Homework 1

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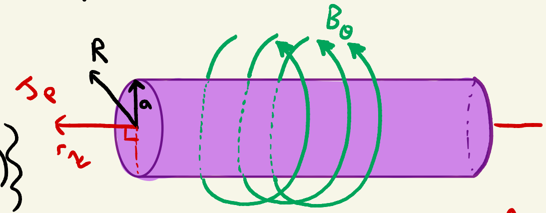
Setup

The magnetic field around the unstable pinch can be given by

$$B(\theta) = \frac{C_0}{2} + \sum_{m=1} \left\{ C_m \cos(m\theta) + S_m \sin(m\theta) \right\}$$

Z-pinch



$$J_p = \begin{cases} \hat{z}\, J_p, & r<a \\ 0, & r>a \end{cases}$$

The Fourier components are given by:

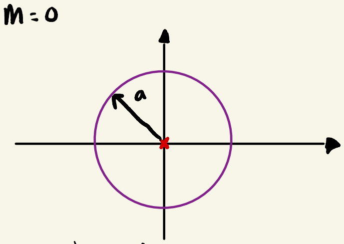
$$C_m = \frac{1}{\pi}\int_0^{2\pi} B(\theta)\cos(m\theta)\,d\theta \approx \frac{1}{\pi}\sum_j B(\theta_j)\cos(m\theta_j)\,\Delta\theta_j$$

$$S_m = \frac{1}{\pi}\int_0^{2\pi} B(\theta)\sin(m\theta)\,d\theta \approx \frac{1}{\pi}\sum_j B(\theta_j)\sin(m\theta_j)\,\Delta\theta_j$$

where the sums are approximations made by descrete b-dot probes at position θ_j seperated by an angular seperation of $\Delta\theta_j$.

The first few of these modes are sketched below:

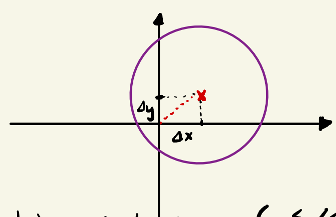
$m=0$



background circle.

$B(\theta) = \frac{C_0}{2}$

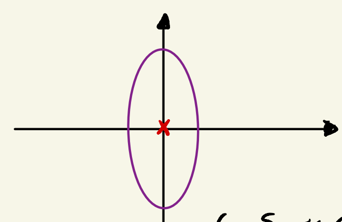
$m=1$



Shifted circle (as long as $C_1, S_1 \ll C_0$)

$B_1(\theta) \propto C_1\cos(\theta) + S_1\sin(\theta)$

$m=2$

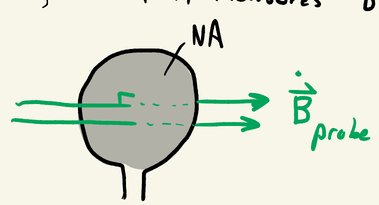


$C_2, S_2 \ll C_0$.

$B_2(\theta) \propto C_2\cos(2\theta) + S_2\sin(2\theta)$

The B-dot probe's working principle is to measure the change in flux through it. Assuming the probe's area remains constant, then it measures $\dot{B}$:

$$V = NA\dot{B}_{probe}$$



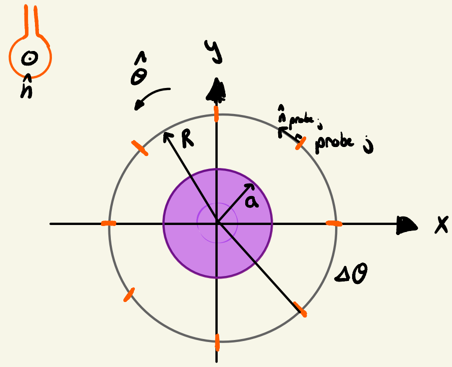
Assume the z pinch is intially stable: $B_0(\theta) = \frac{C_0}{2}$, $B_{m>0} = 0$. Then at $t=0$, the initial condition on all higher modes $B_{m>0} = 0$ at $t=0$. Then,

$$B_{probe} = \sum_{t_i} \frac{V_{probe}}{NA}\Delta t_i$$

where we assume we can digitally sample in time at $\Delta t \ll t_{istability}$, the charachteristic instability timescale. So, we can now measure B_{probe} through each b-dot probe by knowing each Votage V_s at time $s\Delta t$.

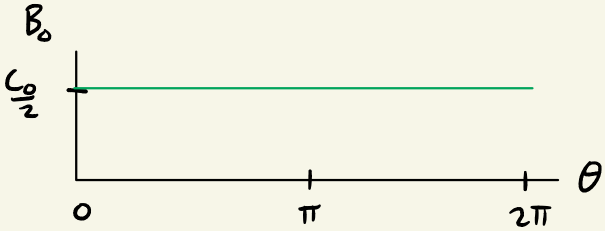
Now we are set up to measure the different modes by discretly placing B-dot's round the Z-pinch with azimuthal spacing $\Delta\theta$, the $\hat{n}$ probe normal vector alligned with $\hat{\theta}$.

General depiction: probes are alligned so $\hat{n} \parallel \hat{\theta}$.



Question 1

Only 1 probe is required because B_0 is independent of θ: so measuring at any θ will do.



This also jibes with the Nyquist frequency limit: our spatial sampling must be greater than twice the angular mode. So, $2^*0 = 0$, so we need ≥ 1 probe the satisfy the Nyquist inequality. $m=0$ is a Sausage instability: J is constricted or broadend to move through a changing area normal to its flow.

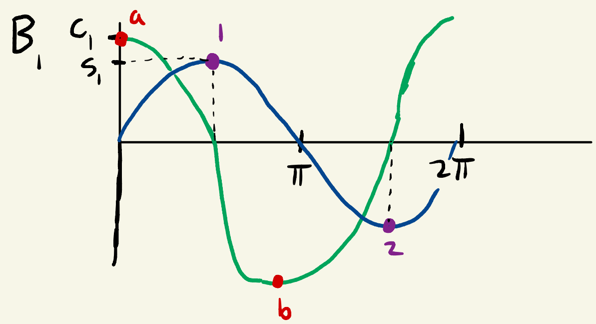
$$C_0 = \frac{1}{\pi}\int_0^{2\pi} B_0 \cos(0)\, d\theta = \frac{1}{\pi}(B_0 2\pi) = 2B_0$$

$$\frac{C_0}{2} = B_0.$$

$C_0 = 2B_0$ so the first fourier coefficient can be determind based on the b-dot measurment.

Question 2 . what is the minimal arrangment of b-dot probes to necessary to sense the $m=1$ displacement mode?

plot for $S_1=0$



Nyquist requirment is one samples at spatial frequency k greater than twice the spatial frequency of the largest spatial frequency we want to measure. Hen for $m=1$, we need to sample at minimum $1 \cdot 2+1=3$ to resolve a spatial frequency of 2π radians per cycle.

For example: The red probe locations a, $\theta=0$ and b, $\theta=\pi$ show that they can resolve the cos componant of B_1. But! at these locations sine is always zero, so probes a and b are insensitive to the sine componant of B_1

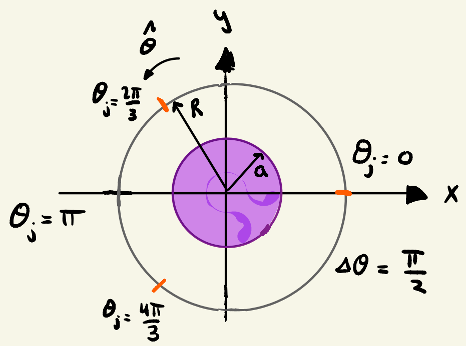
The purple probe locations 1, $\theta=\frac{\pi}{2}$ and 2, $\theta=\frac{3\pi}{2}$ are insensitive to the cosine componant of B_1 and can resolve the sin componants. So two probes are just not enough.

$$\boxed{\text{So we need 3 probes at } \theta=0, \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \text{ to resolve the } m=1 \text{ mode}}$$

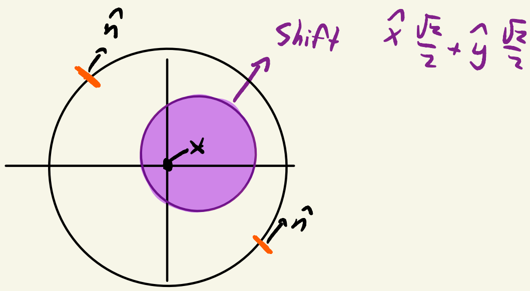
The Fourier componants are:

$$C_1 \approx \frac{1}{\pi}\sum_j B(\theta_j)\cos(1\theta_j)\Delta\theta_j$$

$$S_1 \approx \frac{1}{\pi}\sum_j B(\theta_j)\sin(1\theta_j)\Delta\theta_j$$



Needing 3 probes makes sense. $m=1$ is a shift, so if we only had two probes, π apart, then if the plasma shifts parallel to the normal vector of the two b-dot's, we would miss its motion:

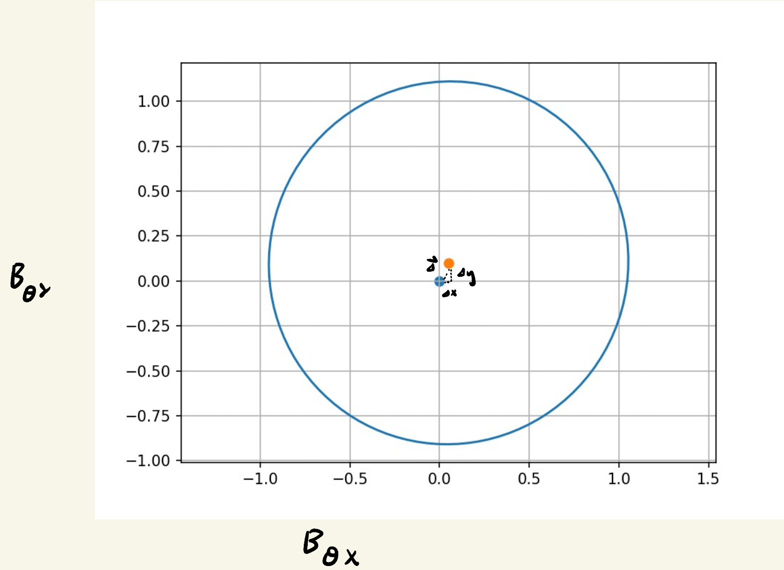


Two probes π appart miss motion parallel to thier normal vectors $\hat{n}$.

As an example, I will define $B_\theta(\theta) = \frac{\mu_0 I}{2\pi R} \frac{1}{\left[\left(\sin\theta - \frac{\Delta_y}{R}\right)^2 + \left(\cos\theta - \Delta x/R\right)^2\right]^{1/2}}$

which is an $m=1$ plasma purtibation where the plasma collumb has been shifted by $x = \Delta_x$ and $y = \Delta_y$. let $\frac{\mu_0 I}{2\pi} = 1$, $R = 1$, $\Delta_x = 0.05$, $\Delta y = 0.1$

This produces the following field:



Now imagine I have 3 probes at $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

The probes would measure (based on my above function)

$\theta_1 = 0$ $\quad B(0) = 1.0468$

$\theta_2 = 2\pi/3$ $\quad B(2\pi/3) = 1.0604$

$\theta_3 = 4\pi/3$ $\quad B(4\pi/3) = 0.8995$

Now I can numerically reconstruct C_1 and S_1 with $\Delta\theta_j = 2\pi/3$:

$$C_1 \approx \frac{1}{\pi} \sum_j B(\theta_j) \cos(1\theta_j)\, \Delta\theta_j = 0.0445$$

(very close to the actual $C_1 = 0.05$, $S_1 = 0.1$)

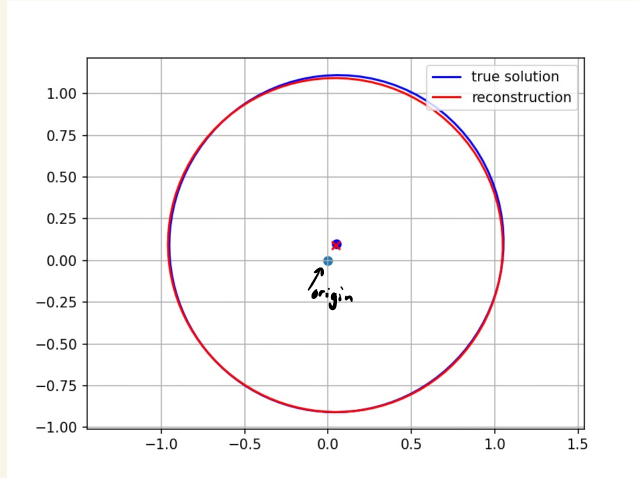
$$S_1 \approx \frac{1}{\pi} \sum_j B(\theta_j) \sin(1\theta_j)\, \Delta\theta_j = 0.0928$$

Using these, along with $C_0 = 2\frac{\mu_0 I}{2\pi R} = 2$ I can plug these into my formula:

$B(\theta) = \frac{C_0}{2} + C_1 \cos(1\theta) + S_1 \sin(1\theta)$ and plot this versus the true function above:

The predicted shift is $\vec{\Delta} = \frac{2C_1}{C_0}\hat{x} + \frac{2S_1}{C_0}\hat{y}$.

3 probes: closely predicts shift (dark blue dot, red x)

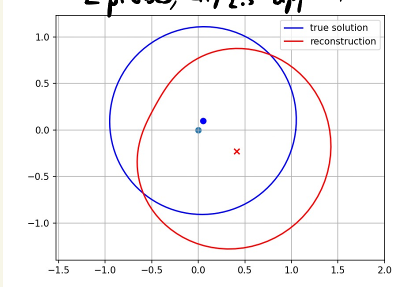


As you can see, the reconstruction nearly exactly predicts the center of the shift (red x overlaps dark blue circle). The light blue circle is 0,0 for reference.

If I only had 2 probes, say $\frac{2\pi}{2.5}$ appart instead of $\frac{2\pi}{3}$, I get

2 probes: not enough!

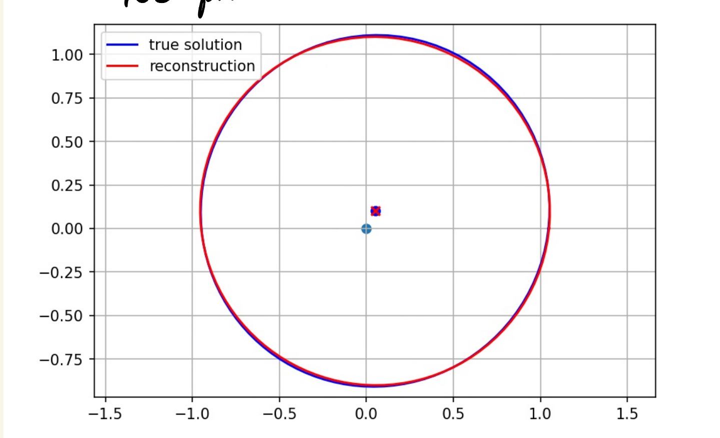
2 probes, $2\pi/2.5$ appart



Completly wrong displacment direction.

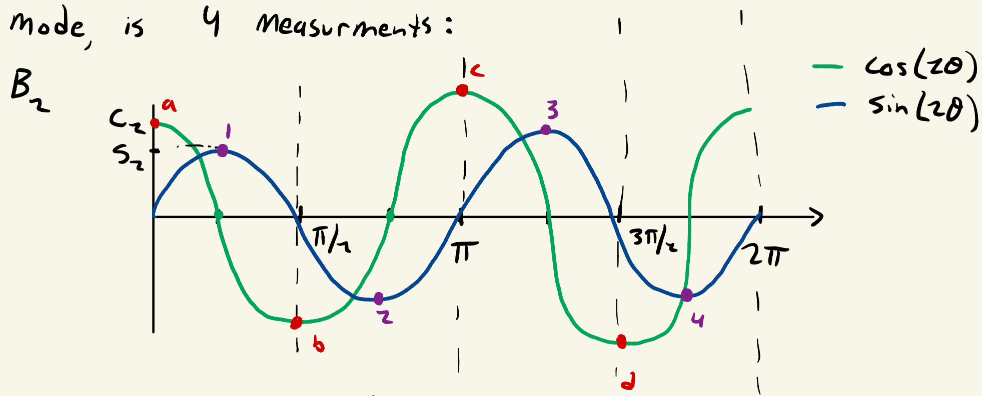
If I have 100 probes: I do slightly better than three. Three is enough for $m=1$.

100 probes



Question 3: What is the minimal number of b-dot probes to sense the $m=2$ (elongation) mode?

The same logic holds. We need, for both the sin and cos componants, twice as many measurments as the spatial frequency which we want to measure. Which, for the $m=2$ mode, is 4 measurments:



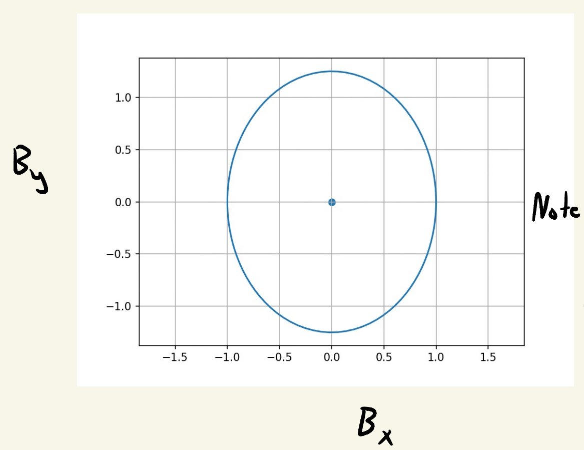
As can be seen, we need greater than 4 probe locations, a, b, c, d, 1, 2, 3, 4 to measure both the sin and cos components of B_2. $2*(m=2)+1 = 5$ probes

So: $\boxed{\text{Use 5 probes spaced } 2\pi/5 \text{ appart to measure } B_2(\theta).}$

As an example, I will create the elongated referense field:

$$B_\theta(\theta) = \frac{\mu_0 I}{2\pi R} \frac{1}{\left[(a_s \sin\theta)^2 + (\cos\theta)^2\right]^{1/2}}$$

which looks like:



Note $C_0 = 2\sqrt{B_\theta(0) B_\theta(\pi/2)}$

$C_0 = 2.236$, which could be measured as I described earlier.

Now imagine I have 5 probes at $\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$.

The probes would measure (based on my above function):

See next page

$\theta_1 = 0 \qquad B(\theta_1) = 1$

$\theta_2 = \frac{2\pi}{5} \qquad B(\theta_2) = 1.21$

$\theta_3 = \frac{4\pi}{5} \qquad B(\theta_3) = 1.068$

$\theta_4 = \frac{6\pi}{5} \qquad B(\theta_4) = 1.068$

$\theta_5 = \frac{8\pi}{5} \qquad B(\theta_5) = 1.21$

Now I can numerically reconstruct C_2 and S_2 with $\Delta\theta_j = \frac{2\pi}{5}$:

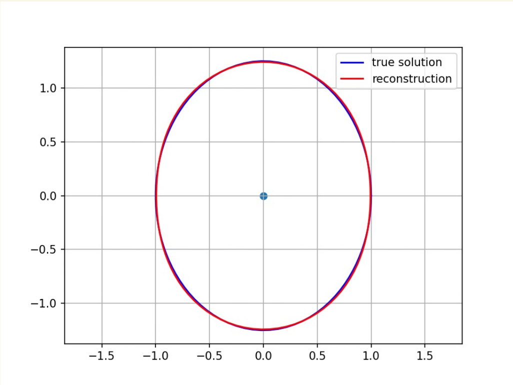
$$C_2 \approx \frac{1}{\pi} \sum_j B(\theta_j) \cos(2\theta_j) \Delta\theta_j = -0.1239$$

$$S_2 \approx \frac{1}{\pi} \sum_j B(\theta_j) \sin(2\theta_j) \Delta\theta_j = 1.413$$

Using these, along with $C_0 = 2.236$ I can plug these into my formula:

$B(\theta) = \frac{C_0}{2} + C_2 \cos(2\theta) + S_2 \sin(2\theta)$ and plot this versus the true function above:

These yeild the following:



which is a very good approximation.

So: $\boxed{\text{Use 5 probes spaced } 2\pi/5 \text{ appart to measure } B_2(\theta).}$

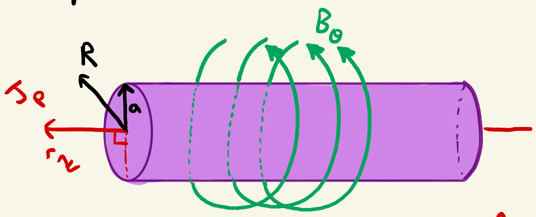
Question 4

For higher order modes, the Nyquist limit still holds. So, for Mode m, I need $2m+1$ evenly spaced b-dots to detect the mode with no alliesing, etc.

Question 5 If the assumption of uniform current density is removed, do my results still hold?

Now I have

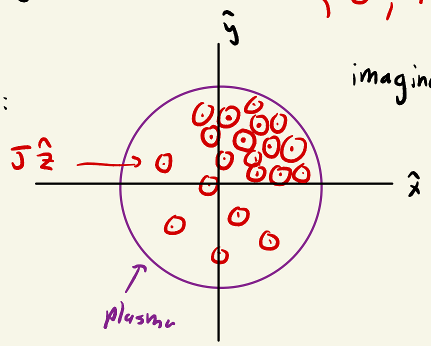
Z-pinch



Let $\vec{J}$ be in $\hat{z}$ and vary in r,θ as $f(r,\theta)$.

$$J_p = \begin{cases} \hat{z}\, f(r,\theta)\, J_p, & r \leq a \\ 0, & r > a \end{cases}$$

For example:



imagine if $J\hat{z}$ was localized in the upper right as shown. The cylidrical symetry of the problem is broken, and we can expect $\vec{B}$ outside the plasma to be $\vec{B} = B_r\hat{r} + B_\theta\hat{\theta}$ by symetry braking.

So, our b-dots, although still on a loop $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc}$ fully surrounding I_{enc}, we can no longer say $\vec{B}\cdot d\vec{\ell} = B_{tot}\, r\, d\hat{\theta}$, instead $\vec{B}\cdot d\vec{\ell} = B_\theta\, r\, d\hat{\theta}$.

So, we can only infer information about $B_\theta\hat{\theta}$ from our b-dots (assuming we don't allign them to the local $\vec{B}$, and leave their normal vectors pointing in $\hat{\theta}$), and not the total $\vec{B}$.