MODULE-III
MAXWELL'S EQUATIONS and EM WAVES

INTRODUCTION:

Electromagnetics is the subject that deals with the theory and applications of electric and magnetic fields. The concept of electromagnetics is of prime importance in almost all fields of engineering, especially electromagnetic communication which forms the foundation of Mobile communication, optical fiber communication etc. In all these applications, Electric and Magnetic field properties are utilized to transfer the information.

Fundamentals of vectors:

Physical quantities are divided into two main categories.

(i) Scalars and (ii) Vectors

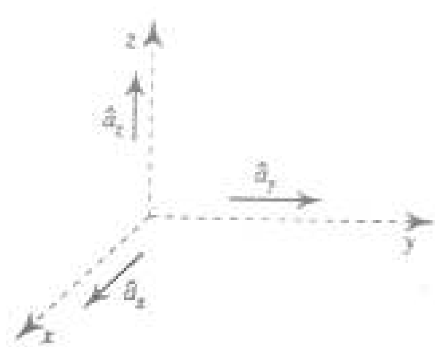
Scalars are those physical quantities which possess only magnitude and no direction in space. **Eg**.: Mass, time, volume, work etc.

Vectors are those physical quantities which have both magnitude and direction. **Eg**.: Force, electric field, acceleration etc.

A vector can be represented by a directed line segment. The length of the line segment is proportional to the magnitude of the vector and its inclination is along the direction of action of the vector quantity.

Unit Vector is a vector with a magnitude of unity. It always indicates just the direction.

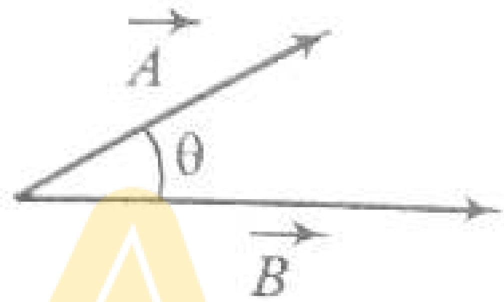
Base Vectors:



Generally, a coordinate system is used to represent the orientation of a vector quantity. In a Cartesian coordinate system, the position or orientation is represented by arbitrarily choosing an origin and three coordinates (X, Y and Z) mutually perpendicular to each other from that origin. The unit vectors along these coordinate axes are known as base vectors. The base vectors along X, Y and Z coordinates can be represented as $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$ respectively. In terms of the base vectors, $\vec{A}$ can be written as

$$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$$

<u>Scalar product or Dot Product</u>:



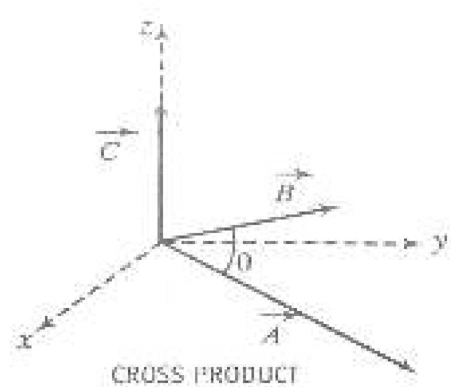
Dot Product of two vectors is defined as the product of their magnitudes and cosine of the smaller angles between them.

If $\vec{A}$ & $\vec{B}$ are any two vectors inclined at an angle θ, their dot product is given by $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|cos\theta$

The dot product of two vectors is a scalar quantity.

<u>Vector product or Cross Product</u>:

Cross Product of two vectors $\vec{A}$ & $\vec{B}$ is a vector $\vec{C}$ whose magnitude is equal to the product of their magnitudes and sine of the smaller angles between them.



If $\vec{A}$ & $\vec{B}$ are any two vectors inclined at an angle θ, their cross product is given by $\vec{A} \times \vec{B} = \vec{C} = |\vec{A}||\vec{B}|sin\theta\ \hat{n}$

Where $\hat{n}$ is a unit vector in the direction of $\vec{C}$. The cross product of two vectors is also a vector quantity. The direction of $\vec{C}$ is perpendicular to the plane containing $\vec{A}$ & $\vec{B}$ which can be found by right hand rule.

In terms of components of $\vec{A}$ & $\vec{B}$, the vector $\vec{C}$ can be expressed as a third order determinant,

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vector Operator, ∇ (Del):

'Del' is a vector differential operator represented by the symbol V (Nabla). It is an instruction to differentiate the function that follows it (i.e. to its right). In a Cartesian coordinate system, del operator is given by

$$\nabla = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)$$

When applied to a function defined on a one dimensional domain, del denotes its standard derivative as defined in calculus. When applied to a field (a function defined on a multi-dimensional domain), del may denote the **gradient** of a scalar field, the **divergence** of a vector field, or the **curl** (rotation) of a vector field, depending on the way it is applied.

Gradient:

Consider a scalar filed function which depends on more than one variables such as temperature 'T' in a room. The change in T with different variables is given by

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$$

The above equation indicates the variation of T when the variables x, y, and z are varied by a small magnitude dx, dy and dz respectively.

$$dT = \left(\frac{\partial T}{\partial x}\hat{a}_x + \frac{\partial T}{\partial y}\hat{a}_y + \frac{\partial T}{\partial z}\hat{a}_z\right).\left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z\right)$$

$$dT = \left(\overrightarrow{\nabla T}\right).\overrightarrow{dl}$$

$$dT = \left|\overrightarrow{\nabla T}\right|\left|\overrightarrow{dl}\right|cos\theta$$

Where ∇T is known as gradient of T and read as 'grad T'.

Gradient of a scalar function is a vector quantity. Gradient vector always points in a direction showing maximum change of the scalar function.

Divergence:

This operation is used to evaluate the amount of physical quantity emerging or diverging from a small volume.

Divergence of a vector field $\vec{A}$ at a given point P is the outward flux per unit volume as the volume goes to zero about P. It is given by

$$div\vec{A} = \lim_{\Delta v \to 0} \frac{Outward\ Flux\ of\ \vec{A}}{\Delta v}$$

$$div\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{A}.\overrightarrow{dS}}{\Delta v}$$

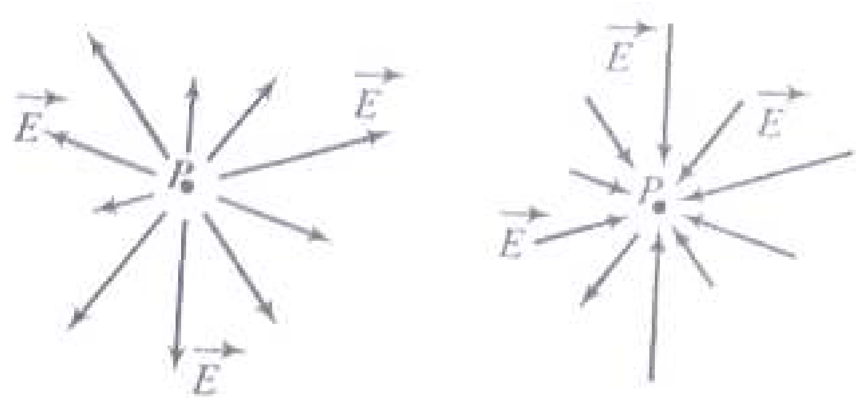
$$\nabla.\vec{A} = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right).\left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right)$$

$$\text{divergence of } \vec{A} = \nabla.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The divergence of vector function is scalar quantity. If the divergence of a vector field is zero such field is known as **solenoidal** field.

Physical significance of divergence:

The divergence of the vector field $\vec{A}$ at a given point is a measure of how much the field diverges or emerges or spreads from that point. i.e. it is a measure of the number of lines of force representing the vector field $\vec{A}$ originating from that point.



If there are positive electric charges at a point, then large number of field lines diverge from that point (positive divergence). If there are negative charges at a point, then field lines will converge at the point or negative divergence.

Curl:

The curl of the vector field $\vec{A}$ at a given point 'P' is a measure of the circulation or rotation of the field around the point 'P'.

It is defined at a point 'P' as the maximum circulation of a vector field $\vec{A}$ per unit area as the area shrinks to zero around 'P'. Curl of $\vec{A}$ is a vector with its direction perpendicular to the area around 'P' when the area oriented for maximum circulation.

$$Curl\ \vec{A} = \lim_{\Delta S \to 0} \frac{Maximum\ circulation\ around\ P}{\Delta S}$$

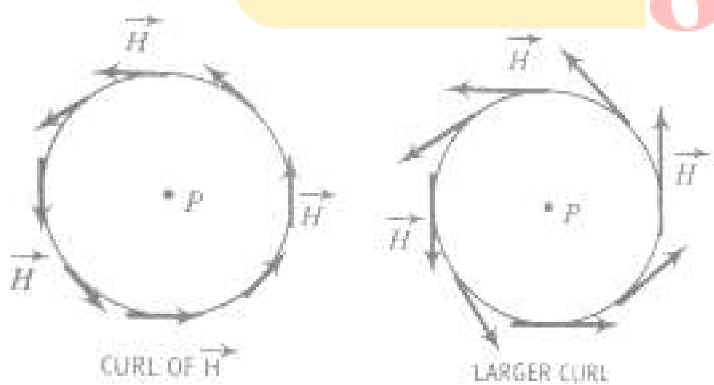
$$\nabla \times \vec{A} = \left(\lim_{\Delta S \to 0} \frac{\oint \vec{A} . dl}{\Delta S} \right) \hat{a}_n$$

Where $\hat{a}_n$is the unit vector perpendicular to ΔS.

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

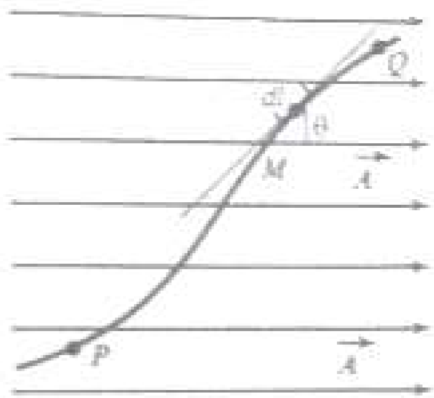
$$\nabla \times \vec{A} = \hat{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - \hat{a}_y \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) + \hat{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Physical significance of Curl:



Curl measures the tendency of a vector filed to swirl around or to rotate around a point. Greater magnitudes to a vector field around a point means more vigorous or larger circulation of that field about that point (Fig below).

Line integral:



Consider a linear path PQ in a vector field $\vec{A}$. Consider a small elementary length 'dl' of this line as shown in fig. Let 'θ' be the angle made by tangent to 'dl' at M w.r.t. the field $\vec{A}$. Then the dot product,

$$\vec{A}.\vec{d} = Adlcos\theta$$

Considering the line 'PQ' to be made up of a number of elemental lengths 'dl' and if we integrate the all the dot products $\vec{A}.\vec{dl}$ between P and Q then it is known as line integral.

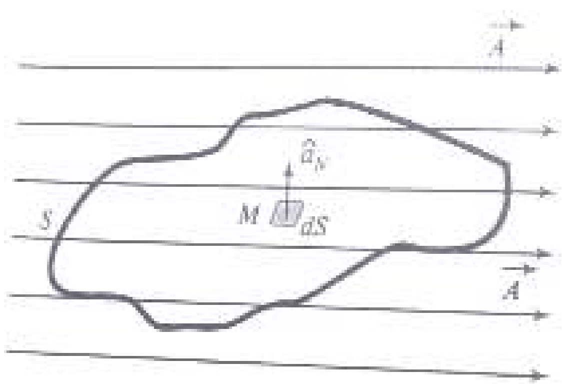
i.e. Line integral of the path PQ = $\int_P^Q \vec{A}.\vec{dl}$

For closed path **L**,

Line integral = $\oint_L \vec{A}.\vec{dl}$

This is known as closed contour integral. It is also called circulation of $\vec{A}$ around **L**.

Surface integral:



Consider a surface 'S' in a vector field $\vec{A}$. The surface 'S' can be imagined to be made up of a number of infinitely small surface elements of area 'dS'. If $\hat{a}_n$ is the unit vector normal (perpendicular) to 'dS', then in the vector field the surface element acts as a vector given by $\overrightarrow{dS} = dS\hat{a}_n$. The surface integral is given by

$$\emptyset = \int_S \vec{A}.\overrightarrow{dS}$$

Here the dot product represents the outward flux through the surface element 'dS' and ∅ represents the total outward flux through the surface 'S'. If the surface is a closed surface then the unit vector $\hat{a}_n$ is chosen outward at every point and the surface integral is given by

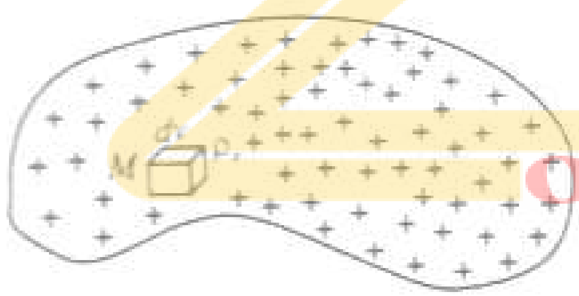
$$\emptyset = \oint_S \vec{A}.\overrightarrow{dS}$$

Volume integral:

Consider a closed surface 'S' enclosing a volume 'V' in a vector field $\vec{A}$. Then the volume integral is given by

$$Volume\ integral = \int_V TdV$$

Where 'T' is a scalar quantity.



Consider a continuous distribution charges in a volume. Let ρ_v be the charge density (charge per unit volume) within the volume. Consider an elementary volume 'dV' at 'M', then

$$Volume\ integral = \int_V \rho_v dV$$

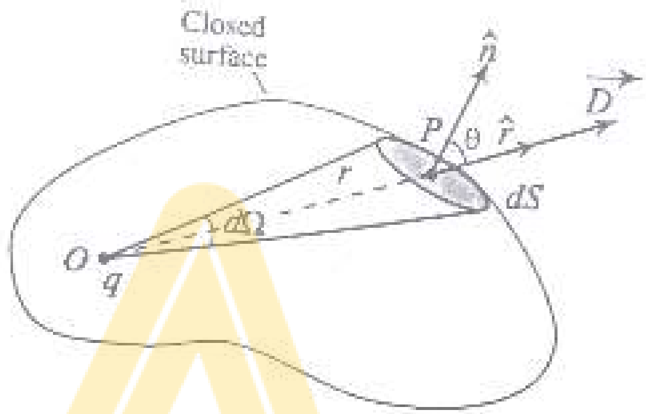
The quantity on the right hand side represents the total charge enclosed by the volume 'V'.

Gauss Law in Electrostatics

States that "*the total electric flux through a closed surface is equal to* $\left(\frac{1}{\varepsilon_0}\right)$*times the total charge enclosed by the surface*".

OR

"*Total flux density through a closed surface is equal to the total charge enclosed by the surface*".



Let an electric charge 'q' be surrounded by a closed surface of arbitrary shape. This surface is known as 'Gaussian Surface'. Let the flux density passing through the surface element 'dS' be $\vec{D}$. Then the total flux through the closed surface is

$$\emptyset = \oint_S \vec{D}.\overrightarrow{dS}$$

According to Gauss law, this total flux is equal to the total charge enclosed.

$$\therefore \qquad \oint_S \vec{D}.\overrightarrow{dS} = q \text{ --------- (1)}$$

This is the **integral form** of Gauss law.

For a distribution charges throughout the volume of the closed surface if ρ_v is the charge density then total charge enclosed by the surface is

$$q = \int_V \rho_v dV$$

$$q = \rho_v \int_V dV$$

$$q = \rho_v V \text{ ----------- (2)}$$

From equations (1) and (2)

$$\oint_S \vec{D}.\vec{dS} = \rho_v V$$

$$\frac{1}{V}\oint_S \vec{D}.\vec{dS} = \rho_v$$

As the volume shrinks to zero,

$$\lim_{V\to 0}\frac{\oint_S \vec{D}.\vec{dS}}{V} = \lim_{V\to 0}\rho_v$$

i.e. $$\nabla.\vec{D} = \rho_v \quad \text{(By definition of divergence)}$$

This is the **differential form** of Gauss law.

Gauss Law in Magneto statics

Consider a closed of arbitrary shape in a magnetic field. Magnetic flux lines always form a closed loop. In the case of a closed surface in a magnetic field, every line that enters the surface, there must always be a line emerging out of the surface. Therefore total flux over the closed surface must be equal to zero.

i.e. $$\phi = \int \vec{B}\bullet \vec{ds} = 0$$

In differential form (vector form) we can write

$$\nabla.\vec{B} = 0$$

Where $\vec{B}$ is the magnetic flux density.

Gauss Divergence Theorem:

States that, "the surface integral of the normal component of the flux density (or any vector) over any closed surface is equal to the volume integral of the divergence of the flux density throughout the volume enclosed by the surface".

Mathematically,

$$\oint_S \vec{D}.\vec{dS} = \int_V \nabla.\vec{D}dV$$

Proof:

Consider a Gaussian surface in certain region of space enclosing some charges. Let 'dV' be an elemental volume within the surface enclosing a charge 'dQ'. Then the charge density is given by

$$\rho_v = \frac{dQ}{dV}$$

If 'Q' is the total charge enclosed by the Gaussian surface, then

$$Q = \int dQ = \int_V \rho_v dV$$

From Gauss law,

$$\nabla.\vec{D} = \rho_v$$

$$\therefore \quad Q = \int_V \nabla.\vec{D}\,dV \quad \text{------ (1)}$$

Also from Gauss law,

$$\oint_S \vec{D}.\vec{dS} = Q \quad \text{------ (2)}$$

From equations (1) and (2),

$$\oint_S \vec{D}.\vec{dS} = \int_V \nabla.\vec{D}dV$$

Equation of Continuity:

States that "the amount current diverging from an volume element (closed surface) is equal to the time rate of decrease of charge within the volume of closed surface"

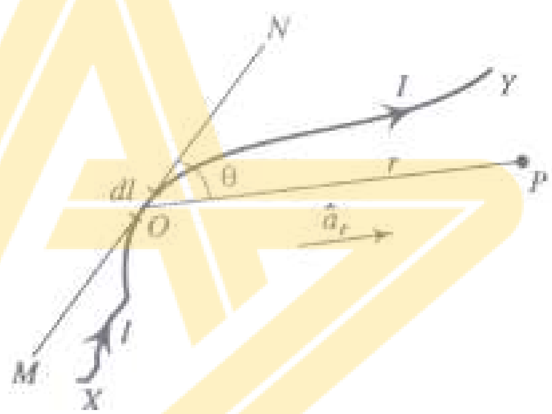
Vectorially, $$\nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

Where $\vec{J}$ is the current density, ρ_v is the charge density.

Note: This is a fundamental equation in fluid dynamics or in situations which involves a conserved quantity such as mass, charge etc.

Biot-Savart's Law

This law gives the magnetic flux density at a point due to a current carrying conductor.



Consider a conductor XY carrying a current I. Consider an elemental length 'dl' of the conductor at 'O'. Let 'P' be a point at a distance 'r' from 'O'. Let 'θ' be the angle made by the tangent to 'dl' with the line joining the point 'P' and the current element 'dl'. Then according Biot-Savart's law

*The magnitude of the flux density '**dB**' at a point due to a current element is directly proportional to the product of the current (I), the length of the current element (dl) and the sine of the angle (sinθ) between the current element and the position vector to the point from the current element and it is inversely proportional to the square of the distance between the point and the current element.*

i.e. $$dB \propto \frac{Idlsin\theta}{r^2}$$

or $$dB = \frac{\mu_0}{4\pi}\frac{Idlsin\theta}{r^2}$$

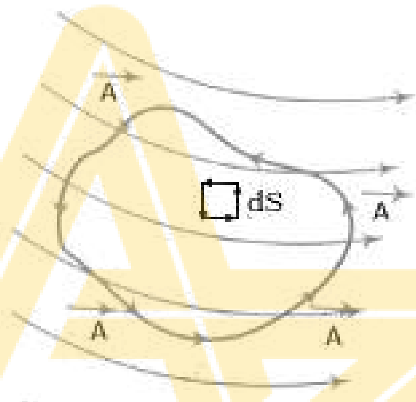
In vector form,

$$\overrightarrow{dB} = \frac{\mu_0 I(\overrightarrow{dl}\times\vec{r})}{r^3}$$

The direction of this field is perpendicular the plane containing 'dl' and 'r'. Here $\overrightarrow{dl}$ is directed along the direction of the current.

Stoke's Theorem:

States that '*the net circulation of a vector field $\overrightarrow{A}$ over some open surface 'S' is equal to the line integral of $\overrightarrow{A}$ along the closed contour 'C' which bounds 'S'.*



In the figure shown, 'S' is an open surface of arbitrary shape in a vector field $\overrightarrow{A}$ and 'C' is the closed curve/path that bounds 'S'.

The circulation of the field $\overrightarrow{A}$ over a surface element, 'dS' is given by $(\nabla \times \vec{A}).dS$. Therefore total curl (circulation) over the entire surface is given by

$$\int_S (\nabla \times \vec{A}).\overrightarrow{dS}$$

Dividing the boundary of the surface into a number of small length elements 'dl' in the vector field $\overrightarrow{A}$ and integrating the dot product $\vec{A}.\overrightarrow{dl}$ over the closed path we get

$$\int_C \vec{A}.\overrightarrow{dl}$$

According Stoke's theorem, $\int_S (\nabla \times \vec{A}).\overrightarrow{dS} = \int_C \vec{A}.\overrightarrow{dl}$

Current Density '$\vec{J}$'

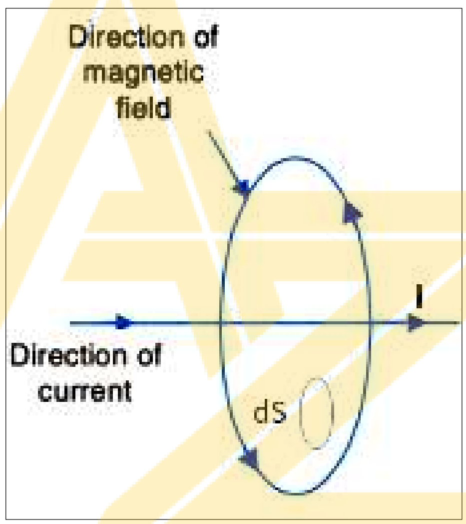
It defined as *the current per unit area of cross of an imaginary surface held normal to the direction of current in a current carrying conductor.*

$$\vec{J} = \frac{I}{A}$$

Where 'A' is the area of cross section normal to the current. In terms of charge density $\vec{\boldsymbol{J}} = \boldsymbol{\rho}\vec{\boldsymbol{v}}$ where $\vec{v}$ is the velocity of the charges at a given point.

Ampere's Law:

States that 'the line integral of magnetic field taken about any closed path is equal to μ_0 times the current enclosed by that path.



Consider a closed loop in a magnetic field enclosing certain current I. Consider an elemental surface area 'dS' normal to the current. If $\vec{J}$is the current density then the total current enclosed by the surface/loop is

$$I = \int_S \vec{J}.\overrightarrow{dS} \text{ ---- (1)}$$

The line integral of the magnetic field over the entire closed loop is $\oint \vec{B}.\overrightarrow{dl}$

∴ According to Ampere's law, $\oint \vec{B}.\overrightarrow{dl} = \mu_0 I$

i.e. $$\oint \vec{B}.\overrightarrow{dl} = \mu_0 \int_S \vec{J}.\overrightarrow{dS} \quad \text{--- (2)}$$

Applying Stokes theorem,

$$\int_S (\nabla \times \vec{B}).\overrightarrow{dS} = \oint \vec{B}.\overrightarrow{dl}$$

∴ (2) becomes $\int_S (\nabla \times \vec{B}).\overrightarrow{dS} = \mu_0 \int_S \vec{J}.\overrightarrow{dS}$

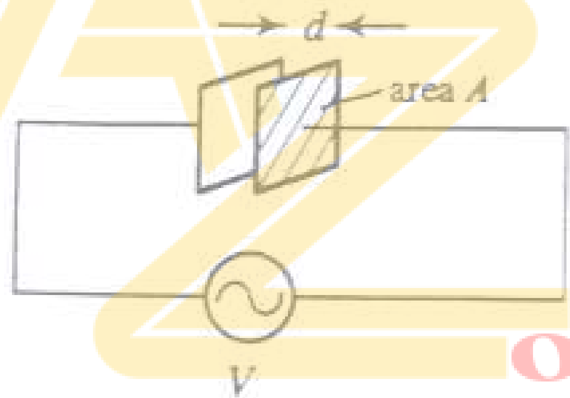
Differentiating both sides

$$\nabla \times \vec{B} = \mu_o \vec{J}$$

This is known as **Ampere's Circuital law** in **differential form**.

Displacement Current:

When a dielectric material is subjected to an electric field, it gets polarized. Even though charges cannot flow freely in a dielectric, the charges in molecules can move a little under the influence of an electric field. The positive and negative charges in molecules separate under the applied field, causing an increase in the state of polarization. A changing state of polarization in the dielectric due to the displacement of charges within the molecules of a dielectric give rise to a current known as **displacement current.**



Consider a parallel plate capacitor as shown in the figure. Let 'V' be the potential difference applied across the plates of the capacitor. Due to the applied potential difference, the charges will accumulate on the plates of the capacitor.

If 'dq' is the charge stored on the plates of the capacitor in time 'dt', then

Displacement current, $i_d = \frac{dq}{dt}$ ---------- (1)

We have, $q = CV$

For a parallel plate capacitor, $C = \frac{\varepsilon A}{d}$

$\therefore \qquad q = \frac{\varepsilon A V}{d}$

$$q = \varepsilon A\vec{E} \quad (Since\ \vec{E} = \frac{V}{d})$$

Equation (1) becomes

$$i_d = \frac{\partial q}{\partial t} = \varepsilon A \frac{\partial \vec{E}}{\partial t}$$

Displacement current density is

$$\vec{J_d} = \frac{i_d}{A} = \frac{\partial(\varepsilon\vec{E})}{\partial t}$$

i.e. $\quad \vec{J_d} = \frac{\partial(\vec{D})}{\partial t}$ (Since $\vec{D} = \varepsilon\vec{E},\ is\ the\ electric\ flux\ density$)

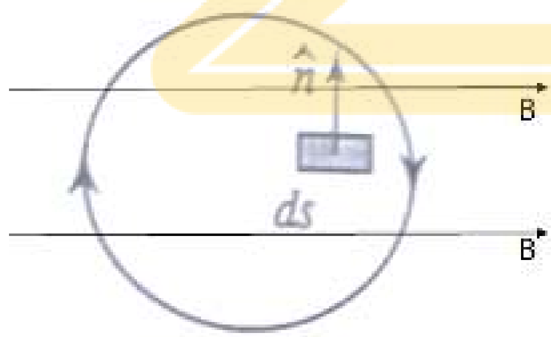
Faraday's Law of Electromagnetic Induction:

Through various experiments Faraday proved that a change in magnetic flux linked with a closed circuit results in an emf induced in the circuit which give rise to the flow of current. The law states that

"*the magnitude of the induced emf in a circuit is equal to the rate of change of magnetic flux through it. The induced emf will be in a direction which opposes the change which causes it*."

i.e. $\quad emf = -\frac{d\emptyset}{dt}$ ------ (1)

where $'\emptyset'$ is the flux linkage with the circuit. Negative sign is in accordance with the Lenz's law.



Consider a stationary loop of a conducting material placed in a changing magnetic field $\vec{B}$. The normal component of the magnetic flux through an elemental surface area dS is

$$d\emptyset = \vec{B}.\overrightarrow{dS}.$$

The total flux over any surface 'S' in a magnetic field is given by

$$\emptyset = \int_S \vec{B}.\overrightarrow{dS} \quad \text{-------- (2)}$$

By definition, emf is the work done per unit charge in moving the charges around closed path. The corresponding force $\vec{E}$ must be acting tangential to the closed loop at every point on it. Total work done by this force is

$$Workdone = \oint_L \vec{E}.\overrightarrow{dl}$$

Where is $'\boldsymbol{dl}'$ the elemental length of the loop.

$$\therefore \quad emf = \oint_L \vec{E}.\overrightarrow{dl} \quad \text{---------- (3)}$$

From equation (1) and (3),

$$\oint_L \vec{E}.\overrightarrow{dl} = -\frac{d\emptyset}{dt}$$

$$\oint_L \vec{E}.\overrightarrow{dl} = -\frac{\partial}{\partial t}\left(\int_S \vec{B}.\overrightarrow{dS}\right) \quad \text{[From (2)]}$$

$$\oint_L \vec{E}.\overrightarrow{dl} = -\int_S \frac{\partial(\vec{B}.\overrightarrow{dS})}{\partial t}$$

$$\oint_L \vec{E}.\overrightarrow{dl} = -\int_S \frac{\partial \vec{B}}{\partial t}.\overrightarrow{dS}. \quad \text{--------- (4)}$$

This is the **integral form** of Faraday's law.

Using Stoke's theorem,

$$\oint_L \vec{E}.\overrightarrow{dl} = \int_S \left(\nabla \times \vec{E}\right).\overrightarrow{dS}$$

$$-\int_S \frac{\partial \vec{B}}{\partial t}.\overrightarrow{dS} = \int_S \left(\nabla \times \vec{E}\right).\overrightarrow{dS} \qquad \text{[From (4)]}$$

Differentiating both sides

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{----------- (5)}$$

This is the **differential form** Faraday's law.

<u>Ampere-Maxwell's law</u>:

The Ampere law is given by

$$\nabla \times \vec{B} = \mu_o \vec{J}$$

which holds good under static conditions. But for time varying fields, the law does not holds good.

Taking divergence on both sides of above equation

$$\nabla.\left(\nabla \times \vec{B}\right) = \mu_0\left(\nabla.\vec{J}\right)$$

From the rules of vector analysis, the divergence of a curl for any vector field is equal to zero. Therefore

$$\nabla.\vec{J} = 0$$

But according to the equation continuity

$$\nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

Thus, Amperes law yields a result which is not in accordance with equation of Continuity. Maxwell fixed this contradiction by adding a term to include time-varying fields. The modified Ampere's law is given by

$$\nabla \times \vec{B} = \mu_o\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \qquad [\ \frac{\partial\ \vec{D}}{\partial t} \text{is the displacement current density}]$$

This equation is known as **Ampere-Maxwell's law**. It States that "the line integral of magnetic field taken about any closed path is equal to μ_0 times the sum of the current enclosed by that path and the rate change of electric flux through any surface bounded by that path".

Maxwell's equations

For **time-varying fields**, the Maxwell's equations in differential form are

1. $\nabla.\vec{D} = \rho_v$
2. $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
3. $\nabla.\vec{B} = 0$
4. $\nabla \times \vec{B} = \mu_0\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$

For **static fields**, the Maxwell's equations in differential form are

1. $\nabla.\vec{D} = \rho_v$
2. $\nabla \times \vec{E} = 0$
3. $\nabla.\vec{B} = 0$
4. $\nabla \times \vec{B} = \mu_0\vec{J}$

Maxwell's equations in word form:

1. The total electric flux density through a closed surface enclosing a volume is equal to the total charge within the volume.

2. The total magnetic flux emerging through any closed surface is zero.

3. The magneto motive force around a closed path is equal to the sum of conduction current $(\vec{J})$ and the rate of change of electric displacement $(\vec{D})$ through the surface bounded by the path.

4. The electromotive force around a closed path is equal to the rate of change of magnetic flux through the surface bounded by the path.

ELECTROMAGNETIC WAVES

Wave Equation in differential form in terms of Electric field

For time-varying fields we have the Maxwell's equations in differential form given by

$$\nabla.\vec{D} = \rho_v \text{ ----------- } (1)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{ ----------- } (2)$$

$$\nabla.\vec{B} = 0 \text{ ----------- } (3)$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \text{ ------- } (4)$$

Consider a non-conducting medium of permittivity $'\boldsymbol{\varepsilon}'$ and permeability$'\boldsymbol{\mu}'$. For such a medium $\sigma = 0$ which means that $\vec{J} = 0$. Therefore the fourth Maxwell's equation reduces to

$$\nabla \times \vec{B} = \mu \left(\frac{\partial \vec{D}}{\partial t}\right)$$

$$\nabla \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t} \text{ --------- } (5)$$

Taking curl on both sides of equation (2)

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\frac{\partial\left(\nabla \times \vec{B}\right)}{\partial t} \text{ ----------- } (6)$$

From the rules of vector analysis,

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla.\vec{E}\right) - \nabla^2\vec{E}$$

Where ∇^2 is the Laplacian operator.

From equation (1) we have $\nabla.\vec{E} = \frac{\rho_v}{\varepsilon}$.

For non-conducting medium $\rho_v = 0$ and hence $\nabla.\vec{E} = 0$

$$\therefore \qquad \nabla \times \left(\nabla \times \vec{E}\right) = -\nabla^2\vec{E}$$

Equation (6) becomes

$$-\nabla^2\vec{E} = -\frac{\partial\left(\nabla \times \vec{B}\right)}{\partial t}$$

$$\nabla^2\vec{E} = \frac{\partial\left(\mu\varepsilon\frac{\partial\vec{E}}{\partial t}\right)}{\partial t}$$

$$\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

This is the 3-D wave equation in $\vec{E}$

For free space;

$$\nabla^2\vec{E} - \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0 \text{ --------- } (7)$$

In terms of magnetic field, the wave equation in free space is

$$\nabla^2\vec{B} - \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2} = 0 \text{ ------ } (8)$$

The classical wave equation is represented as

$$\nabla^2\varphi - \frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2} = 0 \text{ --------- } (9)$$

From equation (7) and (9) we have the velocity of the EM wave

$$v = c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Plane EM Wave:

Plane electromagnetic waves are the electromagnetic waves in which electric and magnetic fields are uniform over a plane perpendicular to the direction of propagation. They possess variation only in the direction of propagation.

An electromagnetic wave propagation along x-axis is said to be a plane wave if the electric field (and magnetic field) is independent of y and z-axes but is a function of x and t (time) only. For such a there will be no field component in the direction of propagation.

For a plane electromagnetic wave of wavelength λ propagating along positive x-axis, the electric field $(\vec{E}_y)$ and magnetic field $(\vec{B}_z)$ components varying along y and z directions respectively at any instant of time 't' can be written as

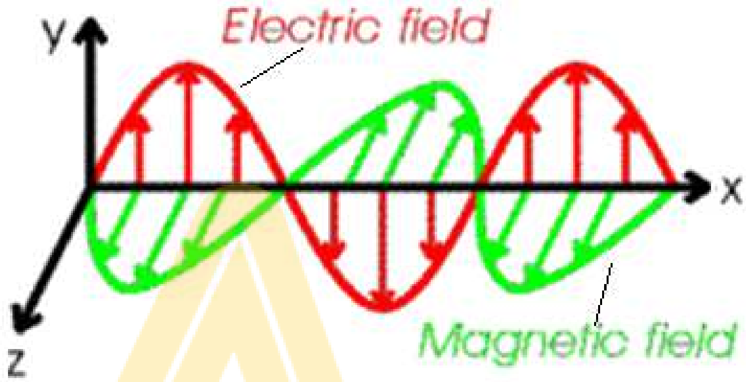
$$\vec{E}_y = A\cos\left[\frac{2\pi}{\lambda}(x - ct)\right]\hat{a}_y$$

and $\vec{B}_z = \frac{1}{c} A\cos\left[\frac{2\pi}{\lambda}(x - ct)\right] \hat{a}_z$; where A is the amplitude of the wave and c is the velocity of the wave in vacuum.

The ratio of the magnitudes of $\vec{E}_y$ and $\vec{B}_z$ in the above equations is

$$c = \frac{E_y}{B_z}$$

TRANSVERSE NATURE OF EM WAVES:



In an electromagnetic wave, electric and magnetic field vectors are perpendicular to each other and at the same time are perpendicular to the direction of propagation of wave. This nature of electromagnetic wave is known as Transverse nature.

Consider a uniform plane wave propagating along X direction in a medium where there are no free charges.

From Maxwell's equation

$$\nabla.\vec{D} = 0 \text{ (Since } \rho_v = 0)$$

i.e $$\nabla.\varepsilon\vec{E} = 0$$

$$\nabla.\vec{E} = 0$$

For free space, $$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \text{ ------- (1)}$$

As the wave is propagating along X direction, E_y and E_z remain constant along Y and Z direction respectively.

$\therefore$ $$\frac{\partial E_y}{\partial y} = 0 = \frac{\partial E_z}{\partial z}$$

From (1) $$\frac{\partial E_x}{\partial x} = 0$$

$$\Rightarrow \qquad E_x = \text{Constant}$$

For a plane wave E_x cannot be a constant as it is periodic.

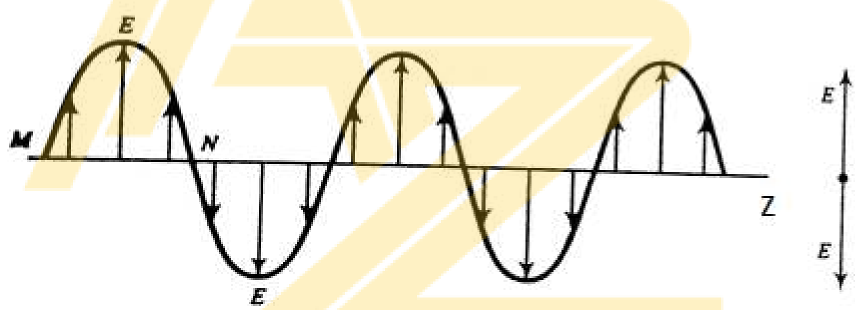
$$\therefore \qquad E_x = 0$$

This means that a uniform plane wave progressing in the x direction has no X component of electric field. It indicates that electric field is perpendicular to the direction of propagation. Thus electromagnetic wave has transverse nature.

A similar analysis would show that the X component of magnetic field is also zero.

POLARIZATION OF EM WAVES:

The phenomenon by which the vibrations of the electric field vector of an EM wave to a particular plane is called **polarization**.



Consider an EM wave, the electric field of which is varying with distance along z direction as shown in the figure. If the variation of electric field is observed along the direction of propagation, tip of electric field vector appears to trace a straight line along vertical direction. In this vibration of E vector is confined to a single plane perpendicular to direction of propagation. Such a wave is known as a **plane polarized wave**.

Types of Polarization:

Depending on how the electric field is oriented, we classify polarized light into three types:

Linearly Polarized:

When the variation of $\vec{E}$ is projected on a plane perpendicular to the direction of propagation, then if the tip of E – vector traces a **straight line**, it is known linearly polarized light.

Circularly Polarized:

When the variation of $\vec{E}$ is projected on a plane perpendicular to the direction of propagation, then if the tip of E – vector traces a **circle,** it is known as circularly polarized light.

Elliptically Polarized:

When the variation of $\vec{E}$ is projected on a plane perpendicular to the direction of propagation, then if the tip of E – vector traces an **ellipse,** it is known as elliptically polarized light.

Consider a wave propagating along z-direction has its $\vec{E}$-vector inclined at an angle θ with the x-axis. Its components $\vec{E}_x$ and $\vec{E}_y$ are directed along the x and y axes.

Linear Polarization:

If **components $\vec{E}_x$ and $\vec{E}_y$ are in phase** and of equal or different amplitudes. The resultant $\vec{E}$ varies along with $\vec{E}_x$ and $\vec{E}_y$ but inclined at an angle θ with x-axis. If the variation is viewed from a point to which z-axis normal, then tip of $\overrightarrow{E}$ will trace a straight line inclined to x-axis. In this case the electric field of light is confined to a single plane and traces a **straight line** on plane perpendicular to the direction of propagation, it is known **linearly polarized** light.

Let $E_1 and E_2$ be the amplitudes of $\vec{E}_x$ and $\vec{E}_y$ and δ be the phase difference them. For linearly polarized light δ=0 and it can be shown that

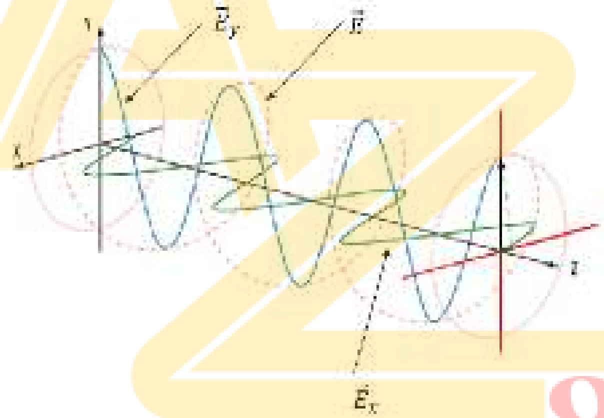
$$E_y = \left(\frac{E_2}{E_1}\right) E_x$$

$$E_y = mE_x \text{ (Since the ratio } \frac{E_2}{E_1} = m\text{, is a constant)}$$

The above equation represents a straight line.

Circular Polarization:

If the electric field of light consists of two linear components $\vec{E}_x$ and $\vec{E}_y$ that are perpendicular to each other, **equal in amplitude**, but have a phase difference **of π/2**. The resulting electric field rotates in a circle around the direction of propagation.



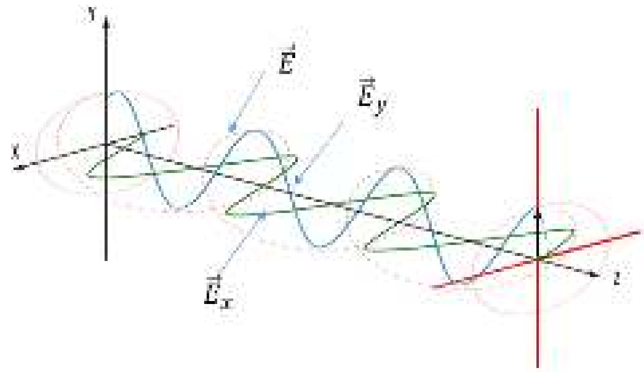
For circularly polarized light δ= π/2 and $E_1 = E_2$, and it can be shown that

$$E_x^2 + E_y^2 = E_1^2$$

which is the equation for a circle.

Elliptical Polarization:

This results from two light waves that are linearly polarized and having **unequal amplitudes and a phase difference that is not equal to π/2**. This results in a light wave with electric vector that both rotates and changes its magnitude. An elliptical shape can be traced out by the tip of the resultant electric field vector, and therefore it is referred to as elliptical polarization.



For elliptically polarized light $\delta \neq 0$ and $E_1 \neq E_2$. For $\delta = \pi/2$ can be shown that

$$\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1$$

which represents an ellipse.

Important Questions:

1. Explain the terms gradient of a scalar, divergence and curl of a vector. Derive Gauss divergence theorem.
2. What is displacement current? Obtain the expression for displacement current.
3. Give the four Maxwell's equations in differential form in vacuum and hence derive the EM wave equation in terms of electric field using Maxwell's equations.
4. Describe the concept of divergence. What is its physical significance? Derive Gauss divergence theorem
5. State and explain Gauss divergence theorem. Mention Stokes theorem.
6. Explain transverse nature of EM waves.
7. Explain Gauss law in electrostatics. Express the same in differential form.
8. Explain Gauss law in magnetism. Express the same in differential form.
9. Explain briefly the Faraday's law of electromagnetic induction. Express the same in the differential form of Maxwell's equation in the case of time varying fields.
10. Explain briefly the Ampere's law. Give the equation for the same in the differential form of Maxwell's equation for static fields.

11 Discuss Maxwell-Ampere law.

MODULE -III

OPTICAL FIBERS

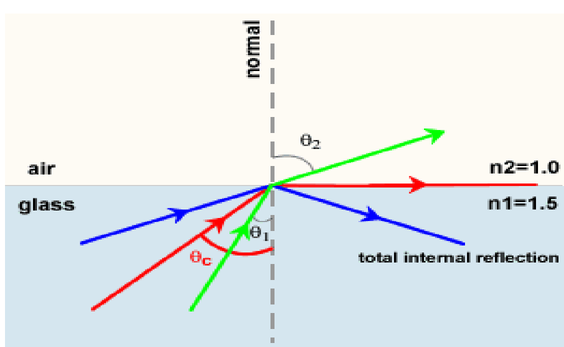
Introduction

A conventional method of long distance communication uses radio waves (10^6 Hz) and micro waves (10^{10} Hz) as carrier waves. A light beam acts as carrier wave which is capable of carrying far more information since optical frequencies are extremely large (10^{15} Hz).

Optical fibers are essentially light guides used in optical communication as waveguides. With the development of laser and flexible fiber, optical fibers are being used extensively for various communication applications. The principle behind the transmission of light waves in an optical fiber **is total internal reflection**. Fiber optic communication has significant advantages over the transmission by conventional coaxial cables. The loss of signal strength is considerably less in optical fibers and hence permits transmission over long distances. Use of light waves in place of radio and microwave has improved the speed of communication.

Basic principle – Total Internal Reflection

The basic principle of optical fiber is multiple total internal reflections. When a ray of light travels from denser to a rarer medium, at an angle of incidence greater than critical angle Φc, the ray is not refracted but it is reflected into the same denser medium. This property is called **total internal reflection**. Light signals are transmitted through optic fibers by multiple total internal reflections.

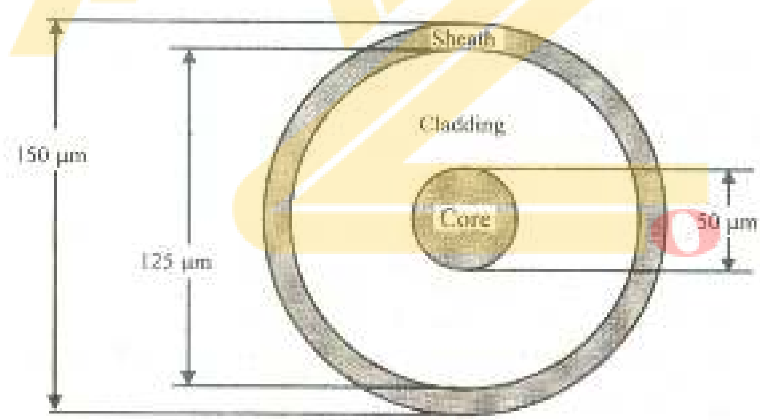


[Light rays incident on a high to low refractive index interface (e.g. glass–air): (a) refraction; (b) the limiting case of refraction showing the critical ray at an angle φ_c; (c) total internal reflection where $\varphi > \varphi_c$]

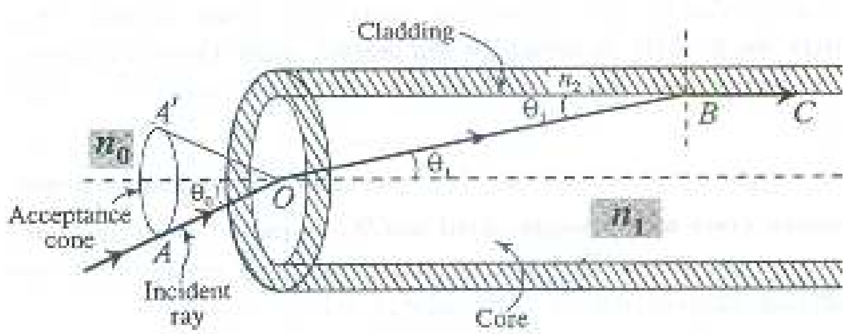
OPTICAL FIBER

An optical fiber is a transparent fiber made of glass or plastic. It is designed to guide light waves along its length. An optical fiber works on the principle of total internal reflection. When light enters one end of the fiber, it undergoes successive total internal reflections from sidewalls and travels down the length of the fiber along a zigzag path.

A practical optical fiber has in general three coaxial regions. The innermost region is the light guiding region known as the **core.** It is surrounded by a coaxial region known as **cladding**. The outermost region is called the **sheath**. The refractive index (RI) of core is always greater than that of the cladding. The purpose of cladding is to make the light to be confined to the core. The sheath protects the cladding and the core from contamination, abrasions, harmful influence of moisture. It also provides a mechanical strength to the fiber.



Angle of Acceptance and numerical Aperture:



Consider an optical fiber into which light is launched at one end from a medium of RI $\mathbf{n_o}$. Let $\mathbf{n_1}$ be the RI of core and $\mathbf{n_2}$ be that of the cladding. Assume that a ray of light enters the fiber at an angle θ_0 with respect to the axis of the fiber. The light ray refracts at an angle θ_1 and strikes the core – cladding interface at an angle of $(90-\theta_1)$. If $(90-\theta_1)$ is greater than the critical angle for the core – cladding interface, the ray undergoes total internal reflection at the interface.

It is clear from the figure that any light ray which enters the core at an angle less than θ_0 will undergo total internal reflection at core – cladding interface and propagates along the fiber. The angle θ_0 is called the **acceptance angle.** ***"It is the maximum angle that a light ray can have relative to the axis of the fiber and propagates down the fiber"***. *Sine of the acceptance angle θ_0, $\sin\theta_0$ is called the* ***numerical aperture (NA)*** *of the fiber.* It represents the light gathering capacity of the optical fiber.

Applying Snell's law at 'O'

$$\frac{\sin\ \theta_0}{\sin\ \theta_1} = \frac{n_1}{n_0} \rightarrow (1)$$

Applying Snell's law at 'B'

$$n_1 \sin(90-\theta_1) = n_2 \sin 90$$

$$\cos\theta_1 = \frac{n_2}{n_1} \rightarrow (2)$$

Rewriting equation (1),

$$\sin\theta_0 = \frac{n_1}{n_0}\sin\theta_1$$

$$= \frac{n_1}{n_0}\sqrt{1-\cos^2\theta_1}$$

$$= \frac{n_1}{n_0}\sqrt{1-\cos^2\theta_1}$$

$$\sin\theta_0 = \frac{\sqrt{n_1^2 - n_2^2}}{n_0} \rightarrow (3)$$

If the medium surrounding the fiber is air, then $n_0 = 1$

$$\therefore \sin\theta_0 = \sqrt{n_1^2 - n_2^2}$$

If θ_i is the angle of incidence of an incident ray w.r.t. the axis of the fiber, then ray will be able to propagate,

if $\quad \theta_i \langle \theta_0$

if $\quad \sin\theta_i \langle \sin\theta_0$

or $\quad \boxed{\sin\theta_i \langle N.A.}$

This is the **condition for propagation.**

<u>Fractional Index Change</u> (Δ)**:**

It is the ratio of the RI difference between the core and cladding to the RI of core of an optical fiber,

i.e. $\quad \Delta = \dfrac{n_1 - n_2}{n_1};$

Where; $n_1 \rightarrow$ RI of core, $n_2 \rightarrow$ RI of cladding.

<u>Relation between N.A. and</u> Δ**:**

We have numerical aperture,

$$N.A. = \sqrt{n_1^2 - n_2^2}$$

$$= \sqrt{(n_1 + n_2)(n_1 - n_2)}$$

But $\quad \Delta = \dfrac{n_1 - n_2}{n_1}$

i.e. $\quad n_1\Delta = n_1 - n_2$

$\therefore \quad N.A. = \sqrt{(n_1 + n_2)n_1\Delta}$

Since $n_1 \approx n_2$; $\quad n_1 + n_2 \approx 2n_1$

Hence $\quad N.A. = \sqrt{2n_1 n_1 \Delta} = n_1\sqrt{2\Delta}$

i.e. $\quad \boxed{N.A. = n_1\sqrt{2\Delta}}$

Modes of propagation:

Light propagates as an electromagnetic wave through an optical fiber. All waves having ray directions above the critical angle will be trapped within the fiber due to total internal reflection. But all such waves do not propagate along the fiber. There is certain ray directions allowed for the propagation. These allowed ray directions or possible number of path of light in optical fibers is known as modes of propagation. The paths are zigzag paths excepting the axial direction. The number of modes that a fiber will support depends on the diameter of the core and wavelength of the wave being transmitted.

Types of optical fibers:

Depending on the RI profile and number of modes that a fiber can support, we have three types of optical fibers. They are

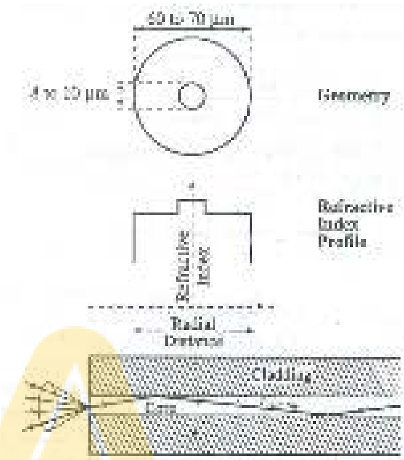
1. Step index single mode fiber
2. Step index multi mode fiber
3. Graded index multi mode fiber

Note: Refractive index profile is a curve which represents the variation of refractive index with respect to the radial distance from the axis of the fiber.

1. Step index single mode fiber:

A step index single mode fiber has a core diameter of about 8 to 10 μm and external diameter of cladding is 60 to 70 μm. The RI of the core has a uniform value. The cladding also has a uniform RI but slightly lesser than that of the core. The RI of the fiber changes abruptly at the core – cladding interface. Hence it is called a step index fiber. This fiber can support only one mode of propagation along its axis. Hence it is called a single mode fiber.

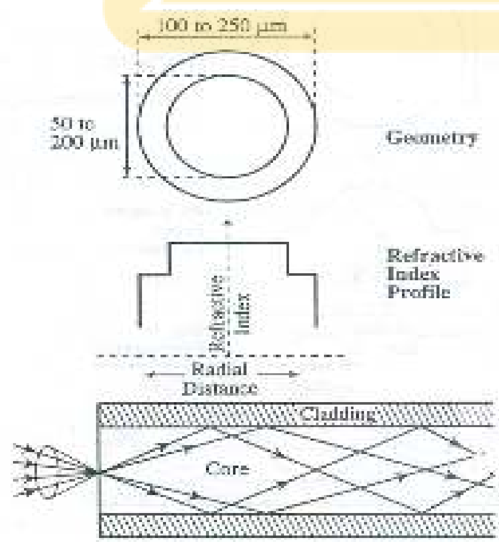
Due to narrow diameter of the fiber only laser can be used as the source of light with these fibers. There is no intermodal dispersion in the fiber. These are widely used in submarine cable systems.



2. Step index multi mode fiber:

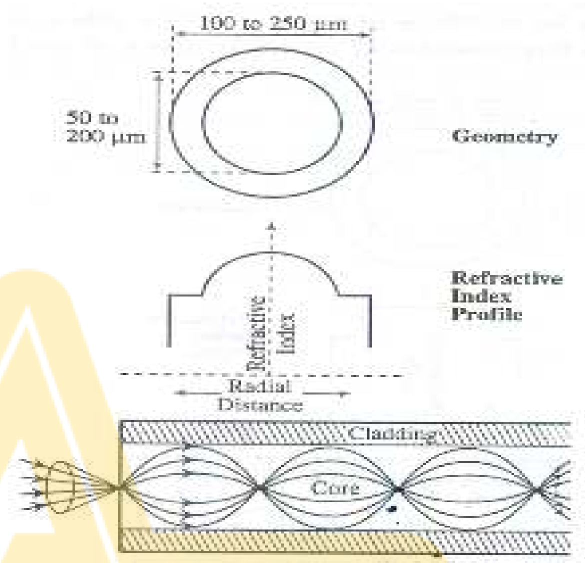
This fiber has a core diameter of 100 μm. The RI remains uniform in the core and cladding region. But the RI changes abruptly at the core – cladding interface. Because of larger diameter, this fiber allows many modes to propagate through it.

The step index multimode fiber can accept either a laser or LED as source of light. It is used in data links which has lower band width requirements.



3. Graded Index multimode fiber:

It is a multimode fiber with a core consisting of concentric layers of different refractive indices. Therefore RI of the core decreases with distance from the fiber axis. The RI of the cladding remains uniform. The RI profile and the modes of propagation are shown in fig. such a RI profile causes a periodic focusing of light propagating through the fiber. Either a laser or a LED can be the source for these fibers.



Normalized frequency (V – number):

It is the relation between fiber size, the refractive indices and the wavelength of light propagating through the fiber. It is given by,

$$\boxed{V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}}$$

Where, $d \rightarrow$ diameter of the core; $n_1 \rightarrow$ RI of the core; $n_2 \rightarrow$ RI of the cladding; $\lambda \rightarrow$ Wavelength of light.

Since $\sqrt{n_1^2 - n_2^2} = N.A.$ we can write

$$V = \frac{\pi d}{\lambda} NA$$

$$V = \frac{\pi d}{\lambda} n_1 \sqrt{2\Delta}$$

The number of modes supported by a fiber is given by $M_N = \frac{V^2}{2}$

Attenuation:

The loss of power suffered by the optical signal as it propagates through the fiber is called **attenuation.**

The attenuation or fiber loss is due to the following factors:

1. Absorption losses
2. Scattering losses
3. Radiation or bending losses

1. Absorption Losses:

The loss of signal strength occurs due to absorption of photons during its propagation. Photons are absorbed by

a) Impurities in the silica glass of which the fiber is made of.

b) Intrinsic absorption by the glass material itself.

a. Absorption by impurities:

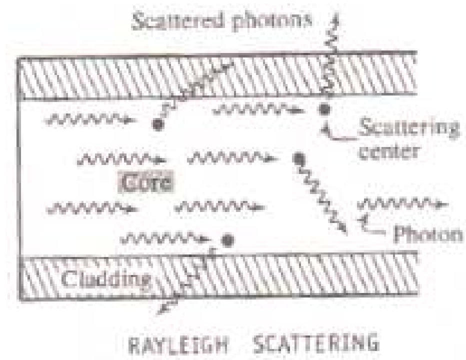
The impurities that are generally present in fiber glass are iron, chromium, cobalt, copper etc. During signal propagation when photons interact with these impurities, the electrons absorb the photons and get excited to higher energy levels. Later these electrons give up their absorbed energy in the form of light photons. But this is of no use, since these photons differ in wavelength and phase.

b. Intrinsic Absorption

The fiber material itself has a tendency to absorb light energy however small it may be. Hence there will be a loss and is known as intrinsic absorption.

2. Scattering Losses:

The optical power is lost due to the scattering of photons. This scattering is due to the non uniformity in the density of the fiber material, which leads to the variation in the RI of the fiber. Structural inhomogenities and defects created in the fiber can also cause scattering.



The loss of light energy by scattering is found to be wavelength dependent. It decreases with increase in the wavelength of light to be transmitted through the fiber.

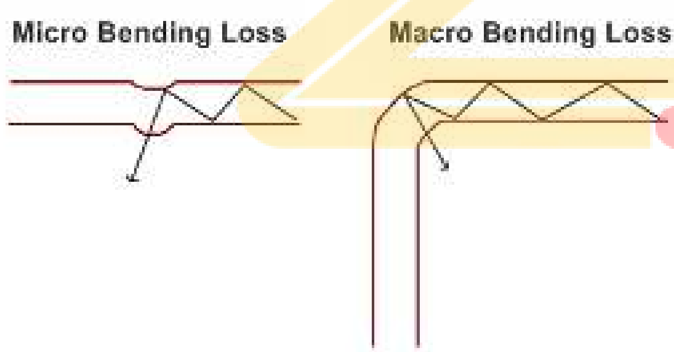
3. **Bending Losses (Radiation Losses):**

Radiation losses occur due to bending of fiber. There are two types of bends

a) Microscopic bends

b) Macroscopic bends

Microscopic bends are caused during manufacturing as well as due to the applied stress on the fiber. Macroscopic bending arises during the installation of the fiber. At the point of a bend, light will escape to the surrounding medium due to the fact that the angle of incidence at that point becomes lesser than the critical angle.



To minimize these losses, the optical fiber has to be laid without sharp bends and they should be freed from the external stresses by providing mechanical strength through external coverage.

Attenuation Coefficient:

When light travels in a material medium there will always be a loss in its intensity with distance travelled. This loss takes place according to **Lambert's law.**

According to **Lambert's law**, "the rate of decrease of intensity of light with distance travelled in a homogeneous medium is proportional to the initial intensity". If 'P' is the initial intensity and 'L' is the distance propagated in the medium, then

$-\frac{dP}{dL} \; \alpha \; P$ (Negative sign indicates that it is decrease in intensity)

Or $-\frac{dP}{dL} = \alpha P$ ----- (1) where α is a constant called the attenuation coefficient, or simply attenuation.

From (1), $\frac{dP}{P} = -\alpha dL$

Integrating both sides, $\int \frac{dP}{P} = -\alpha \int dL$ ----- (2)

Considering a fiber of length 'L' and taking Pin as the initial intensity of light launched into the fiber and Pout as the intensity of light received at the other end of the fiber, equation (2) can be written as

$$\int_{P_{in}}^{P_{out}} \frac{dP}{P} = -\alpha \int_0^L dL$$

$$\left[\log P\right]_{P_{in}}^{P_{out}} = -\alpha \left[L\right]_0^L$$

$$\log\left[\frac{P_{out}}{P_{in}}\right] = -\alpha L$$

Or, $\alpha = -\frac{1}{L}\log\left[\frac{P_{out}}{P_{in}}\right]$ --------- (3)

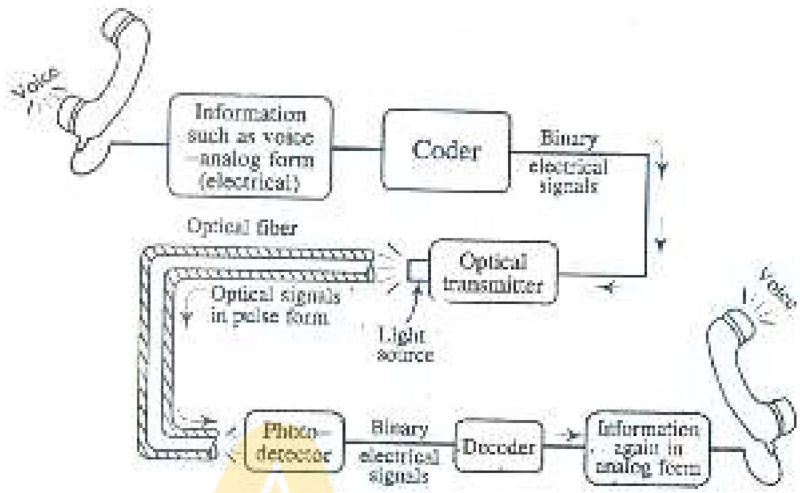
From equation (3) it is clear that the loss light intensity in an optical fiber is logarithmic in nature.

Unit of attenuation is Bel/unit length (B). But generally it is expressed in terms of decibel/kilometers or dB/km.

Hence; $\alpha = -\frac{10}{L}\log\left[\frac{P_{out}}{P_{in}}\right]$ dB/km

Application:

Point – point communication system using optical fibers:



In a point - point communication system, we have analog information such as voice of a telephone user. The voice gives rise to electrical signals in analog form coming out of the transmitter section of the telephone. With the help of a coder, the analog signal is converted into binary data. The binary data in the form of a stream of electrical pulses are converted into pulses of optical power by modulating the light emitted by an optical source such as a laser diode or LED. This unit is called optical transmitter, from which the optical power is launched into the fiber.

During the propagation of the signal, attenuation or losses occurs. This may reach a limiting stage beyond which it may not be possible to retrieve the information from the light signal. Hence a repeater is needed in the transmission path. A repeater consists of a receiver and a transmitter. The receiver converts the optical signal into corresponding electrical signal and then it is amplified. These electrical signals are again converted into optical signals and fed into the optical fiber.

At the receiving end the optical signal from the fiber is fed into a photo detector. Hence signal is converted to pulses of electric current. This is then fed to a decoder which converts the binary data into an analog signal, which will be the same information such as voice; which was there at the transmitting end.

Advantages of optical fibers:

1. Optical fibers can carry very large amounts of information, i.e. these support transmission of signals over a large bandwidth.
2. Attenuation is very low.
3. Power required for optical fiber communication is very less.
4. The information or signals are fully secured.
5. The rate of information transfer is very high.
6. Because of their compactness and lightweight, fibers are much easier to transport.
7. Since number of repeaters required in optical fiber communication are less and raw materials are abundantly available, optical fibers are less expensive.

Disadvantages of optical fibers:

1. Fibers cannot be bent too sharply, for sharper bends either the fiber gets broken or light fails to undergo Total Internal Reflection.
2. The optic connectors used to connect two fibers (splicing) are highly expensive.
3. Whenever a fiber suffers a line break, a highly skilled workmanship is required to reestablish the connections and is time consuming.
4. Fibers undergo expansion and contraction with temperature that upset some critical alignments which leads to loss in signal power.

IMPORTANT QUESTIONS:

1. Using total internal reflection concept, obtain an expression for the acceptance angle in an optical fiber.
2. Discuss the different types of optical fibers with suitable diagrams.
3. What is attenuation in an optical fiber? Explain the attenuation mechanisms.
4. With a neat figure derive an expression for numerical aperture of an optical fiber and arrive at the condition for propagation.
5. Obtain an expression for attenuation coefficient.
6. Describe the point – point communication system, with the help of a block diagram.

7. Discuss the advantages and dis advantages of an optical communication system over conventional communication system

8. Define the terms :

i) Acceptance angle

ii) Numerical aperture

iii) Fractional index change

iv) Modes of propagation

v) Attenuation