# ADVANCED CALCULUS & NUMERICAL METHODS 18MAT21

## MODULE-V

**IMPORTANT RESULTS:**

1. Divided differences with equal length of interval:
   Suppose $y = f(x)$ be a function, let $y_0, y_1, y_2, y_3 \ldots.. y_n$ be the set of values corresponding to its arguments $x_0, x_1, x_2, x_3 \ldots\ldots x_n$ with the equal length of interval $h$, then
   FORWARD DIVIDED DIFFERENCE TABLE:

| $x$ | $y=f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| $x_0$ | $y_0$ | | | | |
| | | $y_1 - y_0 = \Delta y_0$ | | | |
| $x_1$ | $y_1$ | | $\Delta y_1 - \Delta y_0 = \Delta^2 y_0$ | | |
| | | $y_2 - y_1 = \Delta y_1$ | | $\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$ | |
| $x_2$ | $y_2$ | | $\Delta y_2 - \Delta y_1 = \Delta^2 y_1$ | | $\Delta^3 y_1 - \Delta^3 y_0 = \Delta^4 y_0$ |
| | | $y_3 - y_2 = \Delta y_2$ | | $\Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1$ | |
| $x_3$ | $y_3$ | | $\Delta y_3 - \Delta y_2 = \Delta^2 y_2$ | | |
| | | $y_4 - y_3 = \Delta y_3$ | | | |
| $x_4$ | $y_4$ | | | | |

BACKWARD DIVIDED DIFFERENCE TABLE:

| $x$ | $y=f(x)$ | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ |
|---|---|---|---|---|---|
| $x_0$ | $y_0$ | | | | |
| | | $y_1 - y_0 = \nabla y_1$ | | | |
| $x_1$ | $y_1$ | | $\nabla y_2 - \nabla y_1 = \nabla^2 y_2$ | | |
| | | $y_2 - y_1 = \nabla y_2$ | | $\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$ | |
| $x_2$ | $y_2$ | | $\nabla y_3 - \nabla y_2 = \nabla^2 y_3$ | | $\nabla^3 y_4 - \nabla^3 y_3 = \nabla^4 y_4$ |
| | | $y_3 - y_2 = \nabla y_3$ | | $\nabla^2 y_4 - \nabla^2 y_3 = \nabla^3 y_4$ | |
| $x_3$ | $y_3$ | | $\nabla y_4 - \nabla y_3 = \nabla^2 y_4$ | | |
| | | $y_4 - y_3 = \nabla y_4$ | | | |
| $x_4$ | $y_4$ | | | | |

2. Newton's forward interpolation formula:

$$y(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\frac{p)p-1)(p-2)(p-3)}{4!}\Delta^4 y_0+\ldots\ldots ,$$

Where $p=\dfrac{x-x_0}{h}$

3. Newton's backward interpolation formula:

$$y(x)=y_n+p\nabla y_n+\frac{p(p+1)}{2!}\nabla^2 y_n+\frac{p(p+1)(p+2)}{3!}\nabla^3 y_n+\frac{p)p+1)(p+2)(p+3)}{4!}\nabla^4 y_n+\ldots\ldots$$

, Where $p=\dfrac{x-x_n}{h}$

4. Newton's divided difference formula for unequal intervals:

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+$$
$$(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+\ldots..+(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})f(x_0,x_1,x_2,\ldots x_n)$$

5. Lagrange's interpolation formula for unequal intervals:

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)\ldots.(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots.(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots.(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots.(x_1-x_n)}y_1+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)\ldots.(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots.(x_2-x_n)}y_2+\ldots\ldots.$$

6. Newton Raphson method:
   i) Write the given transcendental equation as $f(x)=0$.
   ii) Choose the nearest root $x_0$ , for which $f(x_0)\langle 0$.
   iii) Do the iterations by using $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)},n=0,1,2,3,\ldots.$
7. Regula-False method.
   i) Write the given transcendental equation as $f(x)=0$.
   ii) Choose the nearest root $x_0$ and $x_1$ , for which $f(x_0)\langle 0$ and $f(x_1)\rangle 0$.
   iii) $x_2=\dfrac{x_1 f(x_0)-x_0 f(x_1)}{f(x_0)-f(x_1)}$
8. Numerical Integration:
   i) Simpson's $1/3 rd$ Rule : $\int_a^b f(x)dx=\dfrac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\ldots..)+4(y_1+y_3+\ldots..y_{n-1})\right]$

      , $h=\dfrac{b-a}{n}$
   ii) Simpson's $3/8 th$ rule :

$$\int_a^b f(x)dx=\frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+\ldots..)+2(y_3+y_6+\ldots..y_{n-3})\right],\ h=\frac{b-a}{n}$$

.

iii) Weddle's Rule:

$$\int_a^b f(x)dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right],\ h = \frac{b-a}{n}$$ and $n$ should be a 6.

## PROBLEMS

1. The area A of a circle of diameter *d* is given for the following values.

| $d$ | 80 | 85 | 90 | 95 | 100 |
|---|---|---|---|---|---|
| $A$ | 5026 | 5674 | 6362 | 7088 | 7854 |

Calculate the area of circle of diameter 105.

**Solution:**

| d=x | y=A | $\nabla y$ | $\nabla^2 y$ | $\nabla^3 y$ | $\nabla^4 y$ |
|---|---|---|---|---|---|
| 80 | 5626 | | | | |
| | | 648 | | | |
| 85 | 5674 | | 40 | | |
| | | 688 | | -2 | |
| 90 | 6362 | | 38 | | 4= $\nabla^4 y_4$ |
| | | 726 | | 2= $\nabla^3 y_4$ | |
| 95 | 7088 | | 40= $\nabla^2 y_4$ | | |
| | | 766= $\nabla y_4$ | | | |
| 100 | 7854= $y_4$ | | | | |

By the Newton's backward interpolation formula , we have

$$y(x) = y_4 + p\nabla y_4 + \frac{p(p+1)}{2!}\nabla^2 y_4 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_4 + \frac{p)p+1)(p+2)(p+3)}{4!}\nabla^4 y_4 ,$$

Where $p = \dfrac{x - x_4}{h}$

$$\therefore p = \frac{105-100}{5} = 1$$

$$\therefore y(105) = 7854 + 766(1) + \frac{(1)(2)(40)}{2} + \frac{(1)(2)(3)(2)}{6} + \frac{(1)(2)(3)(4)(4)}{24}$$

$$\Rightarrow y(105) = 7854 + 766 + 40 + 2 + 4$$

$$\Rightarrow y(105) = 8666$$

Thus the area A corresponding to diameter 105 is 8666

2. From the data given below , find the number of students who obtained i) Less than 45 marks ii) Between 40 and 45 marks.

| Marks | 0-40 | 41-50 | 51-60 | 61-70 | 71-80 |
|---|---|---|---|---|---|
| No.os Students | 31 | 42 | 51 | 35 | 31 |

**Solution:**

Let $x$ be the marks obtained the students and let $y = f(x)$ be the number of students getting less than $x$ marks.

| $x$ | $y=f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| <40 | 31 | | | | |
| | | 42 | | | |
| 50 | 73 | | 9 | | |
| | | 51 | | -25 | |
| 60 | 124 | | -15 | | 37 |
| | | 35 | | 12 | |
| 70 | 159 | | -4 | | |
| | | 31 | | | |
| 80 | 190 | | | | |

By the Newton's forward interpolation formula , we have

$$y(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\frac{p)p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 ,$$

Where $p=\dfrac{x-x_0}{h}$

$$\therefore p=\frac{45-40}{10}=0.5$$

$$\therefore y(45)=31+(0.5)(42)+\frac{(0.5)(0.5-1)(9)}{2}+\frac{(0.5)(0.5-1)(0.5-2)(-25)}{6}+\frac{(0.5)(0.5-1)(0.5-2)(0.5-3)(37)}{24}$$

$$\Rightarrow y(45)=31+21-1.125-1.5625-1.445$$

$$\Rightarrow y(45)=47.8675$$

$$\Rightarrow y(45)=48$$

$\therefore$ The number of students , who obtained less than 45 marks = 48
Since the number of students who obtained less than 40 marks is 31

$\therefore$ The number of students, who obtained marks between 40 to 45 is : 48-31 =17

3. Use appropriate interpolating formula to compute $y(82)$ and $y(98)$ for the data.

| $x$ | 80 | 85 | 90 | 95 | 100 |
|---|---|---|---|---|---|
| $y=f(x)$ | 5026 | 5674 | 6362 | 7088 | 7854 |

**Solution:**

| d=x | y=A | I DD | II DD | III DD | IV DD |
|---|---|---|---|---|---|
| 80 | 5626 | | | | |
| | | 648 | | | |
| 85 | 5674 | | 40 | | |
| | | 688 | | -2 | |
| 90 | 6362 | | 38 | | $4=\nabla^4 y_4$ |
| | | 726 | | $2=\nabla^3 y_4$ | |
| 95 | 7088 | | $40=\nabla^2 y_4$ | | |
| | | $766=\nabla y_4$ | | | |
| 100 | $7854=y_4$ | | | | |

To find $y(82)$:

By the Newton's interpolation, we have

$$y(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\frac{p)p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 ,$$

Where $p=\dfrac{x-x_0}{h}$

$$\therefore p=\frac{82-80}{5}=0.4$$

$$\therefore y(82)=5026+(0.4)(648)+\frac{(0.4)(-0.6)(40)}{2}+\frac{(0.4)(-0.6)(-1.6)(-2)}{6}+\frac{(0.4)(-0.6)(-1.6)(-2.6)(4)}{24}$$

$$\Rightarrow y(82)=5026+(0.4)(648)+(0.4)(-0.6)(20)+(0.4)(-0.6)(-1.6)(-0.3333)+(0.4)(-0.6)(-1.6)(-2.6)(0.1666)$$

$$\Rightarrow y(82)=5280.1$$

To find $y(98)$:

By the Newton's backward formula , we have

$$y(x)=y_4+p\nabla y_4+\frac{p(p+1)}{2!}\nabla^2 y_4+\frac{p(p+1)(p+2)}{3!}\nabla^3 y_4+\frac{p)p+1)(p+2)(p+3)}{4!}\nabla^4 y_4 ,$$

Where $p=\dfrac{x-x_4}{h}$

$$\therefore p=\frac{98-100}{5}=-0.4$$

$$\therefore y(98)=7854+766((-0.4))+\frac{(-0.4)(0.6)(40)}{2}+\frac{(-0.4)(0.6)(1.6)(2)}{6}+$$
$$\frac{(-0.4)(0.6)(1.6)(2.6)(4)}{24}$$
$$\Rightarrow y(98)=7854+(0.4)(766)+(-0.4)(0.6)(20)+(-0.4)(0.6)(1.6)(0.3333)+$$
$$(-0.4)(0.6)(1.6)(2.6)(0.1666)$$
$$y(98)=7547.30$$

4. The table gives the distances in nutrical miles of the visible horizon for the given heights in feet above the earth's surface:

| $x=Height$ | 100 | 150 | 200 | 250 | 300 | 350 | 400 |
|---|---|---|---|---|---|---|---|
| $y=Distance$ | 10.63 | 13.03 | 15.04 | 16.81 | 18.42 | 19.90 | 21.27 |

Find the values of $y$ at i) x=160 ft . ii) x=410ft.

**Solution:**
We can find the distance at both x=160 and x=410 by using Newton's forward and Backward formulas simultaneously.
The divided difference table as follows.

| $x$ | $y$ | I DD | II DD | III DD | IV DD | V DD | VI DD |
|---|---|---|---|---|---|---|---|
| 100 | **10.63** | | | | | | |
| | | **2.4** | | | | | |
| 150 | 13.03 | | **-0.39** | | | | |
| | | 2.01 | | **0.15** | | | |
| 200 | 15.04 | | -0.24 | | **-0.07** | | |
| | | 1.77 | | 0.08 | | **0.02** | |
| 250 | 16.81 | | -0.16 | | -0.05 | | **0.02** |
| | | 1.61 | | 0.03 | | **0.04** | |
| 300 | 18.42 | | -0.13 | | **-0.01** | | |
| | | 1.48 | | **0.02** | | | |
| 350 | 19.90 | | **-0.11** | | | | |
| | | **1.37** | | | | | |
| 400 | **21.27** | | | | | | |

(i) Since $x=160\,ft.$ is near to the $x_0$.
To Find $y(160)$:

$$y(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\frac{p)p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 \quad ,$$

Where $p=\dfrac{x-x_0}{h}$

$$\therefore p=\frac{160-100}{50}=1.2$$

$$\therefore y(160)=10.63+(1.2)(2.4)+\frac{(1.2)(0..2)(-0.39)}{2}+\frac{(1.2)(0..2)(-0.8)(0.15)}{6}+$$
$$\frac{(1.2)(0..2)(-0.8)(-1.8)(-0.07)}{24}+\frac{(1.2)(0..2)(-0.8)(-1.8)(-2.8)(0.02)}{120}+$$
$$\frac{(1.2)(0..2)(-0.8)(-1.8)(-2.8)(-3.8)((0.02)}{720}$$
$$\Rightarrow y(160)=10.63+2.88-0.0468-0.0048-0.001008-0.00016128+0.000102144$$
$$\Rightarrow y(160)=13.4573$$

To find $y(410)$:

$$y(x)=y_n+p\nabla y_n+\frac{p(p+1)}{2!}\nabla^2 y_n+\frac{p(p+1)(p+2)}{3!}\nabla^3 y_n+\frac{p)p+1)(p+2)(p+3)}{4!}\nabla^4 y_n+\ldots\ldots$$

, Where $p=\frac{x-x_n}{h}$

$$\Rightarrow p=\frac{410-400}{50}=0.2$$
$$\therefore y(410)=21.27+0.2(1.7)+\frac{0.2(1.2)(-0.11)}{2}+\frac{0.2(1.2)(2.2)(0.02)}{6}+$$
$$\frac{0.2(1.2)(2.2)(3.2)(-0.01)}{24}+\frac{0.2(1.2)(2.2)(3.2)(4.2)(0.04)}{120}+$$
$$\frac{0.2(1.2)(2.2)(3.2)(4.2)(5.2)(0.02)}{720}$$
$$\Rightarrow y(410)=21.27+0.274-0.0132+0.00176-0.000704+0.002365+0.001025$$
$$\Rightarrow y(410)=21.5352$$

5. Use an appropriate interpolation formula to compute $f(42)$ using the following data.

| $x$ | 40 | 50 | 60 | 70 | 80 | 90 |
|---|---|---|---|---|---|---|
| $y=f(x)$ | 184 | 204 | 226 | 250 | 276 | 304 |

**Solution:**

We can find the solution of $f(42)$ by using Newton's forward interpolation formula , because the value of $x=42$ is near to $x_0$, the formula is

$$y(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\frac{p)p-1)(p-2)(p-3)}{4!}\Delta^4 y_{0+\ldots.}\quad ,$$

Where $p=\frac{x-x_0}{h}$

| $x$ | $y=f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|---|---|---|---|---|---|---|
| 40 | 184 | | | | | |
| | | 20 | | | | |
| 50 | 204 | | 2 | | | |
| | | 22 | | 0 | | |
| 60 | 226 | | 2 | | 0 | |
| | | 24 | | 0 | | 0 |
| 70 | 250 | | 2 | | 0 | |
| | | 26 | | 0 | | |
| 80 | 276 | | 2 | | | |
| | | 28 | | | | |
| 90 | 304 | | | | | |

$$p=\frac{x-x_0}{h}=\frac{42-40}{10}=0.2$$

$$\therefore f(42)=184+(0.2)(20)+\frac{(0.2)(-0.2)(2)}{2}+0$$

$$\Rightarrow f(42)=184+4-0.16$$

$$f(42)=187.4$$

6. Find the equation of the polynomial , which passes through $(4,-43),(7,83),(9,327),(12,1053)$ using Newton's divided difference interpolation formula.

**Solution:**

The Newton's divided difference table for unequal intervals as follows.

| $x$ | $y=f(x)$ | I DD | II DD | III DD |
|---|---|---|---|---|
| 4 | -43= $f(x_0)$ | | | |
| | | 42= $f(x_0,x_1)$ | | |
| 7 | 83 | | 16= $f(x_0,x_1,x_2)$ | |
| | | 122 | | 1= $f(x_0,x_1,x_2,x_3)$ |
| 9 | 327 | | 24 | |
| | | 242 | | |
| 12 | 1053 | | | |

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+$$
$$(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$

$$\Rightarrow f(x)=-43+(x-4)(42)+(x-4)(x-7)(16)+(x-4)(x-7)(x-9)(1)$$

$$f(x)=-43+42x-168+16x^2-176x+448+x^3-16x^2+63x-4x^3+64x-252$$

$$f(x)=x^3-4x^2-7x-15$$

7. Using Newton's divided difference interpolation formula fit a polynomial for the following data.

| $x$ | 2 | 4 | 5 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| $y=f(x)$ | 10 | 96 | 196 | 350 | 868 | 1746 |

**Solution:**

Newton's divided difference table for unequal intervals as follows.

| $x$ | $y=f(x)$ | I DD | II DD | III DD |
|---|---|---|---|---|
| 2 | 10= $f(x_0)$ | | | |
| | | 43= $f(x_0,x_1)$ | | |
| 4 | 96 | | 19= $f(x_0,x_1,x_2)$ | |
| | | 100 | | 2= $f(x_0,x_1,x_2,x_3)$ |
| 5 | 196 | | 27 | |
| | | 154 | | 2 |
| 6 | 350 | | 35 | |
| | | 259 | | 2 |
| 8 | 868 | | 45 | |
| | | 439 | | |
| 10 | 1746 | | | |

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+$$
$$(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$
$$\Rightarrow f(x)=10+(x-2)(43)+(x-2)(x-4)(19)+(x-2)(x-4)(x-6)(2)$$
$$f(x)=10+43x-86+19x^2-114x+152+2x^3-24x^2+88x-96$$
$$f(x)=2x^3-43x^2+9x-20$$

**8.** Use Lagrange's interpolation formula to find $f(4)$ for the following data.

| $x$ | **0** | **2** | **3** | **6** |
|---|---|---|---|---|
| $y=f(x)$ | **-4** | **2** | **14** | **158** |

**Solution:**

Given $x_0=0,\ x_1=2,\ x_2=3,\ x_3=6$

$y_0=-4,\ y_1=2,\ y_2=14,\ y_3=158$

We know that

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$\Rightarrow y(4)=\frac{(4-2)(4-3)(4-6)(-4)}{(0-2)(0-3)(0-6)}+\frac{(4-0)(4-3)(4-6)(2)}{(2-0)(2-3)(2-6)}+$$
$$\frac{(4-0)(4-2)(4-6)(14)}{(3-0)(3-2)(3-6)}+\frac{(4-0)(4-2)(4-3)(158)}{(6-0)(6-2)(6-3)}$$
$$\Rightarrow y(4)=-0.44-2+24.89+17.56$$
$$\Rightarrow y(4)=40.81$$

9. Applying Lagrange's interpolation formula to find $u(4)$ for the data $u_0=707, u_2=819, u_3=866, u_6=966$.

**Solution:**

Let $x_0=0, x_1=2, x_2=3, x_3=6$

$y_0=707, y_1=819, y_2=866, y_3=966$

We know that

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$
$$\Rightarrow y(4)=\frac{(4-2)(4-3)(4-6)(707)}{(0-2)(0-3)(0-6)}+\frac{(4-0)(4-3)(4-6)(819)}{(2-0)(2-3)(2-6)}+$$
$$\frac{(4-0)(4-2)(4-6)(866)}{(3-0)(3-2)(3-6)}+\frac{(4-0)(4-2)(4-3)(966)}{(6-0)(6-2)(6-3)}$$
$$\Rightarrow y(4)=78.56-819+1539.56+107.$$
$$\Rightarrow y(4)=906.45$$

10. Write an interpolating polynomial of the form $x=f(y)$ and hence find $x(5) and y(5)$ for the following data.

| $x$ | 2 | 10 | 17 |
|---|---|---|---|
| $y=f(x)$ | 1 | 3 | 4 |

**Solution:**

Let $x_0=2\ x_1=10\ x_2=17$

$y_0=1, y_1=3, y_2=4,$

By the inverse Lagrange's interpolation formula , we have

$$x(y)=\frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}x_0+\frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}x_1+\frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}x_2$$

$$x(y)=\frac{(y-3)(y-4)}{(1-3)(1-4)}2+\frac{(y-1)(y-4)}{(3-1)(3-4)}10+\frac{(y-1)(y-3)}{(4-1)(4-3)}17$$

$$x(y)=\frac{1}{3}(y^2-7y+12)-5(y^2-5y+4)+\frac{17}{3}(y^2-4y+3)$$

$$x(y)=y^2+1$$

$$x=y^2+1--------------------(1)$$

$$\therefore x(5)=5^2+1=26$$

When $x=5$

$$(1)\Rightarrow 5 \ =y^2+1$$

$$\Rightarrow 5=y^2+1$$

$$y^2=4$$

$$y=\pm 2$$

11. Find the root of the equation $Cosx=xe^x$ using Regula-falsi method.

**Solution:**

*Given*

$$Cosx=xe^x$$

$$\Rightarrow Cosx-xe^x=0$$

$$\therefore f(x)=Cosx-xe^x$$

Let $x_0=0$ , $x_1=0$

$$\Rightarrow f(x_0)=f(0)=1>0$$

$$f(x_1)=f(1)=-2.1779<0$$

The root lies between 0 and 1

By the Regula-falsi method , we have

$$x_2=\frac{x_0f(x_1)-x_1f(x_0)}{f(x_1)-f(x_0)}$$

$$\Rightarrow x_2=\frac{(0)(-2.1779)-(1)(1)}{-2.1779-1}=0.31467$$

$$\therefore f(x_2)=f(0.31467)=0.51987>0$$

$$\therefore x_3=\frac{x_2f(x_1)-x_1f(x_2)}{f(x_1)-f(x_2)}$$

$$\Rightarrow x_3=\frac{(0.31467)(-2.1779)-(1)(0.51987)}{-2.1779-0.51987}=0.44672$$

$$\therefore f(x_3)=f(0.44672)=0.20356>0$$

$$\therefore x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$x_4 = \frac{(0.44672)(-2.17790.) - (1)(0.20356)}{-2.1779 - 0.2.356}$$

$$\therefore x_4 = 0.49402$$

Similarly, we can get

$$\therefore x_5 = 0.50995$$

$\Rightarrow x = 0.5100$ be approximate real root.

12. Solve $xe^x - 2 = 0$ by Regula-falsi method.

**Solution:**

$$xe^x - 2 = 0$$

Given $\Rightarrow f(x) = 0$

$$\therefore f(x) = xe^x - 2$$

Let $x_0 = 0.8$ , $x_1 = 0.9$

$$\Rightarrow f(x_0) = f(0.8) = -0.2195 < 0$$

$$f(x_1) = f(0.9) = 0.2136 > 0$$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x_2 = \frac{(0.8)(0.2136) - (0.9)(-0.2195)}{0.2136 + 0.2195} = 0.8506$$

$$\therefore f(x_2) = f(0.8536) = -0.0087 < 0$$

$$\therefore x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$\Rightarrow x_3 = \frac{(0.8506)(0.2136) - (0.9)(-0.0087)}{0.2136 + 0.0087} = 0.0.8525$$

$$\therefore f(x_3) = f(0.8525) = -0.0004 < 0$$

$$\therefore x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$\Rightarrow x_4 = \frac{(0.8525)(0.2136) - (0.9)(-0.0004)}{0.2136 + 0.0004} = 0.0.8526$$

$$\therefore f(x_4) = f(0.8526) = -0.00002 < 0$$

The real root of the equation is $x = 0.8526$

13. Find the real root of the equation $x \log_{10} x - 1.2 = 0$ by using Regula-falsi method.

**Solution:**

Given $x\log_{10} x - 1.2 = 0$

$\Rightarrow f(x) = 0$

$\therefore f(x) = x\log_{10} x - 1.2$

Let $x_0 = 2.7$ , $x_1 = 2.8$

$\Rightarrow f(x_0) = f(2.7) = -0.0353 < 0$

$f(x_1) = f(2.8) = 0.0520 > 0$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x_2 = \frac{(2.8)(0.0520) - (2.9)(-0.0353)}{0.0520 + 0.0353} = 2.7404$$

$\therefore f(x_2) = f(2.7404) = -0.0002 < 0$

$$\therefore x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$\Rightarrow x_3 = \frac{(2.7404)(0.0520) - (2.9)(-0.0002)}{0.0520 + 0.0002} = 2.7406$$

$\therefore f(x_3) = f(2.7406) = -0.00002 < 0$

$\therefore$

The real root of the equation is $x = 2.7406$

14. Find the real root of the equation $x^3 - 2x - 5 = 0 = 0$ by using Regula-falsi method.

**Solution:**

Given

$x^3 - 2x - 5 = 0 = 0$

$\Rightarrow f(x) = 0$

$\therefore f(x) = x^3 - 2x - 5$

Let $x_0 = 2$ , $x_1 = 2.1$

$\Rightarrow f(x_0) = f(2) = -1 < 0$

$f(x_1) = f(2.1) = 0.061 > 0$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x_2 = \frac{(2)(0.061) - (2.1)(-1)}{0.061 + 1} = 2.0942$$

$\therefore f(x_2) = f(2.0942) = -0.003922 < 0$

$$\therefore x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$\Rightarrow x_3 = \frac{(2.0942)(0.061) - (2.1)(-0.003922)}{0.061 + 0.003922} = 2.0945$$

$$\therefore f(x_3) = f(2.0945) = -0.0005 < 0$$

The real root of the equation is $x = 2.0945$

15. Solve $xe^x - 3 = 0$ by Regula-falsi method. (HW)
16. Use Newton-Raphson method to find a real root of $xSinx + Cosx = 0$ (Or $x\tan x + 1 = 0$) near $x = \pi$ correct to three decimal places.( Convert calculator to radian mode)

**Solution:**

Given

$$xSinx + Cosx = 0 \ , \ x_0 = \pi$$

$$\Rightarrow f(x) = 0$$

$$\therefore f(x) = xSinx + Cosx$$

$$f'(x) = xCosx$$

By the Newton-Raphson method, we know that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1 = \pi - \frac{(-1)}{\pi} = \pi - \frac{1}{\pi} = 2.8246$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_1 = 2.8246 - \frac{(-0.06971)}{(-2.6839)} = 2.8246 - 0.0259 = 2.7987$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\Rightarrow x_3 = 2.7987 - \frac{(-0.00082742)}{(-2.6358)} = 2.7987 - 0.0003139 = 2.7983$$

$\therefore$ The real root of the equation is $x = 2.7983$

17. Using Newton-Raphson method find the real root of the equation $3x = Cosx + 1$.

**Solution:**

Given

$$3x = Cosx + 1 \ , \text{ Let } x_0 = 0.5$$

$\Rightarrow f(x)=0$

$\therefore f(x)=3x-Cosx-1$

$f'(x)=3+Sinx$

By the Newton-Raphson method, we know that

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1=0.5-\frac{(-0.3775)}{(3.4794)}=0.5+0.1085=0.6085$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_2=0.6085-\frac{(0.004993)}{(3.5716)}=0.6085-0.001397=0.6071$$

$$x_3=x_2-\frac{f(x_2)}{f'(x_2)}$$

$$\Rightarrow x_3=0.6071-\frac{(0.000005884)}{(3.5704)}=0.6071+0.000001647=0.6071$$

Clearly $x_2=x_3=0.6071$

$\therefore$ The real root of the equation is $x=0.6071$

18. Using Newton-Raphson method find the real root of the equation $x\log_{10}x=1.2$.

**Solution:**

Given

$x\log_{10}x=1.2$

$\Rightarrow x\log_{10}x-1.2=0$

$\Rightarrow f(x)=0$

$$\therefore f(x)=x\log_{10}x-1.2=x\frac{\log_e x}{\log_e 10}-1.2=(0.4343)x\log x-1.2$$

$f'(x)=(0.4343)(1+\log_e x)$

Let $x_0=2$

By the Newton-Raphson method, we know that

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1=2-\frac{(-0.5979)}{0.7352}=2+0.8132=2.8132$$

$$x_2=x_1-\frac{f(x_1)}{f'(x\ )}$$

$$x_2 = 2.8132 - \frac{0.0636}{0.8834} = 2.8132 - 0.7200 = 2.7412$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 2.7412 - \frac{0.0004831}{0.8691} = 2.7412 - 0.00055 = 2.74064$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

Clearly $x_3 = x_4 = 2.7406$

$\therefore$ The real root of the equation is $x = 2.7406$

19. Using Newton-Raphson method find the cube root of 37.

**Solution:**

$$x = \sqrt[3]{37}$$

$$\Rightarrow x^3 = 37$$

Let $x^3 - 37 = 0$

$$f(x) = 0$$

$$\therefore f(x) = x^3 - 37$$

$$f'(x) = 3x^2$$

Let $x_0 = 3$ , then by the Newton-Raphson method, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1 = 3 - \frac{(-10)}{27} = \frac{91}{27} = 3.3703$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_2 = 3.3703 - \frac{(1.2830)}{(34.0767)} = 3.3703 - 0.03765 = 3.3327$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\Rightarrow x_3 = 3.3327 - \frac{(0.01592)}{(33.3206)} = 3.3327 - 0.0004778 = 3.3322$$

Clearly $x_2 = x_3 = 3.3322$

$$\sqrt[3]{37} = 3.3322$$

20. Use Weddle's rule to evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Cosx dx$ by dividing into 6 equal parts.

**Solution:**

Let I= $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Cosx dx$ , $n=6$

$$\therefore h=\frac{b-a}{n}=\frac{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)}{6}=\frac{\pi}{6}=30^0=0.5238$$

$$\therefore Partition\ \ P=\{a=x_0=-90^0,-60^0,-30^0,0,30^0,60^0,90^0=x_6=b\}$$

| $x$ | $y=Cosx$ | |
|---|---|---|
| -90 | Cos(-90) | $y_0$=0 |
| -60 | Cos(-60) | $y_1$=0.5 |
| -30 | Cos(-30) | $y_2$=0.8660 |
| 0 | Cos(0) | $y_3$=1 |
| 30 | Cos(30) | $y_4$=0.8660 |
| 60 | Cos(60) | $y_5$=0.5 |
| 90 | Cos(90) | $y_6$=0 |

$\therefore$ By the Weddle's rule, we have

$$\int_a^b f(x)dx=\frac{3h}{10}[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6]$$

$$\Rightarrow I=\frac{3\times 0.5238}{10}[0+2.5+0.8660+6+0.8660+2.5+0]$$

$$\Rightarrow I=(0.15714)(12.732)$$

$$\Rightarrow I=2.0007$$

$$\Rightarrow I=2$$

21. Compute the value of $\int_{0.2}^{1.4}(Sinx-\log x+e^x)dx$ by using Simpson's 3/8th rule.

**Solution:**

Let I= $\int_{0.2}^{1.4}(Sinx-\log x+e^x)dx$ $\qquad n=6$ ,

$$\therefore h=\frac{b-a}{n}=\frac{1.4-0.2}{6}=0.2$$

$\therefore Partition\ \ P = \{a = x_0 = 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4 = x_6 = b\}$

| $x$ | $y = Sinx - \log x + e^x$ |
|---|---|
| 0.2 | $y_0$=3.0295 |
| 0.4 | $y_1$=2.7975 |
| 0.6 | $y_2$=2.8976 |
| 0.8 | $y_3$=3.1660 |
| 1 | $y_4$=3.5597 |
| 1.2 | $y_5$=4.0698 |
| 1.4 | $y_6$=4.4042 |

By the Simpson's 3/8th rule , we have

$$\int_a^b f(x)dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$\Rightarrow I = \frac{3 \times 0.2}{8}[(3.0295 + 4.4042) + 3(13.3247) + 2(3.1660)]$$

$$\Rightarrow I = 4.053$$

22. Evaluate $\int_0^3 \frac{dx}{(1+x)^2}$ by using Simpson's 3/8th rule.

**Solution:**

Let I= $\int_0^3 \frac{dx}{(1+x)^2}$ , n=6

$$\therefore h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$

$\therefore Partition\ \ P = \{a = x_0 = 0, 0.5, 1, 1.5, 2, 2.5, 3 = x_6 = b\}$

| $x$ | $y = \frac{1}{(1+x)^2}$ |
|---|---|
| 0 | $y_0$=1 |
| 0.5 | $y_1$=0.4444 |
| 1 | $y_2$=0.25 |
| 1.5 | $y_3$=0.16 |
| 2 | $y_4$=0.3333 |
| 2.5 | $y_5$=0.0816 |
| 3 | $y_6$=0.0625 |

By the Simpson's 3/8th rule , we have

$$\int_a^b f(x)dx = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right]$$

$$\Rightarrow I = \frac{3 \times 0.5}{8}\left[(1 + 0.0625) + 2(0.16) + 3(1.1093)\right]$$

$$\Rightarrow I = (0.1875)(1.0625 + 0.32 + 3.3279)$$

$$\Rightarrow I = (0.1875)(4.7104)$$

$$\Rightarrow I = 0.8832$$

23. Evaluate $\int_4^{5.2} log_e\, xdx$ taking 6 equal parts by applying Weddle's rule.

**Solution:**

Let I= $\int_4^{5.2} log_e\, xdx$ , n=6

$$\therefore h = \frac{b-a}{n} = \frac{5.2-4}{6} = 0.2$$

$$\therefore Partition\ \ P = \{a = x_0 = 4, 4.2, 4.4, 4.6, 4.8, 5, 5.2 = x_6 = b\}$$

| $x$ | $y = \log_e x$ |
|---|---|
| 4 | $y_0$=1.3863 |
| 4.2 | $y_1$=1.4351 |
| 4.4 | $y_2$=1.4816 |
| 4.6 | $y_3$=1.5261 |
| 4.8 | $y_4$=1.5686 |
| 5 | $y_5$=1.6094 |
| 5.2 | $y_6$=1.6487 |

$\therefore$ By the Weddle's rule, we have

$$\int_a^b f(x)dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$\Rightarrow I = \frac{3 \times 0.2}{10}\left[1.3863 + 7.1755 + 1.4816 + 9.1566 + 1.5686 + 8.0470 + 1.6487\right]$$

$$\Rightarrow I = 1.8279$$

24. Use Simpson's $1/3\,rd$ rule to evaluate $\int_0^{\frac{\pi}{2}} \sqrt{Sinx}\,dx$ by dividing into 10 equal parts.

**Solution:**

Let I=$\int_0^{\frac{\pi}{2}} \sqrt{Sinx}dx$ , $n = 10$

$$\therefore h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{10} = \frac{\pi}{20} = 9^0 = 0.1571$$

$$\therefore Partition\ P = \{a = x_0 = 0^0, 9^0, 18^0, 27^0, 36^0, 45^0, 54^0, 63^0, 72^0, 81^0, 90^0 = x_{10} = b\}$$

| $x$ | $y = \sqrt{Sinx}$ |
|---|---|
| 0 | $y_0$=0 |
| 9 | $y_1$=0.3955 |
| 18 | $y_2$=0.5559 |
| 27 | $y_3$=0.6738 |
| 36 | $y_4$=0.7667 |
| 45 | $y_5$=0.8409 |
| 54 | $y_6$=0.8995 |
| 63 | $y_7$=0.9439 |
| 72 | $y_8$=0.9752 |
| 81 | $y_9$=0.9938 |
| 90 | $y_{10}$=1 |

$\therefore$ By the Simpson's $1/3rd$ rule, we have

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0 + y_{10}) + 2(y_2 + y_4 + y_6 + y_8) + 4(y_1 + y_3 + y_5 + y_7 + y_9)]$$

$$\Rightarrow I = \frac{0.1571}{3}[1 + 2(3.1973) + 4(3.8479)]$$

$$\Rightarrow I = (0.0524)(1 + 15.3916 + 6.3946)$$

$$\Rightarrow I = (0.0524)(22.7862)$$

$$\Rightarrow I = 1.1932$$

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