CALCULUS AND LINEAR ALGEBRA

MODULE -03

INTEGRAL CALCULUS

Note:-

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

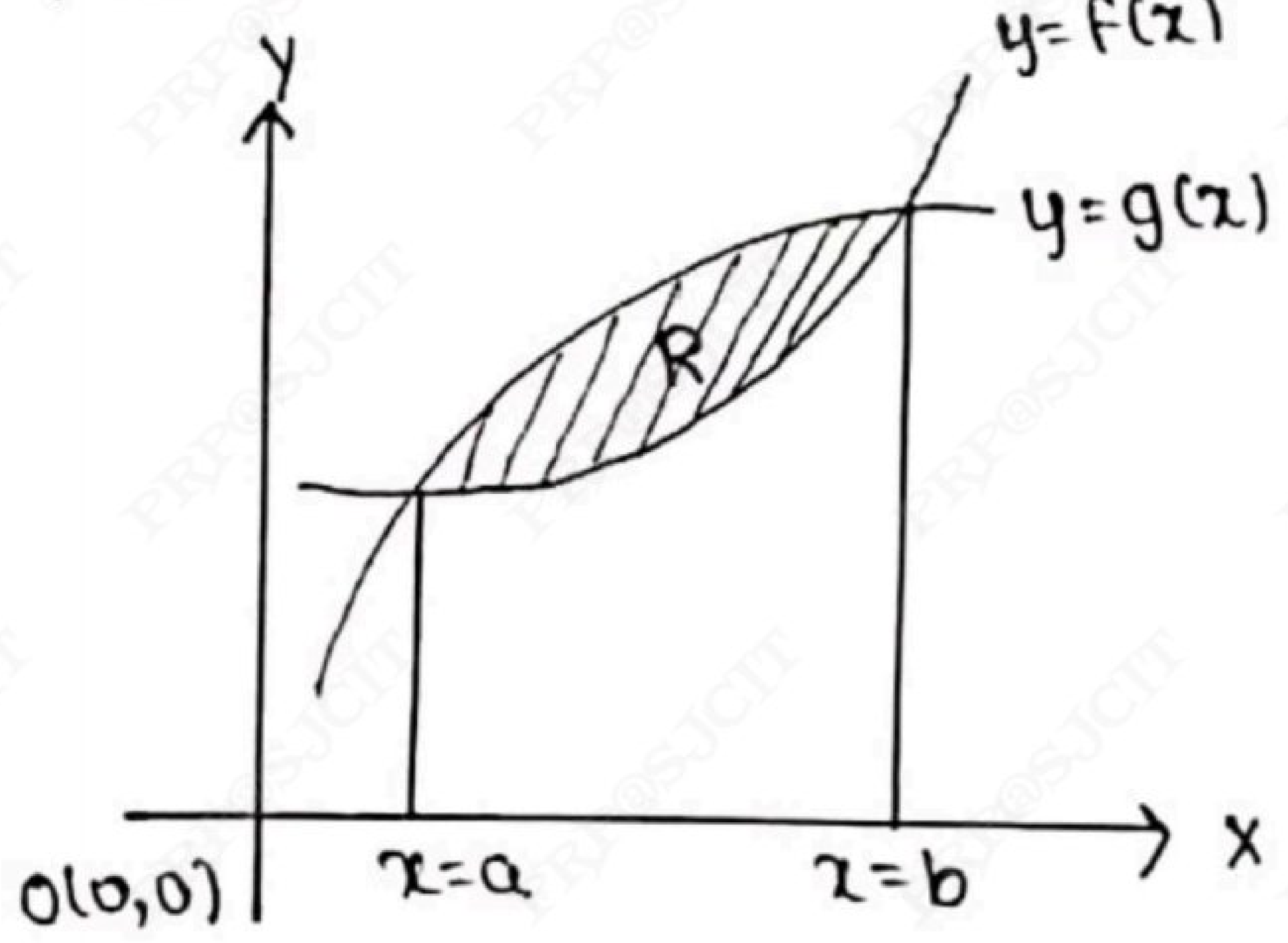
$$\int e^{ax} = \frac{e^{ax}}{a}$$

$$\int u.v \, dx = u\int v \, dx - \int\left(u'\int v \, dx\right) dx$$

Multiple Integrals:

1. Integral $\int_{x=a}^{b} f(x) \, dx$ can be described as the length of a curve $y=f(x)$ from $x=a$ to $x=b$ also it is called the line integral.

2. Integral $\int_{a}^{b}\int_{y=f(x)}^{g(x)} \phi(x,y)\,dy\,dx$ can be described as region of surface bounded between $y=f(x)$, $y=g(x)$ and $x=a$, $x=b$



3. In Integral,

$$I=\int_{x=a}^{b}\int_{y=f(x)}^{g(x)}\int_{z=h_1(x,y)}^{h_2(x,y)} \phi(x,y,z)\,dz\,dy\,dx$$ used to calculate the volume between the mentioned boundaries

1. Evaluate $\int_0^1\int_0^x (x^2+y^2)\,dy\,dx$

Let $I=\int_0^1\int_0^x (x^2+y^2)\,dy\,dx$

$$=\int_{x=0}^{1}\left[\int_{y=0}^{x}(x^2+y^2)\,dy\right]dx$$

$$=\int_{x=0}^{1}\left[x^2y+\frac{y^3}{3}\right]_{y=0}^{x}dx$$

$$=\int_{x=0}^{1}\left[x^3+\frac{x^3}{3}\right]dx$$

$$= \int_0^1 \frac{4x^3}{3}\, dx$$

$$= \frac{4}{3} \int_0^1 x^3\, dx$$

$$= \frac{4}{3}\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3}\left[x^4\right]_0^1$$

$$= \frac{1}{3}[1-0]$$

$$I = \frac{1}{3}$$

2. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$

Let $I = \int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$

$$= \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} xy\, dy\, dx$$

$$= \int_{x=0}^1 x \int_{y=x}^{\sqrt{x}} y\, dy\, dx$$

$$= \int_{x=0}^1 x \int_{y=x}^{\sqrt{x}} y\, dy\, dx$$

$$= \int_{x=0}^1 x \left[\frac{y^2}{2}\right]_x^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_{x=0}^1 x \left[y^2\right]_x^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_{x=0}^{1} x(x-x^2)\,dx$$

$$= \frac{1}{2}\int_{x=0}^{1} (x^2-x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3}-\frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{4-3}{12}\right]$$

$$I = \frac{1}{24}$$

3. Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx$

$$I = \int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left[x^2z+y^2z+\frac{z^3}{3}\right]_{z=-a}^{a} dy\,dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left\{\left[ax^2+ay^2+\frac{a^3}{3}\right]-\left[-ax^2-ay^2-\frac{a^3}{3}\right]\right\} dy\,dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left[2ax^2+2ay^2+\frac{2a^3}{3}\right] dy\,dx$$

$$= 2a\int_{x=-c}^{c}\int_{y=-b}^{b}\left[x^2+y^2+\frac{a^2}{3}\right] dy\,dx$$

$$= 2a\int_{x=-c}^{c}\left[x^2y+\frac{y^3}{3}+\frac{a^2y}{3}\right]_{y=-b}^{b} dx$$

$$= 2a \int_{x=-c}^{c} \left\{ \left[bx^2 + \frac{b^3}{3} + \frac{ba^2}{3} \right] - \left[-\frac{bx^2}{1} - \frac{b^3}{3} - \frac{ba^2}{3} \right] \right\} dx$$

$$= 2a \int_{x=-c}^{c} \left[2bx^2 + \frac{2b^3}{3} + \frac{2ba^2}{3} \right] dx$$

$$= 4ab \int_{x=-c}^{c} \left[x^2 + \frac{b^2}{3} + \frac{a^2}{3} \right] dx$$

$$= 4ab \left[\frac{x^3}{3} + \frac{b^2}{3}x + \frac{a^2 x}{3} \right]_{-c}^{c}$$

$$= 4ab \left\{ \left[\frac{c^3}{3} + \frac{cb^2}{3} + \frac{ca^2}{3} \right] - \left[-\frac{c^3}{3} - \frac{cb^2}{3} - \frac{ca^2}{3} \right] \right\}$$

$$= 4ab \left[\frac{2c^3}{3} + \frac{2cb^2}{3} + \frac{2ca^2}{3} \right]$$

$$= 8abc \left[\frac{a^2}{3} + \frac{b^2}{3} + \frac{c^2}{3} \right]$$

$$= \frac{8}{3} abc \left[a^2 + b^2 + c^2 \right]$$

4. Evaluate $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$

$$I = \int_{z=-1}^{1} \int_{x=0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$$

$$I = \int_{z=-1}^{1} \int_{x=0}^{z} \left[xy + \frac{y^2}{2} + yz \right]_{y=x-z}^{x+z} dx\, dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} \left\{ x(x+z) + \frac{(x+z)^2}{2} + z(x+z) \right] - \left[x(x-z) + \frac{(x-z)^2}{2} + z(x-2) \right] \right\} dx\, dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} \left\{ x(x+z-x+z) + \frac{1}{2}\left[(x+z)^2 - (x-z)^2 + z(x+z) - x+z \right) \right\} dx\, dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{2} (4zx + 2z^2)\, dx\, dz$$

$$= \int_{z=-1}^{1} \left[4z \frac{x^2}{2} + 2z^2 x\right]_{x=0}^{2} dz$$

$$= \int_{z=-1}^{1} (2z^3 + 2z^3) dz$$

$$= \int_{z=-1}^{1} 4z^3 dz$$

$$= 4\left[\frac{z^4}{4}\right]_{-1}^{1}$$

$$= [z^4]_{-1}^{1}$$

$$= (1^4) - (-1)^4$$

$$= 1 - 1$$

$$\boxed{I = 0}$$

5. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$$I = \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y}\, e^{z}\, dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \int_{z=0}^{x+y} e^{z}\, dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left[e^{z}\right]_{z=0}^{x+y} dy\, dx$$

$$= \int_{x=0}^{a} \int_{0}^{x} e^{x+y} \left[e^{x+y} - 1\right] dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} \left[e^{2x+2y} - e^{x+y}\right] dy\, dx$$

$$= \int_{x=0}^{a} e^{2x} \int_{y=0}^{x} e^{2y}\, dy\, dx - \int_{x=0}^{a} e^{x} \int_{y=0}^{x} e^{y}\, dy \cdot dx$$

$$= \int_{x=0}^{a} e^{2x} \int_{y=0}^{x} e^{2y}\, dy\, dx - \int_{x=0}^{a} e^{x} \int_{y=0}^{x} e^{y}\, dy \cdot dx$$

$$= \frac{1}{2} \int_{x=0}^{a} e^{2x} \left[2^{2y} \int_{0}^{x} dx - \int_{x=0}^{a} e^{x} \left[e^{y}\right]_{0}^{x} dx \right.$$

$$= \frac{1}{2} \int_{0}^{a} (e^{4x} - e^{2x})\, dx - \int_{0}^{a} (e^{2x} - e^{x})\, dx$$

$$= \frac{1}{2} \left[\frac{e^{4x}}{4} - \frac{e^{2x}}{2} \right]_{0}^{a} - \left[\frac{e^{2x}}{2} - \frac{e^{x}}{1} \right]_{0}^{a}$$

$$= \frac{1}{2} \left\{ \left[\frac{e^{4a}}{4} - \frac{e^{2a}}{2} \right] - \left[\frac{1}{4} - \frac{1}{2} \right] \right\} - \left\{ \left[\frac{e^{2a}}{2} - \frac{e^{a}}{1} \right] - \left[\frac{1}{2} - 1 \right] \right\}$$

$$= \frac{1}{2} \left\{ \frac{e^{4a}}{4} - \frac{e^{2a}}{2} + \frac{1}{4} \right\} - \left\{ \frac{e^{2a}}{2} - e^{a} + \frac{1}{2} \right\}_{0}$$

$$= \frac{e^{4a}}{8} + \frac{e^{2a}}{4} + \frac{1}{8} - \frac{e^{2a}}{2} + e^{a} - \frac{1}{2}$$

$$= \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} \cdot \frac{3}{8}$$

$$I = \frac{1}{8} \left[e^{4a} - 6a^{2a} + 8e^{a} - 3 \right]$$

6. Evaluate $\displaystyle\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} \int_{0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}}\, dz\, dy\, dx$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}}\, dz\, dy\, dx$$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\left(\sqrt{a^2-x^2-y^2}\right)^2 - z^2}\, dz\, dy\, dx$$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\left(\sqrt{(a^2-x^2-y^2)}\right)^2 - z^2}\, dz\, dy\, dx$$

Let $K = \sqrt{a^2-x^2-y^2}$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{K} \frac{1}{\sqrt{K^2-z^2}}\, dz\, dy\, dz$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \left[\sin^{-1}(z/K)\right]_{z=0}^{K} dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy\, dx$$

$$= \frac{\pi}{2} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} 1\, dy\, dx$$

$$= \frac{\pi}{2} \int_{x=0}^{a} [y]_0^{\sqrt{a^2-x^2}}\, dx$$

$$I = \frac{\pi}{2} \int_0^a \sqrt{a^2-x^2}\, dx$$

Let $x = a\sin\theta \Rightarrow \theta = \sin^{-1}(x/a)$

$dx = a\cos\theta\, d\theta$

U.L $\Rightarrow x = a \Rightarrow \theta = \frac{\pi}{2}$

L.L $\Rightarrow x = 0 \Rightarrow \theta = 0$

$$\therefore I = \frac{\pi}{2} \int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\, .\, a\cos\theta\, d\theta$$

$$= \frac{\pi}{2} \int_0^{\pi/2} a\cos\theta\, .\, a\cos\theta\, d\theta$$

$$= \frac{\pi a^2}{2} \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{\pi a^2}{2} \int_0^{\pi/2} \left(\frac{1+\cos\theta}{2} \right) d\theta$$

$$= \frac{\pi a^2}{4} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{\pi a^2}{4} \left(\frac{\pi}{2} + 0 \right) - (0+0)$$

$$I = \frac{\pi^2 a^2}{8}$$

7. Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz \, dy \, dx$

$$I = \int_0^1 \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz \, dy \, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{\left(\sqrt{a^2-x^2-y^2}\right)^2 - z^2}} \, dz \, dy \, dx$$

let $K = \sqrt{a^2-x^2-y^2}$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{K} \frac{1}{\sqrt{K^2-z^2}} \, dz \, dy \, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} \left[\sin^{-1}(z/K) \right]_{z=0}^{K} dy \, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0) \right] dy \, (dx)$$

$$= \frac{\pi}{2} \int_{x=0}^{1} \int_{y=0}^{\sqrt{a^2-x^2}} 1 \, dy \, dx$$

$$= \frac{\pi}{2} \int_{x=0}^{1} [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{\pi}{2} \int_{0}^{1} \sqrt{a^2-x^2} \, dx$$

Let $x = a \sin\theta \Rightarrow \theta = \sin^{-1}(x/a)$

$dx = a\cos\theta \, d\theta$

UL : $x = 1 \Rightarrow \theta = \frac{\pi}{2}$

LL : $x = 0 \Rightarrow \theta = 0$

$$= \frac{\pi}{2} \int_{0}^{\pi/2} \sqrt{1-\sin^2\theta} \cdot \cos\theta \, d\theta$$

$$= \frac{\pi}{2} \int_{0}^{\pi/2} \cos\theta \cos\theta \, d\theta$$

$$= \frac{\pi}{2} \int_{0}^{\pi/2} \left(\frac{1-\cos\theta}{2} \right) d\theta$$

$$= \frac{\pi}{2} \left[0 + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{\pi}{4} \left[\frac{\pi}{2} + 0 \right] - (0+0)$$

$$I = \frac{\pi^2}{8}$$

8. Evaluate $\displaystyle\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} \int_{0}^{\sqrt{a^2-x^2-y^2}} xyz \, dz . dy \, dx$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyz \, dz \, dy \, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dx\, dy$$

$$= \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\,(a^2 - x^2 - y^2)\, dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (a^2xy - x^3y)\, dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \left[a^2x \cdot \frac{y^2}{2} - x^3\frac{y^2}{2} - \frac{xy^4}{4} \right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \left[\frac{a^2x}{2}(a^2-x^2) - \frac{x^3}{2}(a^2-x^2) - \frac{x}{4}(a^2-x^2) \right] dx$$

$$= \frac{1}{2} \int_{x=0}^{a} (a^2-x^2)\left(\frac{a^2x}{2} - \frac{x^3}{2} \cdot \frac{x(a^2-x^2)}{4} \right) dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \frac{(a^2-x^2)}{4} \left[2a^2x - 2x^3 - a^2x + x^3 \right] dx$$

$$= \frac{1}{8} \int_{0}^{a} (a^2-x^2)(a^2x - x^3)\, dx$$

$$= \frac{1}{2} \int_{0}^{a} (a^4x - a^2a^3 - a^2x^3 + x^5)\, dx$$

$$= \frac{1}{8} \int_{0}^{a} \left[x^5 - 2a^2x^3 + a^4x \right] dx$$

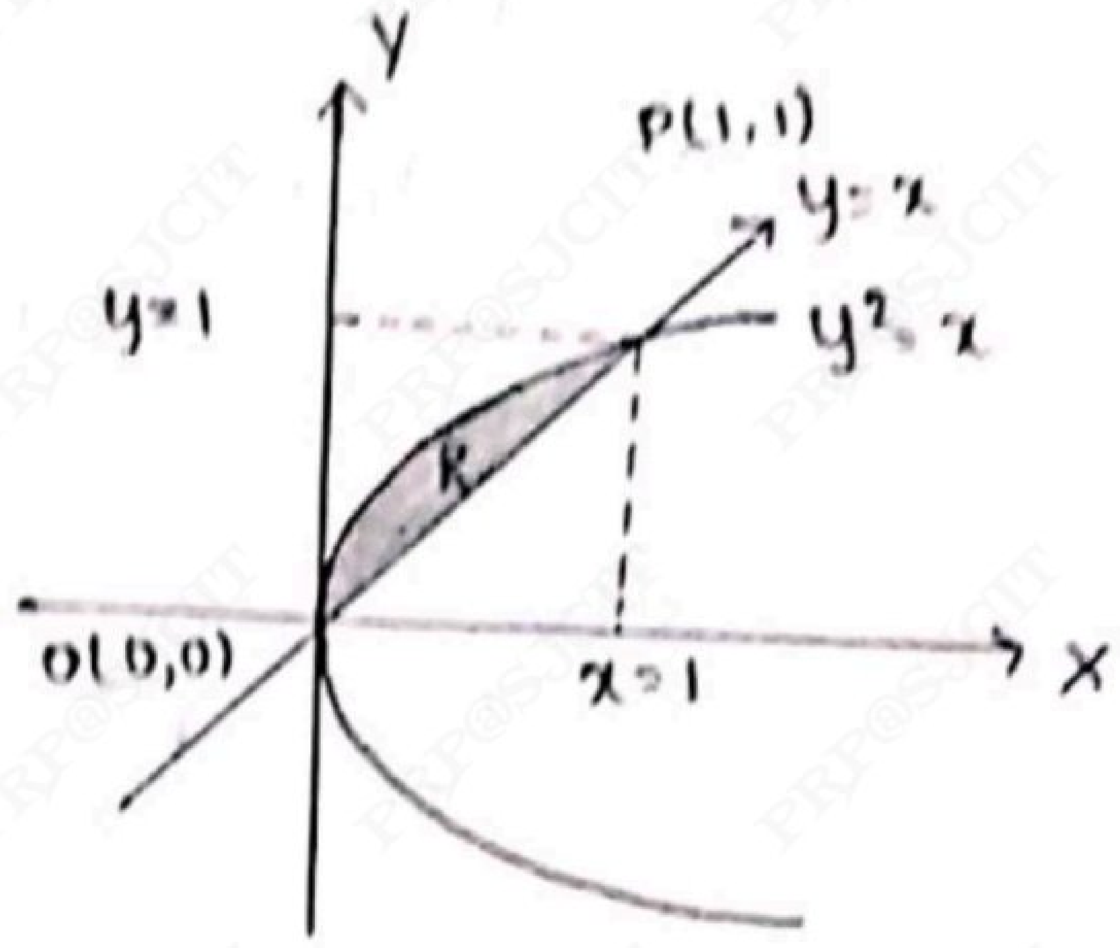
$$= \frac{1}{8} \left[\frac{x^6}{6} - \frac{2a^2x^4}{4} + a^4\frac{x^2}{2} \right]_0^a$$

$$= \frac{1}{8} \left\{ \frac{a^6}{6} - \frac{a^6}{2} + \frac{a^6}{2} \right\} \Rightarrow I = \frac{a^6}{48}$$

⇒ Change of order of Integeration:

1. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$ by change of order of Integeration.

$\Rightarrow I = \int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$



here x varies as $x=0,\ x=1 \rightarrow (1)$

and then y varies as $y=x$

and $y=\sqrt{x} \Rightarrow y^2 = x \rightarrow (2)$

from (1) & (2)

$x = \sqrt{x}$

$\Rightarrow x^2 = x$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x=0,\ x=1$

$\Rightarrow y=0,\ y=1$

∴ By the change of order of Integral

$$I = \int_{y=0}^{1} \int_{x=y^2}^{y} xy\, dx\, dy$$

$$= \int_{y=0}^{1} y \int_{x=y^2}^{y} x\, dx\, dy$$

$$= \int_{y=0}^{1} y \left(\frac{x^2}{2}\right)_{y^2}^{y} dy$$

$$= \int_{y=0}^{1} y \left(\frac{y^2}{2} - \frac{y^4}{2}\right) dy$$

$$= \frac{1}{2}\int_0^1 (y^3 - y^5)\, dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4} - \frac{y^6}{6}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{4} - \frac{1}{6}\right]$$

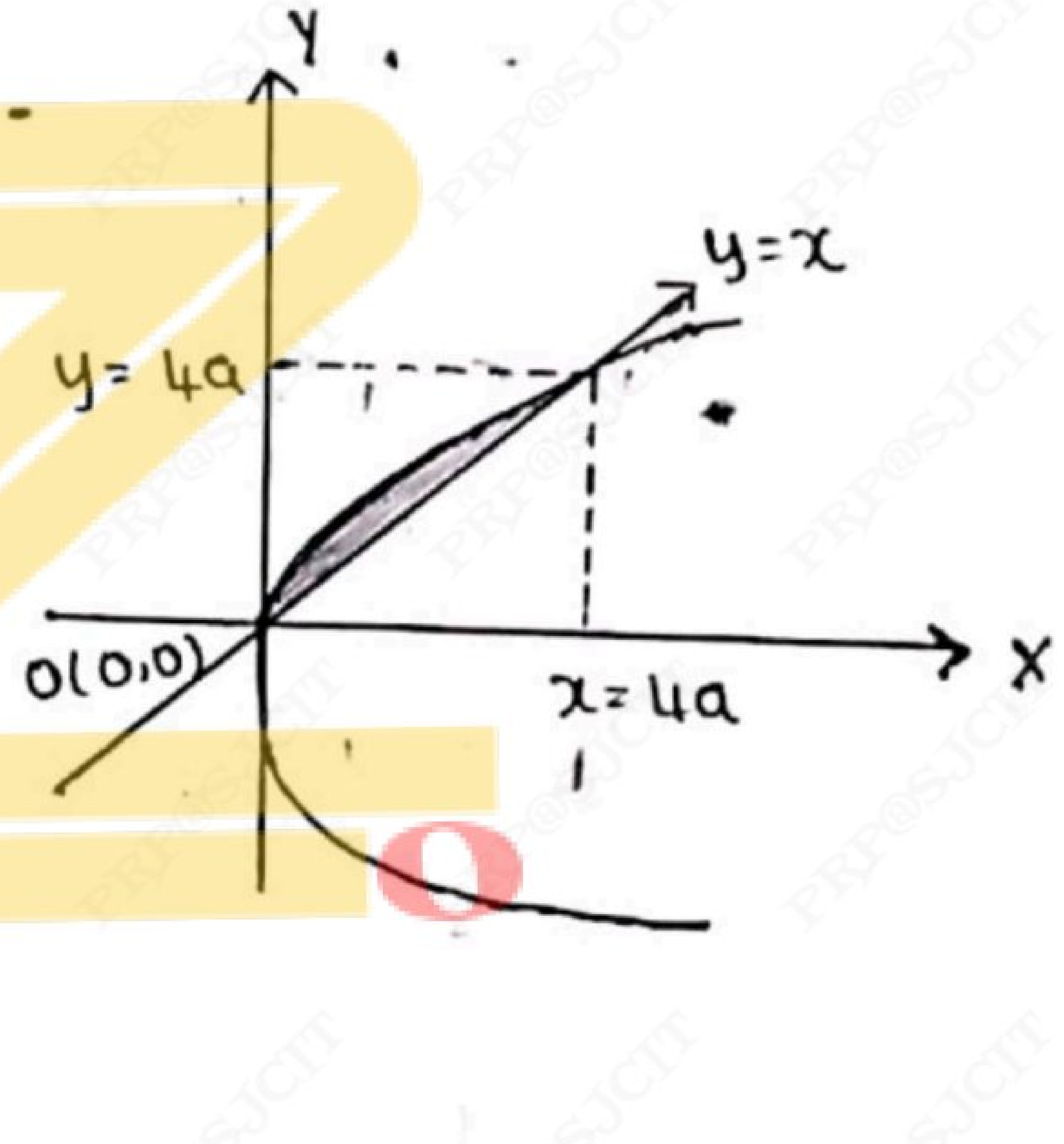
$$= \frac{1}{2}\left[\frac{6-4}{24}\right]$$

$$= \frac{1}{2} \times \frac{2}{24}$$

$$I = \frac{1}{24} \text{ Squnits.}$$

2. Evaluate by change of order of Integtration.

$$\int_0^{4a}\int_x^{2\sqrt{ax}} x^2 \, dy\, dx \quad a > 0.$$



$$I = \int_{x=0}^{4a}\int_{y=x}^{2\sqrt{ax}} x^2 \, dy\, dx \to ①$$

Here, $x=0$, $x=4a$

$y = x$ $\to ①$, $y = 2\sqrt{ax}$ $\Rightarrow y^2 = 4ax$ $\to ②$

From ① and ②

$$x = 2\sqrt{ax}$$

$$\Rightarrow x^2 = 4ax$$

$$\Rightarrow x^2 = 4ax = 0$$

$$\Rightarrow x(x - 4a) = 0$$

$$\Rightarrow x = 0,\ x = 4a$$

$\Rightarrow y=0 \ , \ y=4a$

$$I = \int_{y=0}^{4a} \int_{x=y}^{y^2/4a} x^2 \, dx \, dy$$

$$= \int_{y=0}^{4a} \left[\frac{x^3}{3} \right]_{y}^{y^2/4a} dy$$

$$= \frac{1}{3} \int_{y=0}^{4a} \left[\left(\frac{y^2}{4a} \right)^3 - y^3 \right] dy$$

$$= \frac{1}{3} \int_{y=0}^{4a} \left[\frac{y^6}{64a^3} - y^3 \right] dy$$

$$= \frac{1}{3} \left[\frac{y^7}{7 \times 64a^3} - \frac{y^4}{4} \right]_0^{4a} dy$$

$$= \frac{1}{3} \left[\frac{(4a)^7}{7 \times 64a^3} - \frac{(4a)^4}{4} \right]$$

$$= \frac{1}{3} \left[\frac{4^7 \times a^7}{4^3 \times 7 \times a^3} - \frac{(4)^4 (a^4)}{4} \right]$$

$$= \frac{1}{3} \left[\frac{(4^5)\, a^4 - 7(4^4)(a^4)}{28} \right]$$

$$= \frac{1}{3} \left[\frac{256\, a^4 - 1792 . a^4}{28} \right]$$

$$= \frac{1}{3} . a^4 \frac{[256 - 1792]}{28}$$

$$= \frac{a^4 [-1536]}{28 \times 3}$$

$$= \frac{a^4 [-1536]}{84}$$

$$= a^4 \cdot 18 \cdot 28$$

$$= \frac{1}{3}\left[\frac{(4^5)a^4 - 7(4^4)a^4}{28}\right]$$

$$= \frac{1}{3}\left[4^4\right]\left[\frac{4a^4 - 7a^4}{28}\right]$$

$$= \frac{1}{3}\left[4^4\right]\left[\frac{3a^4}{28}\right]$$

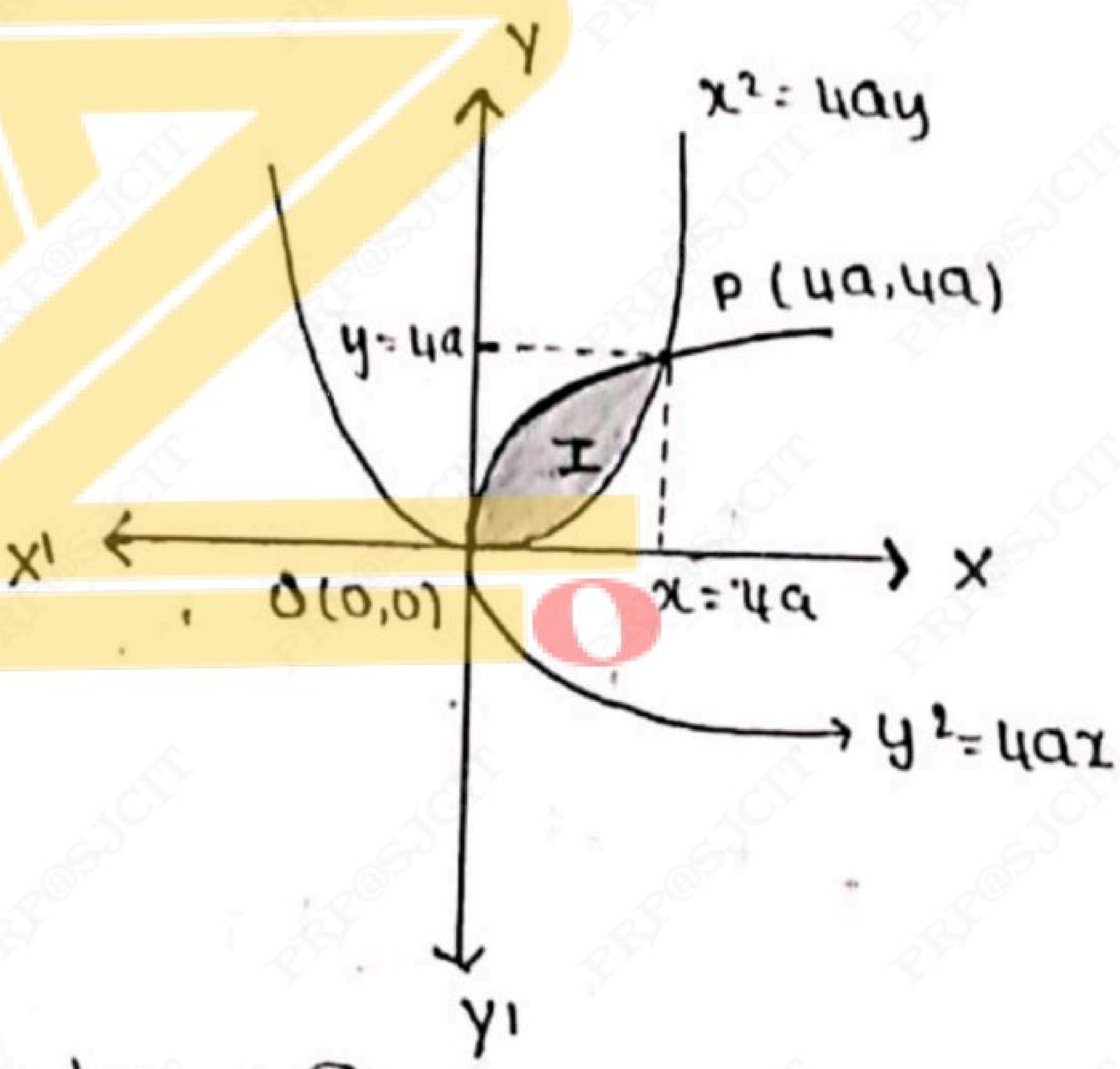
$$= \frac{16 \times 16 \times a^4}{28}$$

$\therefore$

$$I = \frac{64a^4}{7} \text{ sq units}$$

3. Evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx$ by changing the order of integration.

let $I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx$



$$I = \int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx$$

$x = 0$, $x = 4a$ and $y = \frac{x^2}{4a}$

$\Rightarrow x^2 = 4ay$, $y = 2\sqrt{ax} \Rightarrow y^2 = 4a \rightarrow (2)$

From (1) and (2)

$$\frac{x^2}{4a} = 2\sqrt{ax}$$

$$\Rightarrow x^2 = 8a\sqrt{ax}$$

$$x^4 = 64a^2(ax)$$

$$\Rightarrow x^4 - 64a^3x = 0$$

$$\Rightarrow x(x^3 - 64a^3) = 0$$

$$\Rightarrow x = 0, \quad x^3 = 64a^3$$

$$\Rightarrow x^3 = (4a)^3$$

$$\Rightarrow x = 4a$$

where $x = 0 \Rightarrow y = 0$

$x = 4a \Rightarrow y = 4a$

$\therefore$ The change of order of integration we have

$$I = \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} xy \, dx\,dy$$

$$= \int_{y=0}^{4a} y \int_{y^2/4a}^{2\sqrt{ay}} x \, dx\,dy$$

$$= \int_{y=0}^{4a} y \left[\frac{x^2}{2}\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_{y=0}^{4a} \left(y \left[2\sqrt{ay}\right]^2 - \left(\frac{y^2}{4a}\right)^2\right) dy$$

$$= \frac{1}{2}\int_{0}^{4a} y \left[4ay - \frac{y^4}{16a^2}\right] \Big) dy$$

$$= \frac{1}{2}\left\{\frac{4ay^3}{3} - \frac{y^6}{6x/16a^2}\right\}_0^{4a}$$

$$= \frac{1}{2}\left\{\frac{4a}{3}y^3 - \frac{y^6}{96a^2}\right\}_0^{4a}$$

$$= \frac{1}{2}\left[\frac{4a}{3}(4a)^3 - \frac{(4a)^6}{96a^2}\right]$$

$$= \frac{(4a)^4}{2}\left[\frac{1}{3} - \frac{16a^3}{96a^2}\right]$$

$$= \frac{(4a)^4}{2}\left[\frac{1}{3} - \frac{1}{6}\right]$$

$$= \frac{(4a)^4}{2} \times \frac{1}{6}$$

$$= \frac{(4a)^4}{12}$$

$$= \frac{4^4 \times a^4}{12}$$

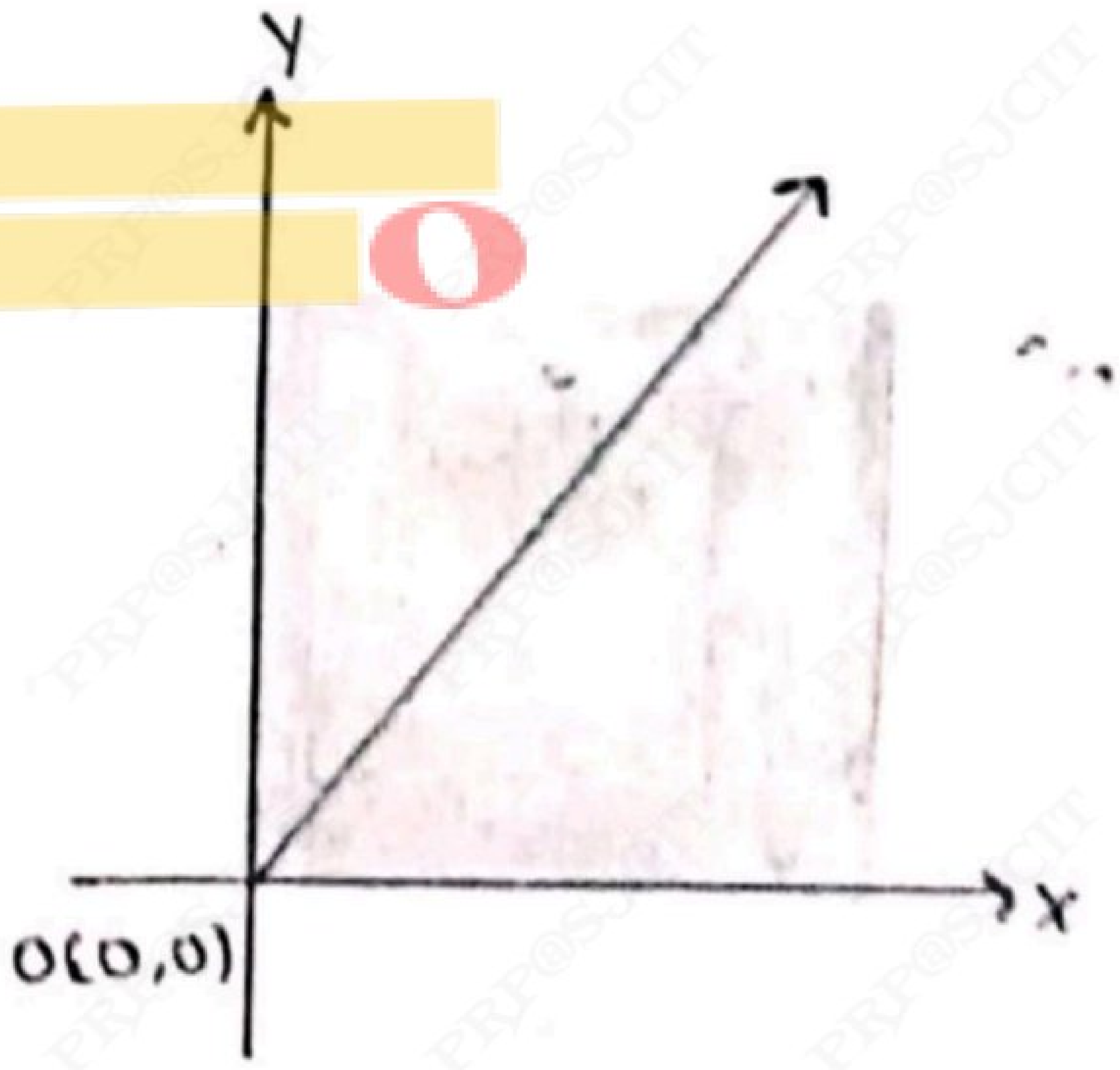
$$= \frac{4 \times 4 \times 4 \times 4 \times a^4}{12}$$

$$I = \frac{64a^4}{3}$$

4. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

$$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx$$

Here x varies from 0 to ∞ and y varies from x to ∞



$\therefore$ By the change of order of integration.

$$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx$$

$$I = \int_{y=0}^{\infty} \int_{x=0}^{y} \left[\frac{e^{-y}}{y}\right] dx\, dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} \int_{x=0}^{y} dx\,dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} [x]_0^y \, dy$$

$$= \int_{y=0}^{\infty} e^{-y} \, dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= [e^{-y}]_0^{y}$$

$$= -[e^{-\infty} - e^{0}]$$

$$= -[0-1]$$

$$= -(-1)$$

$$\Rightarrow I = 1$$

5. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$ using change of order of integration.

let $I = \int_{x=0}^{1} \int_{y=x^2}^{2-x} xy \, dy \, dx$

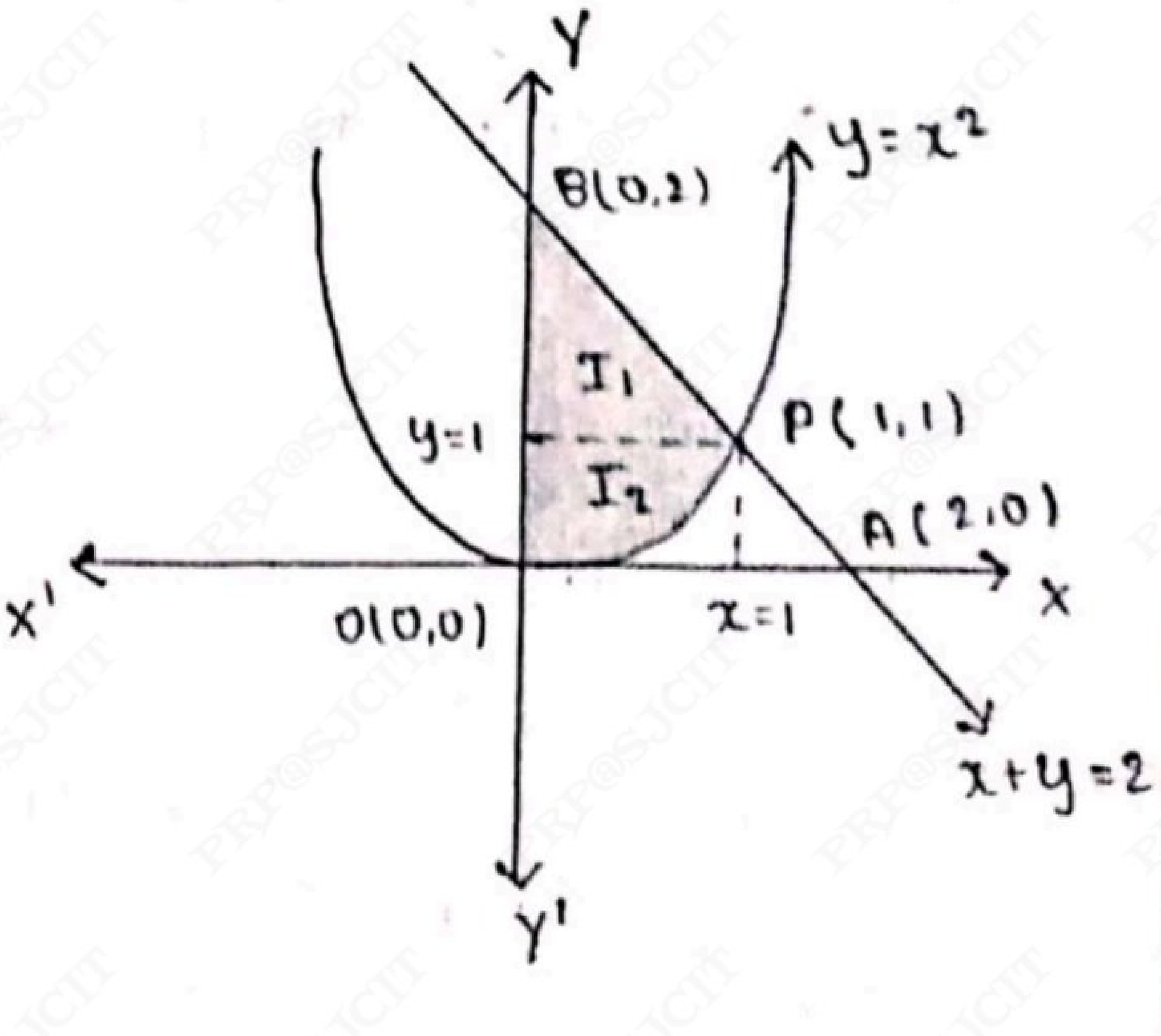
x varies from 0 to 1 and

y varies from $y = 2-x$

$\Rightarrow y + x = 2$

or

$x + y = 2$



$x + y = 2$

$\frac{x}{2} + \frac{y}{2} = 2$

By change of order of integration the boundary region written as in

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy \, dx \, dy$$

$$I = I_1 + I_2$$

$$I_1 = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \cdot dx \, dy$$

$$= \int_{y=0}^{1} y \left[\frac{x^2}{2} \right]_0^{\sqrt{y}} dy$$

$$= \frac{1}{2} \int_0^1 y^2 \, dy$$

$$= \frac{1}{2} \left[\frac{y^3}{3} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{3} - 0 \right]$$

$$I_1 = \frac{1}{6}$$

$$I_2 = \int_{y=1}^{2} \int_{x=0}^{2-y} xy \, dx \, dy$$

$$= \int_{y=1}^{2} y \left[\frac{x^2}{2} \right]_0^{2-y} dy$$

$$= \frac{1}{2} \int_{y=0}^{2} y (2-y)^2 \, dy$$

$$= \frac{1}{2} \int_{y=1}^{y=1} y (y^2 - 4y + 4) \, dy$$

$$= \frac{1}{2}\int_{y=1}^{2} (y^3 - 4y^2 + 4y)\, dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4} - \frac{4y^3}{3} + \frac{4y^2}{2}\right]_1^2$$

$$= \frac{1}{2}\left\{\left[\frac{16}{4} - \frac{32}{3} + \frac{16}{2}\right] - \left[\frac{1}{4} - \frac{4}{3} + \frac{4}{2}\right]\right\}$$

$$= \frac{1}{2}\left[4 - \frac{32}{3} + 8 - \frac{1}{4} + \frac{4}{3} - 2\right]$$

$$= \frac{1}{2}\left[10 - \frac{28}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{120 - 1125 - 3}{12}\right]$$

$$= \frac{1}{2}\left[\frac{5}{12}\right]$$

$$\Rightarrow I_2 = \frac{5}{24}$$

$$I = I_1 + I_2$$

$$I = \frac{1}{6} + \frac{5}{24}$$

$$I = \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8} \,//$$

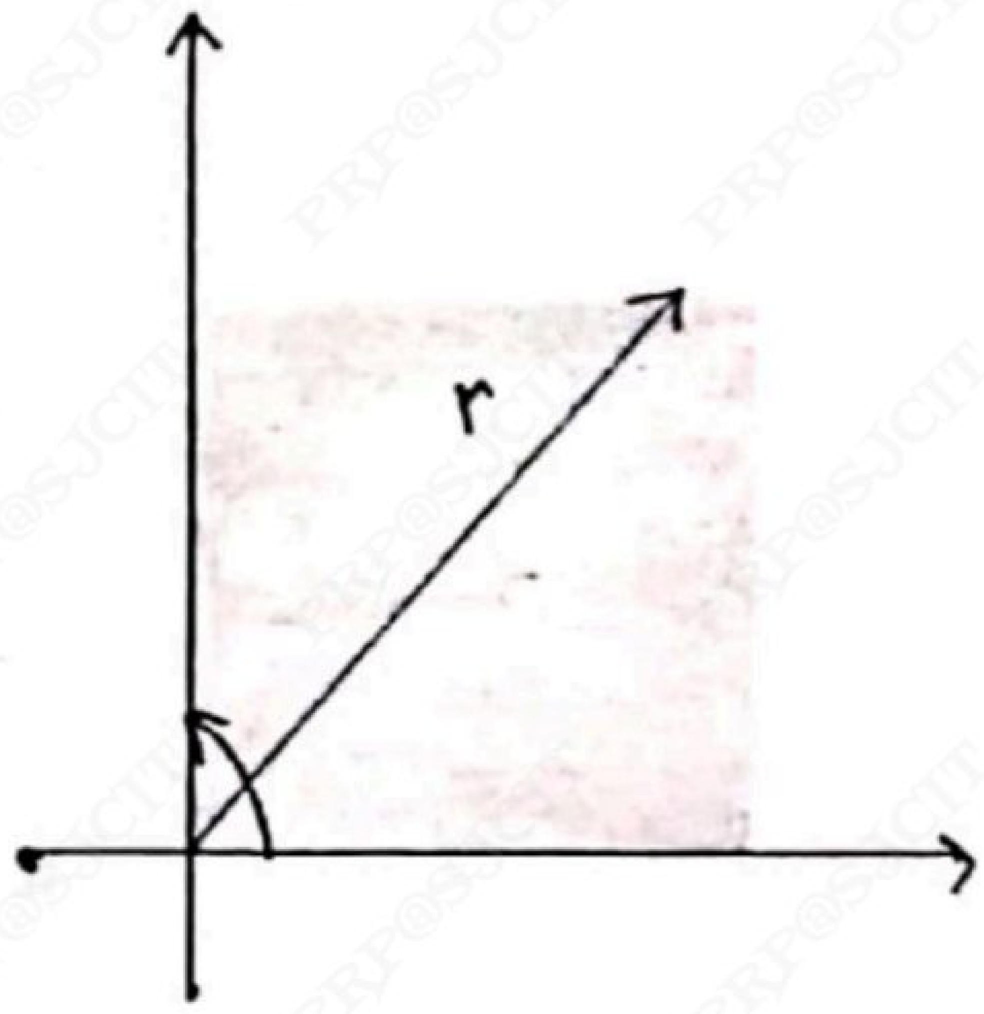
6. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by change into polar co-ordination.

Let $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy \longrightarrow (1)$

$$\Rightarrow I = \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)}\, dy\, dx$$

Let $x = r\cos\theta \ , y = r\sin\theta$

$\Rightarrow dx\,dy = r\,dr\,d\theta$



$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta \rightarrow (2)$$

Let $r^2 = t$

$\Rightarrow 2r\,dr = dt$

$\Rightarrow r\,dr = \frac{1}{2}\,dt$

$\therefore$ UL : $r = \infty \Rightarrow t = \infty$

LL : $r = 0 \Rightarrow t = 0$

$$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t}\, \frac{1}{2}\, dt\, d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t}\, dt . d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} \left[\frac{e^{-t}}{-1}\right]_{t=0}^{\infty} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} \left[e^{-t}\right]_{0}^{\infty} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} \left[e^{-\infty} - e^{0}\right] d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} (0-1)\, d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} 1\, d\theta$$

$$= \frac{1}{2}\left[\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right]$$

$$\Rightarrow I = \frac{\pi}{4}$$

7. Evaluate $\int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-(x^2+y^2)}\, dy\, dx = \frac{\pi}{4}$

$$I = \int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-x^2} \cdot e^{-y^2}\, dy\, dx = \frac{\pi}{4} \longrightarrow (2)$$

$$= \int_{x=0}^{\infty} e^{-x^2}\, dx \int_{y=0}^{\infty} e^{-y^2}\, dy = \frac{\pi}{4} \longrightarrow (3)$$

Let $x = y$

$$(3) \Rightarrow \int_{x=0}^{\infty} e^{-x^2}\, dx \int_{x=0}^{\infty} e^{-x^2}\, dx = \frac{\pi}{4}$$

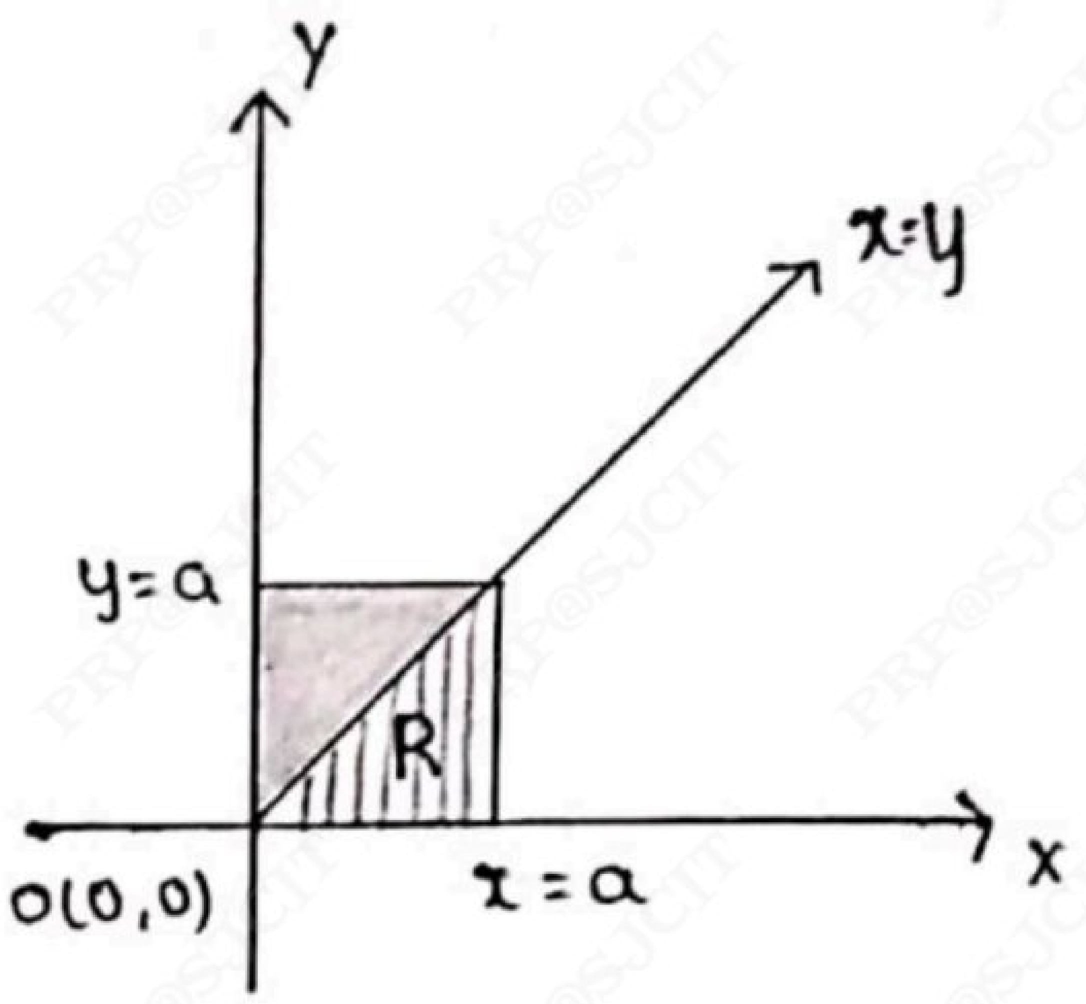
$$\Rightarrow \left[\int_{x=0}^{\infty} e^{-x^2}\, dx\right]^2 = \frac{\pi}{4}$$

$$\Rightarrow \int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

8. Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$

$$I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$$

$$I = \int_{y=0}^{a} \int_{x=y}^{a} \frac{x}{x^2+y^2}\, dx\, dy$$



$\therefore$ By change the order of integration, we have.

$$I = \int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2}\, dy\, dx$$

$$= \int_{x=0}^{a} x \int_{y=0}^{x} \frac{1}{x^2+y^2}\, dy\, dx$$

$$= \int_{x=0}^{a} x\, \frac{1}{x}\left[\tan^{-1}(y/x)\right]_{y=0}^{x} dx$$

$$= \int_{x=0}^{a} \left[\tan^{-1}(y/x)\right]_{y=0}^{x} dx$$

$$= \int_{x=0}^{a} \left[\pi/4 - 0\right] dx$$

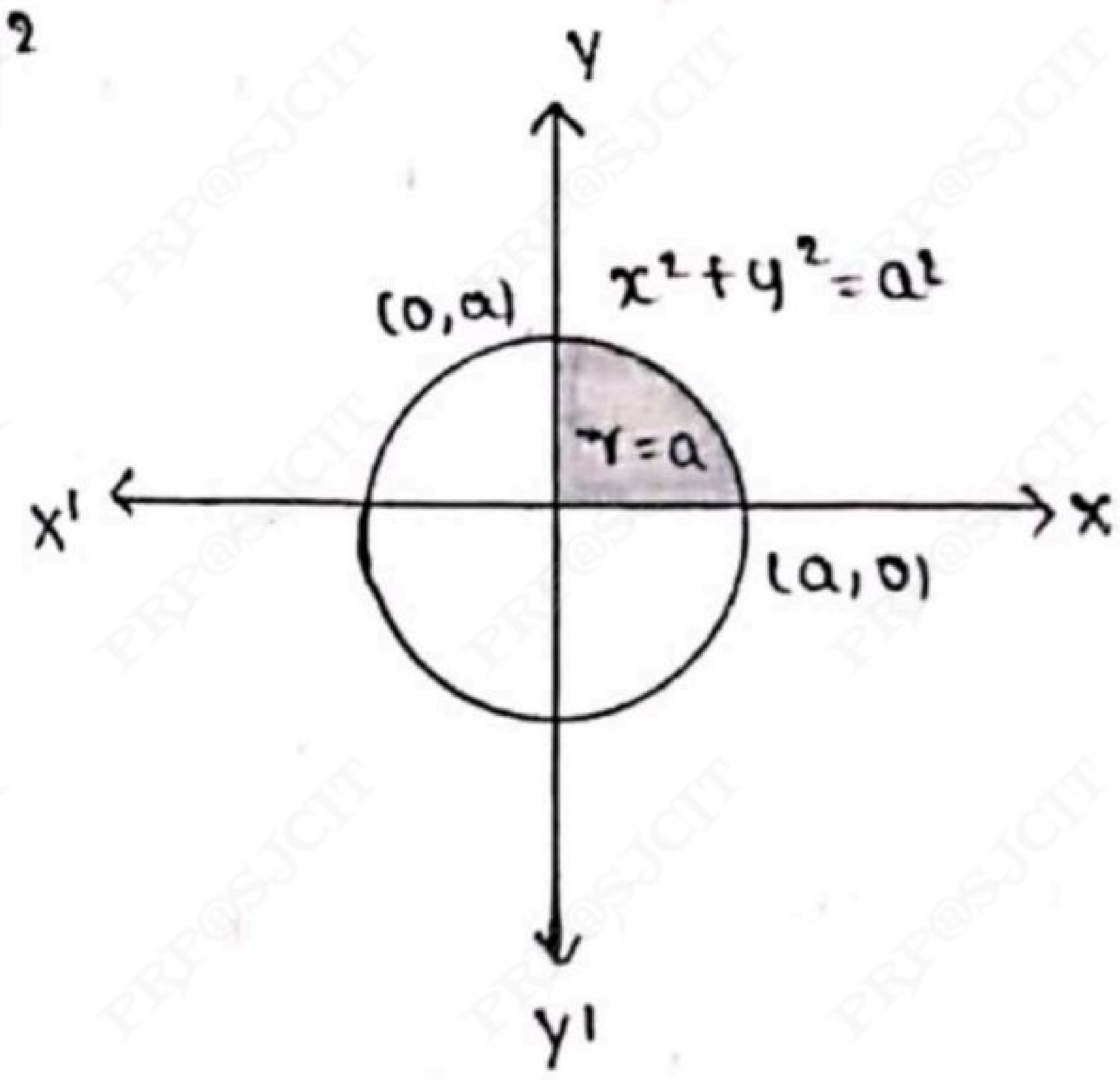
$$= \frac{\pi}{4} \int_{x=0}^{a} 1\, dx$$

$$= \frac{\pi}{4} \left[x\right]_0^a$$

$$I = \frac{\pi a}{4}$$

9. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}$ by change of polar coordinates

Here x varies from 0 to $\sqrt{a^2-y^2}$



$$\Rightarrow x = \sqrt{a^2-y^2}$$

$$x^2 = a^2-y^2$$

$$x^2+y^2=a^2$$

y varies from 0 to a

let $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow dx\,dy = r\,dr\,d\theta$$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\sin\theta \, . \, r \, . \, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^3 \sin\theta\,dr\,d\theta$$

$$\Rightarrow I = \int_{\theta=0}^{\pi/2} \sin\theta\,d\theta \int_{r=\theta}^{a} r^3\,dr$$

$$= -\left[\cos\theta\right]_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a$$

$$= -\left[\left[\cos \pi/2\right] - \cos 0\right]\left[\frac{a^4}{4}\right]$$

$$= (0-1)\frac{a^4}{4}$$

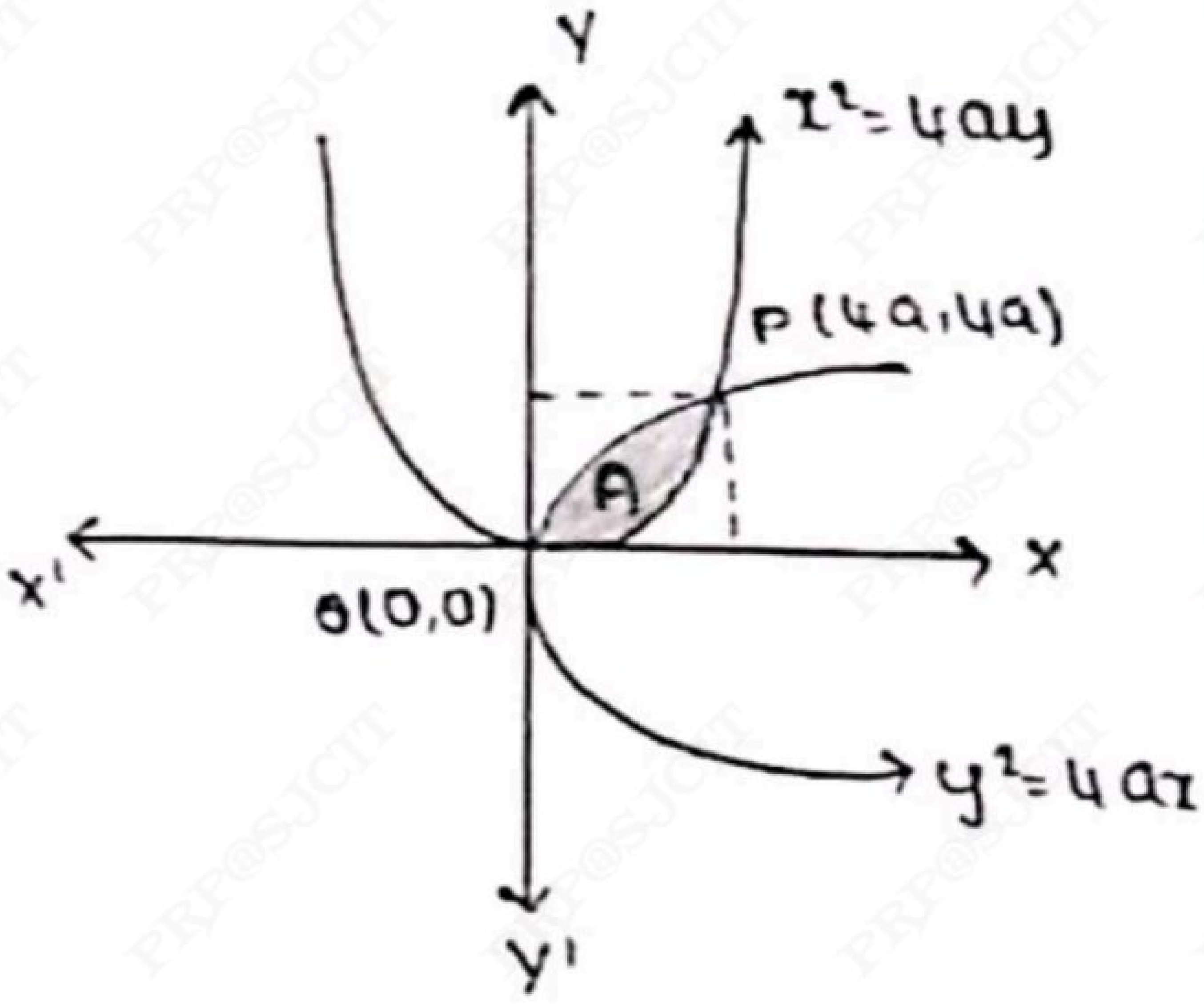
$$I = \frac{a^4}{4}$$

APPLICATIONS

1. Find the area between the parabolas. $y^2=4ax$, $x^2=4ay$

The area between the given two parabolas $y^2=4ax$, $x^2=4ay$ can be evaluated as

$$A = \iint dx\,dy$$



$$A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}}$$

$$= \int_{x=0}^{4a} (y) \Big|_{x^2/4a}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} \left[2\sqrt{a}x - \frac{x^2}{4a} \right] dx$$

$$= 2\sqrt{a} \int_0^{4a} \sqrt{x}\, dx - \frac{1}{4a} \int_0 x^2\, dx$$

$$= 2\sqrt{a} \int_0^{4a} \sqrt{x}\, dx - \frac{1}{4a} \int_0^{4a} x^2\, dx$$

$$= 2\sqrt{a} \left[\frac{x^{3/2}}{3/2} \right]_0^{4a} - \frac{1}{4a} \left[\frac{x^3}{3} \right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3} \left[x^{3/2} \right]_0^{4a} - \frac{1}{12a} \left[x^3 \right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3} (4a)^{3/2} - \frac{1}{12a} (4a)^3$$

$$= \frac{4\sqrt{a}}{3} (4a) 2\sqrt{a} \cdot \frac{1}{12a} 4a \times 4a^2$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$\Rightarrow A = \frac{16a^2}{3} \text{ Sq units.}$$

$$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dx\,dy$$

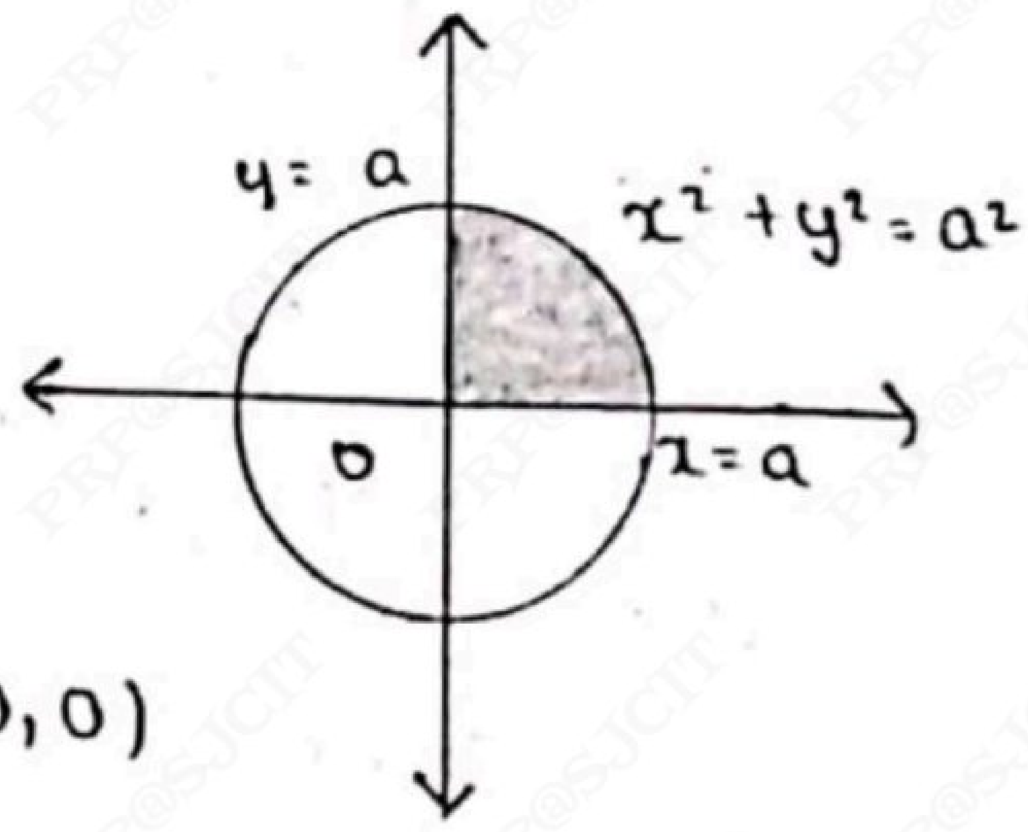
$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (x^2+y^2)\, dx\, dy \Rightarrow ①$$

x varies from $x=0$ to $x=\sqrt{a^2-y^2}$

$$x = \sqrt{a^2-y^2}$$

$$x^2 = a^2-y^2$$

$$x^2+y^2 = a^2$$



It is a circle having centre $(0,0)$ will radius 'a' lies from 0 to a

let $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$I = \int_{\theta=0}^{\pi/2} \int_{0}^{a} r^2(\cos^2\theta + \sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{0}^{a} r^2\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \frac{1}{4}\int_{\theta=0}^{\pi/2} a^4\, d\theta$$

$$= \frac{a^4}{4}\int_{0}^{\pi/2} 1.\, d\theta$$

$$= \frac{a^4}{4}\int_{0}^{\pi/2} 1.\, d\theta$$

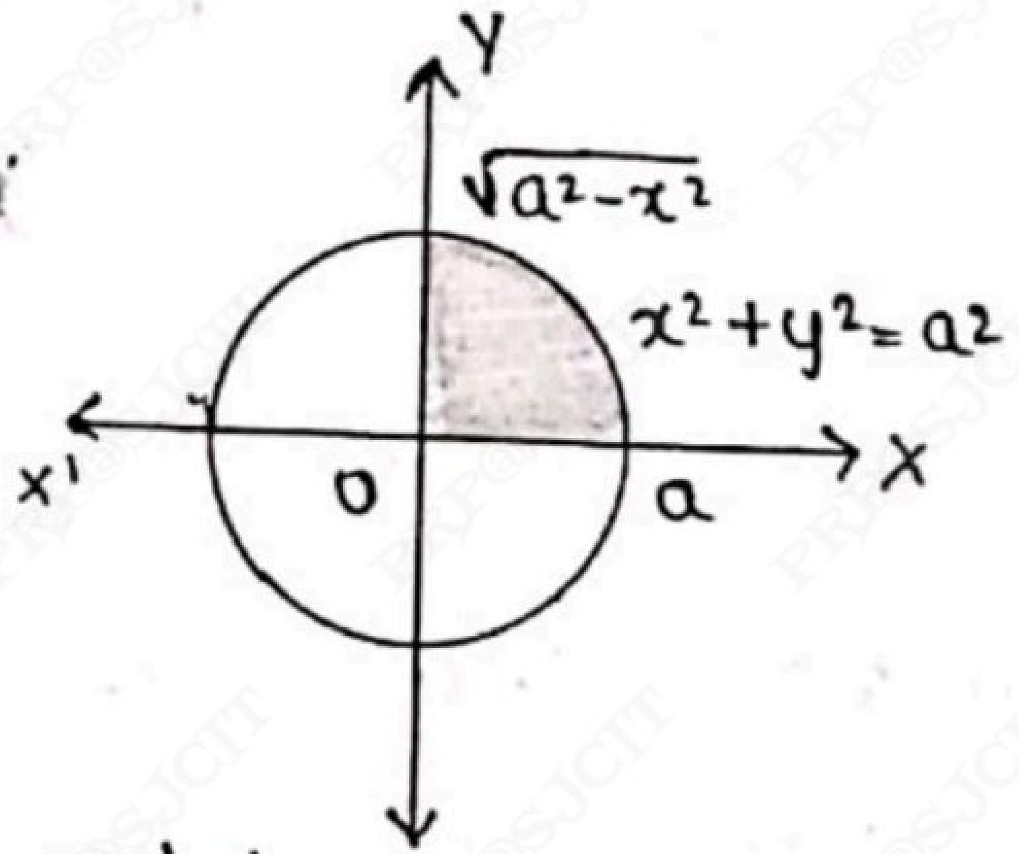
$$= \frac{a^4}{4}[1]_0^{\pi/2}$$

$$I = \frac{a^4}{4}\left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi a^4}{8}$$

2. Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dx\, dy$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dx\, dy$$



x varies from $x=0$, to $x=a$

y varies from $y=0$ to $\sqrt{a^2-x^2}$

Let $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$I = \int_{\theta=0}^{\pi/2} \int_0^a y^2\sqrt{x^2+y^2}\, dx\, dy$$

$$= \int_{\theta=0}^{\pi/2} \int_0^a r^2\sin^2\theta\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\,.\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_0^a r^2\sin^2\theta\,.\,r,\ r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_0^a r^4\ \sin^2\theta\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta \int_0^a r^2\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} -2\sin\theta\cos\theta\left[\frac{r^5}{5}\right]_0^a d\theta$$

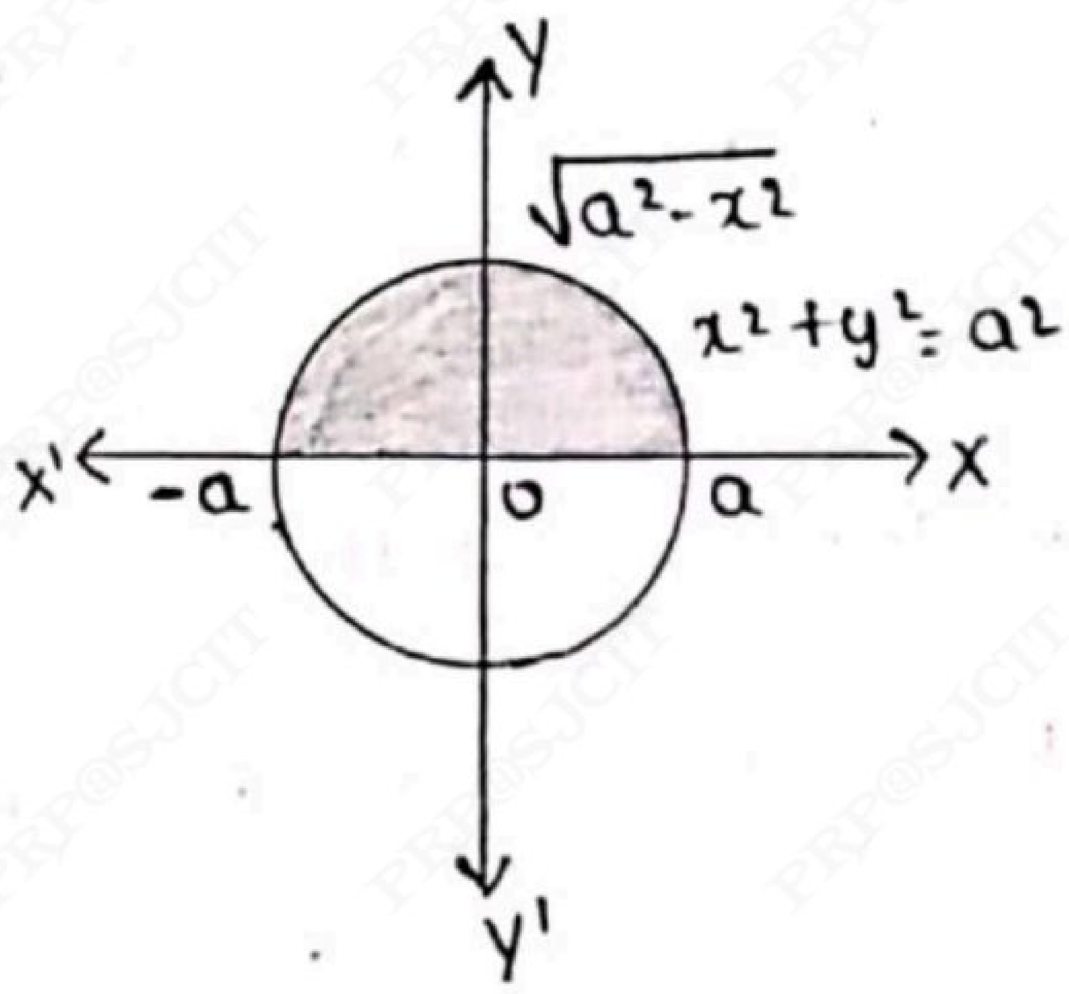
$$= \left[\frac{r^5}{5}\right]_0^a\left[\frac{2-1}{2} - \frac{\pi}{2}\right]$$

$$= \left[\frac{a^5}{5} - 0\right]\left[\frac{1}{2} \cdot \frac{\pi}{2}\right]$$

$$I = \frac{\pi a^5}{20} \text{ sq. units}$$

3. Evaluate change of order of integral $\int_{-a}^{a}\int_{0}^{\sqrt{a^2 x^2}} \sqrt{x^2+y^2}\,dy\,dx$

Let, $I = \int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\; dy\,dx$



Let $x = r\cos\theta$, $y = r\sin\theta$

then x varies from $x = 0$ to a

y varies from $y = 0$ to $y = \sqrt{a^2-x^2}$

$\Rightarrow y = \sqrt{a^2-x^2}$

$x^2 + y^2 = a^2$

$$I = \int_0^{\pi}\int_0^{a} \sqrt{x^2+y^2}\; dy\,dx$$

$$= \int_0^{\pi}\int_0^{a} \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\; r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\int_0^{a} r\; r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{a} r^2\,dr\;d\theta$$

$$= \int_0^{\pi}\left[\frac{r^2}{3}\right]_0^a d\theta$$

$$= \frac{a^3}{3}\left[\pi - 0\right]$$

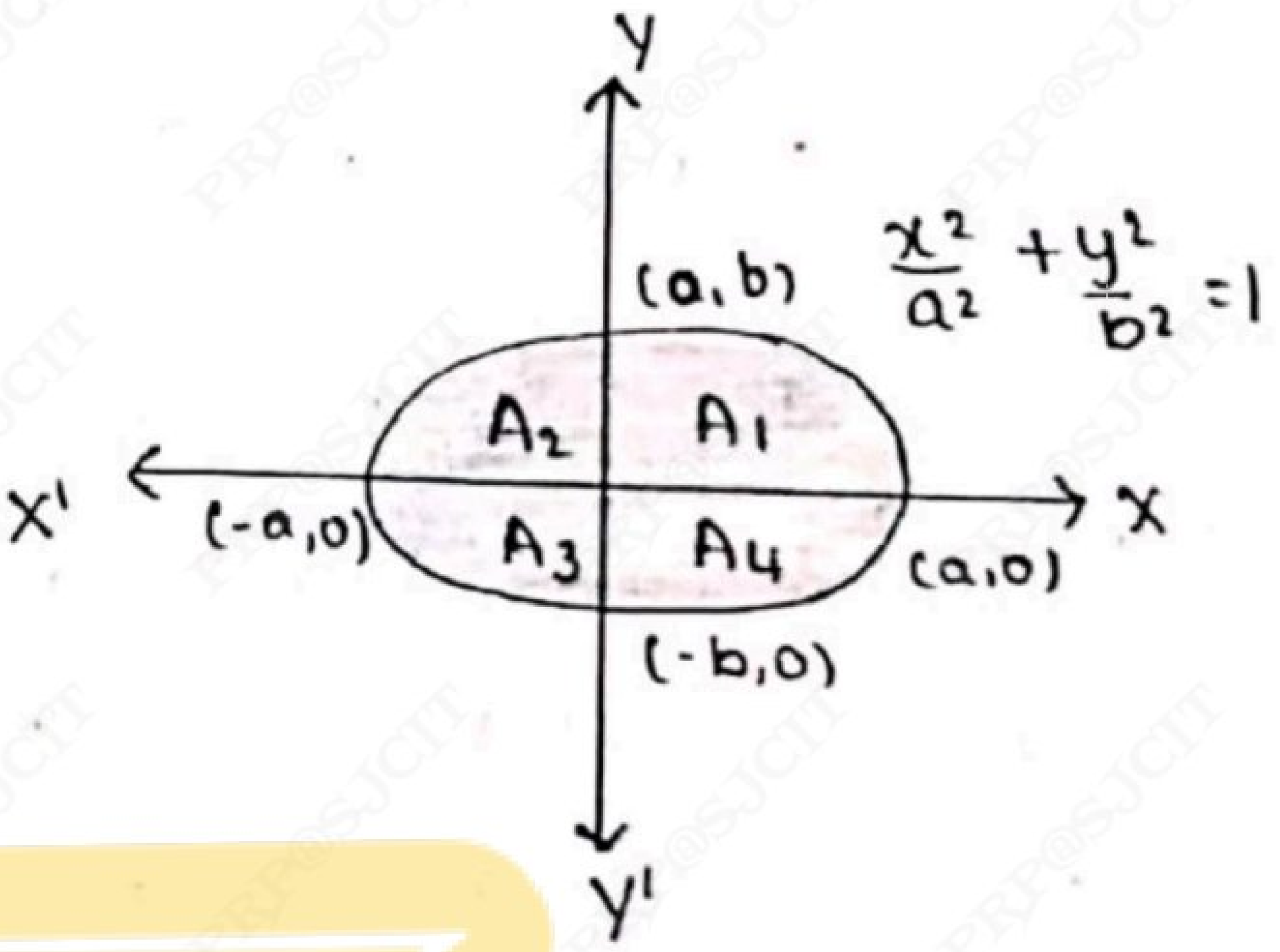
$$I = \frac{\pi a^3}{3}$$

④. use double integration to find the area of a ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and hence find the area of the circle $\frac{x^2}{a^2}+\frac{y^2}{b^2}=a^2$.

$\Rightarrow$ Given the ellipse $x^2+y^2=a^2 \longrightarrow ①$

Covered the area with the co-ordinate axis as shown in the diagram.

$\therefore$ The area of the ellipse is



$A = A_1 4$

$A = 4\iint dx\,dy$

$\Rightarrow A = 4\int_{x=0}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}} 1.\,dy\,dx$

$A = 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\,dx$

$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$A = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx \longrightarrow ②$

Let $x = a\sin\theta \Rightarrow \theta = \sin^{-1}(x/a)$

$\Rightarrow dx = a\cos\theta\,d\theta$

U.L : $x=a \Rightarrow \theta = \frac{\pi}{2}$

L.L : $x=0 \Rightarrow \theta = 0$

$\therefore A = \frac{4b}{a}\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta$

$= \frac{4b}{a}\int_0^{\pi/2} a\sqrt{1-\sin^2\theta}\,.\,a\cos\theta\,d\theta$

$$= 4b\int_0^{\pi/2} a\cos^2\theta\, d\theta$$

$$= 4ab\int_0^{\pi/2}\left[\frac{1+\cos\theta}{2}\right] d\theta$$

$$= 2ab\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 2ab\left[\pi/2 + 0\right]$$

$$\Rightarrow A = 2ab\left(\frac{\pi}{2}\right)$$

$$A = \pi ab \text{ Sq units}$$

when $b=a$, then the given ellipse becomes a circle $x^2+y^2=a^2$ and its area is

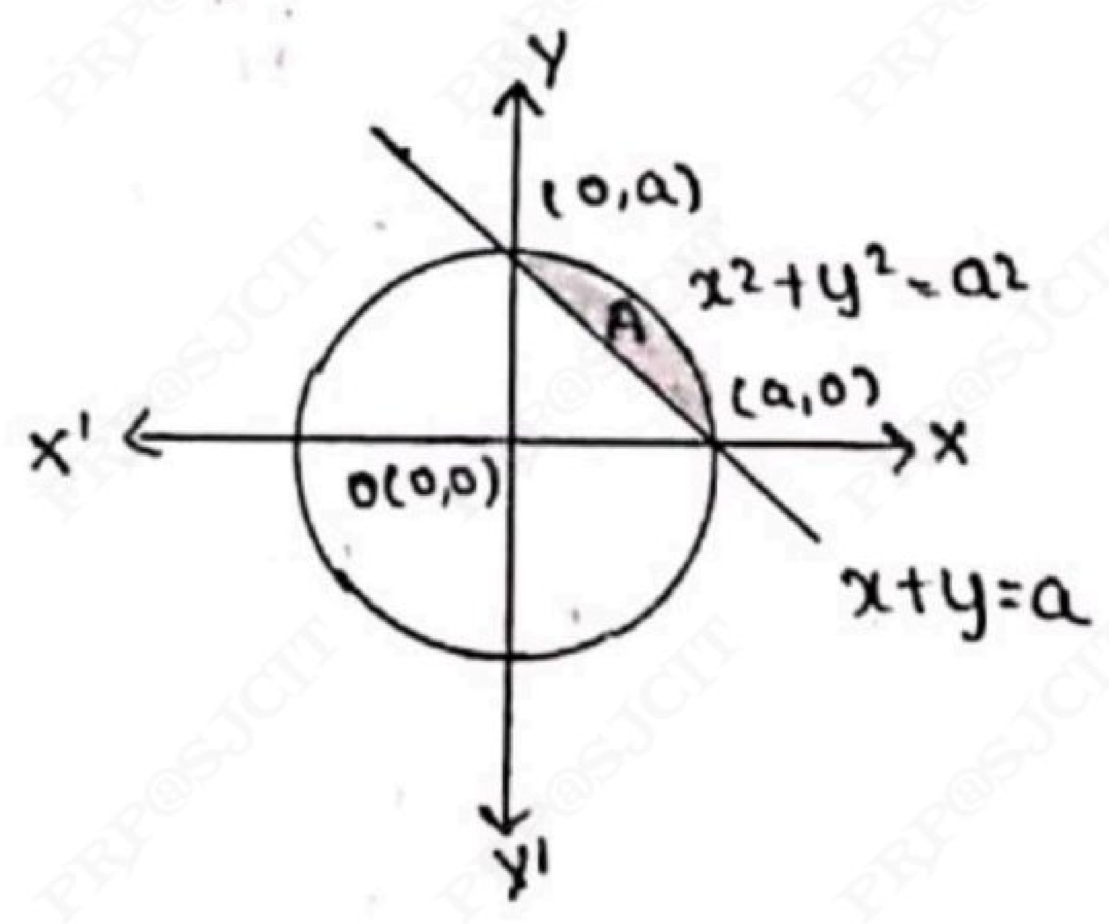
$$A = \pi \times a \times a = \pi a^2 \text{ Sq units}$$

5. Find the area by double integration between the circle $x^2+y^2=a^2$ and to the straight line $x+y=a$

→ Given that the straight line cuts the x axis at $(a,0)$ and y = axis at $(0,a)$

∴ The area between the circle $x^2+y^2=a^2$ and the straight line $x+y=a$ is $A=\iint dx\, dy$

$$A = \int_0^a \int_{y=a-x}^{\sqrt{a^2-x^2}} 1\, dy\, dx$$



$$= \int_0^a [y]_{y=a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left(\sqrt{a^2-x^2} - (a-x)\right) dx$$

$$A = \int_{x=0}^{a} \sqrt{a^2-x^2}\, dx - \int_0^a (a-x)\, dx$$

let $x = a\sin\theta \Rightarrow \theta = \sin^{-1}(x/a)$

$\Rightarrow dx = a\cos\theta$

UL : $x=a \Rightarrow \theta = \frac{\pi}{2}$

LL : $x=0 \Rightarrow \theta = 0$

$$\therefore A = \int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta}\,.\,a\cos\theta\, d\theta - \left[ax - \frac{x^2}{2}\right]_0^a$$

$$= \int_0^{\pi/2} a^2\cos^2\theta\, d\theta - \left\{ \left(a^2 - \frac{a^2}{2}\right) - (0-0) \right\}$$

$$= a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta - \frac{a^2}{2}$$

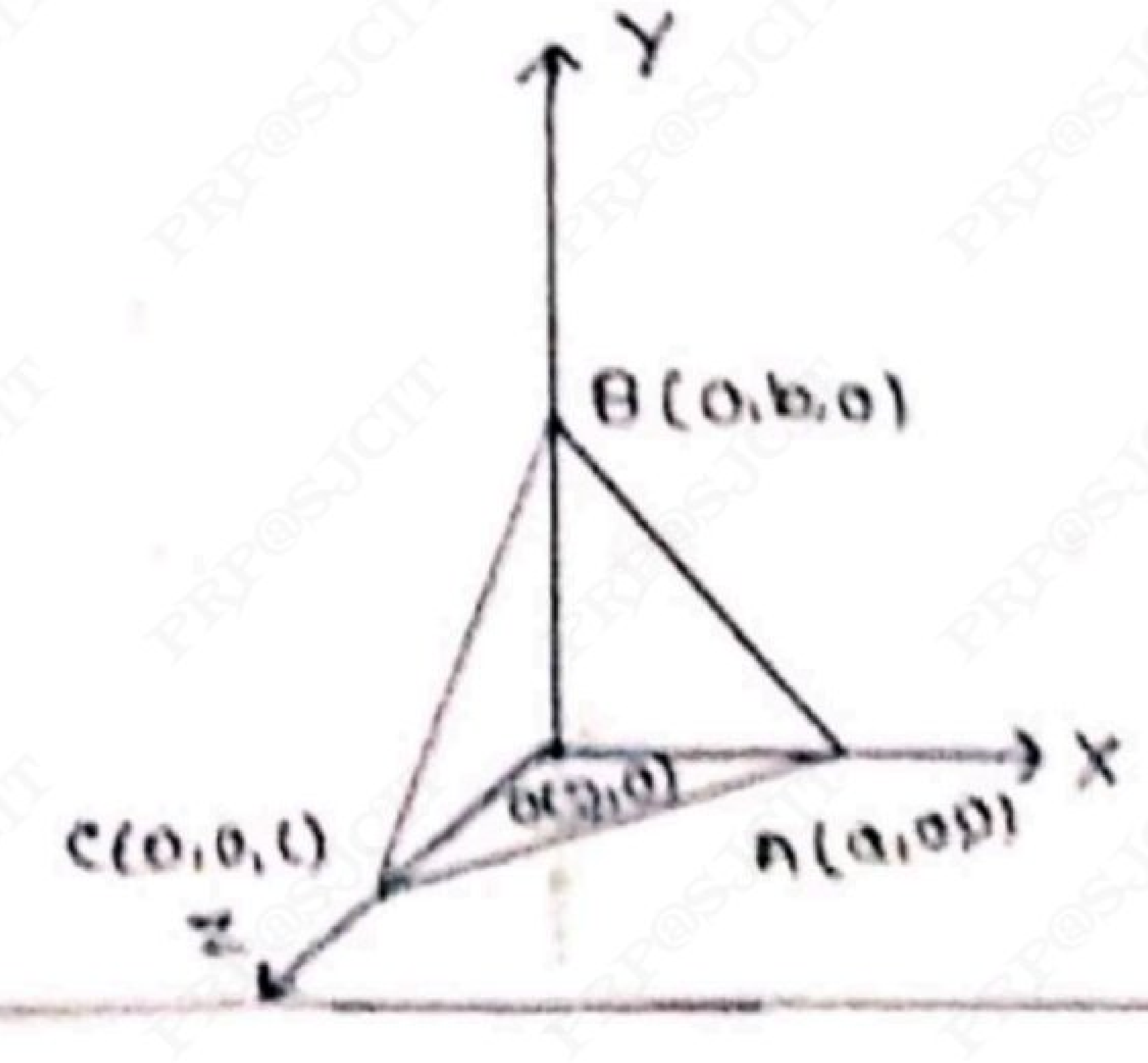
$$= a^2 \left(\frac{2-1}{2}\right) \frac{\pi}{2} - \frac{a^2}{2}$$

$$A = \frac{\pi a^2}{4} - \frac{a^2}{2} \text{ sq units.}$$

$\Rightarrow$ VOLUME OF TETRAHYDRAL

1. Find the volume of tetrahydral bounded by the planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\Rightarrow$ Given tetrahydral bounded by the plan $x=0$, $y=0$, $z=0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$



$\therefore$ The volume $V = \iiint dv$

$$= \int_{x=0}^{a} \int_{y=0}^{b/a(a-x)} \int_{z=0}^{c(1-x/a-y/b)} 1\, dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b/a(a-x)} [z]_0^{c[1-\frac{x}{a}-\frac{y}{b}]} dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b/a(a-x)} c\left[1-\frac{x}{a}-\frac{y}{b}\right] dy\,dx$$

$$= c\int_{x=0}^{a} \left[y-\frac{xy}{a}-\frac{y^2}{2b}\right]_{y=0}^{b/a(a-x)} dx$$

$$= c\int_{x=0}^{a} \left[\frac{b}{a}(a-x)-\frac{x}{a}\cdot\frac{b}{a}(a-x)-\frac{1}{2b}\frac{b^2}{a^2}(a-x^2)\right]dx$$

$$= c\int_0^a \left[\frac{b}{a}(a-x)-\frac{b}{a^2}x(a-x)-\frac{b}{2a^2}(a-x^2)\right]dx$$

$$= c\int_0^a \frac{b}{a^2}(a-x)(a-x)-\frac{1}{2}(a-x)]\,dx$$

$$= c\int_0^a \frac{b}{a^2}(a-x)^2\left[1-\frac{1}{2}\right]dx$$

$$= c\int_0^a \frac{b}{2a^2}(a-x^2)\,dx$$

$$= \frac{bc}{2a^2}\int_0^a (x^2-2ax+a^2)\,dx$$

$$= \frac{bc}{2a^2}\left[\frac{x^3}{3}-ax^2+a^2x\right]_0^a$$

$$= \frac{bc}{2a^2}\left[\frac{a^3}{3}-a^3+a^3\right]$$

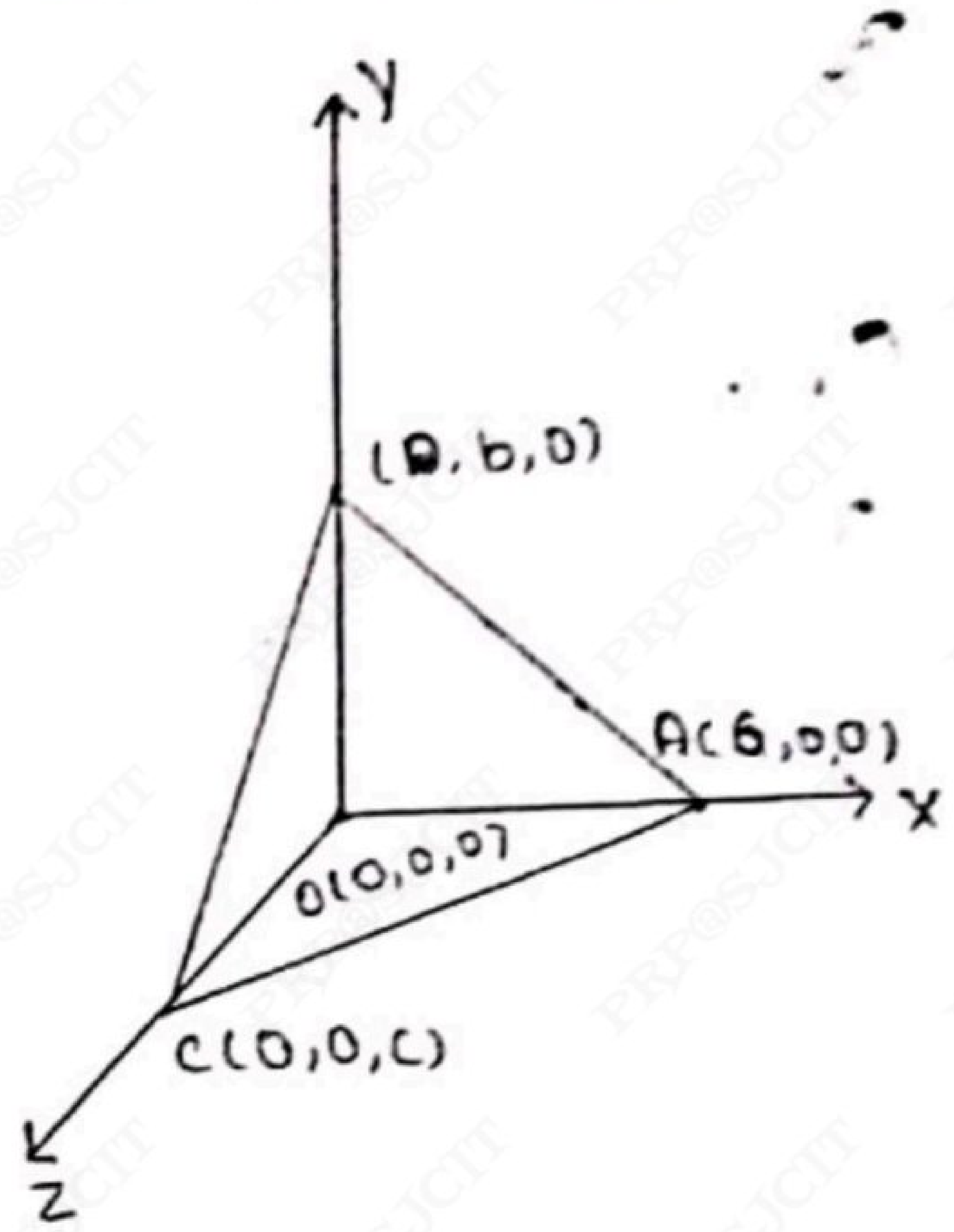
$$= V = \frac{bc}{2a^2}\times\frac{a^3}{3}$$

$\Rightarrow$ $V = \frac{abc}{6}$ cubic meter.

2. Find the volume of the tetrahydral bounded by the planes, $x=0$, $y=0$, $z=0$ and $x+2y+3z=6$.

$\Rightarrow$ Given that the volume of tetrahydral bounded by plans

$x=0,\ y=0,\ z=0$ and $x+2y+3z=6$



$V = \iiint 1\,dV$

$$= \int_{x=0}^{6} \int_{y=0}^{1/2(6-x)} \int_{z=0}^{1/3(6-x-2y)} 1 \cdot dz\,dy\,dx$$

$$= \int_{x=0}^{6} \int_{y=0}^{1/2(6-x)} [z]^{1/3(6-x-2y)} (1)$$

$$= \frac{1}{3} \int_{x=0}^{6} [6y - xy - 2y]\ dy\,dx$$

$$= \frac{1}{3} \int_{x=0}^{6} [6x - xy - y^2]_{y=0}^{1/2(6-x)}$$

$$= \frac{1}{3} \int_{0}^{6} (6-x)\ y - y^2 \int_{y=0}^{1/2(6-x)} dx$$

$$= \frac{1}{3} \int_{0}^{6} \left[(6-x)\frac{(6-x)}{2} - \frac{(6-x)^2}{4} \right] dx$$

$$= \frac{1}{3} \int_{0}^{6} \left[\frac{(6-x)^2}{2} - \frac{(6-x)^2}{4} \right] dx$$

$$= \frac{1}{3} \int_{0}^{6} \left(\frac{1}{2} - \frac{1}{4} \right) (6-x)^2\ dx$$

$$= \frac{1}{3} \int_{0}^{6} \frac{1}{4} (6-x)^2\ dx$$

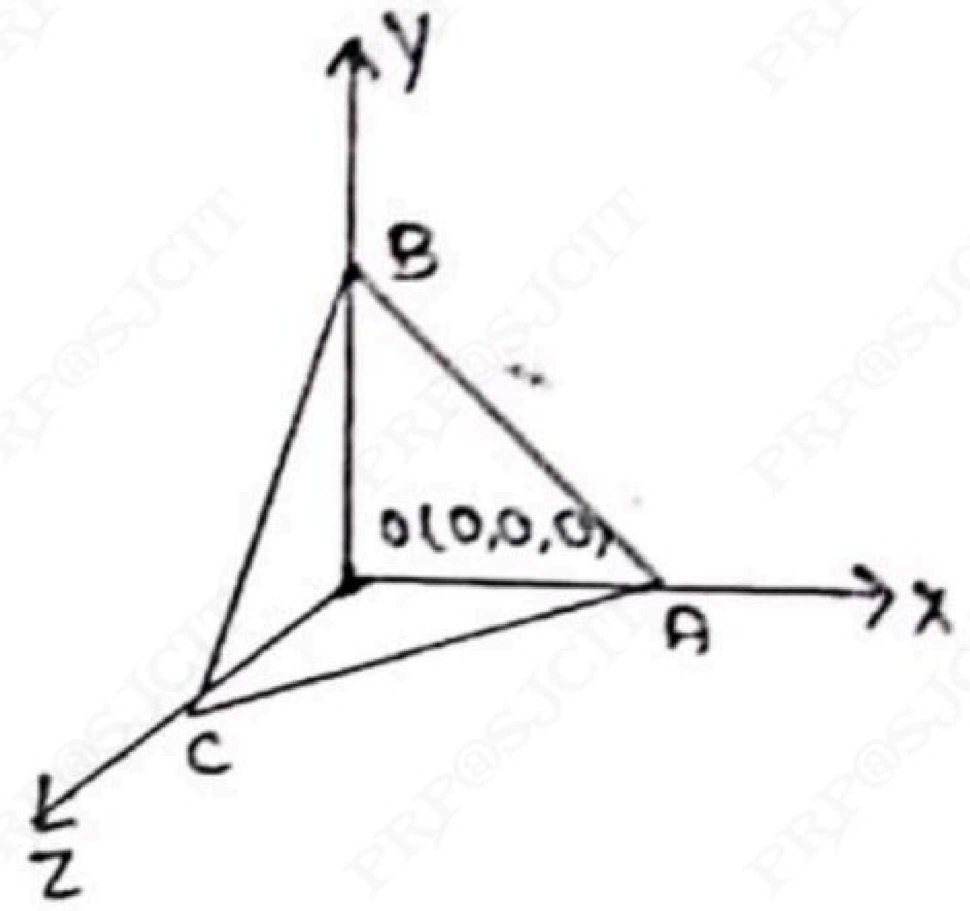
$$= \frac{1}{12} \int_{0}^{6} (x^2 - 12x + 36)\ dx$$

$$= \frac{1}{12} \left[\frac{x^3}{3} - 6x^2 + 36x \right]_0^6$$

$$= \frac{1}{12} \left[\frac{216}{3} - 216 + 216 \right] = \frac{216}{36} \Rightarrow V = 6 \text{ cubic units}$$

3. Find the volume of a tetrahydral bounded by the plane $x=0$, $y=0$, $z=0$ and $x+y+z=1$

Given that, the volume of the tetrahydral $x=0$, $y=0$, $z=0$ and $x+y+z=1$



$$= \int_{x=0}^{1} \int_{y=0}^{(1-x)} \int_{z=0}^{(1-x-y)} 1 \, dz \, dy \, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{(x-1)} [z]_0^{(1-x-y)} \, dy \, dx$$

$$= \int_{x=0}^{1} \int_{0}^{(1-x)} (1-x-y) \, dy \, dx$$

$$= \int_a^1 \left[y - xy - \frac{y^2}{2} \right]_0^{(1-x)}$$

$$= \int_0^1 \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} \right]' dx$$

$$= \frac{1}{2} \int_0^1 (1-x)\left[(1-x) - \frac{1}{2}(1-x)\right] dx$$

$$= \int_0^1 (1-x)^2 \left[1 - \frac{1}{2}\right] dx$$

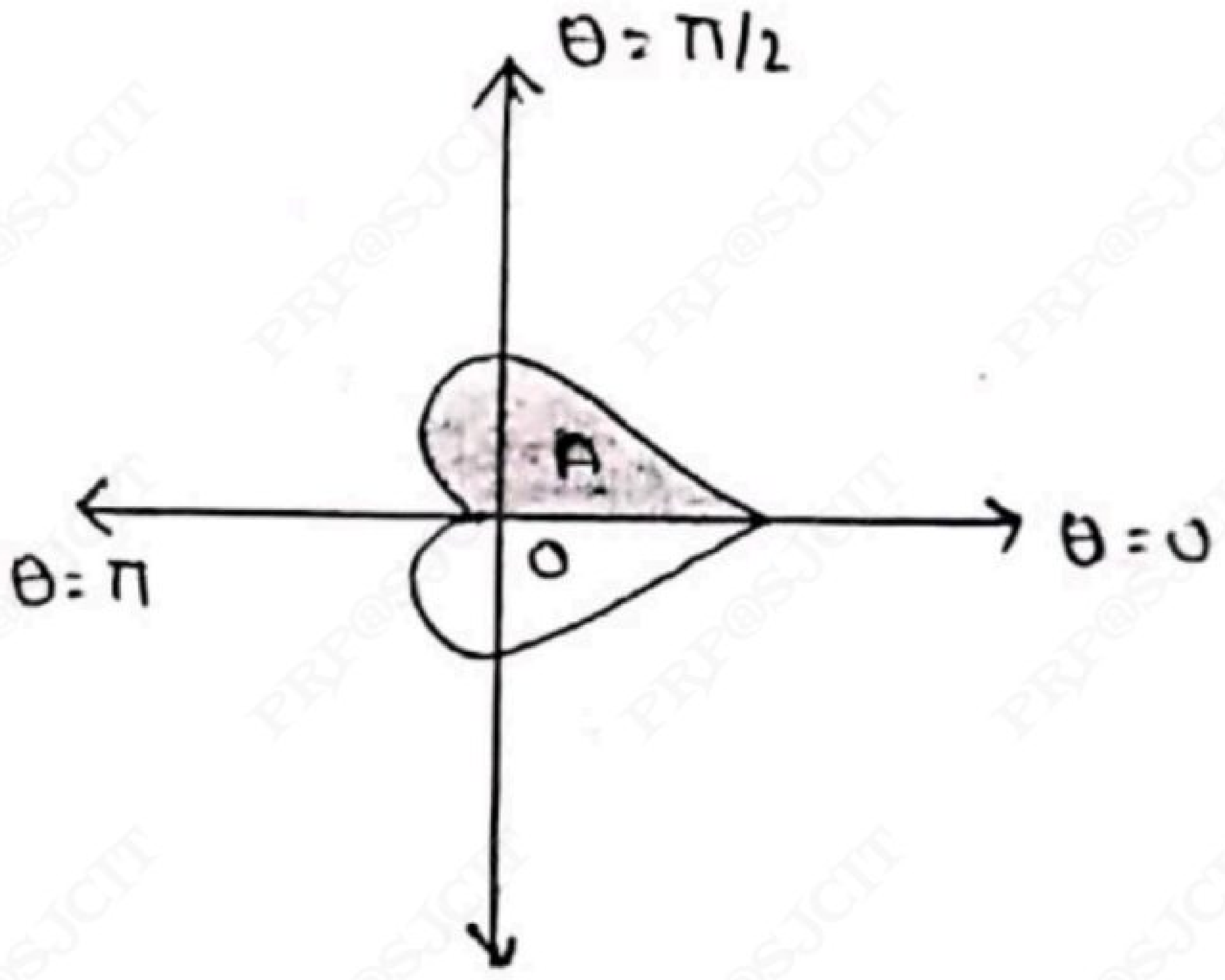
$$= \frac{1}{2} \int_0^1 (1-2x+x^2) \, dx$$

$$= \frac{1}{2} \left[x - \frac{2x^2}{2} + \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} \left[1 - \frac{2(1)^2}{2} + \frac{1^3}{3} \right]$$

$$= \frac{1}{2} \left[1 - 0 - \frac{1}{3} \right] = \frac{1}{6} \text{ cubic units}$$

4. Find the area bounded by $\theta = 0$, $\theta = \pi$ of a cardiac $r = a(1+\cos\theta)$



$$A = \iint dx\,dy$$

$$= \iint r\,dr\,d\theta$$

$$= \int_0^{a} \int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi} \left[r^2/2 \right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi} a^2 (1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\pi} (1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \left[1 + 2\cos\theta + \cos^2\theta\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \left[1 + 2\cos\theta + \frac{1+\cos\theta}{2}\right] d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi} \left[2 + 4\cos\theta + 1 + \cos 2\theta\right] d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi} \left[\cos 2\theta + 4\cos\theta + 3\right] d\theta$$

$$= \frac{a^2}{4}\left[\frac{\sin 2\theta}{2} + 4\sin\theta + 3\theta\right]_0^{\pi}$$

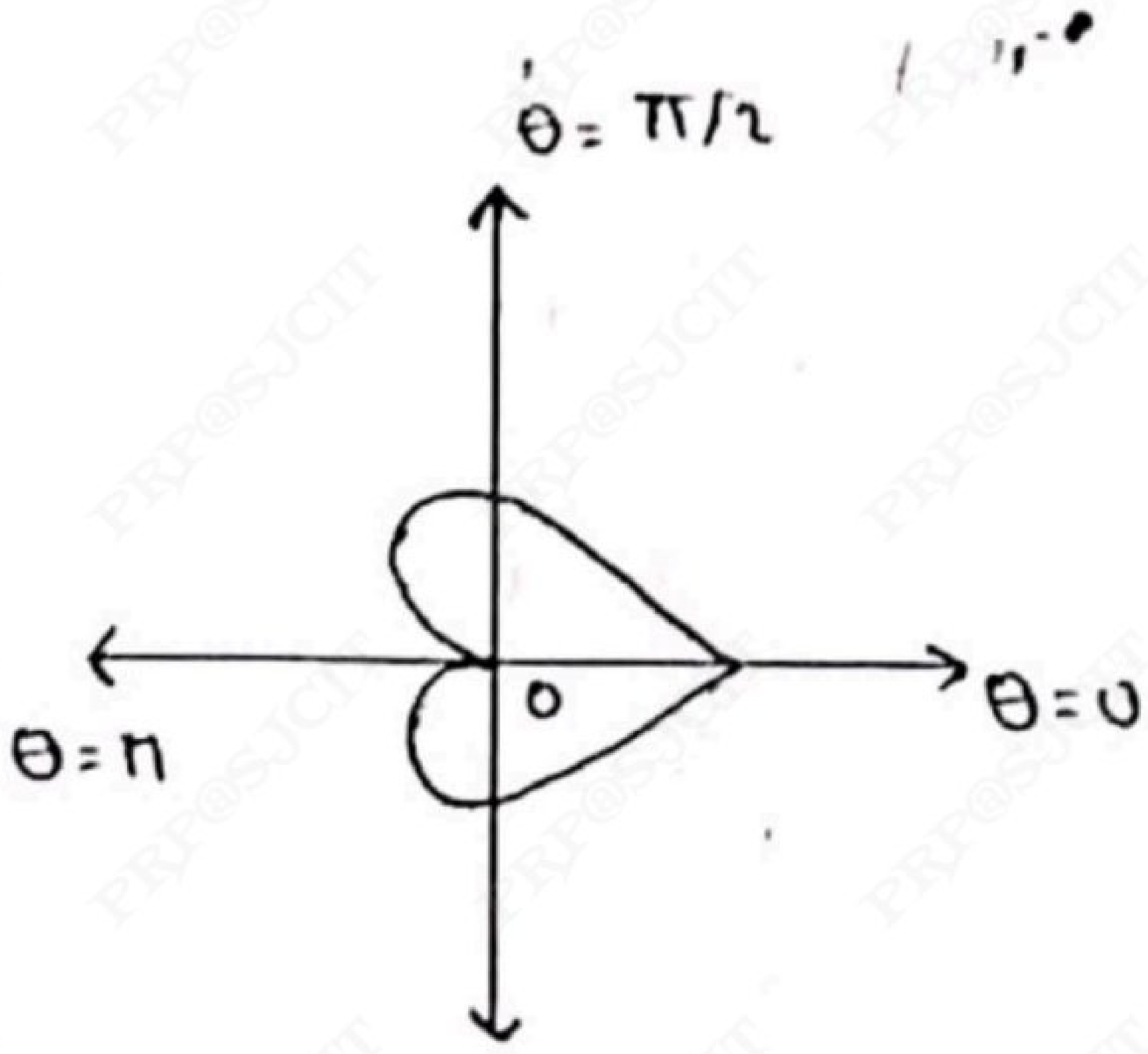
$$= \frac{a^2}{4}\left[(0+0+3\pi) - (0+0+0)\right]$$

$$A = \frac{3a^2\pi}{4} \text{ sq units}$$

5. Find the volume of the cardiac $r = a(1+\cos\theta)$ above the intial line

W.K.T

The volume of the cardiac $r = a(1+\cos\theta)$ above the intial line



$$V = \iint 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{2\pi}{3} \int_{\theta}^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_{\theta=0}^{\pi} (1+\cos\theta)^3 \sin\theta \, d\theta$$

Let $1+\cos\theta = t$

$\Rightarrow -\sin\theta \, d\theta = dt$

$\Rightarrow \sin\theta \, d\theta = -dt$

UL : $\theta = \pi \Rightarrow t = 1+\cos(\pi) = 1-1 = 0$

LL : $\theta = 0 \Rightarrow t = 1+\cos\theta = 1+1 = 2$

$$V = \frac{2\pi a^3}{3} \int_2^0 t^3(-dt)$$

$$= \frac{2\pi a^3}{3} \int_0^2 t^3 dt$$

$$= \frac{2\pi a^3}{3} \left[\frac{t^4}{4}\right]_0^2$$

$$= \frac{2\pi a^3}{4\times 3} \times (2^4 - 0)$$

$$= \frac{2\pi a^3}{4\times 3} \times 2\times 2\times 2\times 2$$

$$A = \frac{8\pi a^3}{3} \text{ cubic units}$$

$\Rightarrow$ <u>Beta - Gamma functions</u>

Defination :- For any $(m,n) > 0$, then the imporper integral can be defined as $\beta(m,n) \int_1^0 x^{n-1}(1-x)^{n-1}\,dx \rightarrow (1)$ is called the Beta function in m and n,

let, $x = \sin^2\theta \Rightarrow \theta = \sin^{-1}(\sqrt{x})$

$\Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

UL : $x = 1 \Rightarrow \theta = \frac{\pi}{2}$

LL : $x = 0 \Rightarrow \theta = 0$

$$(1) \Rightarrow \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} \cdot (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2(m-1)}\theta \cdot \cos^{2(m-1)}\theta \cdot \sin\theta\cos\theta\, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin_\theta^{2m-2+1} \cdot \cos_\theta^{2n-2+1}\, d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \sin\theta^{2m-1} \cdot \cos\theta^{2n-1}\, d\theta \rightarrow (2)$$

let $p = 2m-1$, $q = 2n-1$

$$\Rightarrow m = \frac{p+1}{2},\ n = \frac{q+1}{2}$$

$$(2) \Rightarrow \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta\, d\theta$$

$$\Rightarrow \int_0^{\pi/2} \sin^p\theta \cos^q d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Gamma function :- for every $n>0$ gamma function can be defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \rightarrow (1)$$

let $x = y^2$

$$\Rightarrow dx = 2y\,dy$$

UL : $x = \infty \Rightarrow y = \infty$

LL : $x = 0 \Rightarrow y = 0$

$$\therefore (1) \Rightarrow \Gamma n = \int_0^\infty e^{-y^2} (y^2)^{n-1} 2y\,dy$$

$$= 2\int_0^\infty e^{-y^2} y^{2n-2+1} dy$$

$$\Rightarrow \Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$\Rightarrow \Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx \quad (\because y = x)$$

RELATION BETWEEN BETA AND GAMMA FUNCTION

we know that,

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta\, d\theta \rightarrow (1)$$

$$\Gamma m = 2\int_0^\infty e^{-x^2} x^{2m-1} dx \rightarrow (2)$$

$$\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1} dy \rightarrow (3)$$

$$\Gamma(m+n) = 2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr \rightarrow (4)$$

Let $\Gamma m\,\Gamma n = 2\int_0^\infty e^{-x^2} x^{2m-1} y^{2n-1}\, dx\, dy \rightarrow (5)$

Let $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow dx\, dy = r\, dr\, d\theta$

(5) $\Rightarrow \Gamma m\,\Gamma n = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1}\, r\, dr\, d\theta$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2m-1+2n-1+1} \sin^{2n-1}\theta \cos^{2m-1}\theta\, dr\, d\theta$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2(m+n)-1} \sin^{2n-1}\theta \cos^{2m-1}\theta\, \theta\, dr\, d\theta$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2(m+n)-1} \sin^{2n-1}\theta \cos^{2n-1}\theta\, dr\, d\theta$$

$$\Gamma m\,\Gamma n = \left[2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr\right]\cdot\left[2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta\, d\theta\right]$$

$$\Gamma m\,\Gamma n = (\Gamma(m+n))\, \beta(m,n)$$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} \;//$$

Note:-

1. $\beta(m,n) = \beta(n,m)$
2. $\Gamma(n+1) = n\Gamma n = n!$
3. $\Gamma 1 = 1$
4. $\Gamma(1/2) = \sqrt{\pi}$
5. $\Gamma n\, \Gamma(1-n) = \dfrac{\pi}{\sin(n\pi)}$

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① Show that $\Gamma 1/2 = \sqrt{\pi}$ using Beta-gamma function.

WKT

$$\Gamma m = 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx \rightarrow ①$$

$$\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy \rightarrow ②$$

$$\therefore \Gamma m\, \Gamma n = \left[2\int_0^\infty e^{-x^2} x^{2n-1} dx\right]\left[2\int_0^\infty e^{-y^2} y^{2n-1} dy\right]$$

$$\Rightarrow \Gamma m\, \Gamma n = \int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2} x^{2m-1} y^{2n-1}\, dx\, dy \rightarrow ③$$

Let $m = \frac{1}{2}$, $n = \frac{1}{2}$

$$② \Rightarrow \Gamma 1/2\, \Gamma 1/2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^0 y^0\, dx\, dy$$

$$\Rightarrow (\Gamma 1/2)^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy \rightarrow ④$$

Let $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow dx\,dy = r\, dr\, d\theta$$

$$\therefore ④\ (\Gamma 1/2)^2 = 4\int_{r=0}^\infty \int_{\theta=0}^{\pi/2} e^{-r^2} r\, dr\, d\theta$$

$$= 4\int_{r=0}^\infty r e^{-r^2} dr\, [\theta]_0^{\pi/2}$$

$$= \frac{4\pi}{2}\int_{r=0}^\infty r\, e^{-r^2} dr$$

$$= \pi \int_0^\infty (2r) e^{-r^2} dr$$

Let $r^2 = t$

$$\Rightarrow 2r\, dr = dt$$

$$(\Gamma 1/2)^2 = \pi \int_0^\infty e^{-t}\, dt$$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = \pi\left(\frac{e^{-t}}{-1}\right)^{\infty}$$

$$= -\pi\left[e^{-t}\right]_0^{\infty}$$

$$= \pi\left[e^{-\infty}-1\right]$$

$$= -\pi\left(0-1\right]$$

$$\Rightarrow \left(\Gamma\tfrac{1}{2}\right)^2 = \pi$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi} \;//$$

2. Show that $\int_0^2 (4-x^2)^{3/2}dx = 3\pi$

$$I = \int_0^2 (4-x^2)^{3/2}\,dx$$

Let $x = 2\sin\theta \Rightarrow \theta = \sin^{-1}(x/2)$.

$$\Rightarrow dx = 2\cos\theta\, d\theta$$

UL : $x = 2 \Rightarrow \theta = \sin^{-1}(1) = \frac{\pi}{2}$

LL : $x = 0 \Rightarrow \theta = \sin^{-1}(0) = 0$

$$\therefore I = \int_0^{\pi/2} (4-4\sin^2\theta)^{3/2} \cdot 2\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (4)^{3/2}(\cos^2\theta)^{3/2}\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (2^2)^{3/2}\cos^3\theta\,\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (2^2)^{3/2}\cos^3\theta\,\cos\theta\, d\theta$$

$$I = 16\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$\therefore \int_0^{\pi/2}\cos^4\theta\, d\theta = \int_0^{\pi/2}\sin^0\theta\,\cos^4\theta = \int_0^{\pi/2}\sin^0\theta\,\cos^4\theta\, d\theta$$

$\therefore p = 0,\ q = 4$

$$\int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma 1/2\ \Gamma 5/2}{\Gamma 1/2 + 5/2}$$

$$= \frac{\frac{1}{2}\Gamma 1/2\ (5/2-1)\ \Gamma 5/2-1}{\Gamma 3}$$

$$= \frac{\frac{1}{2}\Gamma 1/2\ (3/2)\ \Gamma 3/2}{2!}$$

$$= \frac{\frac{1}{2}\Gamma 1/2\ (3/2)(3/2-1)\ \Gamma 3/2-1}{2}$$

$$= \frac{\frac{1}{2}\Gamma 1/2\ (3/2)(3/2-1)\ \Gamma 3/2-1}{2}$$

$$= \frac{3}{16}\sqrt{\pi}\ \sqrt{\pi}$$

$$\int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{3\pi}{6}$$

$$\therefore I = 16 \times \frac{3\pi}{16}$$

$$\Rightarrow I = 3\pi$$

3. Show that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta$

$I_1 \times I_2$

$$I_1 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$p = 1/2, \; q = 0$

$$I_1 = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(3/4 ; 1/2\right)$$

$$= \frac{1}{2} \frac{\Gamma 3/4 \; \Gamma 1/2}{\Gamma 3/4 + 1/2}$$

$$= \frac{1}{2} \frac{\Gamma 3/4 \; \Gamma 1/2}{\Gamma 5/4}$$

$$= \frac{1}{2} \frac{\Gamma 3/4 \; \Gamma 1/2}{\frac{1}{4} \Gamma 1/4}$$

$$\Rightarrow I_1 = \frac{2 \Gamma 3/4 \; \Gamma 1/2}{\Gamma 1/4}$$

$$\therefore I_2 = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\sin^{1/2}\theta}.$$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta \, d\theta$$

$p = -1/2, \; q = 0$

$$I_2 = \frac{1}{2} \beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(1/4, 1/2\right)$$

$$= \frac{1}{2} \frac{\Gamma 1/4 \, \Gamma 1/2}{\Gamma 1/4 + 1/2}$$

$$\Rightarrow I_2 = 1/2 \frac{\Gamma 1/4 \, \Gamma 1/2}{\Gamma 3/4}$$

$$I = 2 \frac{\Gamma 2/4 \, \Gamma 1/2}{\Gamma 1/4} \cdot 1/2 \frac{\Gamma 1/4 \, \Gamma 1/2}{\Gamma 3/4}$$

$$\therefore I = \Gamma 1/2 \; \Gamma 1/2$$

$$I = \sqrt{\pi} \sqrt{\pi}.$$

$$I = \pi$$

4. Evaluate $\int_0^1 x^{3/2} (1-x)^{1/2} dx$ by expressing interms of Beta - gamma function.

Let $I = \int_0^1 x^{3/2} (1-x)^{1/2} dx$

Let $x = \sin^2\theta \Rightarrow \theta = \sin^{-1}(\sqrt{x})$

$$dx = 2 \sin\theta \cos\theta \, d\theta$$

UL : $x = 1 \Rightarrow \theta = \pi/2$
LL : $x = 0 \Rightarrow \theta = 0$

$$I = \int_0^{\pi/2} (\sin^3\theta)^{3/2} (1 - \sin^2\theta)^{1/2} \, 2 \sin\theta \cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^3\theta \cos\theta \sin\theta \cos\theta \, d\theta$$

$$I = \int_0^{\pi/2} \sin^4\theta \cos^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^4\theta \cos^2\theta \, d\theta \quad \cdot \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$\therefore p = 4, \; q = 2$

$$\int_0^{\pi/2} \sin^4\theta \cos^2\theta \, d\theta = \frac{1}{2} \beta\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma 5/2 \, \Gamma 3/2}{\Gamma 4}$$

$$= \frac{1}{2} \frac{3/2 \cdot 1/2 \, \Gamma 1/2 \cdot 1/2 \, \Gamma 1/2}{3!}$$

$$= \frac{1}{2} \frac{3/2 \; 1/2 \, \Gamma 1/2 \;\; 1/2 \, \Gamma 1/2}{6}$$

$$= \frac{1}{2^5} \sqrt{\pi} \sqrt{\pi}$$

$$= \frac{\pi}{2^5}$$

$$\therefore I = \frac{\pi}{16}$$

5. Evaluate $\int_0^4 x^{3/2} (4-x)^{5/2} \, dx$ by using Beta-gamma function. Show that $\beta(m,n) = \int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} \, dx$

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} \, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$$

Let $I = \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$

then, $x = \frac{1}{y}$, $y = \frac{1}{x}$

$$dx = \frac{1}{y^2} \, dy$$

U.L : $x = \infty \Rightarrow y = 0$

LL : $x = 1 \Rightarrow y = 1$

$$\therefore I = \int_0^1 \frac{(1/y)^{m-1}}{(1+1/y)^{m+n}} \left(-\frac{1}{y^2}\right) dy$$

$$= \int_0^1 \frac{\frac{1}{y^{m-1}} \frac{1}{y^2}}{\frac{(y+1)^{m+n}}{y^{(m+n)}}} dy$$

$$= \int_0^1 \frac{\frac{1}{y^{m-1}}}{\frac{(1+y)^{m+n}}{y^{m+n}}} dy$$

$$= \int_0^1 \frac{y^{m+n}}{y^{m+1}(1+y)^{m+n}} dy$$

$$= \int_0^1 \frac{y^m y^n}{y^m y^1 (1+y)^{m+n}} dy$$

$$I = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$I = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\therefore \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx \ //$$

6. Show that $\int_0^\infty \frac{e^{-x^2} dx}{\sqrt{x}} \times \int_0^\infty \sqrt{x}\, e^{-x^2} dx = \frac{\pi}{2\sqrt{2}}$

$$I = \int_0^\infty e^{-x^2} \sqrt{x}\, dx \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$$

$$\Rightarrow I = I_1 \times I_2$$

$$I_1 = \int_0^\infty e^{-x^2}\sqrt{x}\,dx \Rightarrow \int_0^\infty e^{-x^2}x^{1/2}\,dx$$

$$= \int_0^\infty e^{-x^2}x^{2n-1}\,dx$$

$$\therefore 2n-1 = \frac{1}{2} \Rightarrow 2n = \frac{3}{2} \Rightarrow n = \frac{3}{4}$$

$$I_1 = \frac{\Gamma n}{2} = \frac{\Gamma 3/4}{2}$$

$$I_2 = \int_0^\infty e^{-x^2}/\sqrt{x}\,dx \Rightarrow \int_0^\infty e^{-x^2}x^{-1/2}\,dx \Rightarrow \int_0^\infty e^{-x^2}x^{2m-1}\,dx$$

$$\therefore 2m-1 = -\frac{1}{2} \Rightarrow 2m = 1-\frac{1}{2} \Rightarrow 2m = \frac{1}{2} \Rightarrow m = \frac{1}{4}$$

$$I_2 = \frac{\Gamma 1/4}{2}$$

$$I = I_1 \times I_2$$

$$I = \frac{\Gamma 3/4}{2} \times \frac{\Gamma 1/4}{2} \Rightarrow \frac{1}{4}\Gamma 3/4\,\Gamma 1/4 \Rightarrow \frac{1}{4}\Gamma 1-1/4\ \Gamma\frac{1}{4}$$

$$\Rightarrow \frac{1}{4}\frac{\pi}{\sin n\pi}$$

$$I = \frac{1}{4}\frac{\pi}{\sin\frac{\pi}{4}}$$

$$I = \frac{1}{4}\frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$= \frac{\sqrt{2}}{4}\pi \Rightarrow \frac{\pi}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$ hence proved.

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