Eigen value & Eigen vector

→ Let $T : V \to V$ be a linear transformation, A non-zero vector $v \in V$ is called eigen vector of T if there exist $\lambda \in F$ such that $T(v) = \lambda v$, $v \neq 0$, λ is called eigen value of T.

$\left[\begin{array}{l} L_A : R^n \to R^n \\ L_A(x) = Ax \end{array} \right]$

→ $A_{n \times n}$

$A \in M_{n\times n}(F)$, A non-zero vector $v \in F^n$ is called eigen vector of A if $Av = \lambda v$, for some $\lambda \in F$.

Ex: Linear transformation (LT)

1. Any LT, is eigen value exist?

Ex: 1. $I : R^2 \to R^2$, $I(a,b) = (a,b)$.

Sol: $I(a,b) = 1.(a,b)$

$a=1, b=0$, $I(1,0) = 1.(1,0) \Rightarrow v = (1,0)$

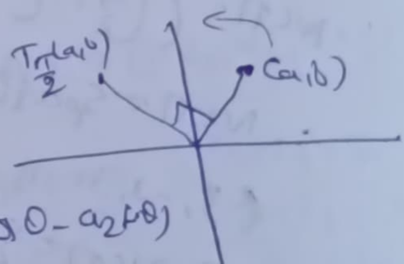
$\therefore$ 1 is eigen value of I, corresponding to eigen vector $v=(1,0)$

Any non-zero vector $v \in R^2$ is an eigen vector w.r.t. eigen value 1.

2. Any L.T, is eigen value exist?

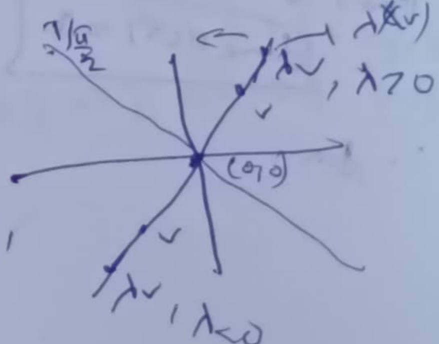
$T(v) = \lambda v$, $v \neq 0$

$T_{\pi/2}$ = rotation of vector by angle $\frac{\pi}{2}$.



$T_{\pi/2}(a,b) = (-b,a)$ $[T_\theta(a_1,a_2) = (a_1\cos\theta - a_2\sin\theta)$

$T_{\frac{\pi}{2}}(v) \neq \lambda v$, $\forall v \neq 0$.



$\therefore T_{\pi/2}$ doesn't have an eigen vector, eigen value.

$T_{\pi/2}(a,b) = (-b,a)$

suppose, λ is an eigen value of T. with eigen vector $v=(a,b)\neq 0$.

$\therefore T(a,b) = (-b,a) = \lambda(a,b)$

$\therefore \lambda a = -b, \ \lambda b = a$

Case I: $\lambda = 0 \Rightarrow (a,b) = (0,0)$

but we want $(a,b) \neq (0,0)$

contradiction false

case II: $\lambda \neq 0$:

$\lambda a = -b, \ \lambda b = a$

$a = -\frac{b}{\lambda} \Rightarrow \lambda b = a \Rightarrow \lambda b = -\frac{b}{\lambda} \Rightarrow \therefore (\lambda^2+1)b = 0$

$\lambda^2+1 = 0$ or $b = 0$.

$\lambda = \pm i$.

but we defined $T: R^2 \to R^2$ over $F=R$

$\therefore \lambda$ should be $\in R$, but we got complex number (i.e. $\Rightarrow \Leftarrow$)

$\therefore$ contradiction fail.

$\Rightarrow a = 0$

$\therefore (a,b) = (0,0)$

$\therefore$ contradiction fail

i.e. $\Rightarrow \Leftarrow$

conclusion: $T_{\pi/2}$ doesn't have eigen vector, eigen value.

$A \in M_{n\times n}(F)$, $I \in M_{n\times n}(F)$

characteristic polynomial, $\det(A-\lambda I) = \lambda^n + a_n\lambda^{n-1} \dots + a_0$

$\det(A-\lambda I) = 0$

eigen values of $A = \{\lambda \in F \mid \det(A-\lambda I) = 0\}$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$

Find eigen values, & eigen vectors.

Sol: $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 2 & 3-\lambda \end{vmatrix}$

$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-2] = 0$

$(1-\lambda)[6-5\lambda+\lambda^2-2] = 0$

$(1-\lambda)[\lambda^2-5\lambda+4] = 0.$

$(1-\lambda)[\lambda^2-\lambda-4\lambda+4] = 0$

$(1-\lambda)[\lambda(\lambda-1)-4(\lambda-1)] = 0$

$(1-\lambda)(\lambda-1)(4-\lambda) = 0.$

$\therefore \lambda = 1, 1, 4.$ = eigen values.

$\underline{\lambda = 1}$: $Ax = 1.x.$ $\quad x = (x, y, z) \in \mathbb{R}^3.$

or $Av = 1.v$

$\therefore (A-\lambda I)v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore (A-I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\left.\begin{matrix} y+z=0 \\ 2y+2z=0 \end{matrix}\right\} \Rightarrow y+z=0 \Rightarrow y=-z \Rightarrow z=-y$

eigen vector corresponds to $\lambda = 1 \Rightarrow \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} / x, y \in \mathbb{R} \right\}$

$x, y \neq 0..$

= span

$\lambda = 4$:

$(A - \lambda I)v = 0 \qquad (A - 4I)v = 0$

$$= \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$\therefore -3x + y + z = 0 \quad , \quad -2y + z = 0 \ , \ 2y - z = 0$

$\therefore 2y = z \Rightarrow \boxed{z = 2y}$ — (1)

$-3x + y + 2y = 0$

$\therefore -3x + 3y = 0 \Rightarrow \boxed{x = y}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y \\ 2y \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

eigen vector corresp to $\lambda = 4$ = $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \middle/ y \in R, y \neq 0 \right\}$

Theorem: $T: V \to V$, $\lambda_1, \lambda_2 \dots \lambda_k$ (all are distinct) then corresponding eigen vectors $v_1, v_2, \dots v_k$ are L.I.

i.e. the eigen vectors corresponding to distinct eigen values are L.I.

Proof: Let $T: V \to V$ be a L.T and $\lambda_1, \lambda_2 \dots \lambda_k$ be an distinct eigen values and $v_1, v_2 \dots v_k$ are corresponding eigen vectors.

To prove $\{v_1 \dots v_k\}$ is L.I.

Induction method:

$1 + 2 + \dots + k = \frac{k(k+1)}{2}$ — (1)

$k = 1$: it true.

Assume (1) is true for $k-1$.

to prove (1) is true for k.

like that, take

proof done by induction method

$\underline{\underline{k=1}}$:

to prove $\{v_1\}$ is L.I.

since, $v_1 \neq 0$ (v_1 is an eigen vector)

$\therefore$ v_1 is L.I.

Assume that $\{v_1, \ldots v_{k-1}\}$ is L.I, to prove $\{v_1, \ldots, v_k\}$ is L.I.

$\therefore$ $d_1 v_1 + d_2 v_2 + \ldots + d_k v_k = 0$ — ①

Apply transformation $(T - \lambda_k I)$ on both sides

$(T-\lambda_k I)(d_1 v_1 + d_2 v_2 + \ldots + d_k v_k) = (T-\lambda_k I)0$

$T(d_1 v_1 + d_2 v_2 + \ldots + d_k v_k) - \lambda_k I(d_1 v_1 + \ldots + d_k v_k) = T(0) - \lambda_k I(0)$

$= 0 - 0 = 0$

$\therefore T(d_1 v_1 + d_2 v_2 + \ldots + d_k v_k) - [\lambda_k d_1 v_1 + \ldots + d_k \lambda_k v_k] = 0$

$d_1 T(v_1) + \ldots + d_k T(v_k) - [\lambda_k d_1 v_1 + \ldots + d_k \lambda_k v_k] = 0$

$[\because T(v_1) = \lambda_1 v_1, \ldots, T(v_k) = \lambda_k v_k.]$

$\therefore d_1 \lambda_1 v_1 + \ldots + d_k \cancel{v_k} v_k - [d_1 \lambda_k v_1 + \ldots + d_k \cancel{\lambda_k v_k}] = 0$

$\therefore d_1(\lambda_1 - \lambda_k)v_1 + d_2(\lambda_2 - \lambda_k)v_2 + \ldots + d_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0$

$[\because \{v_1, \ldots v_{k-1}\}$ are L.I assumed,

$\therefore d_1 v_1 + d_2 v_2 + \ldots + d_{k-1} v_{k-1} = 0]$

$\Rightarrow d_1 = d_2 = \ldots = d_k = 0.$

$\therefore$

$\Rightarrow d_1(\lambda_1 - \lambda_k) = 0 = d_2(\lambda_2 - \lambda_k) = \ldots = d_{k-1}(\lambda_{k-1} - \lambda_k)$

we know that $\lambda_1, \lambda_2 \dots \lambda_R$ are distinct, $\therefore \lambda_i \neq \lambda_R$, $i = 1, 2 \dots R$, $= 1, 2 \dots R-1$.

$[\because \lambda_i - \lambda_R \neq 0]$

since $\lambda_i - \lambda_R \neq 0$, for $i = 1, 2, 3 \dots R-1$ $\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_{R-1} = 0$. — (2)

substitute (2) in (1)

(1) becomes, $\alpha_R v_R = 0$. $[\because v_R \neq 0] \Rightarrow \alpha_R = 0$,

$\therefore \alpha_1 = \alpha_2 = \dots = \alpha_R = 0 \Rightarrow \{v_1, v_2 \dots v_R\}$ is L.I,

Cayley Hamilton theorem

Let $A \in M_{n \times n}(F)$ and $f(t)$ be the characteristic polynomial of A, $f(\lambda) = \det(A - \lambda I) = \det(A - tI)$.

then $f(A) = 0$.

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow (1-\lambda)(2-\lambda) = \lambda^2 - 3\lambda + 2 = f(\lambda)$

$\therefore f(A) = A^2 - 3A + 2I = 0$.

If A is invertible,

$A^{-1}(A^2 - 3A + 2I) = A^{-1} \cdot 0 = 0$

$A - 3I + 2A^{-1} = 0 \Rightarrow \boxed{A^{-1} = \frac{3I - A}{2}}$

Similar matrices (B~A)

Let $A, B \in M_{n\times n}(F)$, B is similar to A if there exist an invertible matrix P such that $\boxed{B = P^{-1}AP}$

i.e. $PB = AP$

Ex: $I \sim P^{-1}IP = P^{-1}P = I$

$O \sim P^{-1}OP = O$.

Properties:

1. If $B\sim A \Rightarrow \boxed{\det B = \det A}$

Proof: $B\sim A$ i.e. $\Rightarrow B = P^{-1}AP$

$\det B = \det(P^{-1}AP)$ [$\because \det(CD) = \det C . \det D$]

$= \det P^{-1} \times \det A \times \det P$ [$\because \det D^{-1} = \frac{1}{\det D}$]

$= \frac{1}{\det P} \times \det A \times \det P$

$\boxed{\det B = \det A}$

2. Let $B\sim A$, then eigen values of A & B are equal.

i.e. $\det(A-\lambda I) = \det(B-\lambda I)$.

Proof: $\det(A-\lambda I) = 0$.

given, $B\sim A$ i.e. $B = P^{-1}AP$

$\det(B-\lambda I) = \det(P^{-1}AP - \lambda I) = \det P^{-1}AP - \det \lambda I$

$= \det P^{-1} \det A . \det P - \lambda \det I$

$= \det A - \lambda \det I$

(or) $= \det(A-\lambda I)$

$= \det(P^{-1}AP - \lambda I) = \det(P^{-1}AP - P^{-1}P\lambda)$ [$\because I = P^{-1}P$]

$= (\det(P^{-1}P(A-\lambda I)))$ [$\because \det(A+B) \neq \det A + \det B$]

[$\because AB \neq BA$]

$= \det(P^{-1}AP - P^{-1}\lambda P) = \det(P^{-1}(A-\lambda I)P))$

$= \det P^{-1} . \det(A-\lambda I) . \det P$

$= \frac{1}{\det P} \times \det(A-\lambda I) \times \det P$

$\boxed{\det(B-\lambda I) = \det(A-\lambda I)}$

property 3: x is an eigen vector of A corresponding to eigen value λ, and $B \sim A$ then $P^{-1}x$ is an eigen vector of B corresponding to same eigen value λ.

sol: given, $Ax = \lambda x$ for $x \neq 0$. — ①

$B = P^{-1}AP$.

find $\Rightarrow$ $By = \lambda y$. $y \neq 0$. $\Rightarrow y = ?$

$\therefore$ ⊕ $B \cdot P^{-1}x = \lambda P^{-1}x$, $P^{-1}x \neq 0$.

$\therefore BP^{-1}x = (P^{-1}AP) \cdot P^{-1}x = P^{-1}AIx$ $(\because PP^{-1} = I)$

$= P^{-1}Ax = P^{-1}(\lambda x)$ $[\because$ from ①$]$

$\boxed{BP^{-1}x = \lambda P^{-1}x}$

$\therefore$ hence proved.

note: $x \neq 0 \Rightarrow P^{-1}x \neq 0$.

assume $\Rightarrow P^{-1}x = 0$.

$P \cdot P^{-1}x = P \cdot 0 \Rightarrow Ix = 0 \Rightarrow x = 0$

$\therefore$ which is contradict to x non zero i.e. $x \neq 0$.

Hence $\Rightarrow$ $\boxed{\text{eigen vectors can be different for similar matrices}}$

4. $B \sim A \Rightarrow B^k \sim A^k$, $k \geq 1$.

Sol:- $B \sim A$ means that $B = P^{-1}AP$.

$\therefore B^2 = P^{-1}AP \cdot P^{-1}AP = P^{-1}AIAP$

$\underbrace{B^2 = P^{-1}A^2P} \Rightarrow B^2 \sim A^2$

Similarly we can prove that $B^3 \sim A^3$.

<u>Theorem</u>:- eigen values of A and A^T are same.

<u>Proof</u>:- Characteristic polynomial of A = Ch. Poly. of A^T.

$\det(A - \lambda I) = \det(A^T - \lambda I)$

$\therefore \det(A^T - \lambda I) = \det(A^T - \lambda AA^T.)$ ✗ $[\because AA^T = I]$

$= \det(IA^{-1}.$ ✗ $[\because \det B = \det B^T]$

$\det(A^T - \lambda I^T) = \det(A - \lambda I)^T$ $[\because (A-B)^T = A^T - B^T$

$= \det(A - \lambda I)$ $[\because \det B^T = \det B]$.

<u>Complex matrices</u>:-

$A \in M_{n \times n}(C)$, $z \in C \Rightarrow z = a + ib$, $\bar{z} = a - ib$.

$A = [a_{ij}]$; $a_{ij} \in C$,

let $A, B \in M_{n \times n}(C)$

1. $\overline{A+B} = \bar{A} + \bar{B}$
2. $\overline{\alpha A} = \bar{\alpha} \cdot \bar{A}$
3. $\bar{A}^T = \overline{(A^T)}$
4. $\overline{AB} = \bar{A} \cdot \bar{B}$

$\left(\because \overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}\right)$
$\overline{z_1 z_2} = \bar{z_1} \cdot \bar{z_2}$

α is complex number

Note: If λ is an eigen value of A, then $\bar{\lambda}$ is an eigen value of $\bar{A}$.

Soln $Ax = \lambda x,\ x \neq 0$

$\overline{Ax} = \overline{\lambda x}$

$\Rightarrow \bar{A}\cdot\bar{x} = \bar{\lambda}\cdot\bar{x},\ \bar{x} \neq 0.$

$\therefore\ \bar{\lambda}$ is an eigen value of $\bar{A}$.

Defn $\Rightarrow$ $A^* = \bar{A}^T$ $\qquad A = [a_{ij}]$, $A^T = [a_{ji}]$, $\bar{A}^T = [\bar{a}_{ji}]$

then 1. $(A^*)^* = A$

2. $(A+B)^* = A^* + B^*$

3. $(\alpha A)^* = \bar{\alpha}\cdot A^*$

4. $(AB)^* = B^* A^*$

Sol $\Rightarrow$ 1. $(A^*)^* = A$

$(A^*)^* = \left((\bar{A})^T\right)^* = \left(\overline{\bar{A}^T}\right)^T = \left((\bar{\bar{A}})^T\right)^T = (A^T)^T = A$ $\quad [\because z = a+ib,\ \bar{z} = a-ib,\ \bar{\bar{z}} = a+ib = z]$

4. $(AB)^* = (\overline{AB})^T = (\bar{A}\cdot\bar{B})^T = \bar{B}^T\cdot\bar{A}^T$ $\quad [\because (AB)^T = B^T\cdot A^T]$

$= B^*\cdot A^*$

Hermitian matrix: $A \in M_{n\times n}(\mathbb{C})$

$\therefore$ $\boxed{A = A^*} = \bar{A}^T$ $\quad (A = \bar{A}^T)$

A is Hermitian if

A is skew Hermitian if $A = -A^* = -\bar{A}^T$ $\quad (A = -\bar{A}^T)$.

If $A \in M_{n\times n}(\mathbb{R})$ $\quad [\because A = \bar{A}]$

A is symmetric if $A = A^T$

A is antisymmetric if $A = -A^T$.

Ex: $A = \begin{bmatrix} 3 & i \\ -i & -2 \end{bmatrix}$ is ~~skew~~ hermitian

verify $\Rightarrow$ $\bar{A}^T = A$

$\bar{A} = \begin{bmatrix} 3 & -i \\ i & -2 \end{bmatrix}$ $\Rightarrow$ $(\bar{A})^T = \begin{bmatrix} 3 & i \\ -i & -2 \end{bmatrix} = A.$

$[\because$ Hence $\Rightarrow$ $A = (\bar{A})^T]$.

Theorem:

If A is hermitian then, eigen values of A is real. (i.e. $\lambda = \bar{\lambda}$)
$(A = \bar{A}^T)$ $(\lambda \in C)$. — (1)

Proof: Let λ be an eigen value of A, $Ax = \lambda x$, $x \neq 0$.

$a + i0 = a \in R$.

i.e. $z = a + ib$ & $z \in R$, $a, b \in R$

$\therefore z = \bar{z}$

$\therefore a + ib = a - ib \Rightarrow$ $\boxed{b = 0}$

$\therefore Ax = \lambda x$

$\therefore \bar{x}^T A x = \bar{x}^T \lambda x = \lambda \bar{x}^T x$ [~~from (1)~~]

[$\because$ $\lambda \to A$, $\bar{\lambda} \to \bar{A}$, $\bar{\lambda} \to (\bar{A})^T$.]

$\bar{x}^T \bar{A}^T x = \bar{x}^T \bar{A} x$ [$\because \lambda \to A$, $\bar{\lambda} \to \bar{A}$]

$= \bar{x}^T \bar{\lambda} x$

$\therefore$ [$\because \bar{\lambda}$ is an eigen value of $\bar{A}^T$)

$= \bar{\lambda} \bar{x}^T x$.

λ is an eigen value of A then $k.A$ has λ is an eigen value of $k.\lambda$.

Since, $A = \bar{A}^T$

$\therefore \bar{x}^T A x = \bar{x}^T (\bar{A}^T) x$

$\therefore \lambda \bar{x}^T x = \bar{\lambda} \cdot \bar{x}^T \cdot x$.

$\therefore (\lambda - \bar{\lambda}) \bar{x}^T x = 0$. — (2)

Suppose $x \in M_{n \times 1}$, $A = A_{n \times n}$.

$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\bar{x}^T = (\bar{x}_1 \; \bar{x}_2 \; \bar{x}_3)$

$\therefore \bar{x}^T x = (\bar{x}_1 \; \bar{x}_2 \; \bar{x}_3)_{1\times3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1} = |x_1|^2 + |x_2|^2 + |x_3|^2$. [$\because z\bar{z} = |z|^2$]

From (2) $\Rightarrow$ $(\lambda - \bar{\lambda}) \bar{x}^T x = 0$.

$\bar{x}^T x \neq 0$.

Since, $x \neq 0 \Rightarrow$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$\therefore \bar{x}^T x \neq 0 \Rightarrow \lambda - \bar{\lambda} = 0$.

$\therefore \boxed{\lambda = \bar{\lambda}}$

$\therefore \lambda$ is real. $\Rightarrow$ Hence proved.