Mathematical Methods for Power Engineering Assignment - 1	Name:- Syed Abdul Khader Jeelan Arafath Roll No:- M220396EE Class:- M.Tech 1st year 1st sem

Q1) Let $V = \{(a_1, a_2) : a_1, a_2 \in R\}$ for $(a_1, a_2), (b_1, b_2) \in V$ and $\alpha \in R$

define $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$

$\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2)$

Is V is a Vector Space over R with these operations?

Sol:- Here $V = \{(a_1, a_2) : a_1, a_2 \in R\}$

Let $u = (a_1, a_2)$, $v = (b_1, b_2)$, $w = (c_1, c_2)$ be vectors in V and α be any Scalar then,

1. Closure property for addition:-

$u + v = (a_1, a_2) + (b_1, b_2)$

$= (a_1 + 2b_1, a_2 + 3b_2)$

Since $a_1, a_2, b_1, b_2 \in R$. So, $a_1 + 2b_1$ and $a_2 + 3b_2$ also $\in R$

Thus, $u + v \in V$. Hence, V is closed under addition operation

2. Closure property for Scalar multiplication:-

$\alpha u = (\alpha)(a_1, a_2)$

$\alpha u = (\alpha a_1, \alpha a_2)$

Since $a_1, a_2 \in R$ and α is also Real. So, αa_1 & $\alpha a_2 \in R$

Thus, $\alpha u \in V$. Hence, V is closed under Scalar multiplication

3. Commutative Law for addition

$u + v = (a_1, a_2) + (b_1, b_2)$

$= (a_1 + 2b_1, a_2 + 3b_2)$ (by definition in question)

$$v+u = (b_1, b_2) + (a_1, a_2)$$
$$= (b_1+2a_1,\ b_2+3a_2) \quad \text{(by definition in question)}$$

$$u+v \neq v+u$$

Therefore addition operation is not Commutative

4. Associative Law for addition

$$u+(v+w) = (a_1,a_2) + [(b_1,b_2)+(c_1,c_2)]$$
$$= (a_1,a_2) + [(b_1+2c_1,\ b_2+3c_2)]$$
$$= (a_1+2(b_1+2c_1),\ a_2+3(b_2+3c_2))$$
$$= (a_1+2b_1+4c_1,\ a_2+3b_2+9c_2)$$

$$(u+v)+w = [(a_1,a_2)+(b_1,b_2)] + (c_1,c_2)$$
$$= [(a_1+2b_1,\ a_2+3b_2)] + (c_1,c_2)$$
$$= (a_1+2b_1+2c_1,\ a_2+3b_2+3c_2)$$

$$u+(v+w) \neq (u+v)+w$$

Therefore, addition operation is not associative

By 3 & 4, we can conclude that V is not a Vector Space.

Q2) Let $V = M_{n\times n}(R)$ be set of all $n\times n$ Matrices whose Entries from R. Verify the following subsets $W \subseteq V$ are Subspace or not?

a) $W = \{A \in M_{n\times n}(R) \mid A = A^T\}$

b) $W = \{A \in M_{n\times n}(R) \mid A = -A^T\}$

c) $W = \{A \in M_{n\times n}(R) \mid \text{trace}(A) = 0\}$

a) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid A = A^T\}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$ be two

Vectors in W

Given $A = A^T$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{nn} \end{bmatrix}$$

Similarly $B = B^T$

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & & \vdots \\ b_{1n} & b_{2n} & & b_{nn} \end{bmatrix}$$

$$(A+B) = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & \cdots & a_{n1}+b_{n1} \\ a_{12}+b_{12} & a_{22}+b_{22} & \cdots & a_{n2}+b_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n}+b_{1n} & a_{2n}+b_{2n} & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \vdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{bmatrix}$$

$= A^T + B^T$

$= (A+B)$ $\quad (\because A^T = A,\ B^T = B)$

Thus, $(A+B)^T = (A+B)$ (symmetric)

Hence, $A+B \in W$

Let K be any Scalar, then

$$(KA) = \begin{bmatrix} Ka_{11} & Ka_{12} & \cdots & Ka_{1n} \\ Ka_{21} & Ka_{22} & \cdots & Ka_{2n} \\ \vdots & & & \\ Ka_{n1} & Ka_{n2} & \cdots & Ka_{nn} \end{bmatrix}$$

$$(KA)^T = \begin{bmatrix} Ka_{11} & Ka_{21} & \cdots & Ka_{n1} \\ Ka_{12} & Ka_{22} & \cdots & Ka_{n2} \\ \vdots & \vdots & & \vdots \\ Ka_{1n} & Ka_{2n} & \cdots & Ka_{nn} \end{bmatrix}$$

$$(KA)^T = K \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

$$= KA^T \qquad (\because A^T = A)$$

$$= KA$$

Thus, $(KA)^T = KA$

Hence, $KA \in W$

Therefore, $W = \{ A \in M_{n\times n}(R) \mid A = A^T \}$ is a Subspace

b) $W = \{ A \in M_{n\times n}(R) \mid A = -A^T \}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & & b_{nn} \end{bmatrix}$ be two

Vectors in W

Given $A^T = -A$

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} = -\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Illy $B^T = -B$

$$\begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{bmatrix} = -\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$(A+B) = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & \cdots & a_{n1}+b_{n1} \\ a_{12}+b_{12} & a_{22}+b_{22} & \cdots & a_{n2}+b_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n}+b_{1n} & a_{2n}+b_{2n} & & a_{nn}+b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \vdots & & & \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{bmatrix}$$

$= A^T + B^T$

$= -A - B$ $\left(\because \text{Given } A^T = -A,\ B^T = -B\right)$

$= -(A+B)$

Thus, $(A+B)^T = -(A+B)$. Hence $A+B \in W$
(Skew Symmetric)

Let K be any scalar, then

$$(KA) = \begin{bmatrix} Ka_{11} & Ka_{12} & \cdots & Ka_{1n} \\ Ka_{21} & Ka_{22} & \cdots & Ka_{2n} \\ \vdots & \vdots & & \\ Ka_{n1} & Ka_{n2} & \cdots & Ka_{nn} \end{bmatrix}$$

$$(KA)^T = \begin{bmatrix} Ka_{11} & Ka_{21} & \cdots & Ka_{n1} \\ Ka_{12} & Ka_{22} & \cdots & Ka_{n2} \\ \vdots & \vdots & & \\ Ka_{1n} & Ka_{2n} & \cdots & Ka_{nn} \end{bmatrix} = K\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

$= K(A)^T$ $(\because A^T = -A)$

$= K(-A) = -(KA)$

Thus $(KA)^T = -(KA)$

Hence, $KA \in W$

Therefore, $W = \{A \in M_{n\times n}(\mathbb{R}) \mid A = -A^T\}$ is a subspace

c) $W = \{A \in M_{n\times n}(\mathbb{R}) \mid \text{trace}(A) = 0\}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ & $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$ be two vectors of W.

Therefore

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = 0$$

$$tr(B) = b_{11} + b_{22} + \cdots + b_{nn} = 0$$

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$

$$tr(A+B) = (a_{11}+b_{11}) + (a_{22}+b_{22}) + \cdots + (a_{nn}+b_{nn})$$

$$= (a_{11}+a_{22}+\cdots+a_{nn}) + (b_{11}+b_{22}+\cdots+b_{nn})$$

$$= 0 + 0 \qquad (\because A, B \in W \quad tr(A)=0 \;\&\; tr(B)=0)$$

$$tr(A+B) = 0$$

Hence, $A+B \in W$

Let K be any scalar then,

$$KA = K\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} Ka_{11} & Ka_{12} & \cdots & Ka_{1n} \\ Ka_{21} & Ka_{22} & \cdots & Ka_{2n} \\ \vdots & & & \vdots \\ Ka_{n1} & Ka_{n2} & \cdots & Ka_{nn} \end{bmatrix}$$

$$tr(KA) = Ka_{11} + Ka_{22} + \cdots + Ka_{nn}$$

$$= K(a_{11}+a_{22}+\cdots+a_{nn}) = K(0)$$

$$tr(KA) = 0 \qquad \left(\because \begin{array}{c} A \in B \\ tr(A)=0 \end{array}\right)$$

Hence, $KA \in W$

Therefore $W = \{A \in M_{n\times n}(\mathbb{R}) \mid \text{trace}(A) = 0\}$ is a subspace

Q3) Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and u, v, w be distinct vectors in V.

prove that

a) $\{u, v\}$ is Linearly Independent $\Leftrightarrow$ $\{u+v, u-v\}$ is Linearly Independent.

b) $\{u, v, w\}$ is Linearly Independent $\Leftrightarrow$ $\{u+v, v+w, w+u\}$ is Linearly Independent c> Verify $\{(1,2,3,4), (0,5,-1,2), (1,0,2,3)\} \subseteq R^4$ is L.I or not ?

Sol:

a> Given $\{u, v\}$ is Linearly Independent

To Show,

$\{u+v, u-v\}$ is L.I or not

$a_1(u+v) + a_2(u-v) = 0$

$a_1u + a_1v + a_2u - a_2v = 0$

$(a_1+a_2)u + (a_1-a_2)v = 0$

Since $\{u, v\}$ are L.I then

$a_1 + a_2 = 0 \rightarrow (1)$

$a_1 - a_2 = 0 \rightarrow (2)$

By Solving (1) & (2), we get

$a_1 = 0, \ a_2 = 0$

Therefore, $\{u+v, u-v\}$ is Linearly Independent

b> Given $\{u, v, w\}$ is Linearly Independent

To Show,

$\{u+v, v+w, w+u\}$ is L.I (or) not

$a_1(u+v) + a_2(v+w) + a_3(w+u) = 0$

$a_1u + a_1v + a_2v + a_2w + a_3w + a_3u = 0$

$(a_1+a_3)u + (a_1+a_2)v + (a_2+a_3)w = 0$

Since $\{u, v, w\}$ are L.I then

$a_1 + a_3 = 0 \rightarrow (1)$

$a_1 + a_2 = 0 \rightarrow (2)$

$a_2 + a_3 = 0 \rightarrow (3)$

$(3) \Rightarrow a_2 = -a_3$

$(2) \Rightarrow a_1 - a_3 = 0 \Rightarrow a_1 = a_3$

$(3) \Rightarrow a_3 + a_3 = 0$

$2a_3 = 0 \Rightarrow a_3 = 0$

$a_1 = 0$

$a_2 = 0$

Therefore, $\{u+v, v+w, w+u\}$ is Linearly Independent c)

To Show $\{(1,2,3,4), (0,5,-1,2), (1,0,2,3)\} \subseteq R^4$ is L.I or not

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & -1 & 2 \\ 1 & 0 & 2 & 3 \end{bmatrix} \quad R_3 \to R_1 - R_3$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & -1 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix} \quad R_3 \to 5R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & -1 & 2 \\ 0 & 0 & 7 & 1 \end{bmatrix}$$

Rank = 3

Therefore $\{(1,2,3,4), (0,5,-1,2), (1,0,2,3)\}$ is L.I

(or)

$a_1(1,2,3,4) + a_2(0,5,-1,2) + a_3(1,0,2,3) = (0,0,0,0)$

$a_1 + a_3 = 0 \rightarrow a_1 = -a_3 \rightarrow (1)$

$2a_1 + 5a_2 = 0 \rightarrow a_1 = -5/2\, a_2 \rightarrow (2)$

$3a_1 - a_2 + 2a_3 = 0 \rightarrow (3)$

$4a_1 + 2a_2 + 3a_3 = 0 \rightarrow (4)$

(3) $\Rightarrow$ $3(-5/2\, a_2) - a_2 + 2a_3 = 0 \Rightarrow \frac{-17}{2} a_2 + 2a_3 = 0$

(4) $\Rightarrow$ $4\left(-\frac{5}{2} a_2\right) + 2a_2 + 3a_3 = 0 \Rightarrow -8a_2 + 3a_3 = 0 \rightarrow a_3 = \frac{8}{3} a_2$

Sub $a_3 = \frac{8}{3} a_2$ in $\frac{-17}{2} a_2 + 2a_3 = 0$

then

$$-\frac{17}{2} a_2 + 2\left(\frac{8}{3} a_2\right) = 0$$

$$\frac{-51a_2 + 32a_2}{6} = 0$$

$$-19a_2 = 0$$

$$a_2 = 0$$

from (2), $a_1 = -5/2(0) = 0$

from (1), $a_3 = 0$

So, $a_1 = a_2 = a_3 = 0$

$\{(1,2,3,4), (0,5,-1,2), (1,0,2,3)\}$ is L.I

4) Let U and W be the Subspaces of R^4 generated by
$(1,4,-1,3)$, $(1,5,0,5)$, $(3,10,-5,5)$ and
$(1,4,0,6)$, $(1,2,-1,5)$, $(2,2,-3,9)$
Find $\dim(U+W)$ and $\dim(U\cap W)$

Sol:-
Given U is generated by $(1,4,-1,3)$, $(1,5,0,5)$, $(3,10,-5,5)$

$$U = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 1 & 5 & 0 & 5 \\ 3 & 10 & -5 & 5 \end{bmatrix} \underset{\substack{R_2\to R_2-R_1 \\ R_3 - R_3-3R_1}}{\sim} \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -4 \end{bmatrix} \underset{R_3\to R_3+2R_2}{\sim} \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 2

Basis of $U = \{(1,4,-1,3),(0,1,1,2)\}$

$\boxed{\dim(U) = 2}$

Given W is generated by $(1,4,0,6)$, $(1,2,-1,5)$, $(2,2,-3,9)$

$$W = \begin{bmatrix} 1 & 4 & 0 & 6 \\ 1 & 2 & -1 & 5 \\ 2 & 2 & -3 & 9 \end{bmatrix} \underset{\substack{R_2\to R_2-R_1 \\ R_3\to R_3-2R_1}}{\sim} \begin{bmatrix} 1 & 4 & 0 & 6 \\ 0 & -2 & -1 & -1 \\ 0 & -6 & -3 & -3 \end{bmatrix} \underset{R_3\to R_3-3R_2}{\sim} \begin{bmatrix} 1 & 4 & 0 & 6 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 2

Basis of $W = \{(1,4,0,6),(0,-2,-1,-1)\}$

$\boxed{\dim(W) = 2}$

$$U+W = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 1 & 5 & 0 & 5 \\ 3 & 10 & -5 & 5 \\ 1 & 4 & 0 & 6 \\ 1 & 2 & -1 & 5 \\ 2 & 2 & -3 & 9 \end{bmatrix} \underset{\substack{R_2\to R_2-R_1 \\ R_3\to R_3-3R_1 \\ R_4\to R_4-R_1 \\ R_5\to R_5-R_1 \\ R_6\to R_6-2R_1}}{\sim} \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & -2 & 0 & 2 \\ 0 & -6 & -1 & 3 \end{bmatrix} \underset{\substack{R_3\to R_3+2R_2 \\ R_5\to R_5+2R_2 \\ R_6\to R_6+6R_2}}{\sim}$$

$$\sim \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 5 & 15 \end{bmatrix} \underset{\substack{\cancel{R_4\leftrightarrow R_4\leftrightarrow R_4} \\ R_5\to R_5-2R_4 \\ R_6\to R_6-5R_4}}{\sim} \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3 $\Rightarrow$ Basis of $U+W = \{(1,4,-1,3),(0,1,1,2),(0,0,1,3)\}$

$\boxed{\dim(U+W) = 3}$

$\dim(U+W) = \dim(U) + \dim(W) - \dim(U\cap W)$

$\dim(U\cap W) = 2+2-3 = 1 \Rightarrow \boxed{\dim(U\cap W) = 1}$

5) Let $T: R^3 \to R^4$ be the Linear mapping defined by

$T(x, y, z) = (x+2y-z,\ y+z,\ x+y-2z,\ -x+6z)$

Find the basis and dimension of $R(T)$ and $N(T)$

Sol:- If $T: R^3 \to R^4$ is a linear Transformation

then $R(T) = \{ \beta \in R^4 ;\ \beta = T(\alpha) \text{ for Some } \alpha \in R^3 \}$

Standard basis of $R^3 \to \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$

$T(1,0,0) = (1, 0, 1, -1)$

$T(0,1,0) = (2, 1, 1, 0)$

$T(0,0,1) = (-1, 1, -2, 6)$

To check whether $(1,0,1,-1)$, $(2,1,1,0)$, $(-1,1,-2,6)$ is L.I or not

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & 0 \\ -1 & 1 & -2 & 6 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 5 \end{bmatrix} \quad R_3 \to R_3 - R_2 \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Rank $= 3$

$\therefore (1,0,1,-1),\ (2,1,1,0),\ (-1,1,-2,6)$ is L.I

Thus, Basis of $R(T) \to \{(1,0,1,-1),\ (2,1,1,0),\ (-1,1,-2,6)\}$

$\dim(R(T)) = 3$

Let any $(x, y, z) \in N(T)$

By definition of Nullspace

$\cancel{N(T) \neq \{T(x,y,z) \neq}$

$N(T) = \{ (x,y,z) \in R^3 \mid T(x,y,z) = 0_v \in R^4 \}$

$T(x,y,z) = (0,0,0,0)$

$(x+2y-z,\ y+z,\ x+y-2z,\ -x+6z) = (0,0,0,0)$

$x+2y-z=0$

$y+z=0 \rightarrow y=-z$

$x+y-2z=0$

$-x+6z=0 \rightarrow x=6z$

$N(T) = \{(6z,-z,z) \mid z \in R\}$

Basis of $N(T) = \{(6,-1,1)\}$

$\dim(N(T)) = 1$

Q6> Let $T: R^3 \rightarrow R^2$ be Linear map defined by

$T(x,y,z) = (2x+y-z,\ 3x-2y+4z)$

Find the matrix of T, relative to the basis

$\beta = \{(1,1,1),(1,1,0),(1,0,0)\}$, $\gamma = \{(1,3),(1,4)\}$

Sol:- $T: R^3 \rightarrow R^2$

$T(x,y,z) = (2x+y-z,\ 3x-2y+4z)$

Given Basis $\beta = \{(1,1,1),\ (1,1,0),\ (1,0,0)\}$

$\gamma = \{(1,3),\ (1,4)\}$

$T(1,1,1) = (2,5) = a_1(1,3) + a_2(1,4)$

$(2,5) = [(a_1+a_2),\ (3a_1+4a_2)]$

$a_1+a_2 = 2 \rightarrow ①$

$3a_1+4a_2 = 5 \rightarrow ②$

Solve ① × 3, ②

$3a_1+3a_2 = 6$

$3a_1+4a_2 = 5$

$-a_2 = 1$

$\boxed{a_2=-1}$

② ⟹ $3a_1-4=5$

$3a_1 = 9$

$\boxed{a_1=3}$

So, $T(1,1,1) = 3(1,3) - 1(1,4)$

$T(1,1,0) = (3,1) = a_1(1,3) + a_2(1,4)$

$(3,1) = [(a_1+a_2), (3a_1+4a_2)]$

$a_1 + a_2 = 3 \rightarrow ①$

$3a_1 + 4a_2 = 1 \rightarrow ②$

Solve ① × 3 , ②

$3a_1 + 3a_2 = 9$

$3a_1 + 4a_2 = 1$

$- \quad - \quad -$

$-a_2 = 8$

$\boxed{a_2 = -8}$

② ⇒ $3a_1 - 32 = 1$

$3a_1 = 33$

$\boxed{a_1 = 11}$

Thus, $T(1,1,0) = 11(1,3) - 8(1,4)$

$T(1,0,0) = (2,3) = a_1(1,3) + a_2(1,4)$

$(2,3) = [(a_1+a_2), (3a_1+4a_2)]$

$a_1 + a_2 = 2 \rightarrow ①$

$3a_1 + 4a_2 = 3 \rightarrow ②$

Solve ① × 3 , ②

$3a_1 + 3a_2 = 6$

$3a_1 + 4a_2 = 3$

$- \quad - \quad -$

$-a_2 = 3$

$\boxed{a_2 = -3}$

② ⇒ $3a_1 - 12 = 3$

$3a_1 = 15$

$\boxed{a_1 = 5}$

Thus, $T(1,0,0) = 5(1,3) - 3(1,4)$

Matrix of $T = \begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix}$

7) Let $T: R^3 \to R^4$ be Linear map defined by

$T(a,b,c) = (a+b,\ b+c,\ c,\ a+b+c)$

Find $N(T)$ and rank T

Sol:- If $T: R^3 \to R^4$ is a Linear Transformation

then $R(T) = \{\beta \in R^4;\ \beta = T(\alpha) \text{ for some } \alpha \in R^3\}$

Standard Basis of $R^3 \to \{(1,0,0), (0,1,0), (0,0,1)\}$

$$T(1,0,0) = (1,0,0,1)$$
$$T(0,1,0) = (1,1,0,1)$$
$$T(0,0,1) = (0,1,1,1)$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}_{R_2 \to R_2 - R_1} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{Rank} = 3$$

$\{(1,0,0,1), (1,1,0,1), (0,1,1,1)\}$ is L.I

Basis for $R(T) = \{(1,0,0,1), (1,1,0,1), (0,1,1,1)\}$

Rank $T = \dim(R(T)) = 3$

Let any $(a,b,c) \in N(T)$

By definition of Nullspace

$N(T) = \{(a,b,c) \in R^3 \mid T(a,b,c) = 0_V \in R^4\}$

$T(a,b,c) = (0,0,0,0)$

$(a+b,\ b+c,\ c,\ a+b+c) = (0,0,0,0)$

$a+b=0$

$b+c=0 \to b=-c \Rightarrow b=0$

$c=0$

$a+b+c=0 \to a+0+0=0$

Therefore $a=0,\ b=0,\ c=0$

$N(T) = \{(0,0,0)\}$

$\dim(N(T)) = 0 = $ Nullity of T

8) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}$. Find a Linear Transformation $T: R^3 \to R^4$

with respect to the standard basis in R^3 & R^4

Sol:-

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}_{4\times 3} \qquad T: R^3 \to R^4$$

Standard Basis for $R^3 \to \{(1,0,0), (0,1,0), (0,0,1)\}$

Standard Basis for $R^4 \to \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

$T(1,0,0) = \{(1,2,0,0)\} = 1(1,0,0,0) + 2(0,1,0,0) + 0(0,0,1,0) + 0(0,0,0,1)$

$T(0,1,0) = \{(1,0,2,6)\} = 1(1,0,0,0) + 0(0,1,0,0) + 2(0,0,1,0) + 6(0,0,0,1)$

$T(0,0,1) = \{(1,1,-1,0)\} = 1(1,0,0,0) + 1(0,1,0,0) - 1(0,0,1,0) + 0(0,0,0,1)$

$$T(x,y,z) = T[x(1,0,0) + y(0,1,0) + z(0,0,1)]$$

$$= xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$$

$$= x(1,2,0,0) + y(1,0,2,6) + z(1,1,-1,0)$$

$$= (x+y+z,\ 2x+z,\ 2y-z,\ 6y)$$

9) Solve the System of Linear Equations

$$x_1 + 2x_2 + 2x_3 = 2$$

$$x_1 + 8x_3 + 5x_4 = -6$$

$$x_1 + x_2 + 5x_3 + 5x_4 = 3$$

Sol:

$$\underbrace{\begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 0 & 8 & 5 \\ 1 & 1 & 5 & 5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}}_{B}$$

Augmented Matrix $[A/B]$ $= \begin{bmatrix} 1 & 2 & 2 & 0 & 2 \\ 1 & 0 & 8 & 5 & -6 \\ 1 & 1 & 5 & 5 & 3 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 2 & 0 & 2 \\ 0 & -2 & 6 & 5 & -8 \\ 0 & -1 & 3 & 5 & 1 \end{bmatrix} \quad R_3 \to 2R_3 - R_2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 2 \\ 0 & -2 & 6 & 5 & -8 \\ 0 & 0 & 0 & 5 & 10 \end{array}\right]$$

Rank (A) = Rank $(A/B) = 3 < n$ ↳ no. of unknowns $= 4\ (x_1, x_2, x_3, x_4)$

So, $3 < 4$

Infinitely many Solutions Exist

$x_1 + 2x_2 + 2x_3 = 2 \to (1)$

$-2x_2 + 6x_3 + 5x_4 = -8 \to (2)$

$5x_4 = 10$

$\boxed{x_4 = 2}$

from (2) $\Rightarrow$ $-2x_2 + 6x_3 + 10 = -8$

$-2x_2 + 6x_3 = -18$

$2x_2 = 18 + 6x_3$

Let $\boxed{x_2 = K}$

then $2K = 18 + 6x_3$

$x_3 = \dfrac{2K - 18}{6} = \dfrac{K-9}{3} \Rightarrow \boxed{x_3 = \dfrac{K-9}{3}}$

① ⇒ $x_1 + 2x_2 + 2x_3 = 2$

$x_1 + 2K + 2\left(\frac{K-9}{3}\right) = 2$

$x_1 = 2 - 2K - 2\left(\frac{K-9}{3}\right)$

$x_1 = (6 - 6K - 2K + 18)/3$

$$\boxed{x_1 = \frac{24 - 8K}{3}}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{24-8K}{3} \\ K \\ \frac{K-9}{3} \\ 2 \end{bmatrix}$$

Verification,

Let $K = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/3 \\ 1 \\ -8/3 \\ 2 \end{bmatrix}$$

$x_1 + 2x_2 + 2x_3 = \frac{16}{3} + 2(1) + 2(-8/3) = 2$

$x_1 + 8x_3 + 5x_4 = \frac{16}{3} + 8\left(\frac{-8}{3}\right) + 5(2) = -6$

$x_1 + x_2 + 5x_3 + 5x_4 = \frac{16}{3} + 1 + 5\left(\frac{-8}{3}\right) + 5(2) = 3$

10> Find Eigen Value and Eigen Vector for

$$A = \begin{bmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{bmatrix}$$

Sol:-

$$A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$$

$|A - \lambda I| = 0$ (Characteristic Equation)

$$\left| \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{vmatrix} = 0$$

$$-\lambda\left[(1-\lambda)(5-\lambda)+2\right] + 2\left[-5+\lambda+2\right] - 3\left[-2-2+2\lambda\right] = 0$$

$$-\lambda\left[5-\lambda-5\lambda+\lambda^2+2\right] + 2\left[-3+\lambda\right] - 3\left[-4+2\lambda\right] = 0$$

$$-\lambda\left[7-6\lambda+\lambda^2\right] + 2\left[-3+\lambda\right] - 3\left[-4+2\lambda\right] = 0$$

$$-7\lambda+6\lambda^2-\lambda^3-6+2\lambda+12-6\lambda = 0$$

$$-\lambda^3+6\lambda^2-5\lambda-6\lambda+6 = 0$$

$$-\lambda^3+6\lambda^2-11\lambda+6 = 0$$

$\lambda = 1$ is one of the Eigen Value

$$\begin{array}{c|cccc} \lambda=1 & -1 & 6 & -11 & 6 \\ & 0 & -1 & 5 & -6 \\ \hline & -1 & 5 & -6 & \boxed{0} \end{array}$$

$$-\lambda^2+5\lambda-6 = 0$$

$$\lambda^2-5\lambda+6 = 0$$

$$\lambda^2-3\lambda-2\lambda+6 = 0$$

$$\lambda(\lambda-3)-2(\lambda-3) = 0$$

$$\lambda = 2,\ \lambda = 3$$

$\lambda = 1,\ \lambda = 2,\ \lambda = 3$ are the Eigen Values

Eigen Vector corresponding to $\lambda = 1$

$$\underbrace{\begin{bmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} -1 & -2 & -3 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -1 & -2 & -3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = r = 2 < n$

$n \to$ no. of unknowns $= 3$
$r \to$ Rank of Matrix $A = 2$

$\rho(A) = 2 < 3$

Infinitely many non Trivial Solutions Exist

$n - r = 3 - 2 = 1 \Rightarrow$ One Independent Solution

$-x_1 - 2x_2 - 3x_3 = 0$

$2x_2 + 2x_3 = 0 \Rightarrow x_2 = -x_3$

$-x_1 + 2x_3 - 3x_3 = 0$

$x_1 = -x_3$

Let $x_1 = k$ then $x_3 = -k$

& $x_2 = k$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Eigen Value Corresponding to $\lambda=2$

$$\underbrace{\begin{bmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to 2R_2 - R_1$ & $R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & -2 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rank(A) $\to \rho(A) = r = 2 < n$

$\rho(A) = 2 < 3$

Infinitely many non Trivial solutions Exist

$n - r = 3 - 2 = 1 \Rightarrow$ one Independent solution

$-2x_1 - 2x_2 - 3x_3 = 0$

$x_3 = 0$

$-2x_1 - 2x_2 = 0$

$x_1 + x_2 = 0$

$x_1 = -x_2$

Let $x_1 = k$ then $x_2 = -k$

$x_3 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Eigen Value Corresponding to $\lambda=3$

$$\underbrace{\begin{bmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to 3R_2 - R_1$ & $R_3 \to 3R_3 + 2R_1$

$$\begin{bmatrix} -3 & -2 & -3 \\ 0 & -4 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$

$$\begin{bmatrix} -3 & -2 & -3 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = r = 2 < n$

$\rho(A) = 2 < 3$

Infinitely many non Trivial solutions Exist

$n - r = 3 - 2 = 1 \Rightarrow$ one Independent Solution

$-3x_1 - 2x_2 - 3x_3 = 0$

$-4x_2 = 0 \Rightarrow x_2 = 0$

$-3x_1 - 2(0) - 3x_3 = 0$

$-x_1 - x_3 = 0$

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

Let $x_1 = k$ then $x_3 = -k$

$x_2 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} k \\ 0 \\ -k \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$