Nonlinear Programming

An optimization problem is a problem, where we maximize/minimize an objective function subject to some given conditions. If the objective function or the conditions are nonlinear, then we say the optimal problem is a nonlinear programming problem or nonlinear optimization problem. We know that for LPP, the optimal solution is achieved at some extremal points of the feasible region. It may not be true for nonlinear programming. For example, consider the problem

$$\begin{aligned} &\text{maximize } z = x_1x_2 \\ &\text{subject to } 4x_1 + x_2 \leq 8, \qquad x_1, x_2 \geq 0. \end{aligned}$$

The maximum value of k for which the parabola $x_1x_2 = k$ has a common point with the feasible region is 4. The parabola $x_1x_2 = 4$ touches the region at $(1, 4)$, which is not an extreme point.

Let $f : \mathbb{R}^n \to \mathbb{R}$. The function f has global minimum at $\mathbf{x}^*$ if $f(\mathbf{x}) \geq f(\mathbf{x}^*)$ for all $\mathbf{x} \in \mathbb{R}^n$, and has global maximum at $\mathbf{x}^*$ if $f(\mathbf{x}) \leq f(\mathbf{x}^*)$ for all $\mathbf{x} \in \mathbb{R}^n$.

We say $f(\mathbf{x})$ has a local minimum at $\mathbf{x} = \mathbf{x}^*$ if there exists $r > 0$ such that $f(\mathbf{x}) \geq f(\mathbf{x}^*)$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\| \mathbf{x} - \mathbf{x}^* \| < r$. We say $f(\mathbf{x})$ has a local maximum at $\mathbf{x} = \mathbf{x}^*$ if there exists $r > 0$ such that $f(X) \leq f(\mathbf{x}^*)$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\| \mathbf{x} - \mathbf{x}^* \| < r$.

$n = 1$. We say $f(x)$ has a local minimum at $x = x^*$ if there exists $r > 0$ such that $f(x) \geq f(x^*)$ for all $x^* - r < x < x^* + r$. We say $f(x)$ has a local maximum at $x = x^*$ if there exists $r > 0$ such that $f(x) \leq f(x^*)$ for all $x^* - r < x < x^* + r$.

$n = 2$. We say $f(x_1, x_2)$ has a local minimum at $(x_1, x_2) = (x_1^*, x_2^*)$ if there exists $r > 0$ such that $f(x_1, x_2) \geq f(x_1^*, x_2^*)$ for all $(x_1, x_2) \in \mathbb{R}^2$ with $(x_1 - x_1^*)^2 + (x_2 - x_2^*)^2 < r^2$. We say $f(\mathbf{x})$ has a local maximum $(x_1, x_2) = (x_1^*, x_2^*)$ if there exists $r > 0$ such that $f(x_1, x_2) \leq f(x_1^*, x_2^*)$ for all $(x_1, x_2) \in \mathbb{R}^2$ with $(x_1 - x_1^*)^2 + (x_2 - x_2^*)^2 < r^2$.

Unconstrained optimization problem. Let $f : \mathbb{R}^n \to \mathbb{R}$. An unconstrained optimization problem is to minimize/maximize $f(\mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Necessary condition: If $\mathbf{x}^* \in \mathbb{R}^n$ is a local minimum or local maximum point of f over $\mathbb{R}^n$, then

$$(\nabla f)(\mathbf{x}^*) = 0, \quad \text{that is,} \quad \frac{\partial f}{\partial x_1}(\mathbf{x}^*) = \frac{\partial f}{\partial x_2}(\mathbf{x}^*) = \cdots = \frac{\partial f}{\partial x_n}(\mathbf{x}^*) = 0.$$

Sufficient condition: If $\mathbf{x}^* \in \mathbb{R}^n$ satisfies $(\nabla f)(\mathbf{x}^*) = 0$ and f is strictly convex (strictly concave) function in a neighbourhood of $\mathbf{x}^*$, then $\mathbf{x}^*$ is a local minimum (local maximum) point of f over $\mathbb{R}^n$.

Consider the Hessian matrix

$$(\nabla^2 f)(\mathbf{x}^*) = \left[\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) (\mathbf{x}^*) \right]_{n \times n}.$$

If the Hessian matrix is positive definite (negative definite), then f is strictly convex (strictly concave).

Note: The points $\mathbf{x}^*$ satisfying $(\nabla f)(\mathbf{x}^*) = 0$ are called critical points.

$n = 1$. Let $f : \mathbb{R} \to \mathbb{R}$. An unconstrained optimization problem is to minimize/maximize $f(x)$, where $x \in \mathbb{R}$.

Necessary condition: If $x^* \in \mathbb{R}$ is a local minimum or local maximum point of f over $\mathbb{R}$, then

$$f'(x^*) = 0.$$

Sufficient condition: Let $f'(x^*) = f''(x^*) = \cdots = f^{(k-1)}(x^*) = 0$, but $f^{(k)}(x^*) \neq 0$. The point x^* is a

(a) local minimum point if $f^{(k)}(x^*) > 0$ and k is even,

(b) local maximum point if $f^{(k)}(x^*) < 0$ and k is even,

(c) neither a local maximum or local minimum if k is odd.

$n = 2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$. An unconstrained optimization problem is to minimize/maximize $f(x, y)$, where $(x, y) \in \mathbb{R}^2$.

Necessary condition: If $(x^*, y^*) \in \mathbb{R}^2$ is a local minimum or local maximum point of f over $\mathbb{R}^2$, then

$$\left(\frac{\partial f}{\partial x}\right)(x^*, y^*) = \left(\frac{\partial f}{\partial y}\right)(x^*, y^*) = 0.$$

Sufficient condition: The point $(x^*, y^*) \in \mathbb{R}^2$ satisfying the necessary condition is a local minimum (local maximum) point if

$$\begin{bmatrix} \left(\frac{\partial^2 f}{\partial x^2}\right)(x^*, y^*) & \left(\frac{\partial^2 f}{\partial x \partial y}\right)(x^*, y^*) \\ \left(\frac{\partial^2 f}{\partial y \partial x}\right)(x^*, y^*) & \left(\frac{\partial^2 f}{\partial y^2}\right)(x^*, y^*) \end{bmatrix} \qquad \text{(Hessian matrix)}$$

is positive definite (negative definite).

If (x^*, y^*) is a critical point, and if for every $r > 0$, there exist $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{R}^2$ satisfying $(\alpha_1 - x_1^*)^2 + (\alpha_2 - x_2^*)^2 < r^2$ and $(\beta_1 - x_1^*)^2 + (\beta_2 - x_2^*)^2 < r^2$ such that $f(\alpha_1, \alpha_2) > f(x_1^*, x_2^*)$ and $f(\beta_1, \beta_2) < f(x_1^*, x_2^*)$, then (x_1^*, x_2^*) is called a saddle point. If the determinant of Hessian matrix is negative, then the point (x^*, y^*) is a saddle point.

Note: A real symmetric matrix A of order n is said to be positive definite (P.D.) if $\mathbf{x}^T A \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq 0$. A real symmetric matrix A of order n is said to be negative definite (N.D.) if $\mathbf{x}^T A \mathbf{x} < 0$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq 0$.

A real symmetric matrix A is positive definite if and only if all eigenvalues of A are positive. A real symmetric matrix A is negative definite if and only if all eigenvalues of A are negative.

A sufficient condition for a real symmetric matrix A to be positive definite is that all leading principal minors are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \quad \cdots, \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} > 0.$$

A sufficient condition for a real symmetric matrix A to be negative definite is that all leading principal minors alternate in sign starting from negative, that is,

$$a_{11} < 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} > 0, \quad \cdots.$$

Example. Find the natures of the extreme points of the function

$$f(x) = 12x^5 - 45x^4 + 40x^3 + 5.$$

Solution. The necessary condition for extreme points is

$$f'(x) = 60x^4 - 180x^3 + 120x^2 = 60x^2(x-1)(x-2) = 0.$$

The extreme points are $x = 0, 1, 2$.

$$f''(x) = 240x^3 - 540x^2 + 240x.$$

Now $f''(1) = -60 < 0$. So $x = 1$ is a local maximum point. Again $f''(2) = 240 > 0$. So $x = 2$ is a local minimum point. Since $f''(0) = 0$, we require

$$f'''(x) = 720x^2 - 1080x + 240.$$

Since $f'''(0) = 240 \neq 0$ and 3 is odd, $x = 0$ is neither a local minimum point nor a local maximum point. $\square$

Example. Find the natures of extreme points of the function

$$f(x, y) = x^3 + y^3 + 2x^2 + 4y^2 + 6.$$

Solution. The necessary conditions for extreme points are

$$\frac{\partial f}{\partial x} = 3x^2 + 4x = 0, \quad \text{and} \quad \frac{\partial f}{\partial y} = 3y^2 + 8y = 0.$$

The extreme points are $(0, 0), (0, -8/3), (-4/3, 0), (-4/3, -8/3)$. The Hessian matrix is

$$H = \begin{bmatrix} 6x + 4 & 0 \\ 0 & 6y + 8 \end{bmatrix}.$$

The leading principal minors of H are $H_1 = 6x + 4$ and $H_2 = (6x + 4)(6y + 8)$.

1. For (0,0), $H_1 = 4, H_2 = 32$. So H is positive definite and hence (0,0) is a local minimum point.

2. For $(0, -8/3)$, $\det H = -32 < 0$. So $(0, -8/3)$ is a saddle point.

3. For $(-4/3, 0)$, $\det H = -32 < 0$. So $(-4/3, 0)$ is a saddle point.

4. For $(-4/3, 8/3)$, $H_1 = -4, H_2 = 32$. So H is negative definite and hence $(-4/3, -8/3)$ is a local maximum point.

$\square$

Note: We have $f\left(-\frac{4}{3},r\right)-f\left(-\frac{4}{3},0\right)=r^3+4r^2>0$ for all $r>0$. Again for every $0<r<2$,

$$\begin{aligned}
f\left(-\frac{4}{3}+r,0\right)-f\left(-\frac{4}{3},0\right) &= \left(-\frac{4}{3}+r\right)^3+2\left(-\frac{4}{3}+r\right)^2-\left(-\frac{4}{3}\right)^3-2\left(-\frac{4}{3}\right)^2 \\
&= r\left[\left(-\frac{4}{3}+r\right)^2-\frac{4}{3}\left(-\frac{4}{3}+r\right)+\left(-\frac{4}{3}\right)^2\right]+2r\left(-\frac{8}{3}+r\right) \\
&= r\left(\frac{16}{9}-\frac{8}{3}r+r^2+\frac{16}{9}-\frac{4}{3}r+\frac{16}{9}-\frac{16}{3}+2r\right) \\
&= r(r^2-2r)<0.
\end{aligned}$$

Therefore every neighbourhood of $\left(-\frac{4}{3},0\right)$ contain two points such that the value of f at one point is larger than $f\left(-\frac{4}{3},0\right)$, and the value of f at other point is smaller than $f\left(-\frac{4}{3},0\right)$. Therefore $\left(-\frac{4}{3},0\right)$ is a saddle point. Similar calculations can be done to show that $\left(0,-\frac{8}{3}\right)$ is a saddle point.

Exercises.

1. Find the critical points and their natures for the following functions.
   (a) $f(x)=x^2-6x^2+9x+5$.
   (b) $f(x)=2+(x-1)^4$.
2. Find two numbers whose difference is 100 and whose product is a minimum.
3. Find two positive numbers whose product is 100 and whose sum is a minimum.
4. Find the point on the parabola $y^2=2x$ that is closest to the point (1,4).
5. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 5. (The word inscribed means that the rectangle has two vertices on the semicircle and two vertices on the diameter.)
6. Find the critical points and their natures for the following functions.
   (a) $f(x,y)=x^3-y^3-2xy+6$.
   (b) $f(x,y)=x^4-2x^2+y^3-3y$.
   (c) $f(x,y)=x^2+y^4+2xy$.
   (d) $f(x,y)=y\cos x$.

Search methods for optimization problems. Consider the following unconstrained minimization problem:

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}).$$

If the optimization problem involves the objective function that are not stated as explicit functions of the design variables or which are too complicated to manipulate, we cannot solve it by using the classical analytical methods.

Basic scheme: A common basic scheme is of the form

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \, \mathbf{d}_k,$$

where $\mathbf{x}_k$ is the current solution, $\mathbf{d}_k$ is the direction of movement from $\mathbf{x}_k$ and α_k is the appropriate step length for movement along the direction $\mathbf{d}_k$.

Now our objective is to find α_k. If $f(\mathbf{x})$ is the objective function to be minimized, the problem of determining α_k reduces to finding the value $\alpha = \alpha_k$ that minimizes $f(\mathbf{x}_{k+1}) = f(\mathbf{x}_k + \alpha \, \mathbf{d}_k) = F(\alpha)$ for fixed values of $\mathbf{x}_k$ and $\mathbf{d}_k$. Now the problem reduces to one-dimensional minimization problem.

Example. Derive the one-dimensional minimization problem for the following case:

$$\text{Minimize } f(x_1, x_2) = (x_1^2 - x_2)^2 + (1 - x_1)^2$$

from the starting point $(-2, -2)$ along the search direction $(1, 0.25)$.

Solution. The new approximation can be expressed as

$$(x_1, x_2) = (-2, -2) + \lambda(1, 0.25) = (-2 + \lambda, -2 + 0.25\lambda).$$

Substituting $x_1 = -2 + \lambda$ and $x_2 = -2 + 0.25\lambda$ in the given problem, we can write the objective function as

$$\begin{aligned} F(\lambda) = f(-2 + \lambda, -2 + 0.25\lambda) &= [(-2 + \lambda)^2 - (-2 + 0.25\lambda)]^2 + [1 - (-2 + \lambda)]^2 \\ &= (\lambda^2 - 4.25\lambda + 6)^2 + (3 - \lambda)^2 \\ &= \lambda^4 - 8.5\lambda^3 + 31.0625\lambda^2 - 57\lambda + 45. \end{aligned}$$

$\square$

Exercises.

1. Derive the one-dimensional minimization problem for the following case:

   $$\text{Minimize } f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$

   from the starting point $(0, 0)$ along the search direction $(-1, 0)$.

2. Derive the one-dimensional minimization problem for the following case:

   $$\text{Minimize } f(x_1, x_2, x_3) = \frac{2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3}{x_1^2 + x_2^2 + 2x_3^2}$$

   from the starting point $(1, 1, 1)$ along the search direction $(-1, 0, 1)$.

Unimodal function: A function $f : [a, b] \to \mathbb{R}$ is unimodal if it has only one peak (maximum) or valley (minimum) in the interval $[a, b]$.

(a) A function $f : [a, b] \to \mathbb{R}$ is unimodal min if there exists $a \le x^* \le b$ such that $f(x_1) > f(x_2)$ for all $x_1 < x_2 \le x^*$ and $f(x_1) < f(x_2)$ for all $x^* \le x_1 < x_2$.

(b) A function $f : [a, b] \to \mathbb{R}$ is unimodal max if there exists $a \le x^* \le b$ such that $f(x_1) < f(x_2)$ for all $x_1 < x_2 \le x^*$ and $f(x_1) > f(x_2)$ for all $x^* \le x_1 < x_2$.

We shall discuss Fibonacci search technique and golden section search technique that are actually based on the assumption that function is unimodal, at least in the given range. This initial range is called the interval of uncertainty and need to be made finer and finer with the iterations. These search techniques involves the functional value at different points in the interval of uncertainty to obtain a finer interval of uncertainty in which minimum/maximum lies.

Fibonacci search method. This is an iterative method that uses the sequence of Fibonacci numbers, (F_n), for placing the experiments. Fibonacci sequence is given by

$$F_0 = 1, F_1 = 1, \text{ and } F_n = F_{n-2} + F_{n-1} \text{ for all } n \ge 2.$$

Let $f(x)$ be our objective function, and let it be unimodal min. Let $[a, b]$ be the initial interval of uncertainty, and let n be the number of experiments to be conducted.

Let $L_0 = b - a$ is the length of the initial interval of uncertainty, and let

$$L_2^* = \frac{F_{n-2}}{F_n} L_0.$$

We place two experiments x_1 and x_2 which are located at a distance of L_2^* from each end of $[a, b]$. Therefore

$$\begin{aligned} x_1 &= a + L_2^* = a + \frac{F_{n-2}}{F_n} L_0, \\ x_2 &= b - L_2^* = b - \frac{F_{n-2}}{F_n} L_0 = a + L_0 - \frac{F_{n-2}}{F_n} L_0 = a + \frac{F_{n-1}}{F_n} L_0. \end{aligned}$$

If $f(x_1) > f(x_2)$, then the next interval will be $[x_1, b]$. If $f(x_1) < f(x_2)$, then the next interval will be $[a, x_2]$. So there will be a smaller interval of uncertainty of length

$$L_2 = L_0 - L_2^* = L_0 - \frac{F_{n-2}}{F_n} L_0 = \frac{F_{n-1}}{F_n} L_0$$

with one experiment left inside it.

Suppose that $[a, x_2]$ is the next interval of uncertainty, which contains the experiment x_1 inside. This experiment will be at a distance of

$$L_2^* = \frac{F_{n-2}}{F_n} L_0$$

from one end a, and at a distance of

$$L_2 - L_2^* = \frac{F_{n-1}}{F_n} L_0 - \frac{F_{n-2}}{F_n} L_0 = \frac{F_{n-3}}{F_n} L_0.$$

from the other end x_2. Now we place the third experiment x_3 in the interval $[a, x_2]$ so that the current two experiments are located at a distance of

$$L_3^* = \frac{F_{n-3}}{F_n} L_0$$

from each end of the interval $[a, x_2]$. Therefore

$$x_3 = x_2 - L_3^* = a + \frac{F_{n-1}}{F_n} L_0 - \frac{F_{n-3}}{F_n} L_0 = a + \frac{F_{n-2}}{F_n} L_0.$$

If $f(x_1) > f(x_3)$, then the next interval will be $[x_1, x_2]$. If $f(x_1) < f(x_3)$, then the next interval will be $[a, x_3]$. So the length of the interval of uncertainty will be reduced to

$$L_3 = L_2 - L_3^* = \frac{F_{n-1}}{F_n} L_0 - \frac{F_{n-3}}{F_n} L_0 = \frac{F_{n-2}}{F_n} L_0$$

with one experiment left inside it.

This process of reducing the size of the interval of uncertainty and placing a new experiment in it can be continued, so that the location of the k-th experiment and the interval of uncertainty at the end of k experiments are, respectively, given by

$$L_k^* = \frac{F_{n-k}}{F_n} L_0,$$
$$L_k = \frac{F_{n-k+1}}{F_n} L_0.$$

The ratio of the interval of uncertainty remaining after conducting k of the n predetermined experiments to the initial interval of uncertainty becomes

$$\frac{L_k}{L_0} = \frac{F_{n-k+1}}{F_n}.$$

For $k = n$, we have

$$\frac{L_n}{L_0} = \frac{1}{F_n},$$

and this reduction ratio will permit us to determine n, the required number of experiments, to achieve any desired accuracy in locating the optimum point.

Note: We observe that $x_3 - a = L_3^* = x_2 - x_1$ which implies $x_3 = a + (x_2 - x_1)$. Similarly, we can calculate the later experiments from earlier experiments. So we can place all the experiments by calculating L_2^* only.

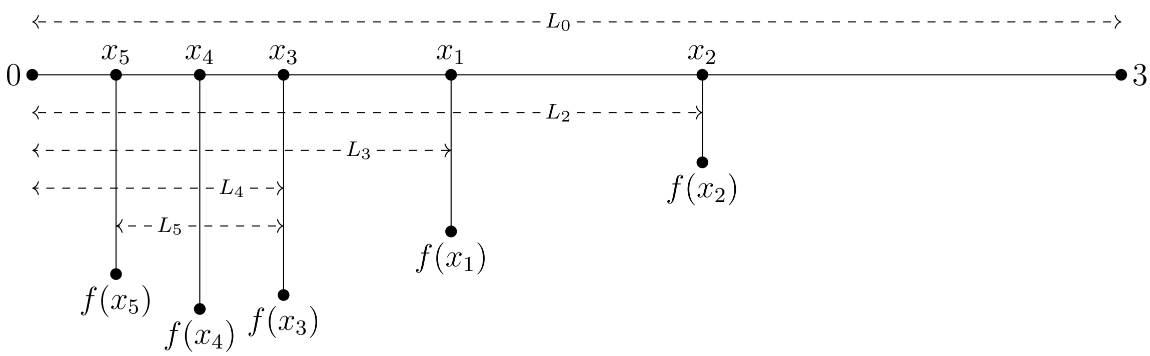
Example. Find the optimal value, interval of uncertainty and reduction ratio for the following optimization problem

$$\text{minimize } f(x) = 0.65 - \frac{0.75}{1+x^2} - 0.65\, x \tan^{-1}\left(\frac{1}{x}\right)$$

using the Fibonacci search method after placing six experiments with initial interval of uncertainty being $[0, 3]$.

Solution. Here $L_0 = 3$ and $n = 6$, which gives

$$L_2^* = \frac{F_4}{F_6} L_0 = \frac{5}{13} 3 = 1.153846.$$



We place two experiments $x_1 = 1.153846$ and $x_2 = 3 - 1.153846 = 1.846154$. Since $f(x_1) = -0.207270 < -0.115843 = f(x_2)$, we have $[0, x_2]$ as the interval of uncertainty.

The third experiment is $x_3 = 0 + (x_2 - x_1) = 0.692308$. Since $f(x_3) = -0.291364 < -0.207270 = f(x_1)$, we have $[0, x_1]$ as the interval of uncertainty.

The fourth experiment is $x_4 = 0 + (x_1 - x_3) = 0.461538$. Since $f(x_3) = -0.291364 > -0.309811 = f(x_4)$, we have $[0, x_3]$ as the interval of uncertainty.

The fifth experiment is $x_5 = 0 + (x_3 - x_4) = 0.230770$. Since $f(x_5) = -0.263678 > -0.309811 = f(x_4)$, we have $[x_5, x_3]$ as the interval of uncertainty.

The final experiment is $x_6 = x_5 + (x_3 - x_4) = 0.461540$. Since $f(x_6) = -0.309810 > -0.309811 = f(x_4)$, we have $[x_5, x_6] = [0.230770, 0.461540]$ as the final interval of uncertainty.

The optimal value is -0.309811 and the reduction ration is $\frac{L_6}{L_0} = \frac{x_6 - x_5}{3} = 0.076923$. $\square$

Note: For every n, we have

$$L_n^* = \frac{1}{2} L_{n-1}.$$

So theoretically, the n-th experiment will be same as the $(n-1)$-th experiment. But in practice, due to round of error, those experiments might be different.

Golden section search method. This method is same as the Fibonacci search method except that in the Fibonacci search method the total number of experiments to be conducted has to be specified before beginning the calculation, whereas this is not required in the golden section search method. In the Fibonacci search method, the location of the first two experiments is determined by the total number of experiments. In the golden section search method, we start with the assumption that we are going to conduct a large number of experiments. If L_0 is the length of the initial interval of uncertainty, then the length of intervals of uncertainty remaining at the end of different number of experiments can be computed as follows:

$$L_2 = \lim_{N\to\infty} \frac{F_{N-1}}{F_N} L_0,$$

$$L_3 = \lim_{N\to\infty} \frac{F_{N-2}}{F_N} L_0 = \lim_{N\to\infty} \frac{F_{N-2}}{F_{N-1}} \frac{F_{N-1}}{F_N} L_0 = \lim_{N\to\infty} \left(\frac{F_{N-1}}{F_N}\right)^2 L_0,$$

and so on. This result can be generalized as

$$L_k = \lim_{N\to\infty} \left(\frac{F_{N-1}}{F_N}\right)^{k-1} L_0.$$

By defining a ratio γ as

$$\gamma = \lim_{N\to\infty} \frac{F_N}{F_{N-1}},$$

from the relation $F_n = F_{n-1} + F_{n-2}$, that is, $\frac{F_N}{F_{N-1}} = 1 + \frac{F_{N-2}}{F_{N-1}}$ we have

$$\gamma = 1 + \frac{1}{\gamma}, \qquad \text{that is, } \gamma = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

Therefore

$$L_k = \left(\frac{1}{\gamma}\right)^{k-1} L_0 \approx (0.618)^{k-1} L_0.$$

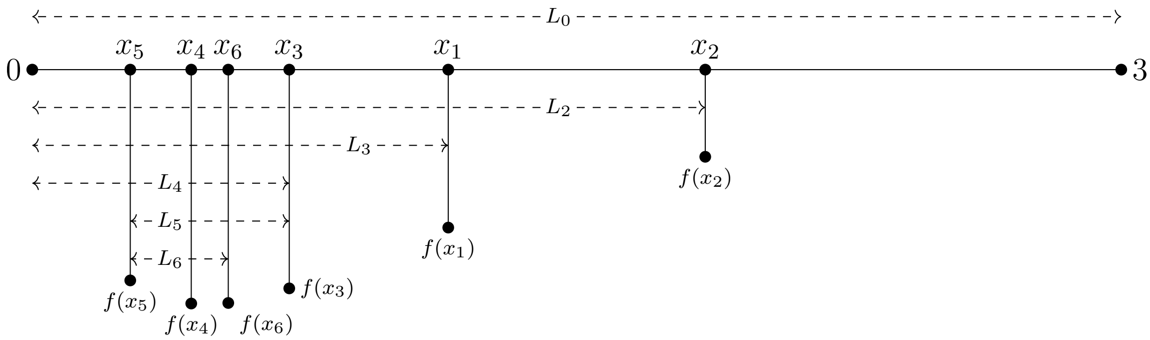
Example. Find the optimal value, interval of uncertainty and reduction ratio for the following optimization problem

$$\text{minimize } f(x) = 0.65 - \frac{0.75}{1+x^2} - 0.65\, x \tan^{-1}\left(\frac{1}{x}\right)$$

using the golden section search method after placing six experiments with initial interval of uncertainty being $[0, 3]$.

Solution. Here $L_0 = 3$, which gives

$$L_2^* = L_0 - L_2 = (1 - 0.618) L_0 = 1.146.$$



We place two experiments $x_1 = 1.146$ and $x_2 = 3 - 1.146 = 1.854$. Since $f(x_1) = -0.208654 < -0.115124 = f(x_2)$, we have $[0, x_2]$ as the interval of uncertainty.

The third experiment is $x_3 = 0 + (x_2 - x_1) = 0.708$. Since $f(x_3) = -0.288943 < -0.208654 = f(x_1)$, we have $[0, x_1]$ as the interval of uncertainty.

The fourth experiment is $x_4 = 0 + (x_1 - x_3) = 0.438$. Since $f(x_3) = -0.288943 > -0.308951 = f(x_4)$, we have $[0, x_3]$ as the interval of uncertainty.

The fifth experiment is $x_5 = 0+(x_3-x_4) = 0.27$. Since $f(x_5) = -0.278434 > -0.308951 = f(x_4)$, we have $[x_5, x_3]$ as the interval of uncertainty.

The sixth experiment is $x_6 = x_5 + (x_3 - x_4) = 0.54$. Since $f(x_6) = -0.308234 > -0.308951 = f(x_4)$, we have $[x_5, x_6] = [0.27, 0.54]$ as the final interval of uncertainty.

The optimal value is -0.308951 and the reduction ration is $\frac{L_6}{L_0} = \frac{x_6 - x_5}{3} = 0.09$. $\square$

Constrained optimization problem with equality constraints (The method of Lagrange's multipliers). Let f and $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, \ldots, m$ be continuously differentiable functions, where $m < n$. Consider the problem

$$\begin{aligned} &\text{minimize/maximize } f(\mathbf{x}) \\ &\qquad\qquad \text{subject to } g_i(\mathbf{x}) = 0 \text{ for } i = 1, 2, \ldots, m. \end{aligned} \tag{CE}$$

We consider the Lagrange's function as

$$L(\mathbf{x}, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}).$$

Necessary condition: Let $\mathbf{x}^* \in \mathbb{R}^n$ be a point of local minimum or local maximum of the problem (CE). Then there is a set of Lagrange's multipliers $\lambda_1^*, \ldots, \lambda_m^*$ such that

$$\left.\begin{array}{c} (\nabla f)(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* (\nabla g_i)(\mathbf{x}^*) = 0, \\ g_i(\mathbf{x}^*) = 0 \text{ for } i = 1, 2, \ldots, m, \end{array}\right\} \tag{CE-N}$$

that is, $\frac{\partial L}{\partial x_i}(\mathbf{x}^*, \lambda^*) = 0$ for $i = 1, 2, \ldots, n$ and $\frac{\partial L}{\partial \lambda_i}(\mathbf{x}^*, \lambda^*) = 0$ for $i = 1, 2, \ldots, m$.

Sufficient condition: Let $(\mathbf{x}^*, \lambda^*) \in \mathbb{R}^n \times \mathbb{R}^m$ exists such that (CE-N) holds, and let $Z(\mathbf{x}^*) = \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{z}^T (\nabla g_i)(\mathbf{x}^*) = 0 \text{ for } i = 1, 2, \ldots, n\}$. The point $\mathbf{x}^*$ is a local minimum point if

$\mathbf{z}^T(\nabla^2_{\mathbf{x}}L)(\mathbf{x}^*,\lambda^*)\,\mathbf{z} > 0$ for all $\mathbf{z} \in Z(\mathbf{x}^*)$ with $\mathbf{z} \neq 0$, and $\mathbf{x}^*$ is a local maximum point if $\mathbf{z}^T(\nabla^2_{\mathbf{x}}L)(\mathbf{x}^*,\lambda^*)\,\mathbf{z} < 0$ for all $\mathbf{z} \in Z(\mathbf{x}^*)$ with $\mathbf{z} \neq 0$, where

$$(\nabla^2_{\mathbf{x}}L)(\mathbf{x}^*,\lambda^*) = \left[\left(\frac{\partial^2 L}{\partial x_i \partial x_j}\right)(\mathbf{x}^*,\lambda^*)\right]_{n\times n}.$$

Note: The sufficient conditions for a maximum or minimum can be stated in terms of the bordered Hessian matrix

$$H_B = \left[\begin{array}{ccc|ccc}
0 & \cdots & 0 & \left(\frac{\partial g_1}{\partial x_1}\right)(\mathbf{x}^*) & \cdots & \left(\frac{\partial g_n}{\partial x_1}\right)(\mathbf{x}^*) \\
\vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & \left(\frac{\partial g_m}{\partial x_1}\right)(\mathbf{x}^*) & \cdots & \left(\frac{\partial g_m}{\partial x_1}\right)(\mathbf{x}^*) \\
\hline
\left(\frac{\partial g_1}{\partial x_1}\right)(\mathbf{x}^*) & \cdots & \left(\frac{\partial g_m}{\partial x_1}\right)(\mathbf{x}^*) & \left(\frac{\partial^2 L}{\partial x_1^2}\right)(\mathbf{x}^*,\lambda^*) & \cdots & \left(\frac{\partial^2 L}{\partial x_1 \partial x_n}\right)(\mathbf{x}^*,\lambda^*) \\
\vdots & & \vdots & \vdots & & \vdots \\
\left(\frac{\partial g_1}{\partial x_n}\right)(\mathbf{x}^*) & \cdots & \left(\frac{\partial g_m}{\partial x_n}\right)(\mathbf{x}^*) & \left(\frac{\partial^2 L}{\partial x_n \partial x_1}\right)(\mathbf{x}^*,\lambda^*) & \cdots & \left(\frac{\partial^2 L}{\partial x_n^2}\right)(\mathbf{x}^*,\lambda^*)
\end{array}\right].$$

Let $H_1, H_2, \ldots, H_{m+n}$ be the leading principal submatrices of H_B. Observe that:

1. The matrices $H_1, \ldots, H_m$ are zero matrices.
2. The matrices $H_{m+1}, \ldots, H_{2m-1}$ have zero determinants.
3. $\det H_{2m} = \pm(\det \widehat{H})^2$, where $\widehat{H}$ is the $k \times k$ submatrix with first k rows and k columns after the zero block.
4. Signs of first $2m$ leading principal minors do not give any information about f.

The sufficient conditions for the maximum and minimum is determined by the signs of the last $(n-m)$ principal minors of matrix H_B.

1. If $\det H_{m+n}$ has sign $(-1)^n$, and $\det H_k$ alternate in sign for $k = 2m+1, 2m+2, \ldots, m+n$, then the critical point is a local maximum point.
2. If $\det H_k$ has sign $(-1)^m$ for $k = 2m+1, 2m+2, \ldots, m+n$, then the critical point is a local minimum point.

Example. Find the extremum values of $f(x,y,z) = x^2+y^2+z^2$ subject to the conditions $g_1(x,y,z) = 3x+y+z-5 = 0$ and $g_2(x,y,z) = x+y+z-1 = 0$.

Solution. The Lagrangian function is given by

$$L(x,y,z,\lambda_1,\lambda_2) = x^2+y^2+z^2+\lambda_1(3x+y+z-5)+\lambda_2(x+y+z-1).$$

Now

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 2x + 3\lambda_1 + \lambda_2 = 0, \tag{1}$$
$$\frac{\partial L}{\partial y} = 0 \Rightarrow 2y + \lambda_1 + \lambda_2 = 0, \tag{2}$$
$$\frac{\partial L}{\partial z} = 0 \Rightarrow 2z + \lambda_1 + \lambda_2 = 0, \tag{3}$$
$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow 3x + y + z - 5 = 0, \tag{4}$$
$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow x + y + z - 1 = 0. \tag{5}$$

From (1), (2) and (3), we have

$$x = -\frac{3\lambda_1 + \lambda_2}{2}, \quad y = z = -\frac{\lambda_1 + \lambda_2}{2}.$$

Then from (4) and (5), we have

$$11\lambda_1 + 5\lambda_2 + 10 = 0, \qquad 5\lambda_1 + 3\lambda_2 + 2 = 0.$$

Solving these we have

$$\lambda_1 = -\frac{5}{2}, \quad \lambda_2 = \frac{7}{2}, \quad x = 2, \quad y = z = -\frac{1}{2}.$$

So the critical point is $\left(2, -\frac{1}{2}, -\frac{1}{2}\right)$.

$$\begin{aligned} L_{xx} = L_{yy} = L_{zz} = 2,\ L_{xy} = L_{yx} = L_{xz} = L_{zx} = L_{yz} = L_{zy} = 0, \\ \frac{\partial g_1}{\partial x} = 3,\ \frac{\partial g_1}{\partial y} = 1,\ \frac{\partial g_1}{\partial z} = 1, \\ \frac{\partial g_2}{\partial x} = 1,\ \frac{\partial g_2}{\partial y} = 1,\ \frac{\partial g_2}{\partial z} = 1. \end{aligned}$$

Therefore the bordered Hessian matrix at $\left(2, -\frac{1}{2}, -\frac{1}{2}\right)$ is

$$H_B = \left[\begin{array}{cc|ccc} 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ \hline 3 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{array}\right].$$

Here $n = 3,\ m = 2$. So $2m + 1 = 5$ and there is only one leading principal submatrix of H_B of order 5 which we need to consider and that leading principal submatrix is H_B itself. Now $\det H_B = 16$ has the sign $(-1)^2 = (-1)^m$. So $\left(2, -\frac{1}{2}, -\frac{1}{2}\right)$ is a point of local minima. The minimum value is $\frac{9}{2}$. $\square$

Example. Find the dimensions of a cylindrical tin (with top and bottom) made up of sheet metal to maximize its volume such that the total surface area is equal to 24π.

Solution. If x_1 and x_2 denote the radius of the base and length of the tin, respectively, the problem can be stated as

$$\begin{aligned} &\text{maximize } f(x_1, x_2) = \pi x_1^2 x_2 \\ &\text{subjet to } 2\pi x_1^2 + 2\pi x_1 x_2 = 24\pi. \end{aligned}$$

Let $g(x_1, x_2) = 2\pi x_1^2 + 2\pi x_1 x_2 - 24\pi$. The Lagrangian function is given by

$$L(x_1, x_2, \lambda) = \pi x_1^2 x_2 + \lambda(2\pi x_1^2 + 2\pi x_1 x_2 - 24\pi).$$

Now

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2\pi x_1 x_2 + 4\pi\lambda x_1 + 2\pi\lambda x_2 = 0 \Rightarrow x_1 x_2 + 2\lambda x_1 + \lambda x_2 = 0, \tag{6}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow \pi x_1^2 + 2\pi\lambda x_1 = 0 \Rightarrow x_1(x_1 + 2\lambda) = 0, \tag{7}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 2\pi x_1^2 + 2\pi x_1 x_2 - 24\pi = 0 \Rightarrow x_1^2 + x_1 x_2 = 12. \tag{8}$$

From (7), we have either $x_1 = 0$ or $x_1 = -2\lambda$. But $x_1 = 0$ does not satisfy (8). So we have $x_1 = -2\lambda$ and $\lambda \neq 0$. Therefore (6) implies

$$-2\lambda x_2 - 4\lambda^2 + \lambda x_2 = 0 \Rightarrow -x_2 - 4\lambda = 0 \Rightarrow x_2 = -4\lambda.$$

Now from (8), we have

$$4\lambda^2 + 8\lambda^2 = 12 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1.$$

Since x_1 and x_2 are lengths, $x_1, x_2 > 0$ and thus

$$\lambda = -1, \quad x_1 = 2, \quad x_2 = 4.$$

So the critical point is $(2, 4)$.

$L_{x_1x_1} = 2\pi x_2 + 4\pi\lambda,\ L_{x_2x_2} = 0,\ L_{x_1x_2} = L_{x_2x_1} = 2\pi x_1 + 2\pi\lambda,\ g_{x_1} = 4\pi x_1 + 2\pi x_2,\ g_{x_2} = 2\pi x_1.$

Therefore the bordered Hessian matrix at $(2, 4)$ is

$$H_B = \left[\begin{array}{c|cc} 0 & 16\pi & 4\pi \\ \hline 16\pi & 4\pi & 2\pi \\ 4\pi & 2\pi & 0 \end{array}\right].$$

Here $n = 2,\ m = 1$. So $2m + 1 = 3$ and there is only one leading principal submatrix of H_B of order ≥ 3 which we need to consider and that leading principal submatrix is H_B itself. Now $\det H_B = 192\pi^3$ has the sign $(-1)^2 = (-1)^n$. So $(2, 4)$ is a point of local maxima. The maximum value is 16π. Therefore the maximum volume of a cylinder with surface area 24π is 16π, and that cylinder has height 4 unit and radius of the base 2 unit. $\square$

Constrained optimization problem with inequality constraints. Let f and $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, \ldots, m$ be continuously differentiable functions. Consider the problem

$$\begin{aligned} \text{minimize/maximize } & f(\mathbf{x}) \\ \text{subject to } & g_i(\mathbf{x}) \leq 0 \text{ for } i = 1, 2, \ldots, m. \end{aligned} \tag{CI}$$

The inequality constraints in (CI) can be transformed to equality constraints by adding nonnegative slack variables, y_i^2, as

$$g_i(\mathbf{x}) + y_i^2 = 0 \text{ for } i = 1, 2, \ldots, m.$$

Now the problem can be solved by the method of Lagrange's multipliers.

Convex programming problem. The optimization problem of the form

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } g_i(x) \leq 0 \text{ for } i = 1, 2, \ldots, m, \end{aligned}$$

is called a convex programming problem if f and g_i are convex functions.

Kuhn-Tucker (KT) conditions. Let f and $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, \ldots, m$ be continuously differentiable functions. Consider the following constrained optimization problem with inequality constraints:

$$\begin{aligned} &\text{minimize } f(\mathbf{x}) \\ &\text{subject to } g_i(\mathbf{x}) \leq 0 \text{ for } i = 1, 2, \ldots, m. \end{aligned} \tag{KT}$$

Necessary condition: Let $\mathbf{x}^* \in \mathbb{R}^n$ be a point of local minimum of the problem (KT). Then there exist multipliers $\lambda_1^*, \ldots, \lambda_m^*$ such that the following conditions hold:

$$\left.\begin{array}{c} (\nabla f)(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i^* \nabla g_i(\mathbf{x}^*) = 0, \\[1em] \lambda_i^* g_i(\mathbf{x}^*) = 0 \text{ for } i = 1, 2, \ldots, m, \\[1em] g_i(\mathbf{x}^*) \leq 0 \text{ for } i = 1, 2, \ldots, m, \\[1em] \lambda_i^* \geq 0 \text{ for } i = 1, 2, \ldots, m. \end{array}\right\} \tag{KT-N}$$

Sufficient condition: Let f and $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, \ldots, m$ be convex functions. If $(\mathbf{x}^*, \lambda^*)$ satisfies the KT conditions (KT-N), then $\mathbf{x}^*$ is a global minimum point of the problem (KT).

Note. Without convexity assumptions on f and g_i, the KT conditions are not sufficient for a point to be a local minimum or global minimum point. For example, consider the problem

$$\begin{aligned} &\text{minimize} \ -x_2 \\ &\text{subject to} \quad x_1^2 + x_2^2 \leq 4, \\ &\qquad\qquad\quad -x_1^2 + x_2 \leq 0. \end{aligned}$$

The point $(0, 0)$ satisfies the KT conditions, but it is neither local minimum nor global minimimum point.