Tutorial 4 (problems 1 to 5)

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1 question-1

Let X,Y be two random variables.
a)If var(X+3Y) = var (X-3Y),must X and Y be uncorelated?

$$\begin{aligned} var(X+3Y) &= var(X-3Y) \\ var(X)+9var(Y)+6Cov(X,Y) &= var(X)+9var(Y)-6Cov(X,Y) \\ 12Cov(X,Y) &= 0 \\ Cov(X,Y) &= 0 \end{aligned}$$

Hence X and Y must be uncorelated.

b)If var(X)=var(Y) must X,Y be uncorelated?

$var(X) = var(Y)$
w.k.t $Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$
By given equation nothing can be concluded about value of Cov(X,Y).hence X and Y may or may not be corelated.

c)Find the inequality between var(X1,X2) vs. var(X1)+var(X2) if $\rho_{X1,X2} > 0$.

Given that $\rho_{X1,X2} > 0$
$Cov(X1,X2)/std(X1)std(X2) > 0$
w.k.t standard deviation is always greater than zero hence Cov(X1,X2)¿0
$var(X1+X2) = var(X1)+var(X2)+2Cov(X1,X2)$
As Cov(X1,X2) is greater than zero,
$var(X1+X2) > var(X1)+var(X2)$

d)Given corelation coefficient of X,Y as $\rho_{X,Y}$,what is the corelation coefficient between X,aY+b?(where a,b are constants with a>0)

$$
\begin{aligned}
\rho_{X,aY+b} &= Cov(X, aY+b)/std(X)std(aY+b) \\
1)Cov(X, aY+b) &= E[(X-\mu_x)(aY+b-a\mu_y-b)] \\
Cov(X, aY+b) &= E[(X-\mu_x)(aY-a\mu_y)] \\
Cov(X, aY+b) &= aE[(X-\mu_x)(Y-\mu_y)] \\
Cov(X, aY+b) &= aCov(X, Y)
\end{aligned}
$$

$$
\begin{aligned}
2)std(aY+b) &= \sqrt{E[(aY+b-a\mu_y-b)^2]} \\
std(aY+b) &= \sqrt{E[a^2(Y-\mu_y)^2]} \\
std(aY+b) &= a\sqrt{E(Y-\mu_y)^2)} \\
std(aY+b) &= a\sqrt{var(Y)} \\
std(aY+b) &= astd(Y)
\end{aligned}
$$

$$
\begin{aligned}
&From \quad 1 \ and \ 2, \\
&\quad 3)\rho_{X,aY+b} = Cov(X, aY+b)/std(X)std(aY+b) \\
&\quad \rho_{X,aY+b} = aCov(X, Y)/astd(X)std(Y) \\
&\quad \rho_{X,aY+b} = Cov(X, Y)/std(X)std(Y) \\
&\quad \rho_{X,aY+b} = \rho_{X,Y}
\end{aligned}
$$

2 question 2

Given X1,X2,..Xn have fixed variance var(Xi) for all i from 0 to n.
$Y = \sum_{i=1}^{n} Xi$,what is the maximum value var(Y) can take if we are allowed to vary the covariances?

Given $Y = \sum_{i=1}^{n} Xi$

$$var(Y) = var(X1 + X2 + X3 + ...)$$
$$var(Y) = \sum_{i=1}^{n} var(Xi) + 2\sum_{i<j} Cov(Xi, Xj)$$
$$var(Y) = \sum_{i=1}^{n} var(Xi) + 2\sum_{i<j} \sqrt{var(Xi)var(Xj)}$$

above equation can be written as w.k.t covariance of two numbers is always less than or equals to modulus of product of their standard deviations.

3 question 3

The neighbourhood bank has 3 tellers two fast and one slow, Time taken to assist the customer is exponentially distributed with $\lambda = 6$ at fast teller and $\lambda = 4$ at slow teller . Jane enters a bank and chooses a teller at random with probability 1/3. Find the pdf of time taken to assist Jane and it's transform(MGF).

Given that the time taken to assist Jane is exponential random variable:
$\lambda_1 = 6e^{-6x}$ (fast teller)
$\lambda_2 = 6e^{-6x}$ (fast teller)
$\lambda_3 = 4e^{-4x}$ (slow teller)
probability of Jane going to any teller is 1/3.

$$P(time\ \ to\ \ complete = x) = \sum_{i=1}^{3}[P(time\ \ to\ \ complete\ \ \ = x|teller = i)P(teller = i)]$$

$$P(x) = (2/3)6e^-6x + (1/3)4e^-4x$$

$MGF : MGF\ of\ exponential\ random\ variable\ is\ of\ the\ form\ \lambda/\lambda - s(where\ \ s < \lambda).$

$$E(e^{sx}) = E[e^{(12/3)se^{-6x}+(4/3)se^{-4x}}]$$
$$MGF = \left\{ \begin{array}{ll} 6/(6-4s/6), & s \leq 12 \\ \text{inf}, & \text{else} \end{array} \right\}$$

($s <= 12$ obtained as $4s/6 < 6$ and $s/3 < 4$).

4 question 4

Let X be the value of number on first die and Y be the sum of two numbers obtained by rolling 2 dice.Compute the joint MGF of X,Y.

X be the random variable representing number on dice after rolling it.
So X is an uniform random variable from 1 to 6.
X∼Unif[1,6]
Y is the sum of numbers obtained on rolling two dice.
let $Y = X + X_1$ where X, X_1 are independent random variables.
MGF:

$$E(e^{s_1x+s_2x+s_2x_1}) = E(e^{(s_1+s_2)x})E(e^{s_2x_1})$$
$$MGF(Y) = \sum_{i=1}^{6} E[e^{(s_1+s_2)i/6}]E[e^{s_2i/6}]$$

5 question 5

Let X be a standard guassian random variable,Z be another random variable independent of X.

$$P_Z(z) = \left\{ \begin{array}{ll} 1/2, & \text{Z = 1} \\ 1/2, & \text{Z = -1} \end{array} \right\}$$

Let Y=ZX.
a)What is the pdf of Y?
Given that X∼N(0,1) and Z is a discrete random variable independent of X.

$$P_Z(z) = \left\{ \begin{array}{ll} 1/2, & \text{Z = 1} \\ 1/2, & \text{Z = -1} \end{array} \right\}$$

$$\begin{aligned} & Given\, Y = ZX \\ a)& P(Y = y) = P(ZX = y) \\ & P(Y = y) = P(X = y|Z = 1)P(Z = 1) + P(X = y|Z = -1)P(Z = -1) \\ & P(Y = y) = f_X(y)1/2 + f_X(y)1/2 \\ & P(Y = y) = f_X(y) \end{aligned}$$

b)Are X and Y uncorelated?

$$\begin{aligned} b)& Cov(X, Y) = E(X)E(Y) \quad (given\, X, Y\, are\, independent\, hence\, Cov(X^2, Y) = E(X^2Z) - E(Z) = 0) \\ & Cov(X, Y) = E(X)E(ZX) \\ & Cov(X, Y) = E(X)^2E(Z) \\ & Cov(X, Y) = 0 \quad (\mu_z = 1(1/2) - 1(1/2) = 0) \end{aligned}$$

Since Cov(X,Y) is zero, X and Y are uncorelated.

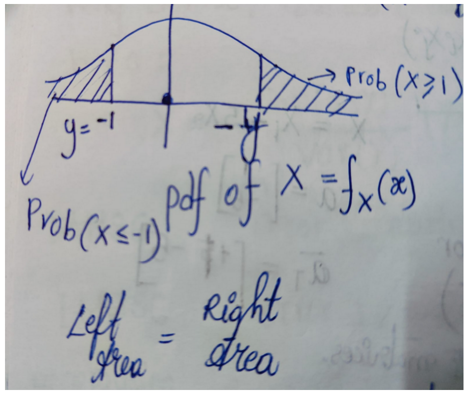

Figure 1: pdf of X as y varies

c)Are X and Y independent?
As y changes, pdf of X is altering from hence X,Y are dependent random variables.(from figure1)

d)Are X and Y jointly guassian?
Let us assume X,Y are jointly guassian:
Any linear combination of them(aX+bY) will also be a jointly guassian random variable.

let a=2,b=3
From above property,$K = 2X + 3Y$ will also be guassian with mean μ_k and variance var(K).

1)$\mu_k = E(2X + 3Y)$

$\mu_k = 2E(X) + 3E(Y) \quad (E(Y) = Cov(ZX) = 0 as Z, X are independent).$

$\mu_k = 0$

2)$\sigma_k = \sqrt{E(2X + 3Y)^2}$

$\sigma_k = \sqrt{E(2X)^2 + E(3Y)^2 + 12Cov(XY)}$

$\sigma_k = \sqrt{0 + 0 + 0}$

$\sigma_k = 0$

As μ_k and σ_k are zero that indicates that K has mean 0,variance 0 i.e; for all values of X,Z it attains same value so it can not be a guassian random variable.
Our assumption is wrong.
X,Y are not jointly guassian.