending moment of the beam $= \frac{Y}{R} I_g$ ------------ (2)

Where R is the radius of curvature of neutral axis at P.

From (1) and (2), $W\,(L - X) = \frac{Y}{R} I_g$ ----------------- (3)

As the moment of the load increases towards the point A, the radius of curvature is different at different points and decreases towards A. For a point Q at a very small distance dx from P. Q is practically same as at P. let $d\theta$ be the angle subtended by P and Q at the centre 'O'

Therefore, $PQ = dx = Rd\theta$

$$\Rightarrow R = \frac{dx}{d\theta} \quad \text{----------------------------} \quad (4)$$

From (3), $W(L - X) = (YI_g)\frac{d\theta}{dx}$

Or $d\theta = \frac{W(L-x)dx}{YI_g}$ ------------------ ----------- (5)

Draw tangents to the neutral axis at P and Q meeting the vertical line BB' at C and D. The angle subtended by them is $d\theta$. The depression dy of Q below P is given by

$$CD = dy = (L - X)d\theta$$

Or $d\theta = \frac{dy}{L-x}$ -------------- (6)

Substituting Eqn. (6) in Eqn.(5), $dy = \frac{W(L-x)^2}{YI_g}dx$ ----------- (7)

Total depression BB[1] of the loaded end, $y = \int_0^L \frac{W(L-x)^2}{YI_g}dx$

$$= \frac{W}{YI_g}\int_0^L (L^2 + x^2 - 2Lx)dx$$

$$y = \frac{W}{YI_g} \times \frac{L^3}{3}$$

For rectangular cross section $I_g = \frac{bd^3}{12}$; b = breadth, d = thickness of beam

Then, the depression, y = $\frac{4mgL^3}{Ybd^3}$ [$\because$W = mg]

And Young's Modulus Y = $\frac{4mgl^3}{ybd^3}$

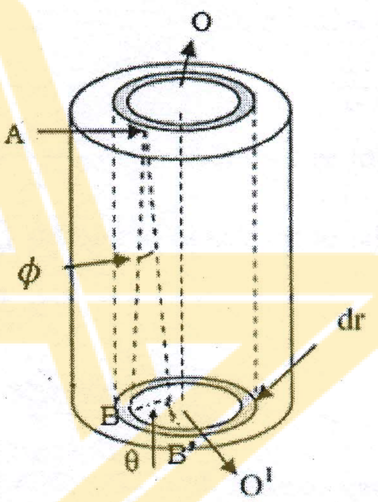
Applications of beam: Beams are used

1. In the fabrication of trolley ways.
2. In the Chassis/ frame as truck beds.
3. In the elevators.
4. In the construction of platform and bridges.
5. Beams are an integral part of Civil engineering structural elements (bridges, dams, multi-storeyed buildings).
6. As girders in buildings and bridges.

Torsion of a cylinder:

A long body which is twisted around its length as an axis is said to be under torsion. The twisting is brought into effect by fixing one end of the body to a rigid support and applying a suitable couple at the other end. The elasticity of a solid, long uniform cylindrical body under torsion can be studied, by imagining it to be consisting of concentric layers of the material of which it is made up of. The applied twisting couple is calculated in terms of the rigidity modulus of the body.

Expression for the Torsion of a cylindrical rod



Consider a long cylindrical rod of length 'L' and radius 'R' rigidly fixed at its upper end. Let OO' be its axis. Imagine the cylindrical rod is made up of thin concentric hollow cylindrical layers each of thickness *'dr'*. When the rod twisted at its lower end, the concentric layers slide one over the other. This movement will be zero at the fixed end and gradually increased along the downward direction. Let us consider one concentric circular layer of radius *'r'* and thickness *'dr'*. Any point 'A' on its uppermost part would remain fixed and a point like 'B' at its bottom moves to 'B''

Now, $\angle BAB' = \phi$, gives the angle of shear.

Since ϕ is small, the movement length $BB' = L\phi$.

Also, if $\angle BOB' = \theta$, the length BB' $= r\theta$.

$$\therefore L\phi = r\theta$$

$$\phi = \frac{r\theta}{L} \text{ --------- } (1)$$

The cross sectional area of the layer under consideration is $2\pi r\,dr$. If 'F' is the shearing force, then the shearing stress T is given by

$$T = \frac{Force}{Area} = \frac{F}{2\pi\, rdr} \text{ ------- (2)}$$

$\therefore$ Rigidity modulus $\eta = \frac{Shearing\ Stress}{Shearing\ Strain}$.

$$\eta = \frac{T}{\phi}$$

$$= \frac{\frac{F}{2\pi r dr}}{\frac{r\theta}{L}}$$

Therefore the force acting on the cylinder,

$$F = \frac{2\pi r^2 \theta \eta dr}{L}$$

Moment of this force the axis OO' of the cylinder

$$= Force \times perpendicular\ distance\ fromthe\ axis$$

$$= \left(\frac{2\pi r^2 \theta \eta dr}{L}\right).r$$

$$= \left(\frac{2\pi r^3 \theta \eta dr}{L}\right)$$

This is only for the one layer of the cylinder.

Therefore, twisting couple acting on the entire cylinder $= \int_0^R \frac{2\pi\eta\theta}{L} r^3 dr$

$$= \frac{2\pi\eta\theta}{L}\left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{\pi\eta\, R^4\theta}{2L}$$

Couple per unit twist is given by C = Total twisting couple / angle of twist.

$$C = \frac{\frac{\pi\eta\, R^4\theta}{2L}}{\theta}.$$

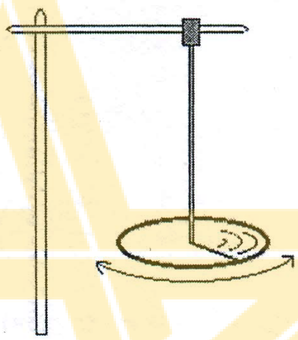
$$C = \left(\frac{\pi\eta\, R^4}{2L}\right)$$

TORSIONAL PENDULUM

A heavy object suspended from end of a fine wire rotating about an axis constitutes a **torsional pendulum**.

Torsion pendulum consists of a heavy metal disc is suspended by means of a wire. When the disc is rotated in a horizontal plane so as to twist the wire, the various elements of the wire undergo shearing strain. The restoring couple of the wire tries to bring the wire back to the original position. As a result, the disc executes to and fro turning with the wire as the axis. These oscillations are known as torsional oscillations.

The oscillations executed by a suspended rigid body due to the twist in the suspension are known as **torsional oscillations**.



The time period of oscillation 'T' for a torsional pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{C}}$$

Where I is the moment of inertia of the rigid body about the axis through the wire, C is the couple per unit twist for the wire.

Applications of Torsional Pendulum:

1. The moment of inertia of irregular rigid bodies can be determined using torsional pendulum.
2. The rigidity modulus of a material can be found by taking the material in the form of wire and setting up of a Torsion pendulum.

****

Important Questions:

1. Explain in brief the factors affecting the elasticity of a material. (4marks)
2. Derive the relation between Y, η *and* σ where the symbols have their usual meaning. (7marks)
3. What are torsional oscillations? Give the expression for time period of torsional oscillations. Mention the applications of torsional oscillations. (5 marks)
4. State and explain Hookes' law. Define elastic and plastic limit (6 marks)
5. Define Poisson's ratio. Mention its limiting values. Obtain the relation between shear strain, elongation strain and compression strain. (10 marks)
6. Explain tensile stress and compressive stress. What are the engineering importance of elastic materials? (6 marks)
7. Define bending moment. Derive the expression for bending moment in terms moment of inertia. (8 marks)
8. Derive the relation between bulk modulus (K), Young's modulus (Y) and Poisson's ratio. What are the limiting values of Poisson's ratio? (8marks)
9. Explain the nature of elasticity with the help of stress-strain diagram.

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