# MODULE – 4

**1.) Define a) Centroid b) centre of gravity c) centroidal axis d) Axis of Reference**

**a) Centroid:**

Centroid is the point where the whole area of the plane figure is assumed to be concentrated. It is represented as ‘C.G’ or ‘ G’.

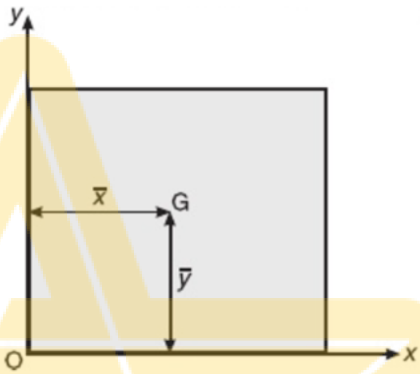


Fig. 1 Centroid of the plane figure.

**b) Centre of gravity:**

It is the point where the whole weight of the body is assumed to be concentrated. It is the point on which the body can be balanced. It is the point through which the weight of the body is assumed to act. This point is usually denoted by ‘C.G.’ or ‘G’.

**c) Centroidal axis:**

The axis which passes through the centroid of the given figure is known as centroidal axis, such as the axis *X* -*X* and the axis *Y* -*Y* shown in Figure 2

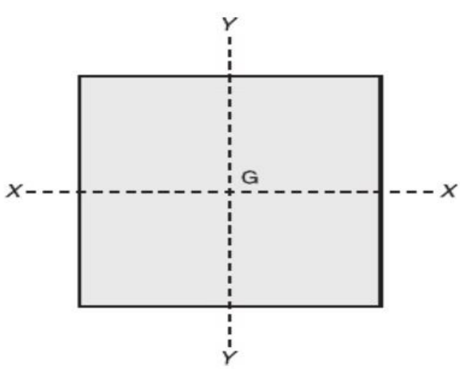


Fig.2 Centroidal axis

d) **Axis of Reference:**

These are the axes with respect to which the centroid of a given figure is determined.

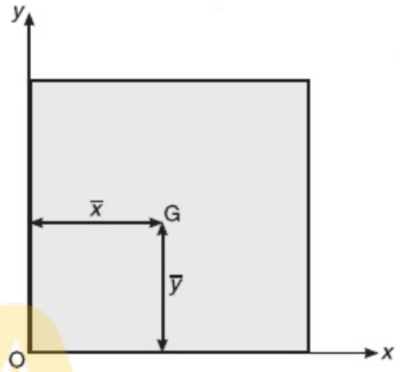


Fig. 3 Axis of Reference

2.) **Define a) Moment of Inertia b) Radius of Gyration c) Polar moment of Inertia.**

a) Moment of Inertia:

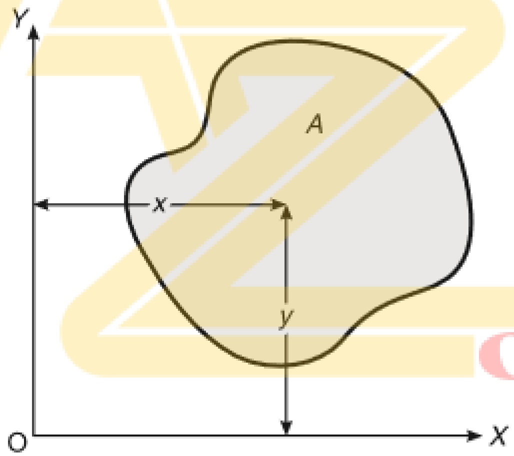


Figure 4. Irregular plane of lamina

Moment of area about the y-axis = first moment of area. If the first moment of area is multiplied by the perpendicular distance x, it gives $Ax^2$ known as the second moment of area or moment of inertia.

b) Radius of Gyration

It is the distance from the given axis where the whole area of a plane figure is assumed to be concentrated so as not to alter the moment of inertia about the given axis. It is denoted as k.

$$k = \sqrt{\frac{I}{A}}$$

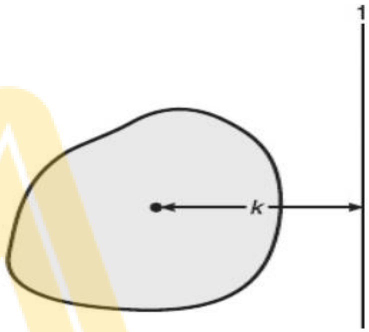


Fig.4.1 Radius of gyration k

c) Polar moment of inertia

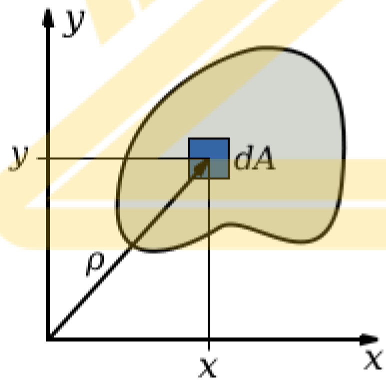


Fig.5 moment of inertia is calculated for an arbitrary shape about an axis O Where ρ is the radial distance to the element dA

The polar moment of inertia, also known as second polar moment of area, is a quantity used to describe resistance to torsional deformation (deflection), in cylindrical objects (or segments of cylindrical object) with an invariant cross-section and no significant warping or out-of-plane deformation.

Polar moment of inertia can be described as the summation of x and y planar moments of inertia, $I_x$ and $I_y$

$$J = I_Z = I_x + I_y$$

**3.) Derive an expression for centroid of**

**a) Rectangle**

**b) Triangle**

**c) Semicircle.**

**a) Rectangle:**

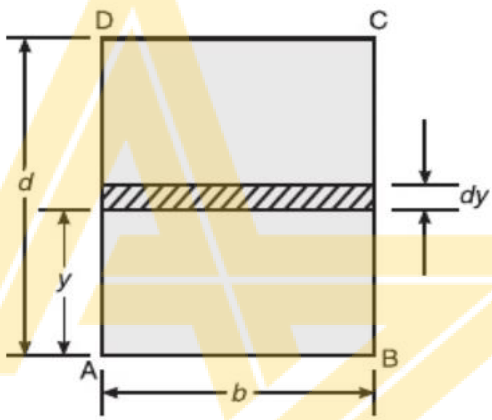


Fig.6 Rectangular lamina

Let us consider a rectangular lamina of area *b X d* as shown in figure. Now consider a horizontal elementary strip of area *b X dy*, which is at a distance y from the reference axis AB.

Moment of area of elementary strip about AB = *b X dy X y*

Sum of moments of such elementary strips about AB is given by

$$\int_0^d b \times dy \times y$$

$$= b\int_0^d y \cdot dy$$

$$= b \times \left[\frac{y^2}{2}\right]_0^d$$

$$= \frac{bd^2}{2}$$

Moment of total area about AB = $bd \times \bar{y}$

Apply the principle of moments about AB,

$$\frac{bd^2}{2} = bd \times \bar{y} \quad \text{or} \quad \bar{y} = \frac{d}{2}$$

By considering a vertical strip, similarly, we can prove that

$$\bar{x} = \frac{b}{2}$$

**b) Triangle**

Consider a triangular lamina of area (1/2) x b x d as shown in figure 7

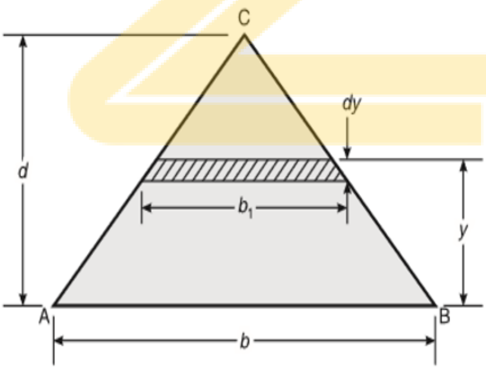


Figure 7. Triangular lamina

Now consider an elementary strip of area $b_1 \times dy$ which is at a distance $y$ from the reference axis AB.

Using the property of similar triangles, we have

$$\frac{b_1}{b} = \frac{d-y}{d}$$

or

$$b_1 = \frac{(d-y)b}{d}$$

Area of the elementary strip $= b_1 \times dy = \dfrac{(d-y)b \cdot dy}{d}$

Moment of area of elementary strip about AB

$$= \text{area} \times y$$

$$= \frac{(d-y)b \cdot dy \cdot y}{d}$$

$$= \frac{b \cdot dy \cdot d \cdot y}{d} - \frac{by^2 \cdot dy}{d}$$

$$= by \cdot dy - \frac{by^2 \cdot dy}{d}$$

Sum of moments of such elementary strips is given by

$$\int_0^d by \cdot dy - \int_0^d \frac{by^2}{d} \cdot dy$$

$$= b \times \left[\frac{y^2}{2}\right]_0^d - \frac{b}{d}\left[\frac{y^3}{3}\right]_0^d$$

$$= \frac{bd^2}{2} - \frac{bd^3}{3d}$$

$$= \frac{bd^2}{2} - \frac{bd^2}{3}$$

$$= \frac{bd^2}{6}$$

Moment of total area about AB $= \dfrac{1}{2} bd \times \bar{y}$

Applying the principle of moments,

$$\frac{bd^2}{6} = \frac{1}{2} \times bd \times \bar{y}$$

$$\therefore \qquad \bar{y} = \frac{d}{3}$$

### c)Semicircle

Consider a semicircular lamina of area $\frac{\pi r^2}{2}$ as shown in Fig. 8. Now consider a triangular elementary strip of area $\frac{1}{2} \times R \times R \times d\theta$ at an angle of $\theta$ from the $x$-axis, whose centre of gravity is at a distance of $\frac{2}{3}R$ from $O$ and its projection on the $x$-axis $= \left(\frac{2}{3}\right)R\cos\theta$.

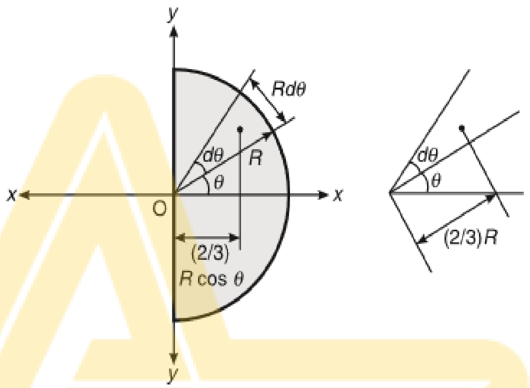


Figure 8. Semicircular lamina

Moment of area of elementary strip about the $y$-axis $= \frac{1}{2} \times R^2 \cdot d\theta \cdot \left(\frac{2}{3}\right)R\cos\theta$

$$= \frac{R^3 \cdot \cos\theta \cdot d\theta}{3}$$

Sum of moments of such elementary strips about the $y$-axis

$$= \int_{-\pi/2}^{\pi/2} \frac{R^3}{3}\cos\theta \cdot d\theta$$

$$= \frac{R^3}{3}[\sin\theta]_{-\pi/2}^{\pi/2}$$

$$= \frac{R^3}{3}\left[\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right] = \frac{2R^3}{3}$$

Moment of total area about the $y$-axis

$$= \frac{\pi R^2}{2} \times \bar{x}$$

Using the principle of moments

$$\frac{2R^3}{3} = \frac{\pi R^2}{2} \times \bar{x}$$

$\therefore$ $$\bar{x} = \frac{2R^3 \times 2}{3R^2\pi}$$

or $$\bar{x} = \frac{4R}{3\pi}$$

**4. Derive an expression for a Moment of Inertia of**
**a) Rectangle**
**b) Triangle**
**c) Circle**
**d) Semicircle.**

## a) Rectangle

Let us consider a rectangular lamina of breadth b and depth d whose moment of inertia is to be determined (Figure 9). Now consider an elementary strip of area b. dy at a distance y from the centroidal x–x axis. The moment of inertia of the strip about the x–x axis = bdy $y^2$. Moment of inertia of the whole figure about the x–x axis

$$= \int_{-d/2}^{d/2} b \cdot dy \times y^2$$

$$= b\left[\frac{y^3}{3}\right]_{-d/2}^{d/2}$$

$$= b\left[\frac{d^3}{24} + \frac{d^3}{24}\right]$$

$$= \frac{b \times d^3}{12}$$

$$\bar{I}_x = \frac{bd^3}{12}$$

$$\bar{I}_y = \frac{db^3}{12}$$

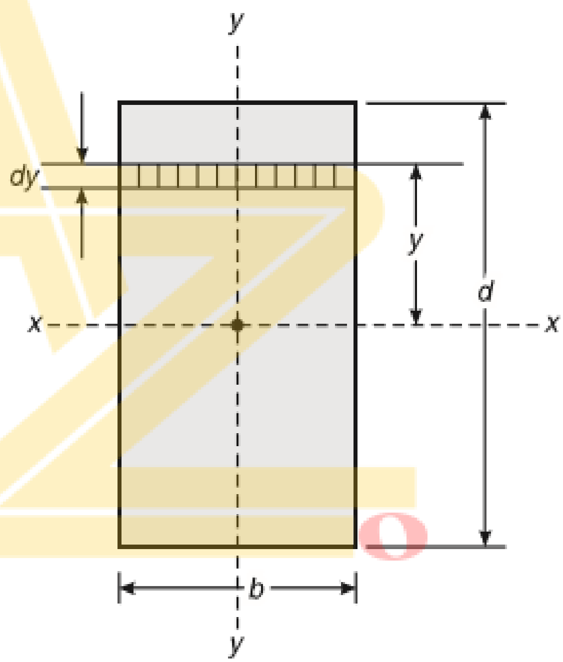


Fig. 9 Rectangular lamina

## b) Triangle

Let us consider a triangular lamina of base b and depth d as shown in Figure 10. Let us consider an elementary strip of area $b_1$dy which is at a distance y from base AB. Using the property of similar triangles,

Using the property of similar triangles,

$$\frac{b_1}{b} = \frac{d-y}{d}$$

$$b_1 = \frac{(d-y)b}{d}$$

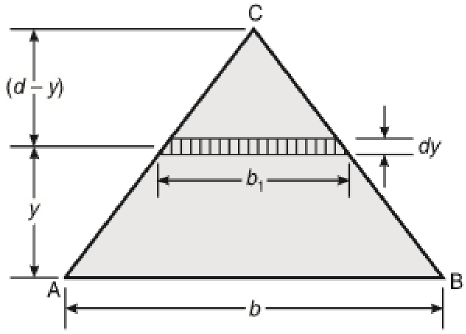


Figure 10. Triangular lamina

Area of the strip = $\dfrac{(d-y)b}{d}\cdot dy$

Moment of inertia of the strip about AB = $\dfrac{(d-y)b}{d}dy \times y^2$

$$= \frac{bdy^2 \cdot dy}{d} - \frac{by^3 \cdot dy}{d}$$

$$= by^2 \cdot dy - \frac{by^3 \cdot dy}{d}$$

Moment of inertia of the whole area about AB,

$$I_{AB} = \int_0^d by^2 dy - \int_0^d \frac{b}{d} y^3 dy$$

$$= b\left[\frac{y^3}{3}\right]_0^d - \frac{b}{d}\left[\frac{y^4}{4}\right]_0^d$$

$$= \frac{bd^3}{3} - \frac{b}{d}\frac{d^4}{4}$$

$$= \frac{bd^3}{3} - \frac{bd^3}{4}$$

$$I_{AB} = \frac{bd^3}{12}$$

Moment of inertia about $x$–$x$ axis is given by

$$I_{AB} = \bar{I}_x + Ay^2$$

$$\bar{I}_x = I_{AB} - Ay^2$$

$$= \frac{bd^3}{12} - \frac{1}{2}bd\left(\frac{1}{3}d\right)^2 = \frac{bd^3}{36}$$

Therefore, the moment of inertia of the triangle about the centroidal $y$-axis = $\dfrac{db^3}{36}$.

**c) Circle**

Let us consider a circular lamina of radius R as shown in Fig. 11

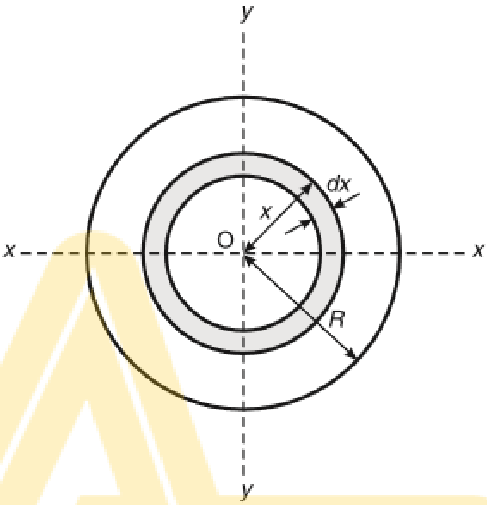


Figure 11 Circular lamina.

Let us choose a circular elementary strip of thickness $dx$ at distance $x$ from the centre.
Area of the strip = $2\pi x dx$.
Moment of inertia about the $z$–$z$ axis = $2\pi x \cdot dx \cdot x^2$
Moment of inertia about the $z$–$z$ axis for whole circle

$$= \bar{I}_z = \int_0^R 2\pi x^3 \cdot dx = 2\pi \left[ \frac{x^4}{4} \right]_0^R$$

$$= \frac{2\pi R^4}{4} = \frac{\pi R^4}{2}$$

For the circular lamina,

$$\bar{I}_x = \bar{I}_y,$$

Using the perpendicular axis theorem, we have

$$\bar{I}_z = \bar{I}_x + \bar{I}_y$$

$$\bar{I}_z = 2\bar{I}_x$$

$$\bar{I}_x = \frac{I_{zz}}{2}$$

$$\bar{I}_x = \frac{\pi R^4}{2 \times 2} = \frac{\pi R^4}{4} = \bar{I}_y$$

**d) Semicircle:**

Let us consider a semicircular lamina of radius R as shown in Figure 12.

Moment of inertia of semicircle about the diametrical axis AB = $\frac{1}{2} \times \frac{\pi R^4}{4} = \frac{\pi R^4}{8}$

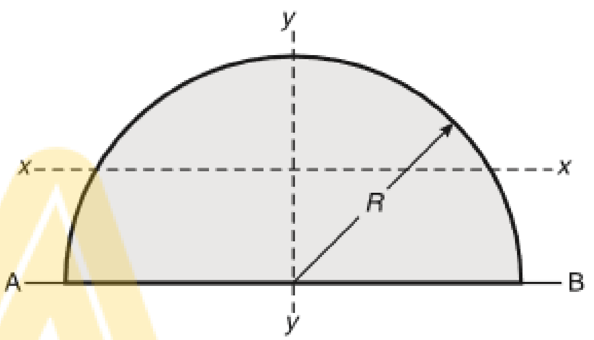


Figure 12. Semicircular lamina

$$I_{AB} = \bar{I}_x + A\bar{y}^2$$

$$\bar{I}_x = I_{AB} - \frac{\pi R^2}{2}\left(\frac{4R}{3\pi}\right)^2$$

$$= \frac{\pi R^4}{8} - \frac{\pi R^2 \times 16R^2}{2 \times 9\pi^2}$$

$$= \frac{\pi R^4}{8} - \frac{8\pi R^4}{9\pi^2}$$

$$= \frac{\pi R^4}{8} - \frac{8R^4}{9\pi}$$

$$\bar{I}_x = 0.11R^4$$

Moment of inertia about the $y$-axis,

$$\bar{I}_y = \frac{\pi R^4}{8} = \frac{1}{2} \times \frac{\pi R^4}{4}$$

**5. State and prove parallel axis theorem.**

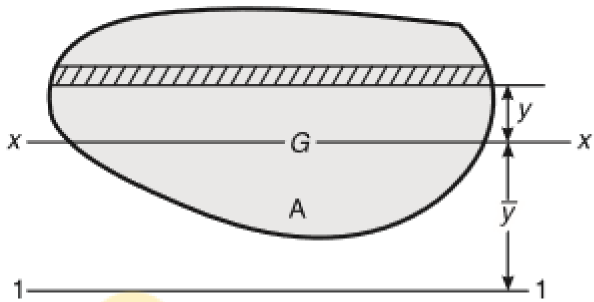


Figure 13: Illustration of parallel axis theorem for moment of inertia about an axis parallel to x–x axis.

This theorem states that the moment of inertia of plane figure about an axis $I_{1-1}$, parallel to the centroidal axis, $I_X$ is equal to sum of moment of inertia about centroidal axis, i.e. $I_X$ and the product of area of the plane figure and square of the distance between the two axes.

***Proof:*** Let us consider a plane figure of total area A as shown in Figure 13. Let $I_x$ be the moment of inertia about the x-axis and $I_{1\text{-}1}$ be the moment of inertia as shown in Figure 1. Let Ix be the moment of inertia about the x-axis and $I_{1-1}$ the moment of inertia about 1–1 axis.

Let us choose an elemental strip of area da at a distance y from the centroidal axis. Moment of inertia of the strip about x–x axis = $da \cdot y^2$

Moment of inertia of the total area about the x–x axis = $Ix = \Sigma da \cdot y^2$

Moment of inertia of the strip about 1–1 axis = $da(y + y)^2$

Moment of inertia of the total area about 1–1 axis

$I_{1-1} = \Sigma da(y^2 + y^2 + 2y\, y)$

$I_{1-1} = \Sigma da y^2 + \Sigma da\, y^2 + 2\, y(\Sigma da y)$

As the distance of C.G. of whole area from the centroidal axis = 0, i.e. y = 0, we get

$I_{1-1} = Ix + A\, y^2$

Similarly, the moment of inertia about an axis $I_{2-2}$ as shown in Figure 14 is given by

$I_{2-2} = Iy + Ax^2$

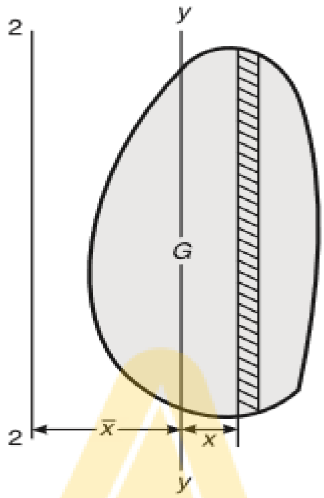


Figure 14: Illustration of parallel axis theorem for moment of inertia about an axis parallel to y – y axis.

**5. State and prove perpendicular axis theorem.**

This theorem states that the moment of inertia of a plane figure about an axis which is perpendicular to the plane of the figure is equal to sum of moment of inertia about two mutually perpendicular axes.

Proof:

Let us consider an irregular figure of total area A as shown in Figure 10.5. Let us choose an elemental strip of area da at a distance x from y-axis, y from x-axis and r from z-axis, respectively. Then, $r^2 = x^2 + y^2$

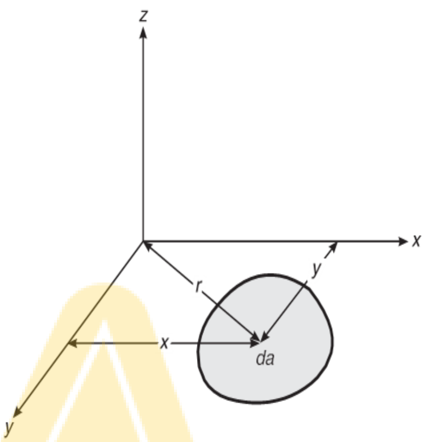


Figure 14. Illustration of perpendicular axis theorem.

Moment of inertia of the strip about x-axis = $da \times y^2$

Moment of inertia of the whole area about the x-axis = $Ix = \Sigma da \cdot y^2$

Similarly, moment of inertia of the strip about y-axis = $da \times x^2$

Moment of inertia of the whole area about y-axis = $I_y = \Sigma da \cdot x^2$

Moment of inertia of the strip about z-axis = $da \times r^2$

Moment of inertia of the whole area about z-axis = $\Sigma da \cdot r^2$

$$= \Sigma da(x^2 + y^2)$$

$$= \Sigma da \cdot x^2 + \Sigma da \cdot y^2$$

$$= I_y + I_x$$

That is, $I_z = I_x + I_y$