Digital Signal Processing

Unit-IV

Realization & Design of Digital Filters

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Introduction

- ✓ Systems may be continuous-time systems or discrete-time systems. Discrete-time systems may be FIR (Finite Impulse Response) systems or IIR (Infinite Impulse Response) systems.
- ✓ FIR systems are the systems whose impulse response has finite number of samples and IIR systems are systems whose impulse response has infinite number of samples.
- ✓ Realization of a discrete-time system means obtaining a network corresponding to the difference equation or transfer function of the system.

- ✓ Filters are of two types—FIR and IIR. The type of filters which make use of feedback connection to get the desired filter implementation are known as recursive filters.
- ✓ Their impulse response is of infinite duration. So they are called IIR filters. The type of filters which do not employ any kind of feedback connection are known as non-recursive filters.
- ✓ Their impulse response is of finite duration. So they are called FIR filters. IIR filters are designed by considering all the infinite samples of the impulse response. The impulse response is obtained by taking inverse Fourier transform of ideal frequency response.

- ✓ There are several techniques available for the design of digital filters having an infinite duration unit impulse response.
- ✓ The popular methods for such filter design uses the technique of first designing the digital filter in analog domain and then transforming the analog filter into an equivalent digital filter because the analog filter design techniques are well developed.

Linear Time-Invariant Systems as Frequency-Selective Filters

- ✓ The term *filter* is commonly used to describe a device that discriminates, according to some attribute of the objects applied at its input, what passes through it.
- ✓ A linear time-invariant system also performs a type of discrimination or filtering among the various frequency components at its input.
- ✓ The nature of this filtering action is determined by the frequency response characteristics $H(\omega)$, which in turn depends on the choice of the system parameters.
- ✓ Thus, by proper selection of the coefficients, we can design frequency-selective filters that pass signals with frequency components in some bands while they attenuate signals containing frequency components in other frequency bands.

- ✓ In general, a linear time-invariant system modifies the input signal spectrum $X(\omega)$ according to its frequency response $H(\omega)$ to yield an output signal with spectrum $Y(\omega) = H(\omega)X(\omega)$.
- ✓ In a sense, $H(\omega)$ acts as a *weighting function* or a *spectral shaping function* to the different frequency components in the input signal.
- ✓ When viewed in this context, any linear time-invariant system can be considered to be a frequency-shaping filter, even though it may not necessarily completely block any or all frequency components.
- ✓ Filtering is used in digital signal processing in a variety of ways, such as removal of undesirable noise from desired signals, spectral shaping such as equalization of communication channels, signal detection in radar, sonar, and communications, and for performing spectral analysis of signals, and so on.

Filtering is a process by which the frequency spectrum of a signal can be modified reshaped or manipulated to achieve some desired objectives. The objectives are

- ❖ To eliminate noise which may be contaminated in a signal.
- ❖ To remove signal distortion which may be due to imperfection transmission channel.
- ❖ To separate two or more distinct signals which were purposely mixed for maximising channel utilization.
- ❖ To resolve signals into their frequency components.
- ❖ To demodulated the signals which were modulated at the transmitter end.
- ❖ To convert digital signals into analog signals.
- ❖ To limit the bandwidth of signals.

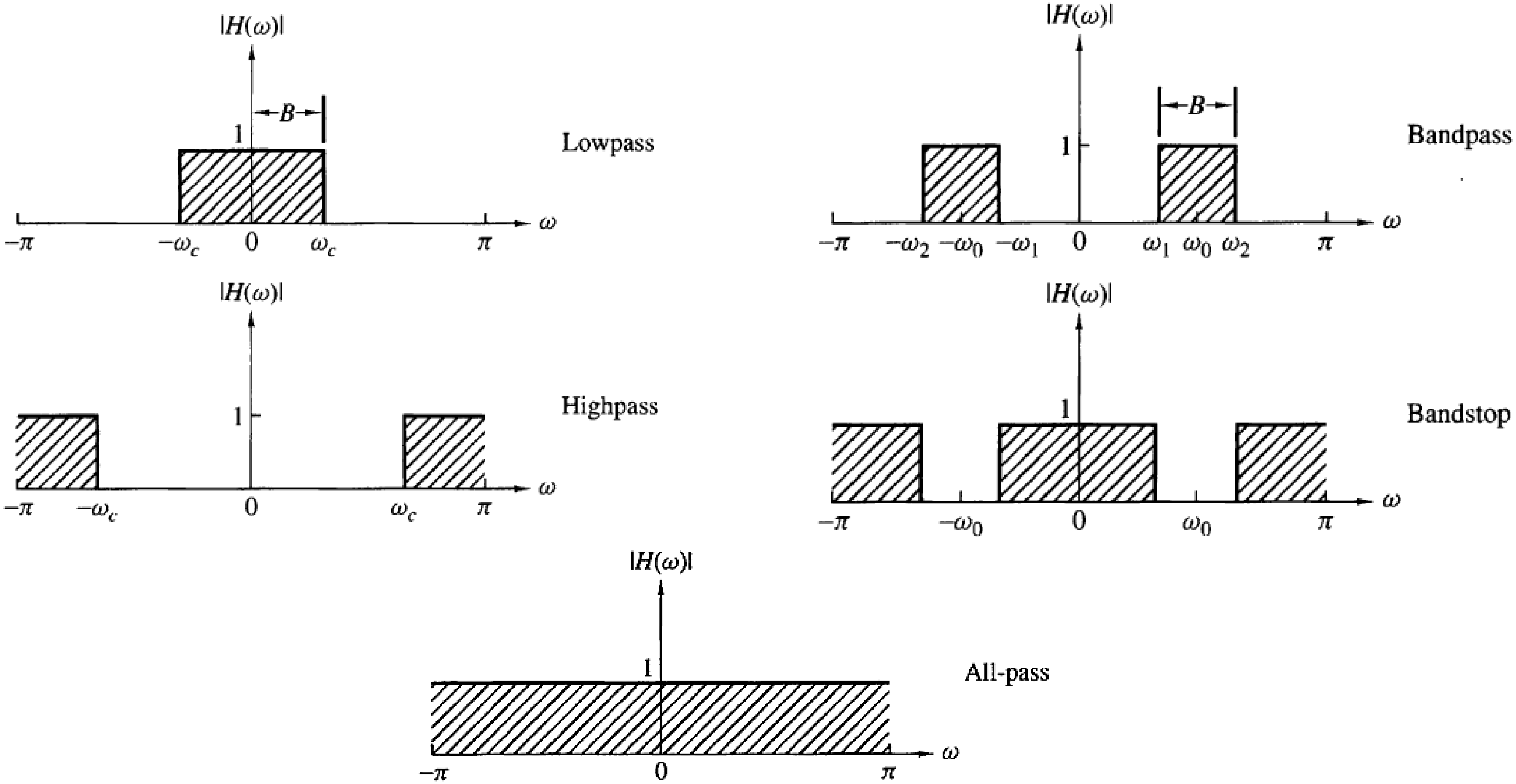
Types of Filters

Analog filters may be classified as either passive or active and are usually implemented with R, L, and C components and operational amplifiers.

Digital filters are implemented using a digital computer or special purpose digital hardware.

Ideal Filter Characteristics

✓ Filters are usually classified according to their frequency-domain characteristics as lowpass, highpass, bandpass, and bandstop or band-elimination filters.



- ✓ Consequently, the filter output is simply a delayed and amplitude-scaled version of the input signal.
- ✓ A pure delay is usually tolerable and is not considered a distortion of the signal. Neither is amplitude scaling. Therefore, ideal filters have a linear phase characteristic within their passband, that is,

$$\Theta(\omega) = -\omega n_0$$

- ✓ The derivative of the phase with respect to frequency has the units of delay. Hence we can define the signal delay as a function of frequency as

$$\tau_g(\omega) = -\frac{d\Theta(\omega)}{d\omega}$$

- ✓ $\tau_g(\omega)$ is usually called the *envelope delay* of the *group delay* of the filter. We interpret $\tau_g(\omega)$ as the time delay that a signal component of frequency ω undergoes as it passes from the input to the output of the system. Note that when $\Theta(\omega)$ is linear, $\tau_g(\omega) = n_o$ = constant.

- ✓ In this case all frequency components of the input signal undergo the same time delay.
- ✓ In conclusion, ideal filters have a constant magnitude characteristic and a linear phase characteristic within their passband.
- ✓ In all cases, such filters are not physically realizable but serve as a mathematical idealization of practical filters.

Comparison between Analog and Digital Filters

Analog Filters

- ✓ In analog filters both input and output are continuous time signals.
- ✓ Implementation of such filters is carried out using passive components and active components.
- ✓ Analog filters operate in infinite frequency range theoretically but limited in practice by finite maximum operating frequencies of the semiconductor devices used.

Digital Filters

- ✓ In analog filters both input and output are discrete time signals.
- ✓ Digital filters are implemented on a digital computer or micro computer using DSP integrated circuits. Three basic elements such as adder, multiplier and delay elements are utilized for implementing digital filters.
- ✓ In case of digital filters, frequency range is restricted to half of the sampling rate. It is also restricted to by maximum computational speed available in a particular application.

Analog Filters

- ✓ Main disadvantages of analog filters are its higher noise sensitivity, non linearities, dynamic range limitations, lack of flexibility in designing and reproductively, errors generated due to drift and variations in the value of active and passive components used in circuits.

- ✓ Analog filters have higher frequency range of operation as well as they can interact directly with real analog world.

Digital Filters

- ✓ Digital filters require additional A/D and D/A converter sections for connecting to the physical analog world.

- ✓ Main advantages of digital filters are that these are insensitive to noise, higher linearity, unlimited dynamic range flexibility in software design, high accuracy, reliability is higher.

Comparison of FIR and IIR Digital Filters

FIR	IIR
1. FIR (Finite Impulse Response) (non-recursive) filters produce zeros.	1. IIR (infinite Impulse Response) (recursive) filters produce poles.
2. In signal processing, a finite impulse response (FIR) filter is a filter whose impulse response (or response to any finite length input) is of finite duration, because it settles to zero in finite time.	2. This is in contrast to infinite impulse response (IIR) filters, which may have internal feedback and may continue to respond indefinitely (usually decaying).
3. Filters combining both past inputs and past outputs can produce both poles and zeros.	3. IIR filters are difficult to control and have no particular phase.
4. FIR filters can be discrete- time or continuous-time, and digital or analog.	4. IIR is derived from analog.
5. FIR filters are dependent upon linear-phase characteristics.	5. IIR filters are used for applications which are not linear.
6. FIR is always stable	6. IIR can be unstable
7. FIR has no limited cycles.	7. IIR filters make polyphase implementation possible.
8. *FIR has no analog history.*	8. IIR filters can become difficult to implement, and also delay and distortion adjustments can alter the poles & zeroes, which make the filters unstable.
9. FIR is dependent upon i/p only.	9. IIR filters are dependent on both i/p and o/p.
10. FIR's delay characteristics is much better, but they require more memory.	10. IIR filters consist of zeros and poles, and require less memory than FIR filters.
11. FIR filters are used for tapping of a higher-order.	11. IIR filters are better for tapping of lower-orders, since IIR filters may become unstable with tapping higher-orders.

Introduction to Z-transform

- ✓ A linear time-invariant discrete-time system is represented by difference equations. The direct solution of higher order difference equations is quite tedious and time consuming. So usually they are solved by indirect methods.
- ✓ The Z-transform plays the same role for discrete-time systems as that played by Laplace transform for continuous-time systems.
- ✓ The Z-transform is the discrete-time counterpart of the Laplace transform.
- ✓ It is the Laplace transform of the discretized version of the continuous-time signal $x(t)$.

- ✓ To solve the difference equations which are in time domain, they are converted first into algebraic equations in *z*-domain using Z-transform, the algebraic equations are manipulated in *z*-domain and the result obtained is converted back into time domain using inverse Z-transform.
- ✓ The Z-transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and the initial conditions can be introduced in the beginning of the process itself.
- ✓ The Z-transform plays an important role in the analysis and representation of discrete-time Linear Shift Invariant (LSI) systems. It is the generalization of the Discrete-Time Fourier Transform (DTFT).

- ✓ The Z-transform may be one-sided (unilateral) or two-sided (bilateral). It is the one-sided or unilateral Z-transform that is more useful, because we mostly deal with causal sequences. Further, it is eminently suited for solving difference equations with initial conditions.
- ✓ The bilateral or two-sided Z-transform of a discrete-time signal or a sequence x(n) is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$ where *z* is a complex variable.

- ✓ The one-sided or unilateral Z-transform is defined as:

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

- ✓ If $x(n) = 0$, for $n < 0$, the one-sided and two-sided Z-transforms are equivalent.

Region of Convergence (ROC)

$$X(z)\Big|_{z=re^{j\omega}} = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\omega n}$$

which is the Fourier transform of the modified sequence $[x(n)r^{-n}]$. If $r = 1$, i.e. $|z| = 1$, $X(z)$ reduces to its Fourier transform. The series of the above equation converges if $x(n)r^{-n}$ is absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} \left|x(n)r^{-n}\right| < \infty$$

- ✓ If the output signal magnitude of the digital signal system, x(n), is to be finite, then the magnitude of its z-transform, X(z), must be finite.
- ✓ The set of z values in the z-plane, for which the magnitude of X(z) is finite, is called the Region of Convergence (ROC).

Important Properties of the ROC for the z-Transform

(i) $X(z)$ converges uniformly if and only if the ROC of the z-transform $X(z)$ of the sequence includes the unit circle. The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin. That is, the ROC of the z-transform of $x(n)$ has values of z for which $x(n)\ r^{-n}$ is absolutely summable.

$$\sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right| < \infty$$

(ii) The ROC does not contain any poles.

(iii) When $x(n)$ is of finite duration, then the ROC is the entire z-plane, except possibly $z = 0$ and / or $z = \infty$.

(iv) If $x(n)$ is a right-sided sequence, the ROC will not include infinity.

(v) If $x(n)$ is a left-sided sequence, the ROC will not include $z = 0$. However, if $x(n) = 0$ for all $n > 0$, the ROC will include $z = 0$.

(vi) If $x(n)$ is two-sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$. That is, the ROC includes the intersection of the ROC's of the components.

(vii) If $X(z)$ is rational, then the ROC extends to infinity, i.e. the ROC is bounded by poles.

(viii) If $x(n)$ is causal, then the ROC includes $z = \infty$.

(ix) If $x(n)$ is anti-causal, then the ROC includes $z = 0$.

Sequence $x(n)$	Z-transform $X(z)$	ROC
1. $\delta(n)$	1	All z
2. $u(n)$	$z/(z-1) = 1/(1-z^{-1})$	$\lvert z \rvert > 1$
3. $u(-n)$	$\dfrac{1}{1-z} = -\dfrac{1}{z-1} = -\dfrac{z^{-1}}{1-z^{-1}}$	$\lvert z \rvert < 1$
4. $u(-n-1)$	$z/(z-1) = 1/(1-z^{-1})$	$\lvert z \rvert < 1$
5. $u(-n-2)$	$z^2/(z-1)$	$\lvert z \rvert < 1$
6. $u(-n-k)$	$z^k/(z-1)$	$\lvert z \rvert < 1$
7. $\delta(n-k)$	z^{-k}	All z except at $z=0$ (if $k>0$) All z except at $z=\infty$ (if $k<0$)
8. $a^n u(n)$	$(z/(z-a)) = 1/(1-az^{-1})$	$\lvert z\rvert > \lvert a\rvert$
9. $-a^n u(-n)$	$a/(z-a)$	$\lvert z\rvert < \lvert a\rvert$
10. $-a^n u(-n-1)$	$z/(z-a) = 1/(1-az^{-1})$	$\lvert z\rvert < \lvert a\rvert$
11. $nu(n)$	$-z/(z-1)^2 = -[z^{-1}/(1-z^{-1})^2]$	$\lvert z\rvert > 1$
12. $na^n u(n)$	$az/(z-a)^2 = az^{-1}/(1-az^{-1})^2$	$\lvert z\rvert > \lvert a\rvert$
13. $-na^n u(-n-1)$	$az/(z-a)^2 = az^{-1}/(1-az^{-1})^2$	$\lvert z\rvert < \lvert a\rvert$
14. $e^{-j\omega n} u(n)$	$z/(z-e^{-j\omega}) = 1/(1-z^{-1}e^{-j\omega})$	$\lvert z\rvert > \lvert 1\rvert$
15. $\cos \omega n\, u(n)$	$\dfrac{z(z-\cos\omega)}{z^2-2z\cos\omega+1} = \dfrac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\lvert z\rvert > 1$
16. $\sin \omega n\, u(n)$	$\dfrac{z\sin\omega}{z^2-2z\cos\omega+1} = \dfrac{z^{-1}\sin\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\lvert z\rvert > 1$
17. $a^n \cos \omega n\, u(n)$	$\dfrac{z(z-a\cos\omega)}{z^2-2az\cos\omega+a^2} = \dfrac{1-z^{-1}a\cos\omega}{1-2az^{-1}\cos\omega+a^2 z}$	$\lvert z\rvert > \lvert a\rvert$
18. $a^n \sin \omega n\, u(n)$	$\dfrac{az\sin\omega}{z^2-2az\cos\omega+a^2} = \dfrac{az^{-1}\sin\omega}{1-2az^{-1}\cos\omega+a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
19. $(n+1)a^n u(n)$	$z^2/(z-a)^2 = 1/(1-az^{-1})^2$	$\lvert z\rvert > \lvert a\rvert$
20. $-nu(-n-1)$	$z/(z-1)^2 = z^{-1}/(1-z^{-1})^2$	$\lvert z\rvert < \lvert 1\rvert$
21. $na^n u(n)$	$z/(z-a)^2$	$\lvert z\rvert > \lvert a\rvert$
22. $[n(n-1)a^{n-2}u(n)]/2!$	$z/(z-a)^3$	$\lvert z\rvert > \lvert a\rvert$
23. $\dfrac{n(n-1)\ldots[n-(k-2)]a^{n-k+1}}{(k-1)!}u(n)$	$z/(z-a)^k$	$\lvert z\rvert > \lvert a\rvert$
24. $1/n,\quad n>0$	$-\ln(1-z^{-1})$	$\lvert z\rvert > \lvert 1\rvert$
25. $n^k a^n,\ \ k<0$	$-\left(-z\dfrac{d}{dz}\right)^n \dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
26. $a^{\lvert n\rvert}$ for all n	$(1-a^2)/[(1-az)(1-az^{-1})]$	$\lvert a\rvert < \lvert z\rvert < \lvert 1/a\rvert$

Advantages of Z-transform

- ✓ The Z-transform converts the difference equations of a discrete-time system into linear algebraic equations so that the analysis becomes easy and simple.
- ✓ Convolution in time domain is converted into multiplication in *z*-domain.
- ✓ Z-transform exists for most of the signals for which Discrete-Time Fourier Transform (DTFT) does not exist.
- ✓ Also since the Fourier transform is nothing but the Z-transform evaluated along the unit circle in the z-plane, the frequency response can be determined.

Inverse Z-transform

✓ The process of finding the time domain signal $x(n)$ from its Z-transform $X(z)$ is called the inverse Z-transform which is denoted as:

$$x(n) = Z^{-1}[X(z)]$$

We have

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} [x(n)\, r^{-n}]\, e^{-j\omega n}$$

✓ Basically, there are four methods that are often used to find the inverse Z-transform. They are:

a) Power series method or long division method
b) Partial fraction expansion method
c) Complex inversion integral method (also known as the residue method)
d) Convolution integral method

Solution of Difference Equations of Digital Filters

- ✓ To solve the difference equation, first it is converted into algebraic equation by taking its Z-transform. The solution is obtained in *z*-domain and the time domain solution is obtained by taking its inverse Z-transform.
- ✓ The system response has two components. The source free response and the forced response.
- ✓ The response of the system due to input alone when the initial conditions are neglected is called the forced response of the system. It is also called the steady state response of the system. It represents the component of the response due to the driving force.

- ✓ The response of the system due to initial conditions alone when the input is neglected is called the free or natural response of the system. It is also called the transient response of the system. It represents the component of the response when the driving function is made zero.
- ✓ The response due to input and initial conditions considered simultaneously is called the total response of the system.
- ✓ For a stable system, the source free component always decays with time. In fact a stable system is one whose source free component decays with time.
- ✓ For this reason the source free component is also designated as the transient component and the component due to source is called the steady state component.
- ✓ When input is a unit impulse input, the response is called the impulse response of the system and when the input is a unit step input, the response is called the step response of the system.

Example: A linear shift invariant system is described by the difference equation $$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n)+x(n-1)$$
with y(–1) = 0 and y(–2) = –1.

Find (a) the natural response of the system (b) the forced response of the system for a step input and (c) the frequency response of the system.

Solution:

(a) The natural response is the response due to initial conditions only. So make $x(n)$ = 0. Then the difference equation becomes

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=0$$

Taking Z-transform on both sides, we have

$$Y(z)-\frac{3}{4}\,[z^{-1}\,Y(z)+y(-1)]+\frac{1}{8}\,[z^{-2}\,Y(z)+z^{-1}y(-1)+y(-2)]=0$$

i.e. $$Y(z)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)-\frac{1}{8}=0$$

$$\therefore\quad Y(z)=\frac{1/8}{1-(3/4)z^{-1}+(1/8)z^{-2}}=\frac{1/8z^2}{z^2-(3/4)z+(1/8)}=\frac{1/8z^2}{[z-(1/2)]\,[z-(1/4)]}$$

The partial fraction expansion of $Y(z)/z$ gives

$$\frac{Y(z)}{z}=\frac{(1/8)z}{[z-(1/2)]\,[z-(1/4)]}=\frac{A}{z-(1/2)}+\frac{B}{z-(1/4)}=\frac{1/4}{z-(1/2)}-\frac{1/8}{z-(1/4)}$$

$$\therefore\qquad Y(z)=\frac{1}{4}\frac{z}{z-(1/2)}-\frac{1}{8}\frac{z}{z-(1/4)}$$

Taking inverse Z-transform on both sides, we get the natural response as:

$$y(n)=\frac{1}{4}\left(\frac{1}{2}\right)^n u(n)-\frac{1}{8}\left(\frac{1}{4}\right)^n u(n)$$

(b) To find the forced response due to a step input, put $x(n) = u(n)$. So we have

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = u(n) + u(n-1)$$

We know that the forced response is due to input alone. So for forced response, the initial conditions are neglected. Taking Z-transform on both sides of the above equation and neglecting the initial conditions, we have

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = U(z) + z^{-1}U(z) = \frac{z}{z-1} + \frac{1}{z-1}$$

i.e.
$$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = \frac{z+1}{z-1}$$

$$\therefore \quad Y(z) = \frac{z+1}{(z-1)\,[1-(3/4)z^{-1} + (1/8)z^{-2})} = \frac{z^2(z+1)}{(z-1)[z^2 - (3/4)z + (1/8)]}$$

$$= \frac{z^2(z+1)}{(z-1)\,[z-(1/2)]\,[z-(1/4)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\therefore \quad \frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)\,[z-(1/2)]\,[z-(1/4)]} = \frac{A}{z-1} + \frac{B}{z-(1/2)} + \frac{C}{z-(1/4)}$$

$$= \frac{16/3}{z-1} - \frac{6}{z-(1/2)} + \frac{5/3}{z-(1/4)}$$

$$\text{or} \quad Y(z) = \frac{16}{3}\left(\frac{z}{z-1}\right) - 6\left[\frac{z}{z-(1/2)}\right] + \frac{5}{3}\left[\frac{z}{z-(1/4)}\right]$$

Taking the inverse Z-transform on both sides, we have the forced response for a step input.

$$y(n) = \frac{16}{3}u(n) - 6\left(\frac{1}{2}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{4}\right)^n u(n)$$

(c) The frequency response of the system $H(\omega)$ is obtained by putting $z = e^{j\omega}$ in $H(z)$.

Here
$$H(z) = \frac{Y(z)}{X(z)} = \frac{z(z+1)}{z^2 - (3/4)z + (1/8)}$$

Therefore,
$$H(\omega) = \frac{e^{j\omega}(e^{j\omega}+1)}{(e^{j\omega})^2 - (3/4)e^{j\omega} + (1/8)}$$

Block diagram representation of linear constant – Coefficient difference equations

- ✓ The convolution sum description of an LTI discrete-time system be used , can in principle, to implement the system.
- ✓ For an IIR finite-dimensional system this approach is not practical as here the impulse response is of infinite length.
- ✓ However, a direct implementation of the IIR finite-dimensional system is practical.
- ✓ Here the input-output relation involves a finite sum of products:

$$y[n] = -\sum_{k=1}^{N} a_k \, y[n-k] + \sum_{k=0}^{M} b_k \, x[n-k]$$

- ✓ On the other hand, an FIR system can be implemented using the convolution sum which is a finite sum of products:

$$y[n]=\sum_{k=0}^{N}h[k]\,x[n-k]$$

- ✓ The actual implementation of an LTI digital filter can be either in software or hardware form, depending on applications.
- ✓ In either case, the signal variables and the filter coefficients cannot represented with finite precision.
- ✓ However, a direct implementation of a digital filter based on either the difference equation or the finite convolution sum may not provide satisfactory performance due to the finite precision arithmetic.
- ✓ It is thus of practical interest to develop alternate realizations and choose the structure that provides satisfactory performance under finite precision arithmetic.

- ✓ A structural representation using interconnected basic building blocks is the first step in the hardware or software implementation of an LTI digital filter..
- ✓ The structural representation provides the key relations between some pertinent internal variables with the input and output that in turn provides the key to the implementation.

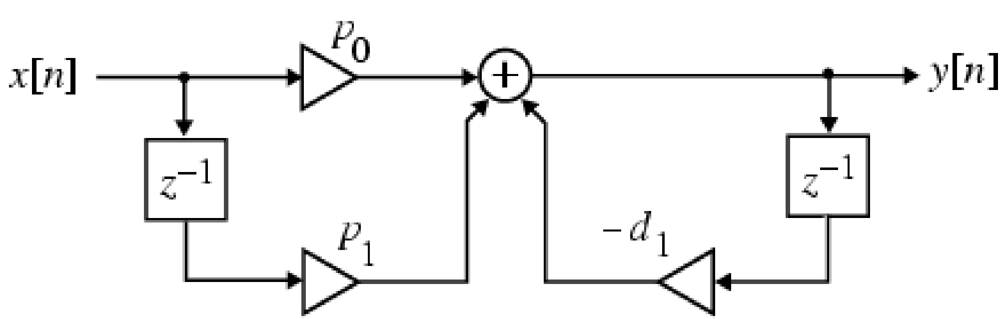
- ✓ In the time domain, the input-output relations of an LTI digital filter is given by the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

or, by the linear constant coefficient difference equation

$$y[n] = -\sum_{k=1}^{N} d_k\, y[n-k] + \sum_{k=0}^{M} p_k\, x[n-k]$$

- ✓ For the implementation of an LTI digital filter, the input-output relationship must be described by a valid computational algorithm
- ✓ To illustrate what we mean by a computational algorithm, consider the causal first-order LTI digital filter shown below



- ✓ The filter is described by the difference equation

$$y[n] = -d_1 y[n-1] + p_0 x[n] + p_1 x[n-1]$$

- ✓ Using the above equation we can compute $y[n]$ for $n \geq 0$ knowing the initial condition $y[-1]$ and the input $x[n]$ for $n \geq -1$:

$$y[0] = -d_1 y[-1] + p_0 x[0] + p_1 x[-1]$$

$$y[1] = -d_1 y[0] + p_0 x[1] + p_1 x[0]$$

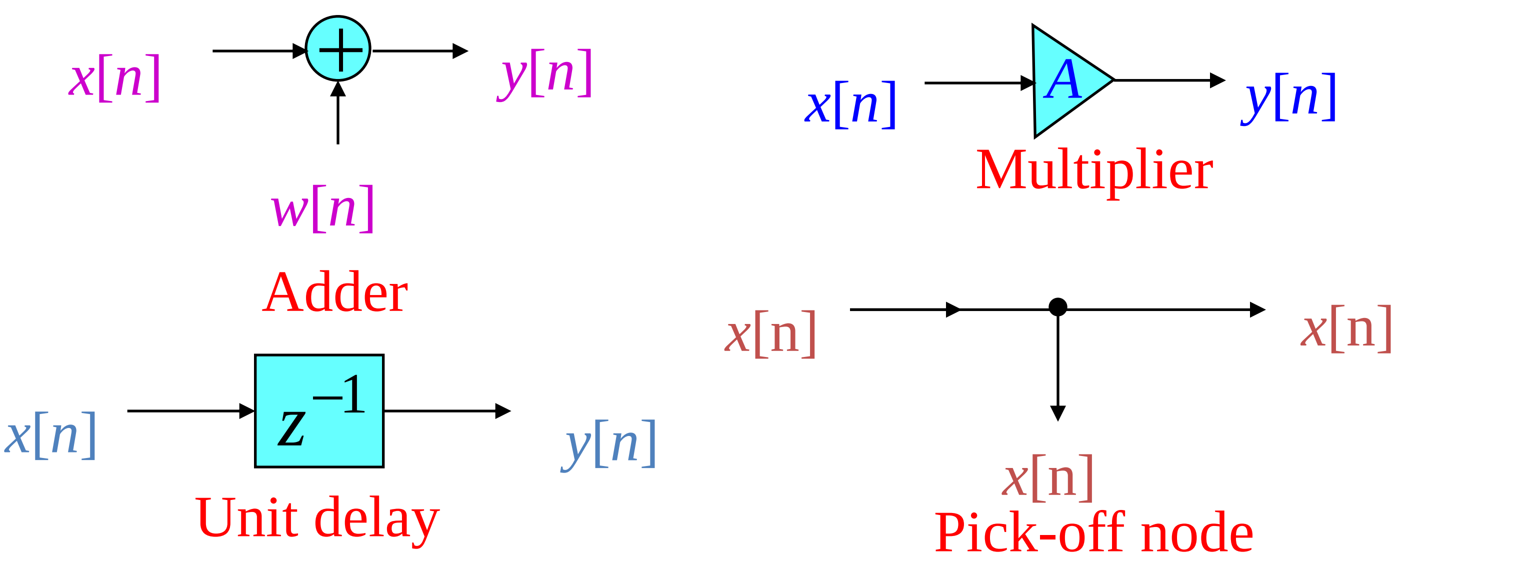
$$y[2] = -d_1 y[1] + p_0 x[2] + p_1 x[1]$$

$$\vdots$$

- ✓ We can continue this calculation for any value of the time index n we desire.

- ✓ Each step of the calculation requires a knowledge of the previously calculated value of the output sample (delayed value of the output), the present value of the input sample, and the previous value of the input sample (delayed value of the input).
- ✓ As a result, the first-order difference equation can be interpreted as a valid computational algorithm.

- ✓ The computational algorithm of an LTI digital filter can be conveniently represented in block diagram form using the basic building blocks shown below



Advantages of block diagram representation

(1) Easy to write down the computational algorithm by inspection

(2) Easy to analyze the block diagram to determine the explicit relation between the output and input

(3) Easy to manipulate a block diagram to derive other “equivalent” block diagrams yielding different computational algorithms

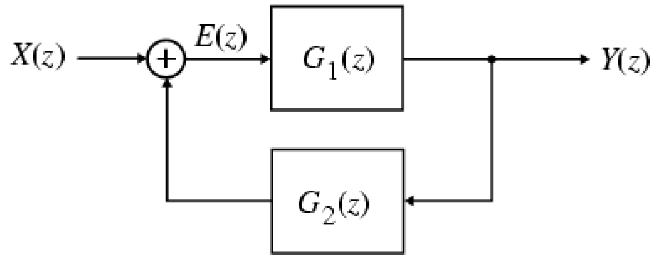
(4) Easy to determine the hardware requirements

(5) Easier to develop block diagram representations from the transfer function directly

Analysis of Block Diagrams

- ✓ Carried out by writing down the expressions for the output signals of each adder as a sum of its input signals, and developing a set of equations relating the filter input and output signals in terms of all internal signals
- ✓ Eliminating the unwanted internal variables then results in the expression for the output signal as a function of the input signal and the filter parameters that are the multiplier coefficients

- Example - Consider the single-loop feedback structure shown below



- The output $E(z)$ of the adder is

$$E(z) = X(z) + G_2(z)Y(z)$$

- But from the figure

$$Y(z) = G_1(z)E(z)$$

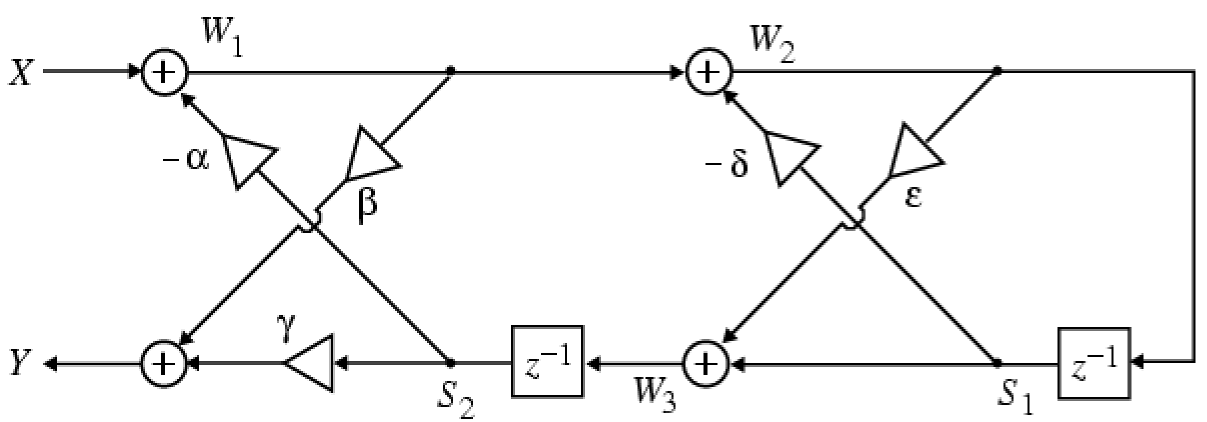
- Eliminating $E(z)$ from the previous two equations we arrive at

$$[1 - G_1(z)G_2(z)]Y(z) = G_1(z)X(z)$$

which leads to

$$H(z) = \frac{Y(z)}{X(z)} = \frac{G_1(z)}{1 - G_1(z)G_2(z)}$$

- <u>Example</u> - Analyze the cascaded lattice structure shown below where the z-dependence of signal variables are not shown for brevity



- The output signals of the four adders are given by

$$W_1 = X - \alpha S_2$$
$$W_2 = W_1 - \delta S_1$$
$$W_3 = S_1 + \varepsilon W_2$$
$$Y = \beta W_1 + \gamma S_2$$

- From the figure we observe

$$S_2 = z^{-1} W_3$$
$$S_1 = z^{-1} W_2$$

- Substituting the last two relations in the first four equations we get

$$W_1 = X - \alpha z^{-1} W_3$$
$$W_2 = W_1 - \delta z^{-1} W_2$$
$$W_3 = z^{-1} W_2 + \varepsilon W_2$$
$$Y = \beta W_1 + \gamma z^{-1} W_3$$

- From the second equation we get $W_2 = W_1 / (1 + \delta z^{-1})$ and from the third equation we get $W_3 = (\varepsilon + z^{-1}) W_2$

- Combining the last two equations we get

$$W_3 = \frac{\varepsilon + z^{-1}}{1 + \delta z^{-1}} W_1$$

- Substituting the above equation in

$$W_1 = X - \alpha z^{-1} W_3, \; Y = \beta W_1 + \gamma z^{-1} W_3$$

we finally arrive at

$$H(z) = \frac{Y}{X} = \frac{\beta + (\beta\delta + \gamma\varepsilon) z^{-1} + \gamma z^{-2}}{1 + (\delta + \alpha\varepsilon) z^{-1} + \alpha z^{-2}}$$

Basic structures of IIR systems

Introduction

- ✓ IIR systems are systems whose impulse response has infinite number of samples. They are designed by using all the samples of the infinite duration impulse response.
- ✓ The convolution formula for IIR systems is given by

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

- ✓ Since this weighted sum involves the present and all the past input samples, we can say that the IIR system has an infinite memory.
- ✓ A system whose output y(n) at time *n* depends on the present input and any number of past values of input and output is called a recursive system.

- ✓ The past outputs are

$$y(n-1),\ y(n-2),\ y(n-3),\ \ldots$$

Hence, for recursive system, the output $y(n)$ is given by

$$y(n) = \mathrm{F}[y(n-1), y(n-2), \ldots, y(n-N), x(n), x(n-1), \ldots, x(n-M)]$$

- ✓ In recursive system, in order to compute $y(n_0)$, we need to compute all the previous values $y(0)$, $y(1)$, $y(2)$, ..., $y(n_0 - 1)$ before calculating $y(n_0)$.
- ✓ Hence, output of recursive system has to be computed in order $[y(0), y(1), y(2), \ldots]$.

Transfer function of IIR systems

✓ In general, an IIR system is described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

i.e. in general, IIR systems are those in which the output at any instant of time depends not only on the present and past inputs but also on the past outputs. Hence, in general, an IIR system is of recursive type.

✓ On taking Z-transform of the above equation for *y*(*n*), we get

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

i.e.

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

- ✓ The system function or the transfer function of the IIR system is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

- ✓ The above equations for $Y(z)$ and $H(z)$ can be viewed as a computational procedure (or algorithm) to determine the output sequence $y(n)$ from the input sequence $x(n)$.
- ✓ The computations in the above equation can be arranged into various equivalent sets of difference equations with each set of equations defining a computational procedure or algorithm for implementing the system.

- ✓ For each set of equations, we can construct a block diagram consisting of delays, adders and multipliers. Such block diagrams are referred to as realization of the system or equivalently as structure for realizing the system.
- ✓ The main advantage of re-arranging the sets of difference equations (i.e. the main criteria for selecting a particular structure) is to reduce the computational complexity, memory requirements and finite word length effects in computations.
- ✓ So the factors that influence the choice of structure for realization of LTI system are: computational complexity, memory requirements and finite word length effects in computations.
- ✓ Computational complexity refers to the number of arithmetic operations required to compute the output value $y(n)$ for the system.

- ✓ Memory requirements refer to the number of memory locations required to store the system parameters, past inputs and outputs and any intermediate computed values.
- ✓ Finite-word-length effects or finite precision effects refer to the quantization effects that are inherent in any digital implementation of the system either in hardware or in software.
- ✓ Although the above three factors play a major role in influencing our choice of the realization of the system, other factors such as whether the structure lends itself to parallel processing or whether the computations can be pipelined may play a role in selecting a specific structure.

The different types of structures for realizing IIR systems are:

- ✓ Direct form-I structure
- ✓ Direct form-II structure
- ✓ Transposed form structure
- ✓ Cascade form structure
- ✓ Parallel form structure
- ✓ Lattice structure
- ✓ Ladder structure

Direct form-I structure

- ✓ Direct form-I realization of an IIR system is nothing, but the direct implementation of the difference equation or transfer function.
- ✓ It is the simplest and most straight forward realization structure available.
- ✓ The difference equation governing the behaviour of an IIR system is

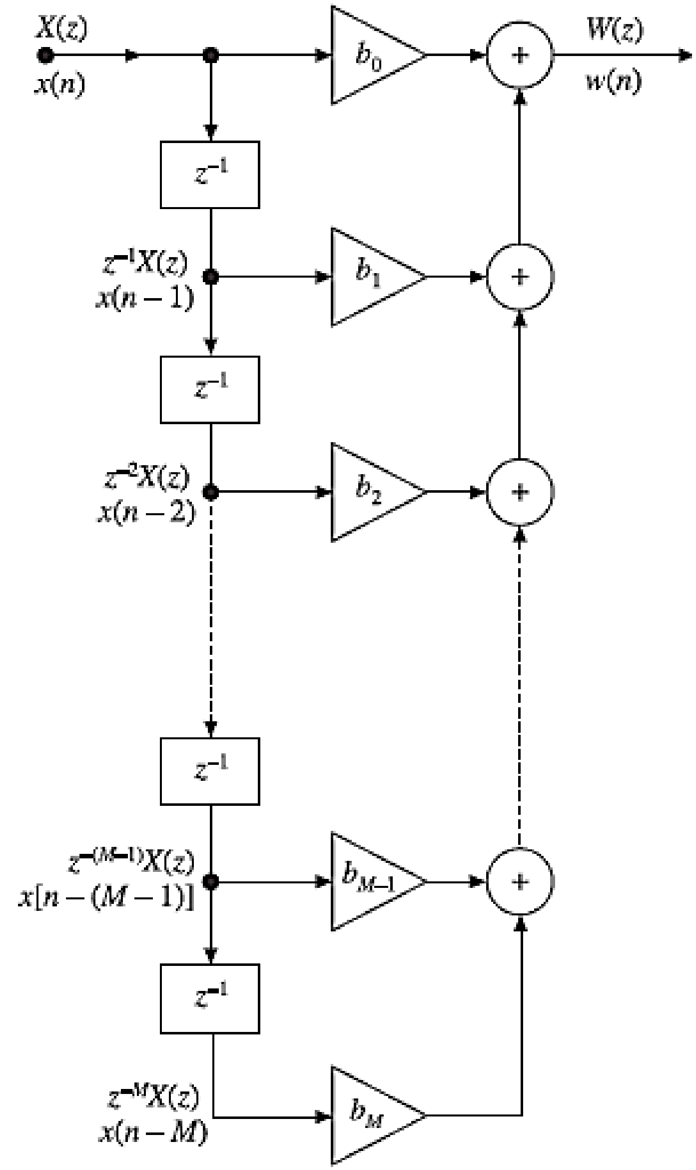
$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

i.e. $y(n) = -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$

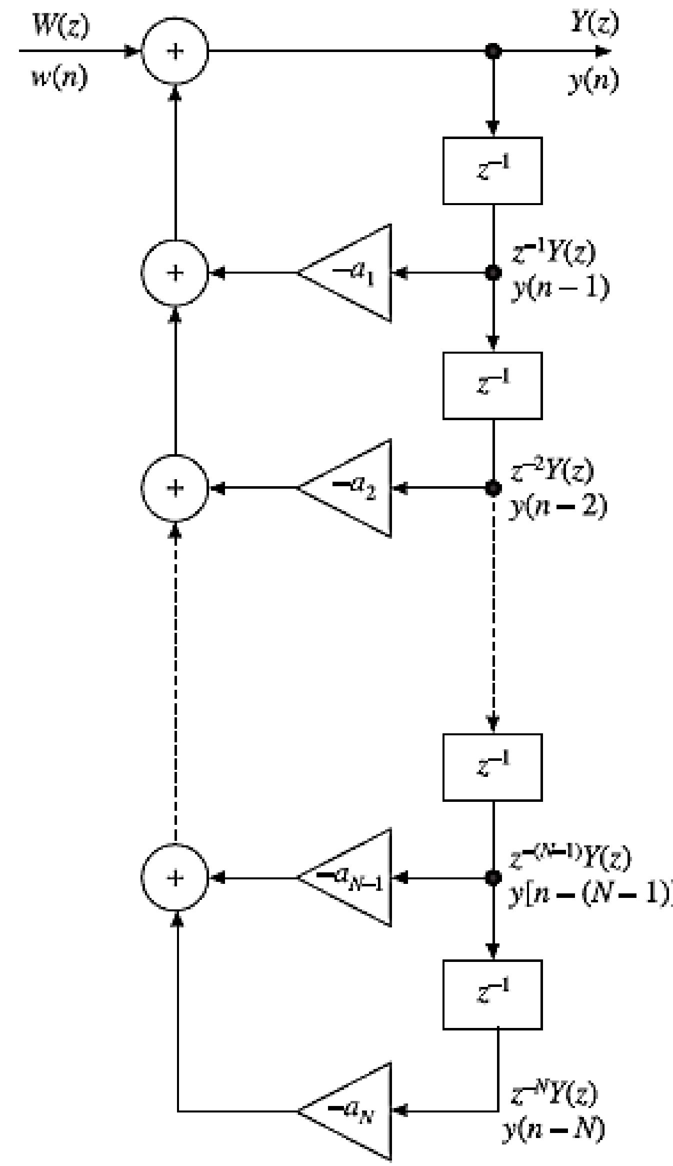
On taking the Z-transform of the above equation for $y(n)$, we get

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

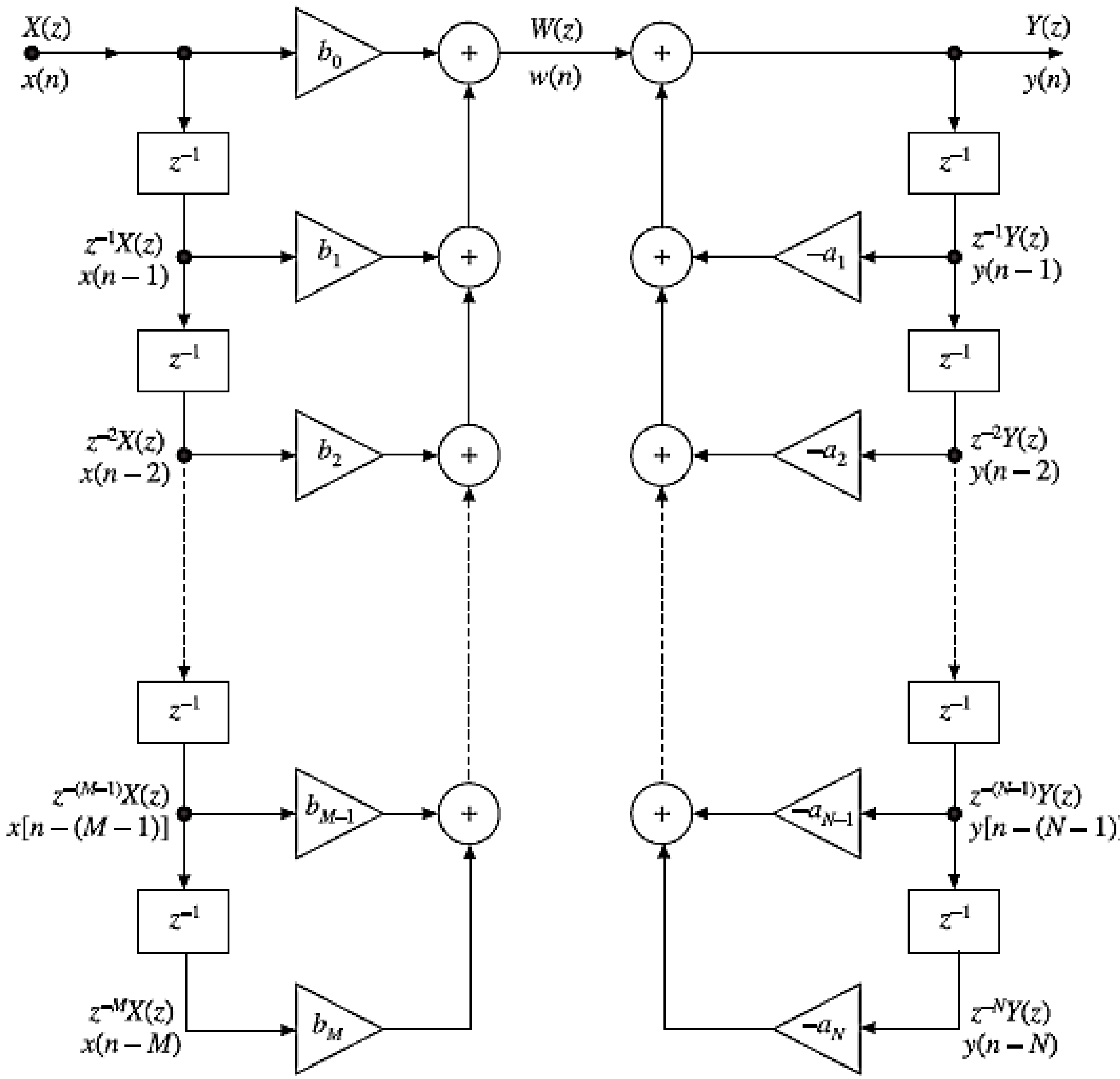
$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$



$$Y(z) = -a_1 z^{-1}Y(z) - a_2 z^{-2}Y(z) - \cdots - a_N z^{-N}Y(z) + b_0 X(z) + b_1 z^{-1}X(z) + \cdots + b_M z^{-M}X(z)$$



The equation for $Y(z)$ [or $y(n)$] can be directly represented by a block diagram as shown in Figure and this structure is called Direct form-I structure.



- ✓ This structure uses separate delays (z^{-1}) for input and output samples. Hence, for realizing this structure more memory is required.
- ✓ The direct form structure provides a direct relation between time domain and z-domain equations.
- ✓ The structure shown in Figure is called a ***non-canonical structure*** because the number of delay elements used is more than the order of the difference equation.
- ✓ If the IIR system is more complex, that is of higher order, then introduce an intermediate variable W(z) so that

$$W(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

or $$w(n) = \sum_{k=0}^{M} b_k x(n-k) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-m)$$

$\therefore$ $$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots + W(z)$$

or $$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \cdots + w(n)$$

- ✓ So, the direct form-I structure is in two parts.
- ✓ The first part contains only **zeros** [that is, the input components either $x(n)$ or $X(z)$] and the second part contains only **poles** [that is, the output components either $y(n)$ or $Y(z)$].
- ✓ In direct form-I, the zeros are realized first and poles are realized second.

Limitations of Direct form-I

- ✓ Since the number of delay elements used in direct form-I is more than (double) the order of the difference equation, it is not effective.
- ✓ It lacks hardware flexibility.
- ✓ There are chances of instability due to the quantization noise.

Example : Realize an IIR system

$$y(n) + 2y(n-1) + 3y(n-2) = 4x(n) + 5x(n-1) + 6x(n-2)$$

using the direct form-I structure.

Solution: Taking Z-transform on both sides of the given difference equation and neglecting initial conditions, we get

$$Y(z) + 2z^{-1}Y(z) + 3z^{-2}Y(z) = 4X(z) + 5z^{-1}X(z) + 6z^{-2}X(z)$$

Therefore, the transfer function of the given IIR system is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{4 + 5z^{-1} + 6z^{-2}}{1 + 2z^{-1} + 3z^{-2}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{4 + 5z^{-1} + 6z^{-2}}{1 + 2z^{-1} + 3z^{-2}}$$

Direct form-II structure

- ✓ The Direct form-II structure is an alternative to direct form-I structure. It is more advantageous to use direct form-II technique than direct form-I, because it uses less number of delay elements than the direct form-I structure.
- ✓ In direct form-II, an intermediate variable is introduced and the given transfer function is split into two, one containing only poles and the other containing only zeros.
- ✓ The poles [that is, the output components $y(n)$ or $Y(z)$ which is the denominator part of the transfer function] are realized first and the zeros [that is, the input components either $x(n)$ or $X(z)$, which is the numerator part of the transfer function] second.

- ✓ If the coefficient of the present output sample $y(n)$ or the non-delay constant at denominator is non unity, then transform it to unity.
- ✓ The systematic procedure is given as follows:

Consider the general difference equation governing an IIR system

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

i.e.
$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) - \cdots - a_N y(n-N)$$
$$+ b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_M x(n-M)$$

On taking Z-transform of the above equation and neglecting initial conditions, we get

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

i.e. $$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

i.e. $$Y(z)[1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}] = X(z)[b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}]$$

i.e. $$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

Let $$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \frac{W(z)}{X(z)}$$

where $$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

and $$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}$$

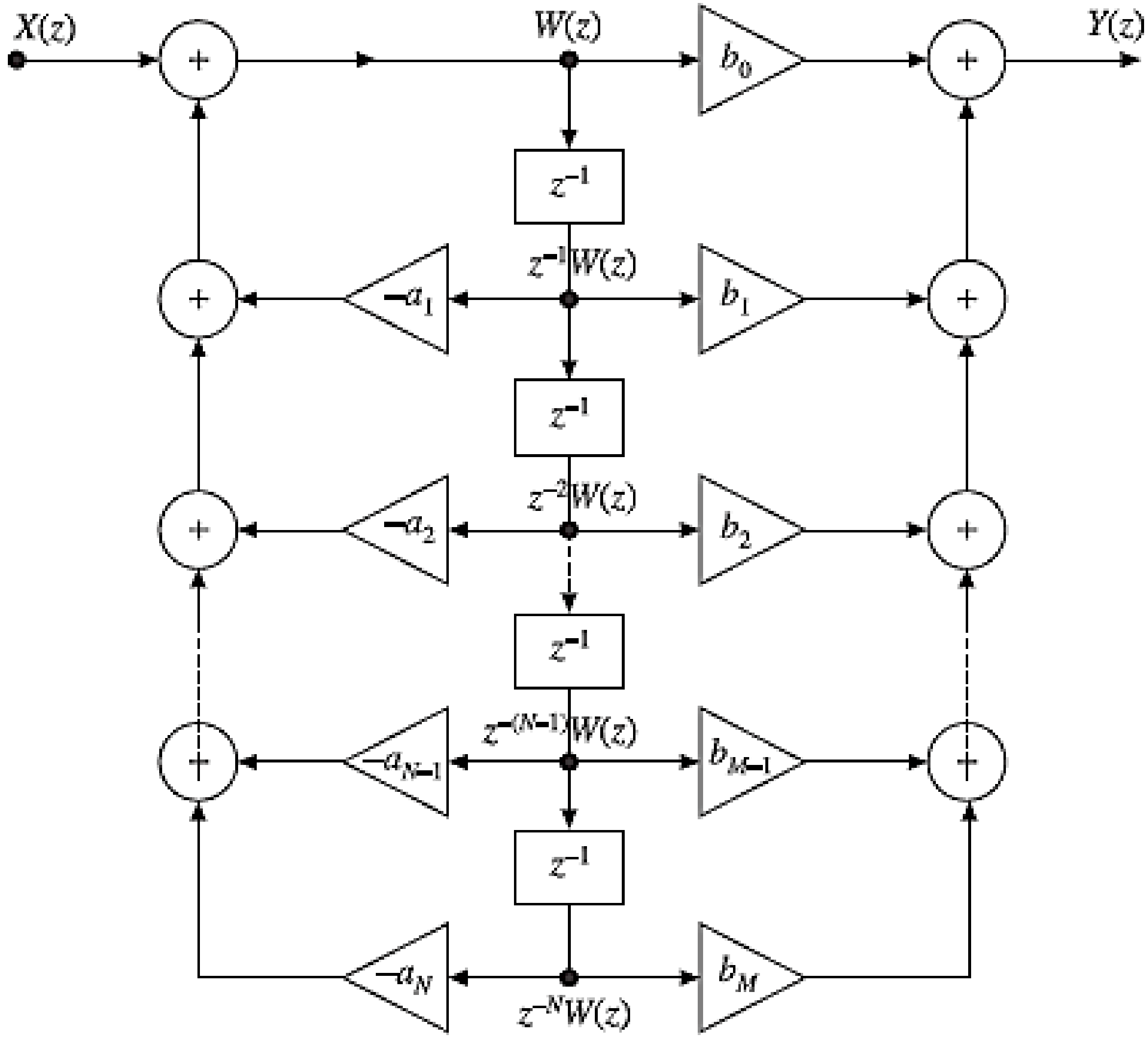
On cross multiplying the above equations, we get

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) + \cdots + a_N z^{-N} W(z) = X(z)$$

$\therefore$ $$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - \cdots - a_N z^{-N} W(z)$$

and $$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + \cdots + b_M z^{-M} W(z)$$

The realization of an IIR system represented by these equations in direct form-II is shown in Figure



- ✓ The number of delay elements used in direct form-II is less than that of direct form-I.

Limitations

- ✓ It also lacks hardware flexibility
- ✓ There are chances of instability due to the quantization noise

- ✓ Since the number of delay elements used in direct form-II is the same as that of the order of the difference equation, direct form-II is called a ***canonical structure***

Direct form-I structure	*Direct form-II structure*
This realization uses separate delays (memory) for both the input and output signal samples.	This realization uses a single delay (memory) for both the input and output signal samples.
For the $(M-1)$th or $(N-1)$th order IIR system, direct form-I requires $M+N-1$ multipliers, $M+N-2$ adders and $M+N-2$ delays.	For the $(M-1)$th or $(N-1)$th order IIR system, direct form-II requires $M+N-1$ multipliers, $M+N-2$ adders and max $[(M-1), (N-1)]$ delays.
It is also called non-canonical, because it requires more number of delays.	It is called canonical, because it requires a minimum number of delays.
It is not efficient in terms of memory requirements compared to direct form-II.	It is more efficient in terms of memory requirements.
Direct form-I can be viewed as two linear time-invariant systems in cascade. The first one is non-recursive and the second one recursive.	Direct form-II can also be viewed as two linear time-invariant systems in cascade. The first one is recursive and the second one non-recursive.

Example: Determine the direct Forms I and II for the second-order filter given by

$$y(n) = 2b \cos\omega_o y(n-1) - b^2 y(n-2) + x(n) - b \cos\omega_o x(n-1)$$

Solution Taking z-transform for the given function, we get

$$Y(z) = 2b\cos\omega_0 z^{-1}Y(z) - b^2 z^{-2}Y(z) + X(z) - b\cos\omega_0 z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - b\cos\omega_0 z^{-1}}{1 - 2b\cos\omega_0 z^{-1} + b^2 z^{-2}}$$

Direct Form I

$$H(z) = H_1(z)\, H_2(z)$$

Therefore, $Y(z) = H_1(z)\, H_2(z) X(z)$

In this form, the intermediate sequence $w(n)$ is introduced between $H_1(n)$ and $H_2(n)$

Let $H_1(z) = 1 - b\cos\omega_0 z^{-1} = \dfrac{W(z)}{X(z)}$

Therefore, $X(z)\,(1 - b\cos\,\omega_0 z^{-1}) = W(z)$

$x(n) - b\cos\,\omega_0\, x\,(n-1) = w(n)$

and

$$H_2(z) = (1 - 2b\cos\omega_0 z^{-1} + b^2 z^{-2})^{-1} = \frac{Y(z)}{X(z)}$$

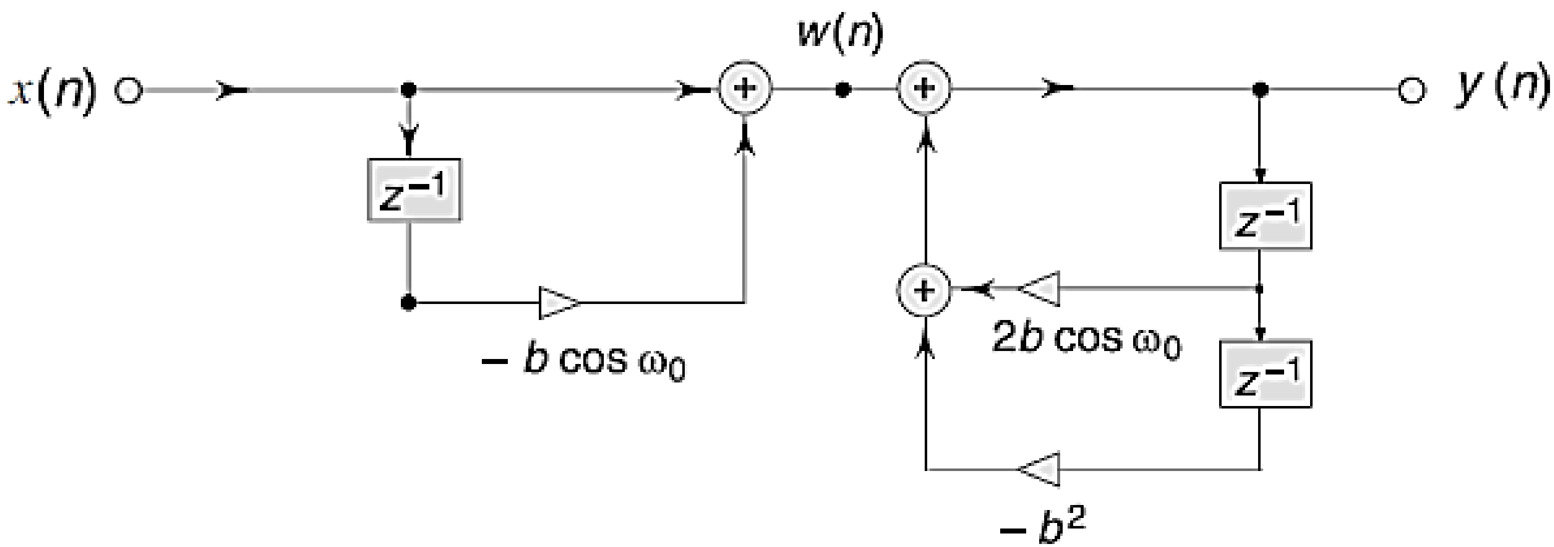
$$Y(z) = \frac{W(z)}{(1 - 2b\cos\omega_0 z^{-1} + b^2 z^{-2})}$$

$$Y(z)[1 - 2b\cos\omega_0 z^{-1} + b^2 z^{-2}] = W(z)$$

The inverse z-transform of this function is

$$y(n) - 2\,b \cos \omega_0\, y\,(n-1) + b^2\, y\,(n-2) = w(n)$$

The direct Form I realisation structure of the above function is shown in Fig.



Direct Form II

$$Y(z) = H_2(z)\, H_1(z)\, X(z)$$

$$H_2(z) = (1 - 2b\cos\omega_0 z^{-1} + b^2 z^{-2})^{-1} = \frac{U(z)}{X(z)}$$

Let $$H_1(z) = 1 - b\cos\omega_0 z^{-1} = \frac{Y(z)}{U(z)}$$

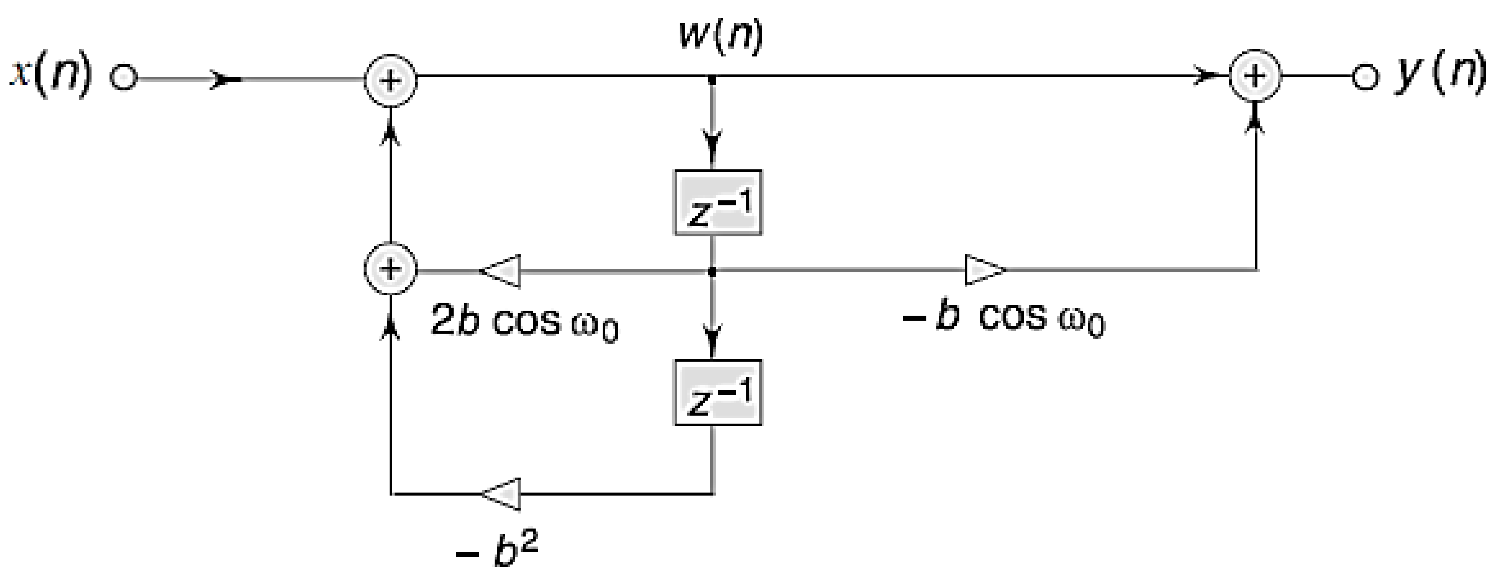
$$X(z) = U(z)\{1 - 2\, b\cos\omega_0 z^{-1} + b^2 z^{-2}\}$$

Hence, $$x(n) = u(n) - 2b\cos\omega_0 u(n-1) + b^2 u(n-2)$$

$$Y(z) = U(z)\{1 - b\cos\omega_0 z^{-1}\}$$

Hence, $$y(n) = u(n) - b\cos\,\omega_0 u(n-1)$$

The Direct Form II realisation structure of the above function is shown in Fig.



Transposed form structure

- ✓ It is practically true that if we reverse the direction of all the branch transmittances and interchange the input and the output in the structure or signal flow graph, the system remains unchanged.
- ✓ The transposed structure or transpose form or reverse structure is obtained by reversing the direction of all branch transmittances and interchanging the input and output in the direct form structure.

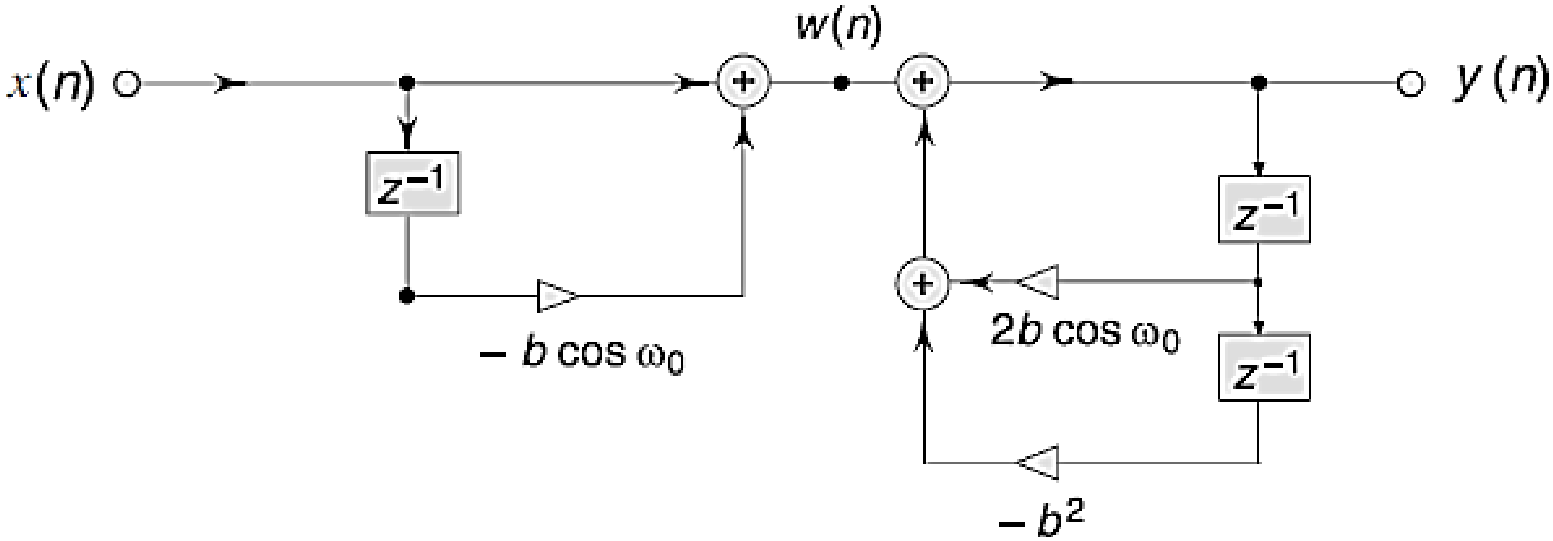
- ✓ The transposed structure remains valid, provided:

1. The branch transmittances are untouched.
2. The direction of all the branches in the structure is reversed.
3. The roles of the input and output are reversed.

By these steps, the system function remains unchanged.

1. First realize the given difference equation or transfer function by using the direct form structure.
2. Reverse or transpose the direction of signal flow and interchange the input and output nodes.
3. Replace the junction points by adders and adders by junction points.
4. Fold the structure, which is the transposed form realization of an IIR system.

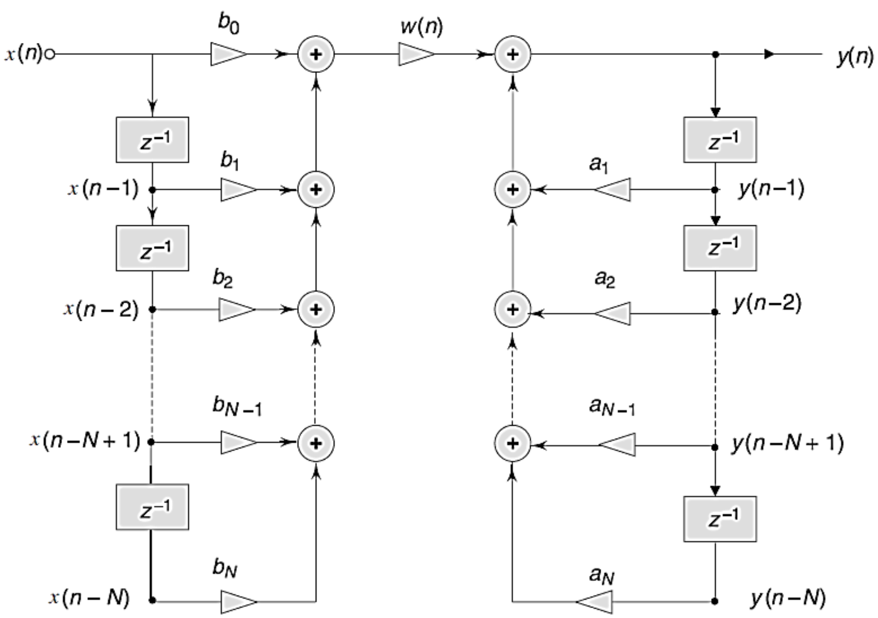
- ✓ In general, the transposed structure realization of an IIR system is advantageous only if it is implemented over a direct form-II structure, because the number of components used or the number of additions and multiplications get reduced when it is used over a direct form-II structure.
- ✓ However, the higher the order of the systems, the better would be its advantages. It has no advantage for a direct form-I structure.

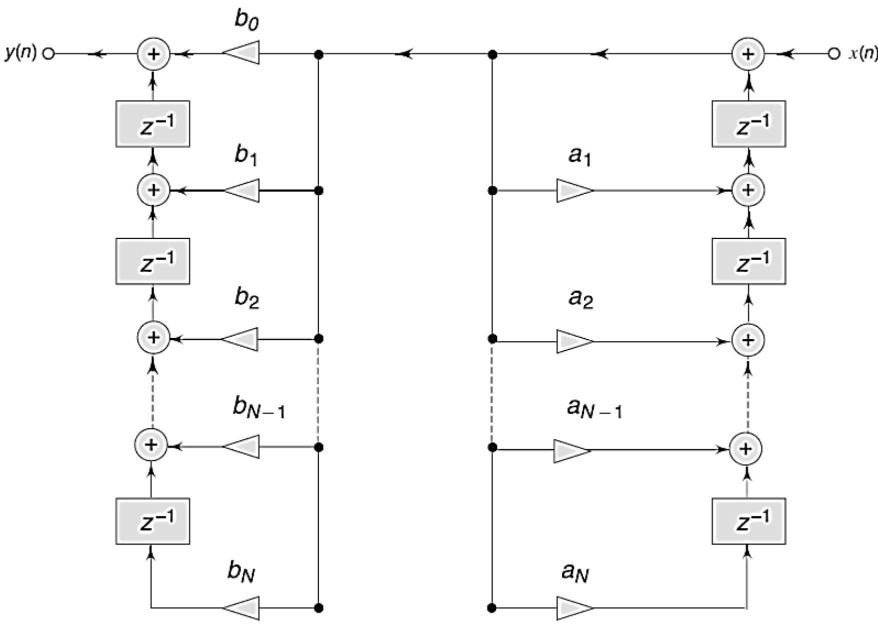

$x(n)$
$w(n)$
$y(n)$
z^{-1}
z^{-1}
z^{-1}
$-b\cos\omega_0$
$2b\cos\omega_0$
$-b^2$

Transposed Structure or Transpose Form or Reverse Structure

- ✓ If two digital filter structures have the same transfer function, then they are called equivalent structures.
- ✓ A simple way to generate an equivalent structure from a given realisation structure is via the transpose operation.
- ✓ The transposed form is obtained by
- ✓ (i) reversing the paths,
- ✓ (ii) replacing pickoff nodes by adders, and vice-versa, and
- ✓ (iii) interchanging the input and output nodes.
- ✓ For a single input-output system, the transposed structure has the same transfer function as the original realisation structure.

- ✓ While using infinite precision arithmetic, any realisation structure will behave identically to any other equivalent structure. But, due to the finite word length limitations, the behaviour of a particular realisation structure is totally different from its equivalent structures.
- ✓ Therefore, it is important to select a structure which has the minimum quantisation effects from the finite word length implementation.
- ✓ In general, the transposed structure realization of an IIR system is advantageous only if it is implemented over a direct form-II structure, because the number of components used or the number of additions and multiplications get reduced when it is used over a direct form-II structure.
- ✓ However, the higher the order of the systems, the better would be its advantages. It has no advantage for a direct form-I structure.





Transposed Direct Form I

Realisation Structure

Example: Realize an IIR system

$$y(n) + 2y(n-1) + 3y(n-2) = 4x(n) + 5x(n-1) + 6x(n-2)$$

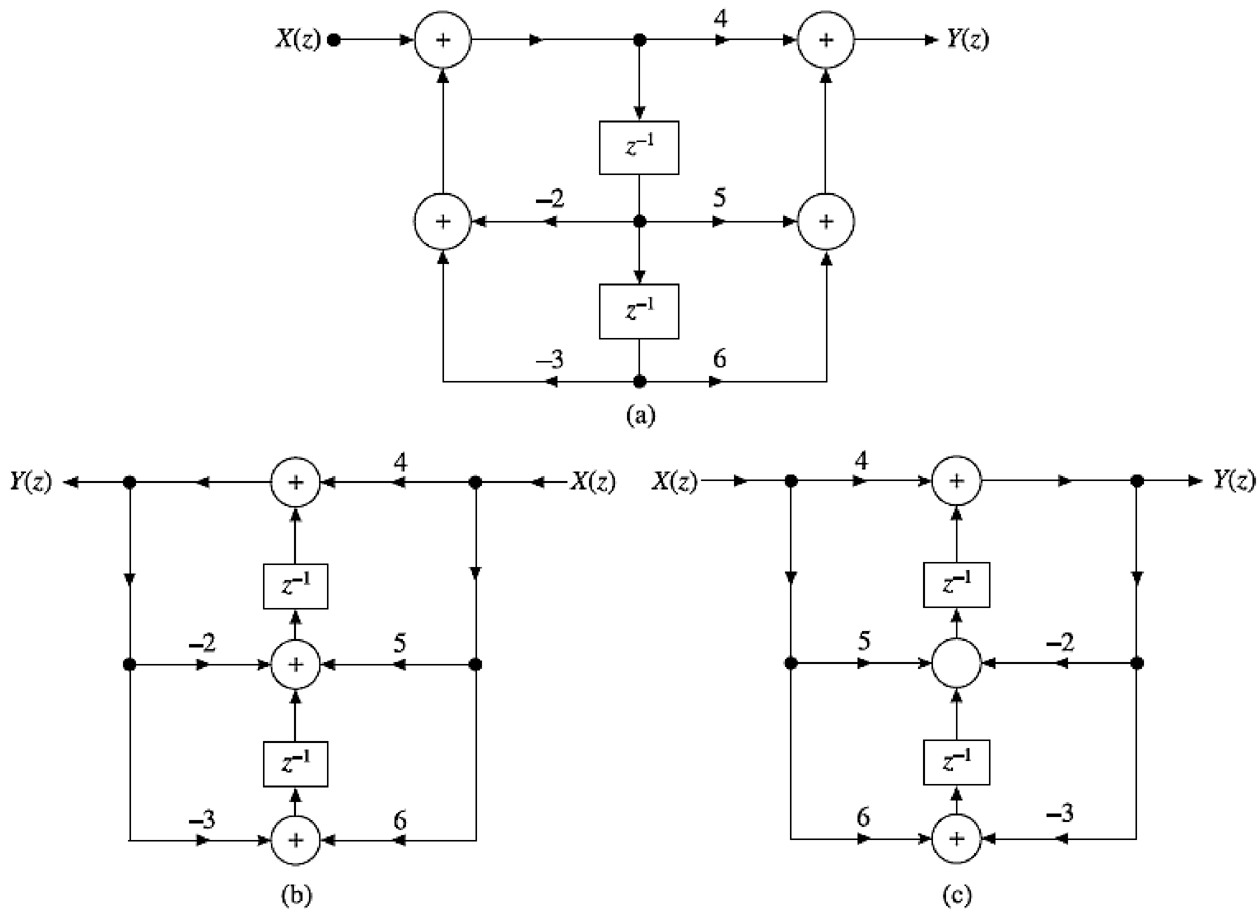
using the transposed form structure.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{4 + 5z^{-1} + 6z^{-2}}{1 + 2z^{-1} + 3z^{-2}}$$

Transfer function of the given IIR system is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{4 + 5z^{-1} + 6z^{-2}}{1 + 2z^{-1} + 3z^{-2}}$$

The direct form-II realization structure, the recovered realization structure and the transposed form realization structure of this system are shown in Figure



Cascade Form Realisation

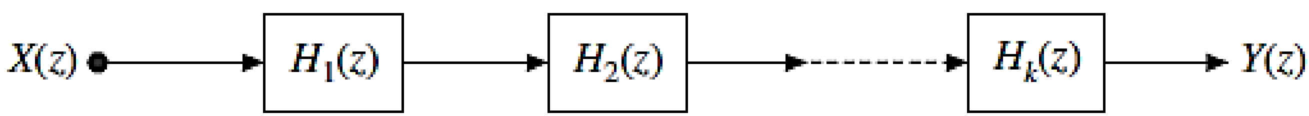
- ✓ The cascade form structure is nothing, but a cascaded or series interconnection of the sub transfer functions or sub system functions which are realized by using the direct form structures (either direct form-I or direct form-II or a combination of both).
- ✓ Hence, in cascade form realization, the given transfer function $H(z)$ is expressed as a product of a number of second order or first order sections as indicated below:

$$H(z) = \frac{Y(z)}{X(z)} = \prod_{i=1}^{k} H_i(z)$$

where $$H_i(z) = \frac{C_{0i} + C_{1i}z^{-1} + C_{2i}z^{-2}}{d_{0i} + d_{1i}z^{-1} + d_{2i}z^{-2}} \quad \text{[second order section]}$$

or $$H_i(z) = \frac{C_{0i} + C_{1i}z^{-1}}{d_{0i} + d_{1i}z^{-1}} \quad \text{[first order section]}$$

- ✓ Each of these sections is realized separately and all of them are connected in cascade (series).
- ✓ Therefore, the cascade form realization is also called a series structure in which one sub transfer function is the input to the other transfer function and so on.
- ✓ The cascade form realization is shown in Figure



The difficulties in cascade structure are:

- ✓ Decision of pairing poles and zeros.
- ✓ Deciding the order of cascading the first and second order sections.
- ✓ Scaling multipliers should be provided between individual sections to prevent the filter variables from becoming too large or too small.

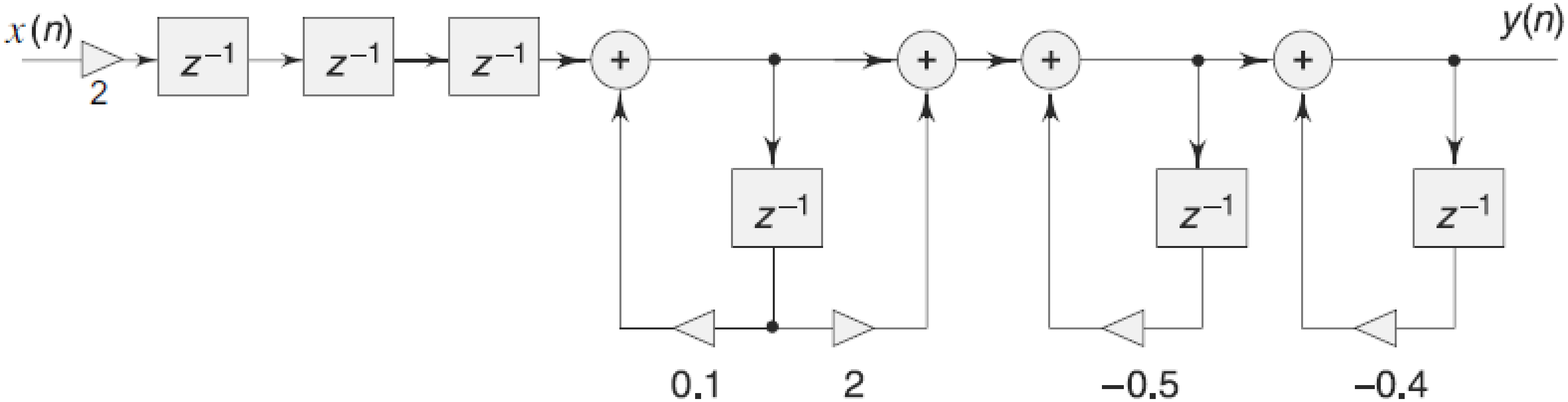
Example: Obtain a cascade realisation of the system characterised by the transfer function

$$H(z) = \frac{2(z+2)}{z(z-0.1)(z+0.5)(z+0.4)}$$

$$H(z) = \frac{2z^{-3}(1+2z^{-1})}{(1-0.1z^{-1})(1+0.5z^{-1})(1+0.4z^{-1})}$$

$$= 2.z^{-3} \cdot \frac{(1+2z^{-1})}{(1-0.1z^{-1})} \cdot \frac{1}{(1+0.5z^{-1})} \cdot \frac{1}{(1+0.4z^{-1})}$$

$$= 2 \cdot H_1(z) \cdot H_2(z) \cdot H_3(z) \cdot H_4(z)$$



x(n)
2
z−1
z−1
z−1
+
+
+
+
y(n)
z−1
z−1
z−1
0.1
2
−0.5
−0.4

Parallel Realisation of IIR Systems

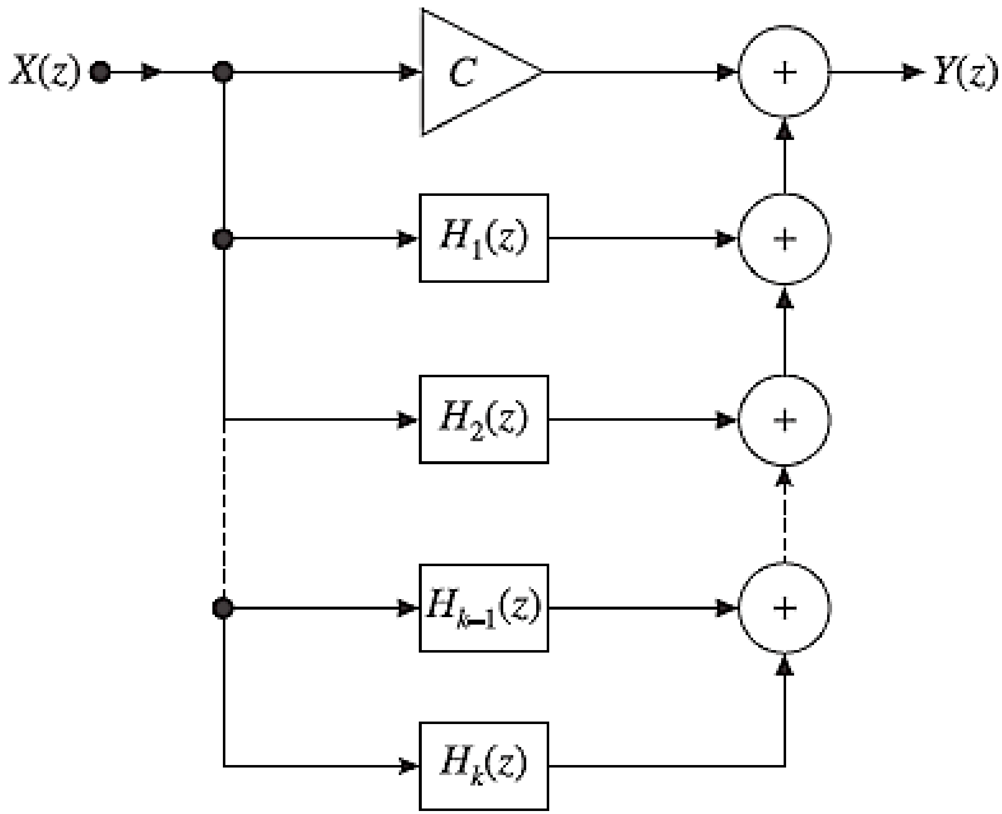
- ✓ Parallel form structure is nothing, but the parallel connection of sub-transfer functions or sub-system functions, which is decomposed by using the partial fraction method.
- ✓ In parallel form realization, by partial fraction expansion, the transfer function *H*(*z*) is expressed as a sum of first and second order sections.

$$H(z) = \frac{Y(z)}{X(z)} = C + \sum_{i=1}^{k} H_i(z)$$

where,
$$H_i(z) = \frac{C_{0i} + C_{1i}z^{-1}}{d_{0i} + d_{1i}z^{-1} + d_{2i}z^{-2}} \quad \text{[second order section]}$$

or
$$H_i(z) = \frac{C_{0i}}{d_{0i} + d_{1i}z^{-1}} \quad \text{[first order section]}$$

✓ Each first order and second order section is realized either in direct form-I structure or in direct form-II structure and the individual sections are connected in parallel to obtain the over all system as shown in Figure.



- ✓ As the filter operation is performed in parallel, i.e. the processing is performed simultaneously, the parallel form structure is used for high speed filtering application.
- ✓ The difficulty with this method is expressing the transfer function in partial fraction form is not easy for higher order systems.

Example: Determine the parallel realisation of the IIR digital filter transfer functions

$$\text{(a)} \quad H(z) = \frac{3(2z^2+5z+4)}{(2z+1)(z+2)} \qquad \text{(b)} \quad H(z) = \frac{3z(5z-2)}{\left(z+\frac{1}{2}\right)(3z-1)}$$

Solution (a) In order to find the parallel realisation, the partial fraction expansion of $H(z)/z$ is first determined, just as we did for inverse z-transforms. This gives

$$F(z) = \frac{H(z)}{z} = \frac{\frac{3}{2}(2z^2+5z+4)}{z\left(z+\frac{1}{2}\right)(z+2)} = \frac{A_1}{z} + \frac{A_2}{\left(z+\frac{1}{2}\right)} + \frac{A_3}{(z+2)}$$

where

$$A_1 = zF(z)\Big|_{z=0} = \left.\frac{\frac{3}{2}(2z^2+5z+4)}{\left(z+\frac{1}{2}\right)(z+2)}\right|_{z=0} = 6$$

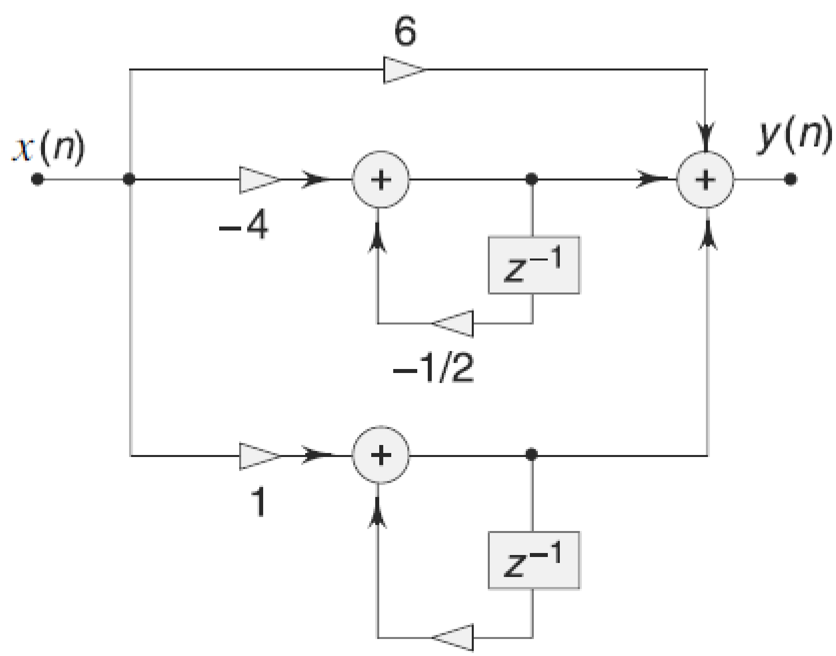
$$A_2 = \left(z+\frac{1}{2}\right)F(z)\Big|_{z=-\frac{1}{2}} = \left.\frac{\frac{3}{2}(2z^2+5z+4)}{z(z+2)}\right|_{z=-\frac{1}{2}} = -4$$

$$A_3 = (z+2)F(z)\Big|_{z=-2} = \left.\frac{\frac{3}{2}(2z^2+5z+4)}{z\left(z+\frac{1}{2}\right)}\right|_{z=-2} = 1$$

Therefore, $$\frac{H(z)}{z} = \frac{6}{z} - \frac{4}{z+\frac{1}{2}} + \frac{1}{(z+2)}$$

Hence, $$H(z) = 6 - \frac{4z}{z+\frac{1}{2}} + \frac{z}{(z+2)} = 6 - \frac{4}{\left(1+\frac{1}{2}z^{-1}\right)} + \frac{1}{1+2z^{-1}}$$

The parallel realisation of this transfer function is shown in Fig.



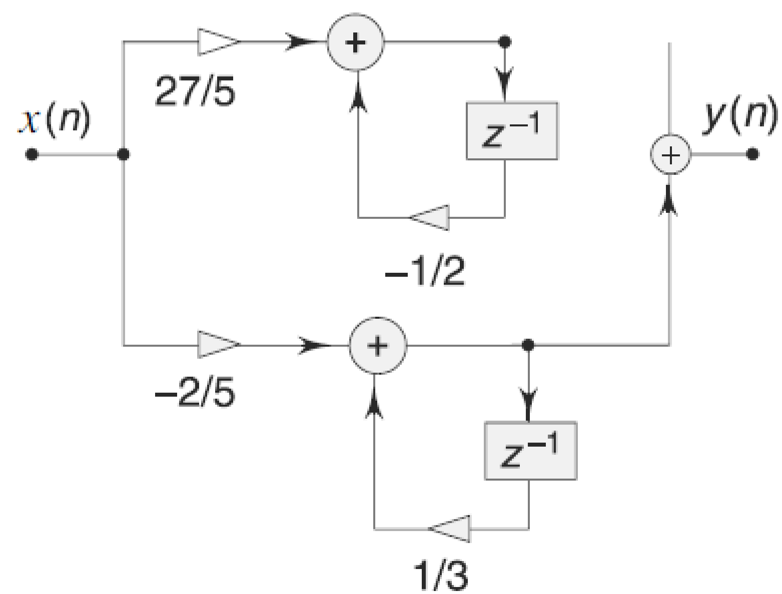
(a)

(b)
$$H(z) = \frac{3z(5z-2)}{(z+\frac{1}{2})(3z-1)} = \frac{z(5z-2)}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$F(z) = \frac{H(z)}{z} = \frac{5z-2}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} = \frac{A_1}{z+\frac{1}{2}} + \frac{A_2}{\left(z-\frac{1}{3}\right)}$$

where
$$A_1 = F(z)\left(z+\frac{1}{2}\right)\Bigg|_{z=-\frac{1}{2}} = \frac{5z-2}{\left(z-\frac{1}{3}\right)}\Bigg|_{z=-\frac{1}{2}} = \frac{27}{5}$$

$$A_2 = F(z)\left(z-\frac{1}{3}\right)\Bigg|_{z=\frac{1}{3}} = \frac{5z-2}{\left(z+\frac{1}{2}\right)}\Bigg|_{z=\frac{1}{3}} = -\frac{2}{5}$$



(b)

Therefore,
$$H(z) = \frac{27}{5}\frac{z}{\left(z+\frac{1}{2}\right)} - \frac{2}{5}\frac{z}{\left(z-\frac{1}{3}\right)}$$

$$= \frac{27}{5}\frac{1}{\left(1+\frac{1}{2}z^{-1}\right)} - \frac{2}{5}\frac{1}{\left(1-\frac{1}{3}z^{-1}\right)}$$

Lattice structure

✓ The IIR system consists of both zeros and poles. Therefore, the poles and zeros will be considered as separate sub-transfer functions in cascade, that is

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}=\left(\sum_{k=0}^{M} b_k z^{-k}\right)\left(\frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}\right)=H_z(z)H_p(z)$$

$$\text{where } H_z(z)=\sum_{k=0}^{M} b_k z^{-k}=\text{Zeros} \quad H_p(z)=\frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}=\text{Poles}$$

The above equation for $H_p(z)$ can be rewritten as

$$H_p(z)=\frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}=\frac{Y_p(z)}{X_p(z)}$$

or

$$X_p(z)=Y_p(z)+\sum_{k=1}^{N} a_k z^{-k} Y_p(z)$$

Taking the inverse Z-transform on both sides and rearranging it, we get

$$y_p(n) = x_p(n) - \sum_{k=1}^{N} a_k y_p(n-k)$$

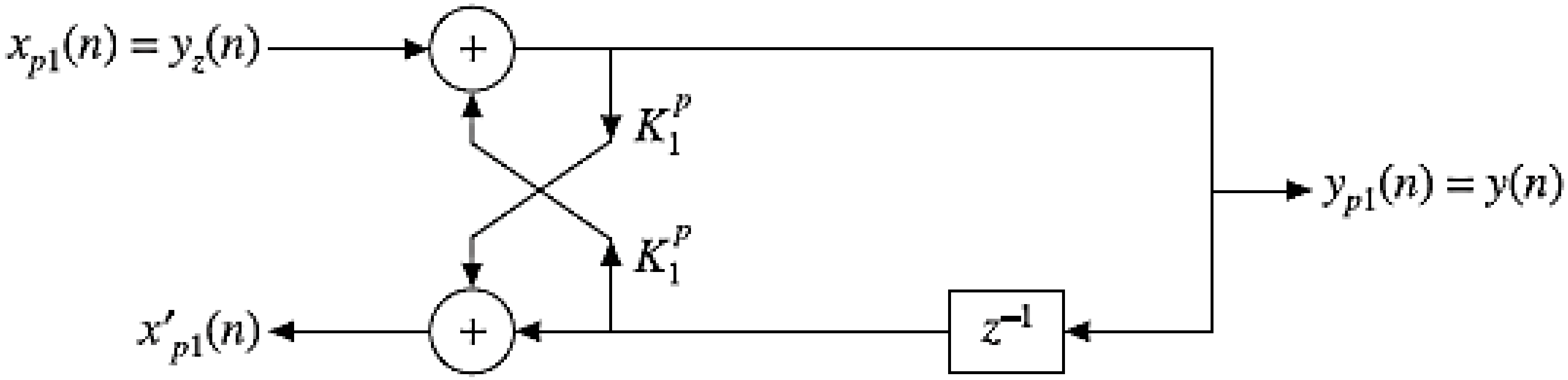
The output of the sub-transfer function $H_z(z)$ is the input to the sub-transfer function $H_p(z)$.

So $y_z(n) = x_p(n)$.

While realizing the IIR system using the lattice structure, the zeros, i.e. $H_z(z)$ should be realized first and then the poles, i.e. $H_p(z)$ which is realized in cascade.

- ✓ For the realization of the poles, a lattice structure consists of two paths, $x_p(n)$ and $x_p{}'(n)$ through which the input $x_p(n)$ or $y_z(n)$ is processed.
- ✓ However, those two different paths are opposite in direction to each other.

A single-stage lattice structure for a pole is shown in Figure.



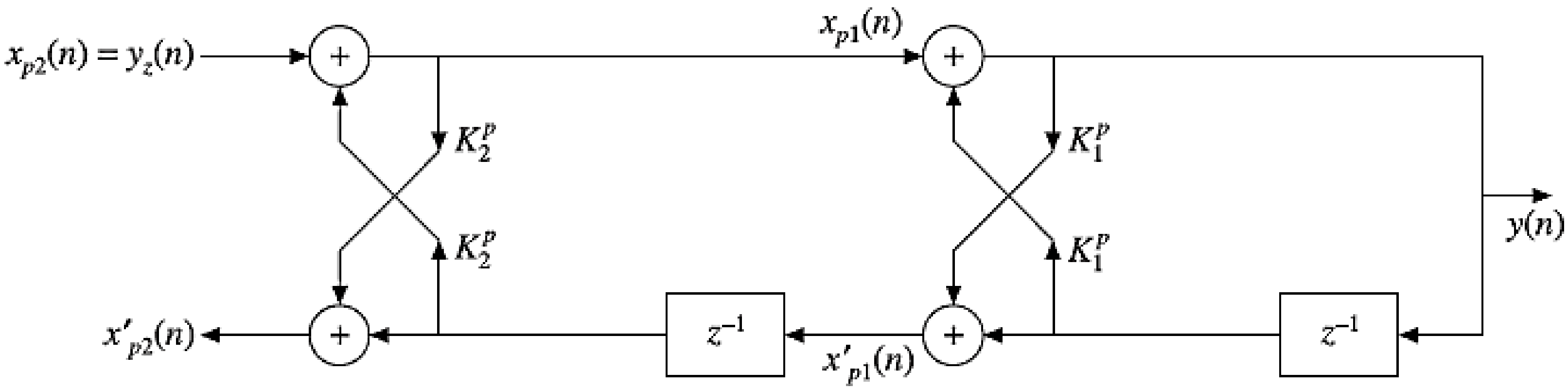
The output from the single-stage lattice structure is

$$y(n) = x_{p1}(n) + K_1^p y(n-1)$$

and the feedback response is

$$x'_{p1}(n) = K_1^p y(n) + y(n-1)$$

Similarly, the two-stage lattice structure for a pole is shown in Figure.



The intermediate output of the two-stage lattice structure is

$$x_{p1}(n) = x_{p2}(n) + K_2^p x'_{p1}(n-1)$$

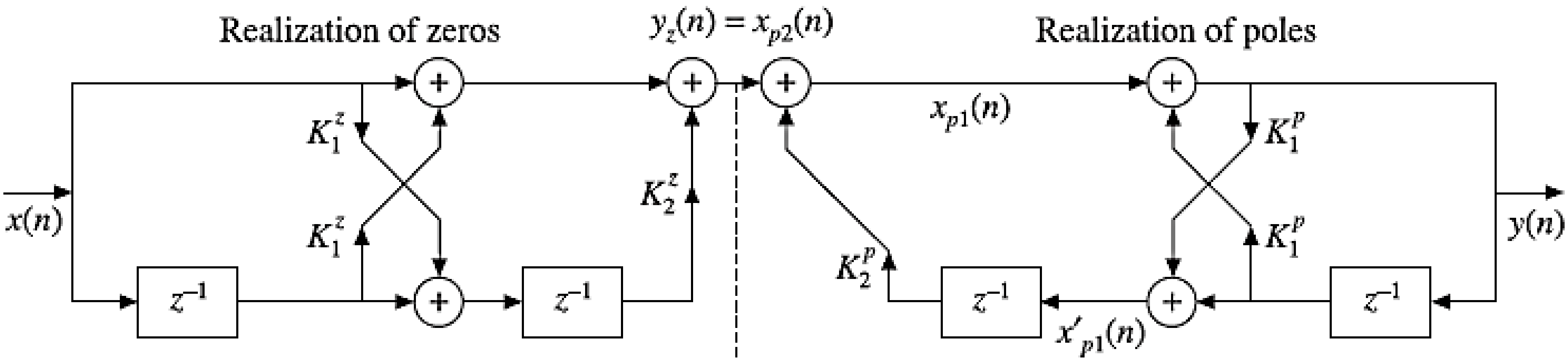
Substituting the value of $x'_{p1}(n)$ in the expression for $x_p(n)$, we have

$$\begin{aligned} x_{p1}(n) &= x_{p2}(n) + K_2^p [K_1^p y(n-1) + y(n-2)] \\ &= x_{p2}(n) + K_1^p K_2^p y(n-1) + K_2^p y(n-2) \end{aligned}$$

Substituting this value of $x_{p1}(n)$ in the expression for $y(n)$, we get

$$\begin{aligned} y(n) &= x_{p2}(n) + K_1^p K_2^p y(n-1) + K_2^p y(n-2) + K_1^p y(n-1) \\ &= x_{p2}(n) + K_1^p [1 + K_2^p] y(n-1) + K_2^p y(n-2) \end{aligned}$$

Therefore, the overall lattice structure for a second order IIR system (both poles and zeros are of second order) can be in general, realized as shown in Figure.



In a similar fashion, the $(M-1)^{th}$ order or $(N-1)^{th}$ order IIR system can be realized by using the lattice structure.

Procedure to realize the lattice structure of IIR systems

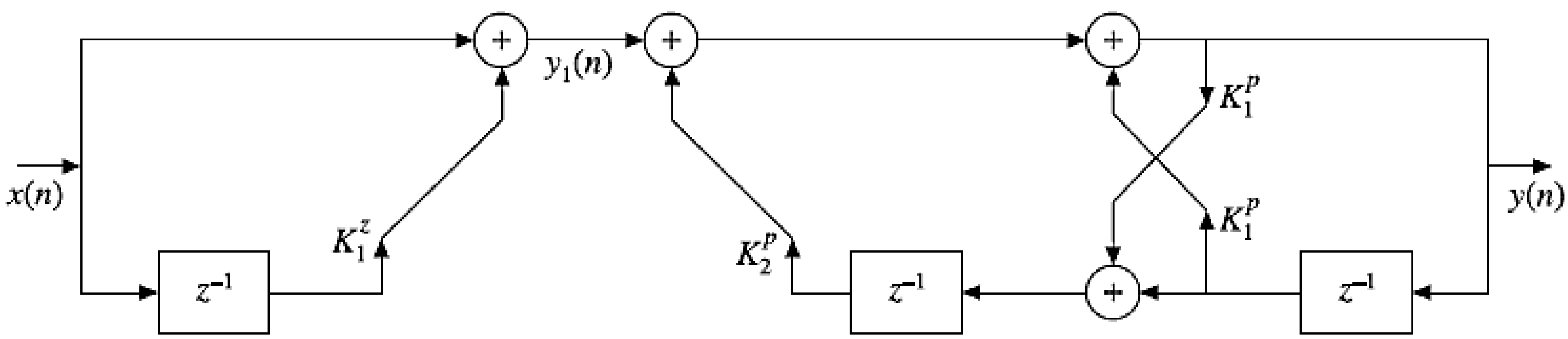
1. Find the order of the difference equation and compare the coefficients with the reflection coefficients $K_1^p, K_2^p, K_3^p, \ldots$ and $K_1^z, K_2^z, K_3^z, \ldots$ of the same order lattice structure output. This applies both for the poles and zeros, separately.
2. Assign the calculated values of the reflection coefficients $K_1^p, K_2^p, K_3^p, \ldots$ and $K_1^z, K_2^z, K_3^z, \ldots$ and construct the structures.
3. Cascade the structures of both zeroes and poles.

EXAMPLE Determine the lattice coefficients corresponding to an IIR filter described by $y(n)-\frac{2}{5}y(n-1)+\frac{1}{5}y(n-2)=x(n)+\frac{1}{4}x(n-1)$ and realize it.

Solution: The given system described by the difference equation

$$y(n)=\frac{2}{5}y(n-1)\;-\;\frac{1}{5}y(n-2)+x(n)+\frac{1}{4}x(n-1)$$

has first order zeroes and second order poles. Hence, the proposed lattice structure is given in Figure .



The output $y(n)$ from Figure is

$$y(n)=y_1(n)+K_1^p[1+K_2^p]y(n-1)+K_2^p y(n-2)$$

i.e.
$$y(n)=x(n)+K_1^z x(n-1)+K_1^p[1+K_2^p]y(n-1)+K_2^p y(n-2)$$

Comparing this equation with the given difference equation, we have

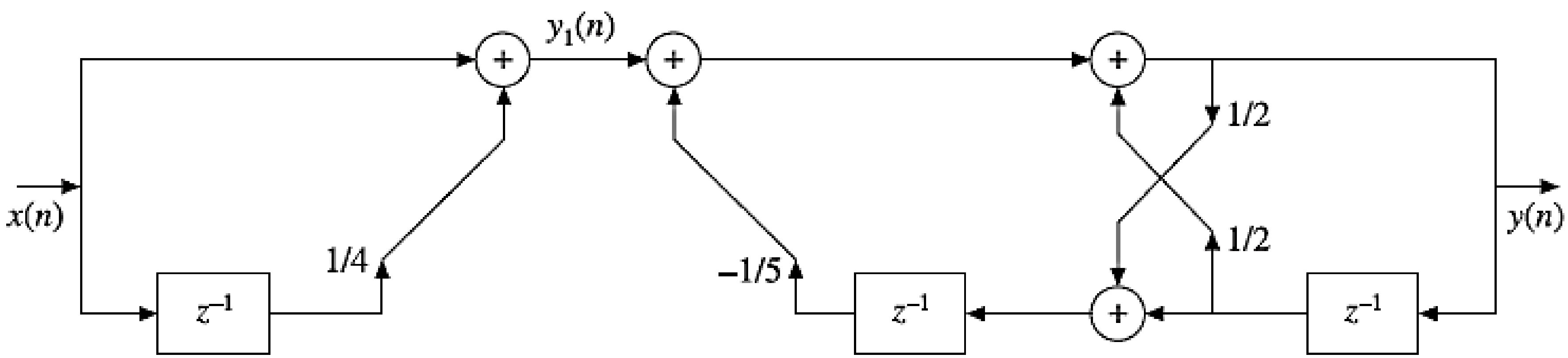
$$K_1^z = \frac{1}{4},\ K_2^p = -\frac{1}{5}$$

and
$$K_1^p\,[1 + K_2^p] = \frac{2}{5}$$

or
$$K_1^p = \frac{2/5}{1 - (1/5)} = \frac{1}{2}$$

Hence, the lattice structure realization of the given IIR filter is shown in Figure .

The realization obtained in Figure is called all-zero-all-pole lattice structure realization.



Ladder structure

- ✓ Ladder structure realization is possible only for an IIR system.
- ✓ In this structure, the numerator polynomial will be divided by the denominator polynomial sequentially and the result substituted at the ladder fashion shown below.
- ✓ Two cases are considered for the realization of the ladder structure.

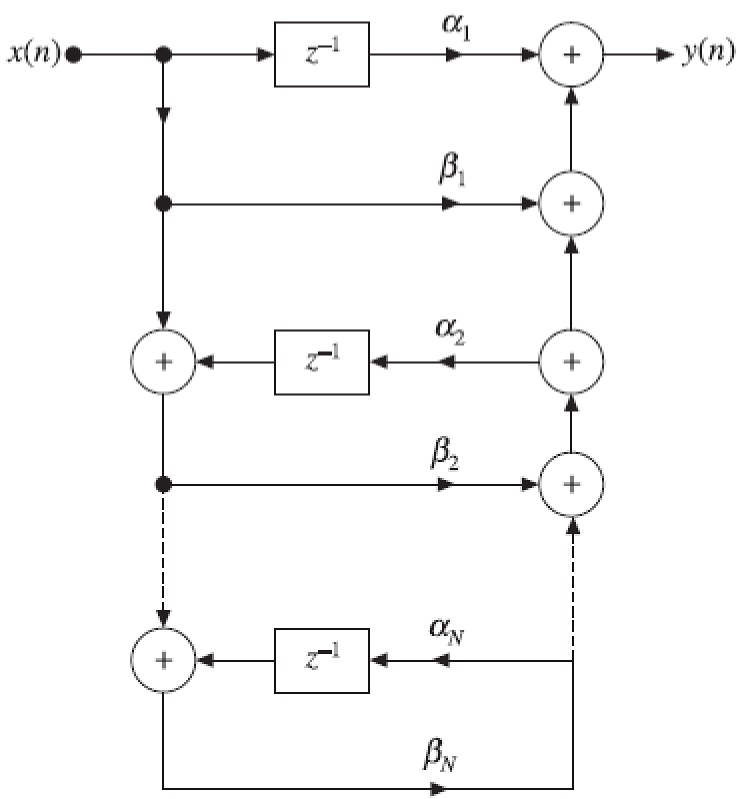
CASE-I: If the negative order of the numerator polynomial is more than the negative order of the denominator polynomial, that is,

$$H(z) = \frac{b_{N+1}z^{-(N+1)} + \cdots + b_3 z^{-3} + b_2 z^{-2} + b_1 z^{-1} + b_0}{a_N z^{-N} + \cdots + a_3 z^{-3} + a_2 z^{-2} + a_1 z^{-1} + a_0}$$

then, the transfer function of a ladder structure is shown below.

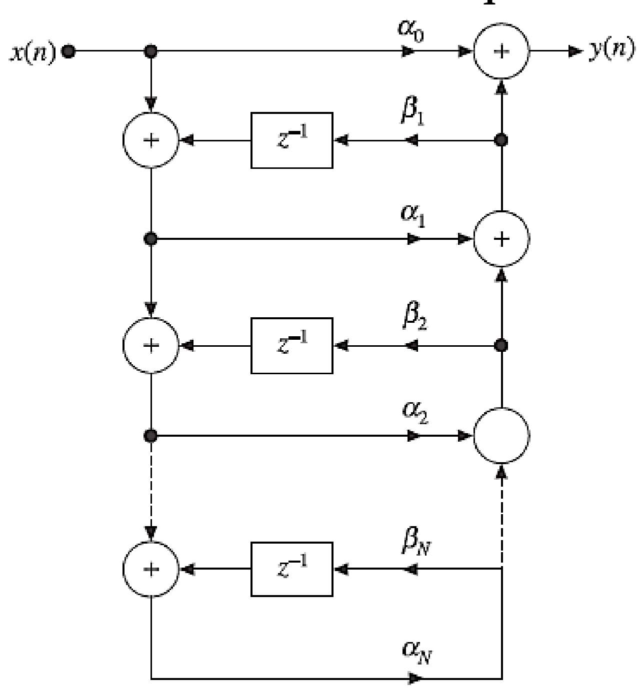
$$H(z)=\alpha_1 z^{-1}+\cfrac{1}{\beta_1+\cfrac{1}{\alpha_2 z^{-1}+\cfrac{1}{\beta_2+\cfrac{1}{\alpha_3 z^{-1}+\cfrac{1}{\beta_3+\cdots}}}}}$$

The ladder structure realization of the above equation for $H(z)$ is shown in Figure.



(a)

CASE-II: If the order of both the numerator and denominator polynomials is equal, that is
$$H(z) = \frac{b_N z^{-N} + \cdots + b_3 z^{-3} + b_2 z^{-2} + b_1 z^{-1} + b_0}{a_N z^{-N} + \cdots + a_3 z^{-3} + a_2 z^{-2} + a_1 z^{-1} + a_0}$$
then, the transfer function for a ladder structure is shown below.
$$H(z) = \alpha_0 + \cfrac{1}{\beta_1 z^{-1} + \cfrac{1}{\alpha_1 + \cfrac{1}{\beta_2 z^{-1} + \cfrac{1}{\alpha_2 + \cfrac{1}{\beta_3 z^{-1} + \cdots}}}}}$$
The ladder structure realization of the above equation for $H(z)$ is shown in Figure.



Procedure to realize the ladder structure of IIR systems

1. Express the numerator and denominator polynomials of the transfer function in descending order of negative powers of z and assess whether it falls under case-I or case-II.
2. Divide the numerator polynomial by the denominator polynomial sequentially and find the quotient at each division.
3. Compare the quotient obtained with that of the general transfer function for the corresponding ladder structure and find the corresponding parameters α and β .
4. Realize the ladder structure with the obtained values of parameters.

EXAMPLE: Realize the IIR filter $H(z) = \dfrac{3z^2 + 5z + 4}{z^2 + 6z + 8}$ using ladder structure.

Solution: Given $H(z) = \dfrac{3z^2 + 5z + 4}{z^2 + 6z + 8} = \dfrac{3 + 5z^{-1} + 4z^{-2}}{1 + 6z^{-1} + 8z^{-2}} = \dfrac{4z^{-2} + 5z^{-1} + 3}{8z^{-2} + 6z^{-1} + 1}$

Here, the negative order of the numerator and denominator is equal. So it falls under case-II.

Performing sequential division operation.

$$
\begin{array}{l}
8z^{-2} + 6z^{-1} + 1)\ \overline{4z^{-2} + 5z^{-1} + 3}\ (1/2 \\
\qquad\qquad\qquad\quad \underline{4z^{-1} + 3z^{-1} + 1/2} \\
\qquad\qquad\qquad\qquad 2z^{-1} + 5/2)\ 8z^{-2} + 6z^{-1} + 1\ (4z^{-1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \underline{8z^{-2} + 10z^{-1}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad -4z^{-1} + 1)\ 2z^{-1} + 5/2\ (-1/2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{2z^{-1} - 1/2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3)\ \overline{-4z^{-1} + 1}\ (-4/3z^{-1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-4z^{-1}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1)\ 3\ (3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \underline{3} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ 0
\end{array}
$$

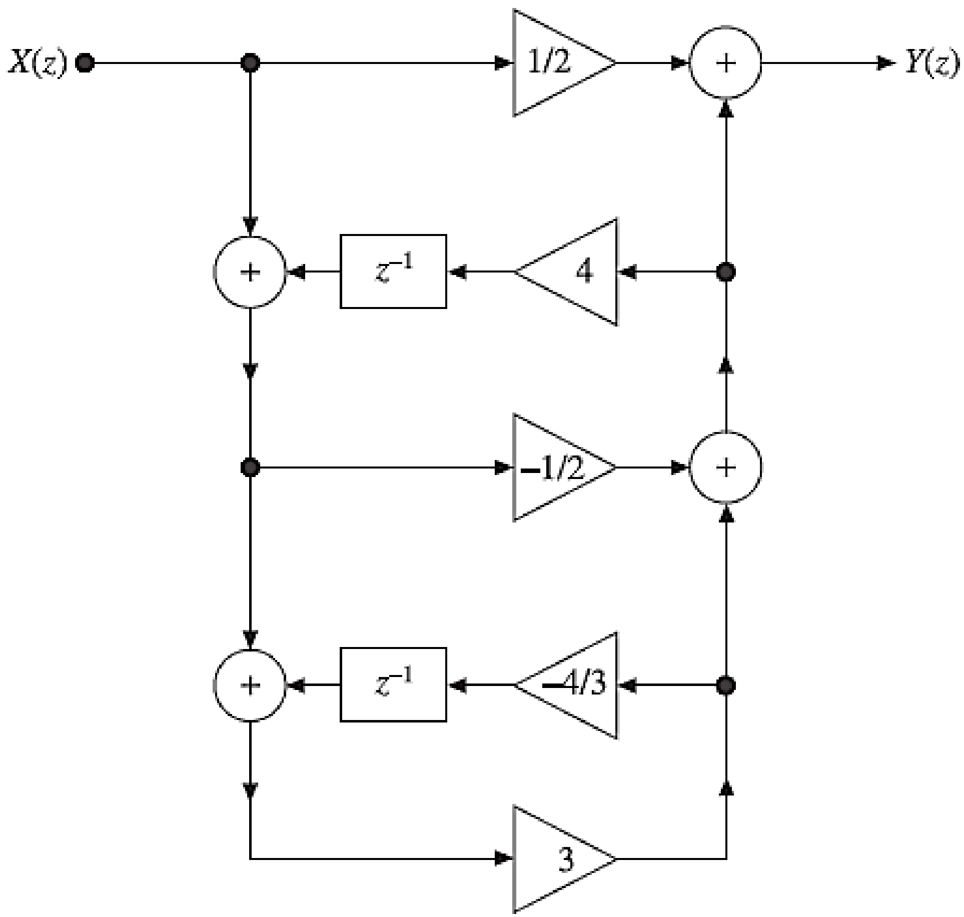
Comparing with $H(z)$ of case-II, we get

$$\alpha_0 = \frac{1}{2},\ \beta_1 = 4,\ \alpha_1 = -\frac{1}{2},\ \beta_2 = -\frac{4}{3} \text{ and } \alpha_2 = 3$$

Hence, the required transfer function for realization is

$$H(z) = \frac{1}{2} + \cfrac{1}{4z^{-1} + \cfrac{1}{-\frac{1}{2} + \cfrac{1}{-\frac{4}{3}z^{-1} + \frac{1}{3}}}}$$

Thus, the realization of the given IIR filter using the ladder form structure is shown in



EXAMPLE: Realize the IIR filter $H(z) = \dfrac{5z^3 + 3z^2 + 4z + 2}{z[2z^2 + 3z + 1]}$ using ladder structure

Solution: Given

$$H(z) = \frac{5z^3 + 3z^2 + 4z + 2}{z[2z^2 + 3z + 1]} = \frac{z^3[5 + 3z^{-1} + 4z^{-2} + 2z^{-3}]}{z^3\,[2 + 3z^{-1} + z^{-2}]} = \frac{2z^{-3} + 4z^{-2} + 3z^{-1} + 5}{z^{-2} + 3z^{-1} + 2}$$

Here the negative order of the numerator polynomial is greater than that of the denominator. So it falls under case-I.

Performing the sequential division operation

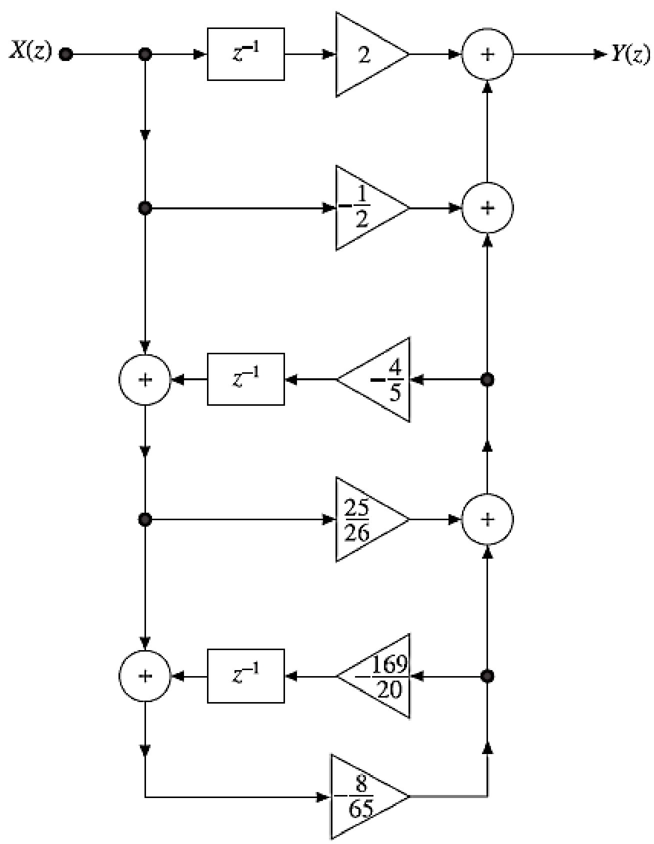
$$\begin{array}{l}
z^{-2} + 3z^{-1} + 2)\; \overline{2z^{-3} + 4z^{-2} + 3z^{-1} + 5}\; (2z^{-1} \\
\qquad\qquad\quad \underline{2z^{-3} + 6z^{-2} + 4z^{-1}} \\
\qquad\qquad\qquad -2z^{-2} - z^{-1} + 5)\; z^{-2} + 3z^{-1} \quad + 2 \;\; (-1/2 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{z^{-2} + 1/2z^{-1} - 5/2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 5/2z^{-1} + 9/2)\; -2z^{-2} - z^{-1} + 5 \quad (-4/5\, z^{-1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-2z^{-2} - 18/5\, z^{-1}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 13/5z^{-1} + 5)\; 5/2z^{-1} + 9/2 \quad (25/26 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{5/2z^{-1} + 125/26} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad -8/13)\; 13/5z^{-1} + 5 \;\; (-169/40z^{-1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{13/5z^{-1}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 5)\; -8/13 \;\; (-8/65 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-8/13} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{\;0\;}
\end{array}$$

Comparing with $H(z)$ of the case-I, we get

$$\alpha_1 = 2, \beta_1 = -\frac{1}{2}, \alpha_2 = -\frac{4}{5}, \beta_2 = \frac{25}{26}, \alpha_3 = -\frac{169}{40}, \beta_3 = -\frac{8}{65}$$

Hence, the required transfer function for realization is

$$H(z) = 2z^{-1} + \cfrac{1}{-\cfrac{1}{2} + \cfrac{1}{-\cfrac{4}{5}z^{-1} + \cfrac{1}{\cfrac{25}{26} + \cfrac{1}{-\cfrac{169}{40}z^{-1} + \cfrac{1}{-\cfrac{8}{65}}}}}}$$



EXAMPLE: Obtain the direct form-I, direct form-II, cascade and parallel form realizations of the LTI system governed by the equation

$$y(n) = -\frac{13}{12}y(n-1) - \frac{9}{24}y(n-2) - \frac{1}{24}y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$$

Solution:

Direct form-I

Given $y(n) = -\frac{13}{12}y(n-1) - \frac{9}{24}y(n-2) - \frac{1}{24}y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$

Taking Z-transform on both sides, we get

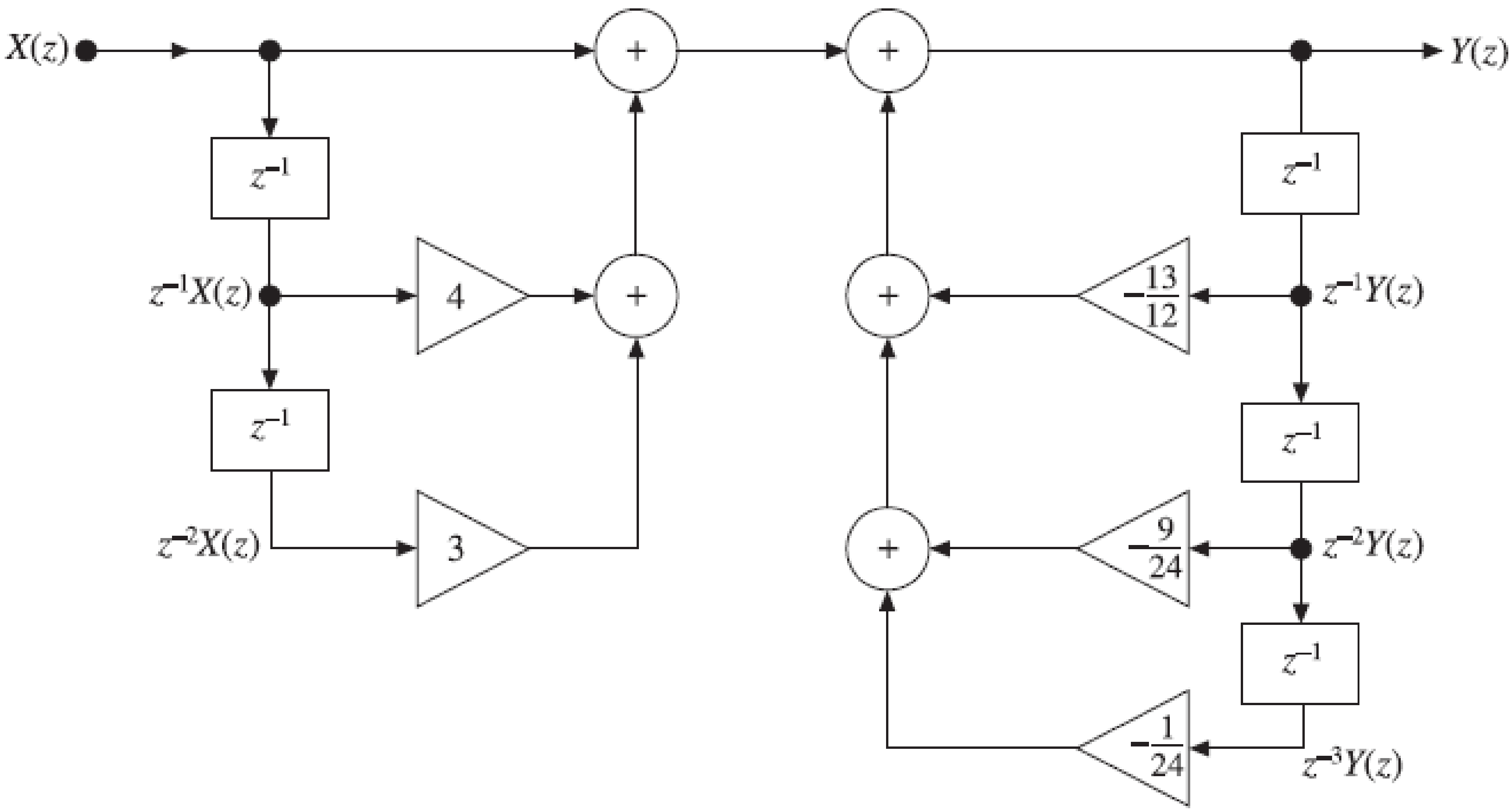
$$Y(z) = -\frac{13}{12}z^{-1}Y(z) - \frac{9}{24}z^{-2}Y(z) - \frac{1}{24}z^{-3}Y(z) + X(z) + 4z^{-1}X(z) + 3z^{-2}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 4z^{-1} + 3z^{-2}}{1 + (13/12)z^{-1} + (9/24)z^{-2} + (1/24)z^{-3}}$$

Let $$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)}\,\frac{W(z)}{X(z)}$$

$$\frac{W(z)}{X(z)} = 1 + 4z^{-1} + 3z^{-2} \qquad \frac{Y(z)}{W(z)} = \frac{1}{1 + (13/12)\, z^{-1} + (9/24)\, z^{-2} + (1/24)\, z^{-3}}$$

The direct form-I structure can be obtained from the above equation as shown in Figure .



Direct form-II

Taking Z-transform of the given difference equation, we have

$$Y(z) = -\frac{13}{12} z^{-1} Y(z) - \frac{9}{24} z^{-2} Y(z) - \frac{1}{24} z^{-3} Y(z) + X(z) + 4z^{-1} X(z) + 3z^{-2} X(z)$$

the transfer function of the system is

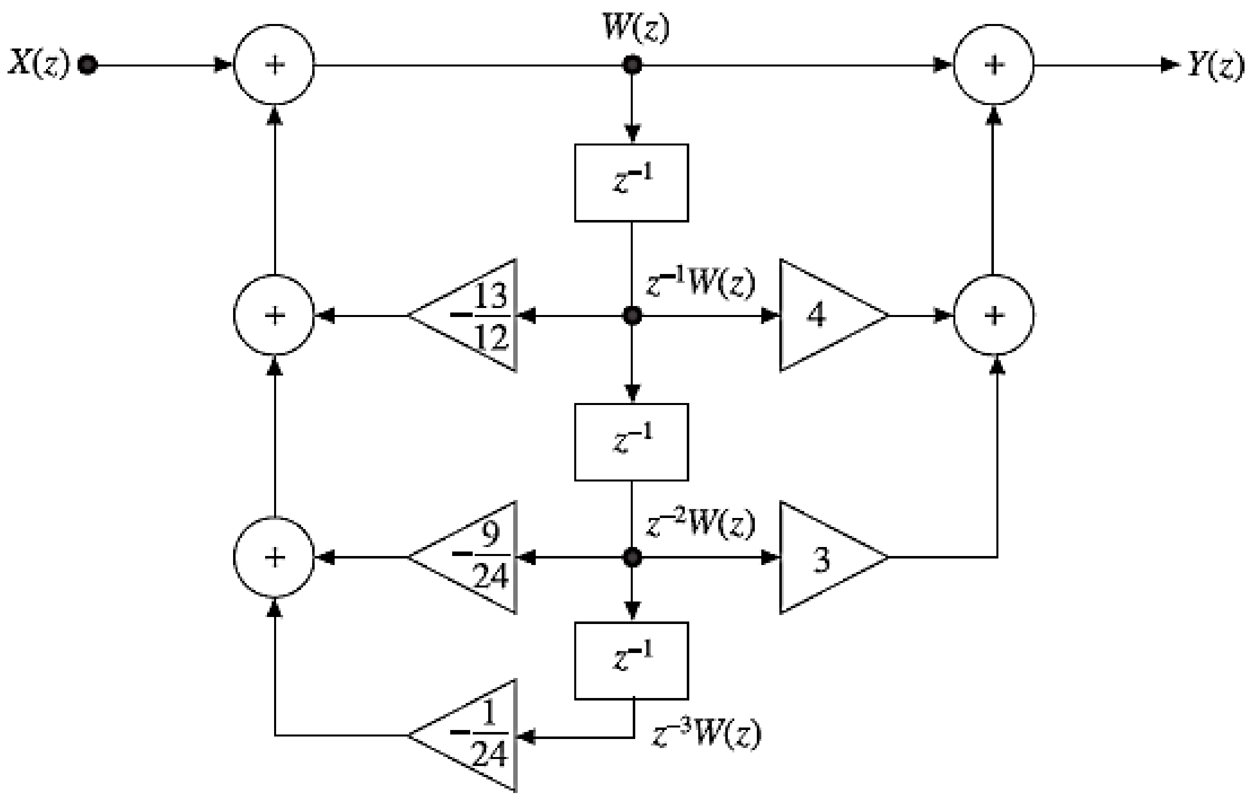
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 4z^{-1} + 3z^{-2}}{1 + (13/12) z^{-1} + (9/24) z^{-2} + (1/24) z^{-3}}$$

Let
$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \frac{W(z)}{X(z)}$$

where
$$\frac{W(z)}{X(z)} = \frac{1}{1 + (13/12) z^{-1} + (9/24) z^{-2} + (1/24) z^{-3}}$$

$$\frac{Y(z)}{W(z)} = 1 + 4z^{-1} + 3z^{-2}$$

The above equations for $W(z)$ and $Y(z)$ can be realized by a direct form-II structure as shown in Figure.



Cascade form

The transfer function is $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1 + 4z^{-1} + 3z^{-2}}{1 + (13/12)\,z^{-1} + (9/24)\,z^{-2} + (1/24)\,z^{-3}}$

Factorizing the numerator and denominator, we have

$$H(z) = \frac{(1 + z^{-1})\,(1 + 3z^{-1})}{[1 + (1/2)\,z^{-1}][1 + (1/3)\,z^{-1}][1 + (1/4)\,z^{-1}]}$$

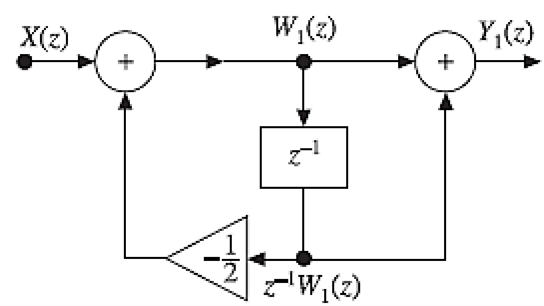
Since there are three first order factors in the denominator of $H(z)$, $H(z)$ can be expressed as product of three sections.

Let $$H(z) = H_1(z)\;\; H_2(z)\;\; H_3(z)$$

where $H_1(z) = \dfrac{1 + z^{-1}}{1 + (1/2)\,z^{-1}}$, $H_2(z) = \dfrac{1 + 3z^{-1}}{1 + (1/3)z^{-1}}$ and $H_3(z) = \dfrac{1}{1 + (1/4)z^{-1}}$

Let $$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{Y_1(z)}{W_1(z)} \cdot \frac{W_1(z)}{X(z)} = \frac{1 + z^{-1}}{1 + (1/2)z^{-1}}$$

where $\dfrac{W_1(z)}{X(z)} = \dfrac{1}{1 + (1/2)\,z^{-1}}$ and $\dfrac{Y_1(z)}{W_1(z)} = 1 + z^{-1}$



(a)

$\therefore$ $$W_1(z) = X(z) - \frac{1}{2}z^{-1}W_1(z)$$

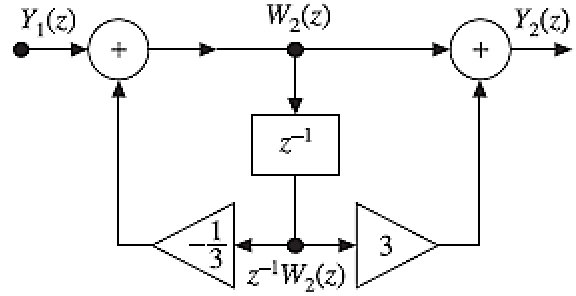
and $$Y_1(z) = W_1(z) + z^{-1}W_1(z)$$

Let $$H_2(z) = \frac{Y_2(z)}{Y_1(z)} = \frac{Y_2(z)}{W_2(z)} \cdot \frac{W_2(z)}{Y_1(z)} = \frac{1 + 3z^{-1}}{1 + (1/3)z^{-1}}$$

where $$\frac{W_2(z)}{Y_1(z)} = \frac{1}{1 + (1/3)z^{-1}} \text{ and } \frac{Y_2(z)}{W_2(z)} = 1 + 3z^{-1}$$

$\therefore$ $$W_2(z) = Y_1(z) - \frac{1}{3} z^{-1} W_2(z)$$

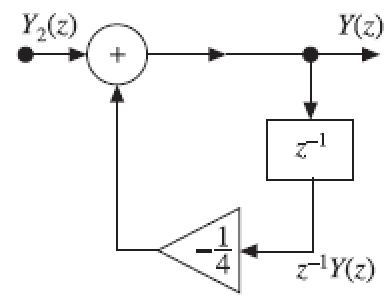
and $$Y_2(z) = W_2(z) + 3z^{-1} W_2(z)$$



(b)

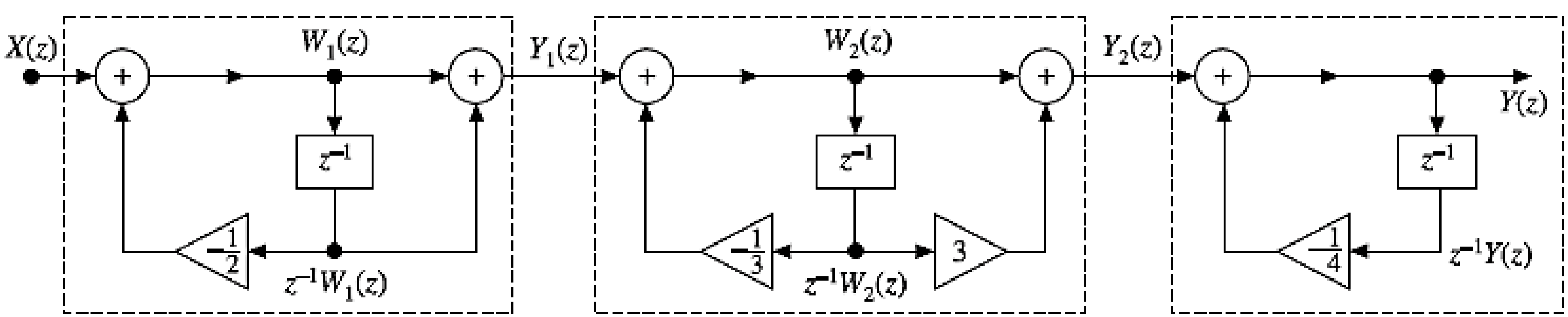
Let $$H_3(z) = \frac{Y(z)}{Y_2(z)} = \frac{1}{1 + (1/4)z^{-1}}$$

$\therefore$ $$Y(z) = Y_2(z) - \frac{1}{4} z^{-1} Y(z)$$



(c)

The cascade structure of the given system is obtained by connecting the individual sections shown in above Figures and is shown in below



Parallel form

Consider the equation $H(z) = \dfrac{(1+z^{-1})(1+3z^{-1})}{[1+(1/2)z^{-1}][1+(1/3)z^{-1}][1+(1/4)z^{-1}]}$

By partial fraction expansion, we have

$$H(z) = \frac{A}{1+(1/2)z^{-1}} + \frac{B}{1+(1/3)z^{-1}} + \frac{C}{1+(1/4)z^{-1}}$$

where, the coefficients A, B and C are

$$A = \frac{(1+z^{-1})(1+3z^{-1})}{\left(1+\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}\Bigg|_{z^{-1}=-2} = \frac{(1-2)(1-6)}{\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)} = \frac{5}{\frac{1}{3}\cdot\frac{1}{2}} = 30$$

$$B = \frac{(1+z^{-1})(1+3z^{-1})}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}\Bigg|_{z^{-1}=-3} = \frac{(1-3)(1-9)}{\left(1-\frac{3}{2}\right)\left(1-\frac{3}{4}\right)} = \frac{16}{-\frac{1}{2}\cdot\frac{1}{4}} = -128$$

$$C = \frac{(1+z^{-1})(1+3z^{-1})}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}\Bigg|_{z^{-1}=-4} = \frac{(1-4)(1-12)}{(1-2)\left(1-\frac{4}{3}\right)} = \frac{33}{-1\cdot-\frac{1}{3}} = 99$$

$\therefore \qquad H(z) = \dfrac{30}{1+(1/2)z^{-1}} - \dfrac{128}{1+(1/3)z^{-1}} + \dfrac{99}{1+(1/4)z^{-1}}$

Let $\qquad H(z) = \dfrac{Y(z)}{X(z)}$

$\therefore \qquad \dfrac{Y(z)}{X(z)} = \dfrac{30}{1+(1/2)z^{-1}} - \dfrac{128}{1+(1/3)z^{-1}} + \dfrac{99}{1+(1/3)z^{-1}}$

$\therefore \qquad Y(z) = \dfrac{30}{1+(1/2)z^{-1}}X(z) - \dfrac{128}{1+(1/3)z^{-1}}X(z) + \dfrac{99}{1+(1/4)z^{-1}}X(z)$

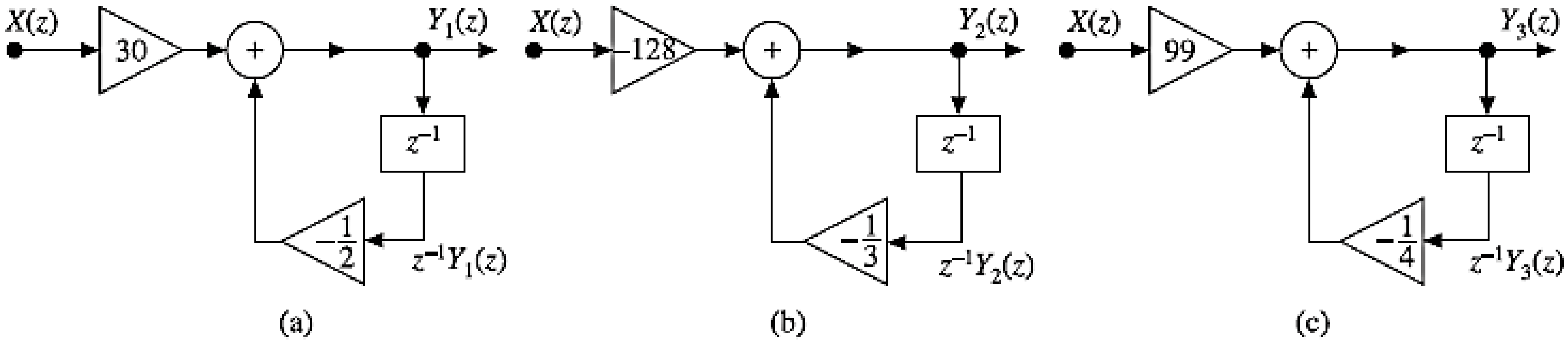
Let $\qquad Y(z) = Y_1(z) + Y_2(z) + Y_3(z)$

where $Y_1(z) = \dfrac{30}{1+(1/2)z^{-1}}X(z)$; $Y_2(z) = -\dfrac{128}{1+(1/3)z^{-1}}X(z)$ and $Y_3(z) = \dfrac{99}{1+(1/4)z^{-1}}X(z)$

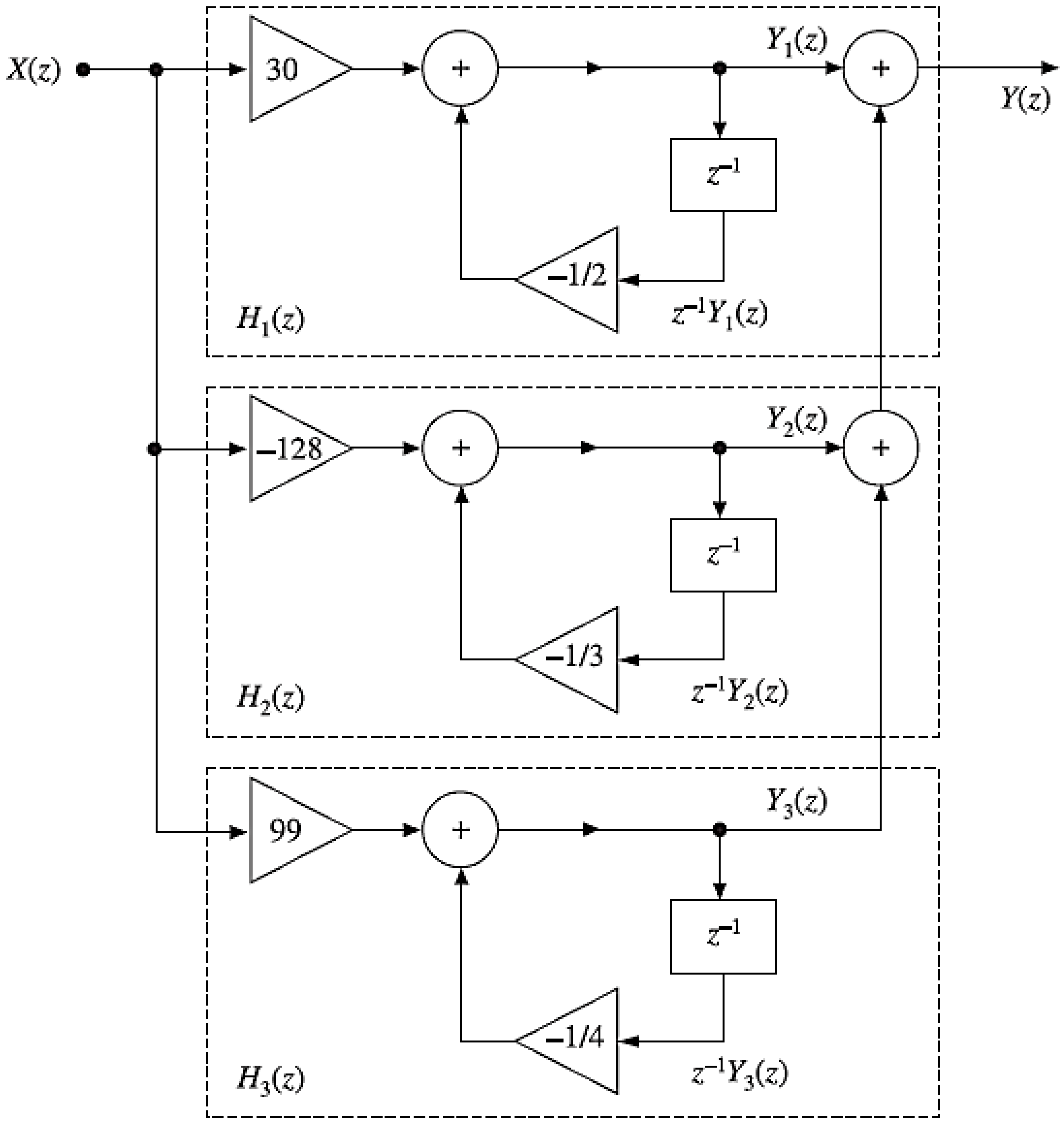
Let $\qquad H_1(z) = \dfrac{Y_1(z)}{X(z)} = \dfrac{30}{1+(1/2)z^{-1}}$; $H_2(z) = \dfrac{Y_2(z)}{X(z)} = -\dfrac{128}{1+(1/2)z^{-1}}$

and $\qquad H_3(z) = \dfrac{Y_3(z)}{X(z)} = \dfrac{99}{1+(1/4)z^{-1}}$

The transfer functions can be realized in direct form-I structures as shown in Figure



The overall structure is obtained by connecting the individual sections in above Figures in parallel as shown in below Figure.



Structures for realization of FIR Systems

- ✓ FIR (Finite duration impulse response) systems are the systems whose impulse response consists of finite number of samples. They are designed by using only a finite number of samples of the infinite duration impulse response.
- ✓ The convolution formula for FIR system is given by

$$y(n) = \sum_{k=0}^{N-1} h(k)\,x(n-k)$$

where $h(n) = 0$ for $n < 0$ and $n \geq N$.

- ✓ This implies that, the impulse response selects only N samples of the input signal.
- ✓ In effect, the system acts as a window that views only the most recent N input signal samples in forming the output.
- ✓ It neglects or simply forgets all prior input samples. Thus, a FIR system has a finite memory of length N samples.

- ✓ A system whose output depends only on the present and past inputs and not on past outputs is called a non-recursive system. Hence, for non-recursive systems, the output $y(n)$ is given by

$$y(n) = \mathrm{F}[x(n), x(n-1), \ldots, x(n-M)]$$

Hence, in general, an FIR system is of non-recursive type.

- ✓ In non-recursive system, $y(n_0)$ can be computed immediately without having $y(n_0 - 1)$, $y(n_0 - 2)$, ... Hence, the output of non-recursive system can be computed in any order.

[i.e. $y(50)$, $y(5)$, $y(2)$, $y(100)$, ...]

Transfer function of FIR Systems

In general, an FIR system is described by the difference equation

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k)$$

i.e. in general, in Finite Impulse Response (FIR) systems, the output at any instant depends only on the present and the past inputs. It does not depend on the past outputs.

- ✓ Taking Z-transform on both sides of equation for $y(n)$, we get

$$y(z) = \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

- ✓ The transfer function of the FIR system is:

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{N-1} b_k z^{-k} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{N-1} z^{-(N-1)}$$

Also, we know that $H(z) = Z[h(n)]$, where $h(n)$ is the impulse response of the FIR system.

Let us replace the index n by k.

$$\therefore \qquad H(z) = \sum_{k=0}^{N-1} h(k) z^{-k} = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots + h(n-1)z^{-(N-1)}$$

On comparing the above two equations for $H(z)$, we can say that $b_k = h(k)$ for $k = 0, 1, 2, ..., (N - 1)$.

- ✓ The above equations for $Y(z)$ and $H(z)$ can be viewed as a computational procedure to determine the output sequence $y(n)$ from the input sequence $x(n)$.
- ✓ These equations can be used to construct the block diagram of the system using delays, adders and multipliers.
- ✓ This block diagram is referred to as realization of the system or equivalently as a structure for realizing the system.

The different types of structures for realizing FIR systems are:

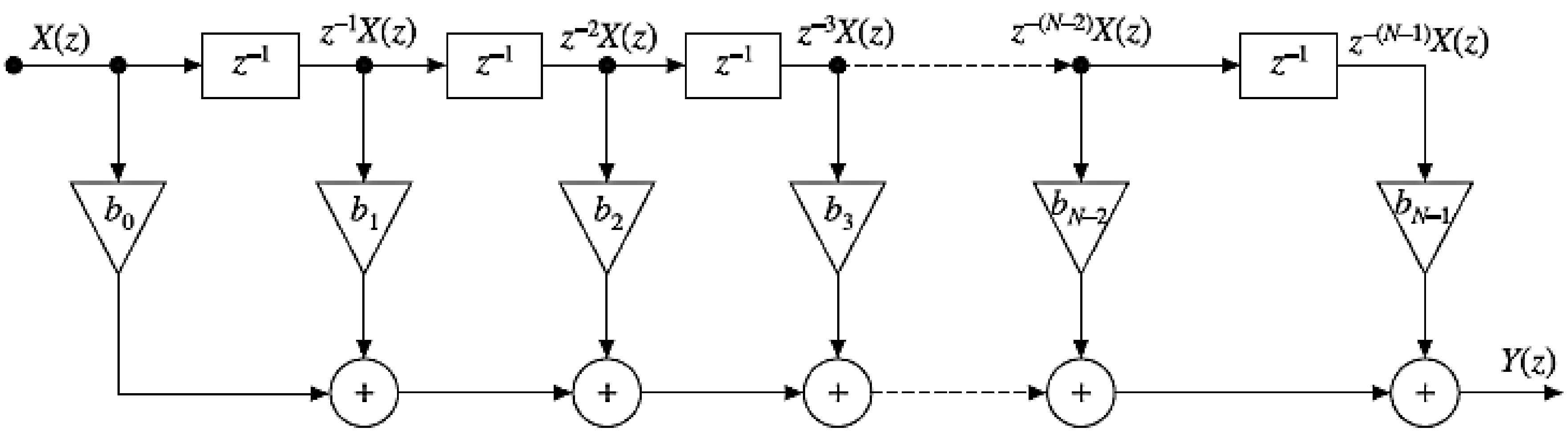
1. Direct form realization
2. Transposed form realization
3. Cascade realization
4. Lattice structure realization
5. Linear phase realization

Direct form realization

- ✓ Since there are no denominator components or poles in an FIR system, the direct form has only one structure which is called *direct form.*
- ✓ It realizes directly either the difference equation or the system function.
- ✓ The direct form structure can be obtained from the general equation for $Y(z)$ governing the FIR system.

$$Y(z) = \sum_{k=0}^{N-1} b_k z^{-k} X(z)$$

$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \cdots + b_{N-2} z^{-(N-2)} X(z) + b_{N-1} z^{-(N-1)} X(z)$$

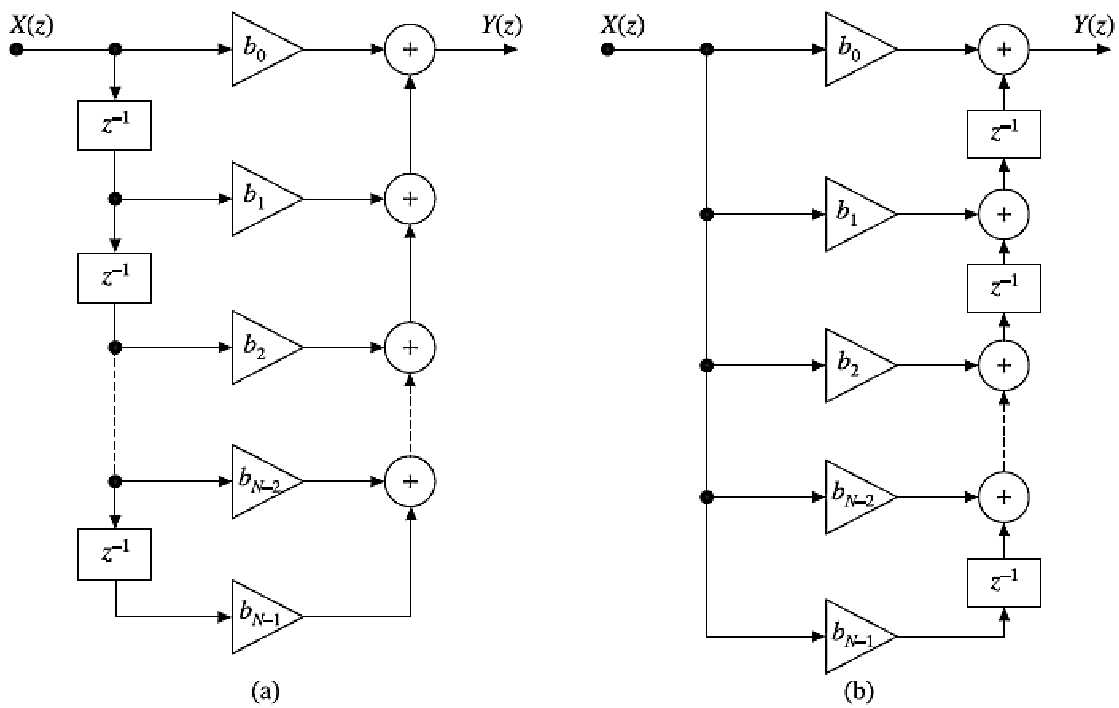


Transposed Form Structure Realization

The transposed form structure realization has already been discussed earlier with respect to IIR systems. The same holds good for FIR systems also. The procedure to realize transposed form is

1. First realize the given difference equation or transfer function by using the direct form structure.
2. Reverse or transpose the direction of signal flow and interchange the input and output nodes.
3. Replace the junction points by adders and adders by junction points.
4. Fold the structure, which is the transposed form realization of an FIR system.

- ✓ In general, the transposed structure realization of an FIR system has no advantages compared to the direct form structure.
- ✓ Whatever be the number of additions, multiplications and storage components needed for the direct form structure, the same number of elements are needed for the transposed structure too.
- ✓ The direct form structure and the transposed form structure of a general *N*th order FIR system are shown in Figure.



EXAMPLE: Realize the second order FIR system

$$y(n) = 2x(n) + 4x(n-1) - 3x(n-2)$$

by using the transposed form structure.

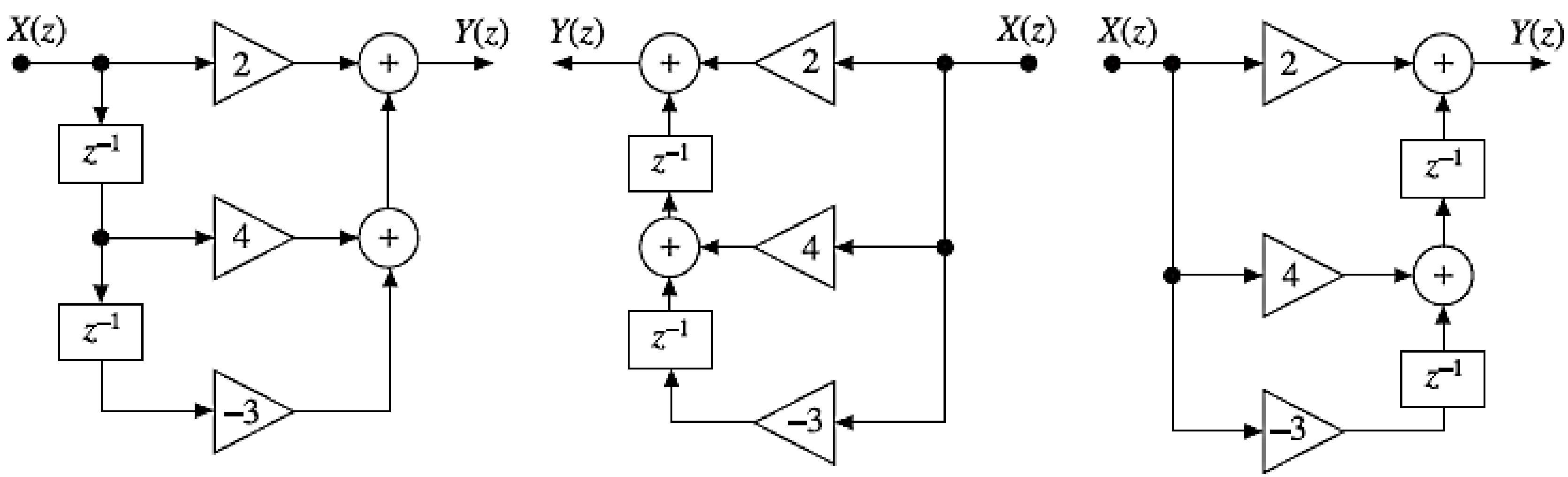
Solution: The given system is:

$$y(n) = 2x(n) + 4x(n-1) - 3x(n-2)$$

Taking Z-transform on both sides, we have

$$Y(z) = 2X(z) + 4z^{-1}X(z) - 3z^{-2}X(z)$$

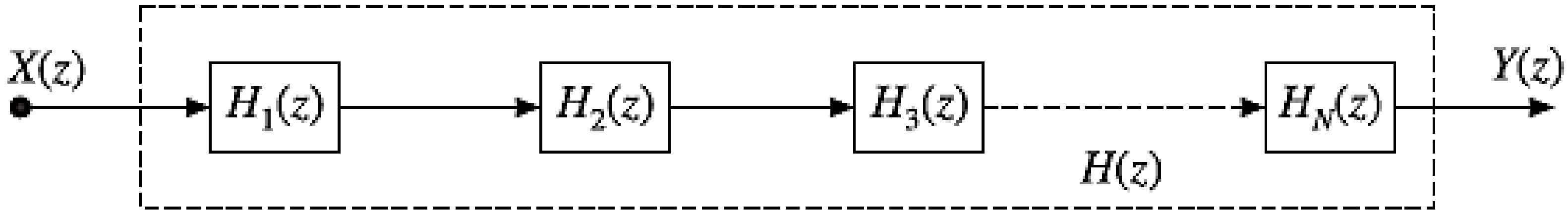
The direct form structure, the recovered realization structure and the transposed structure are shown in Figure.



(a) (b) (c)

Cascade realization

- ✓ The cascade form structure is nothing, but a cascade connection or a series connection of direct form structures.
- ✓ Hence, in cascade structure, the given transfer function $H(z)$ is broken into the product of many sub-transfer functions $H_1(z)$, $H_2(z)$, ..., $H_N(z)$ and each of these sub-transfer functions is realized separately in direct form structure and all these are connected in series or cascade.
- ✓ The block diagram representation of a cascade form structure is shown in Figure.

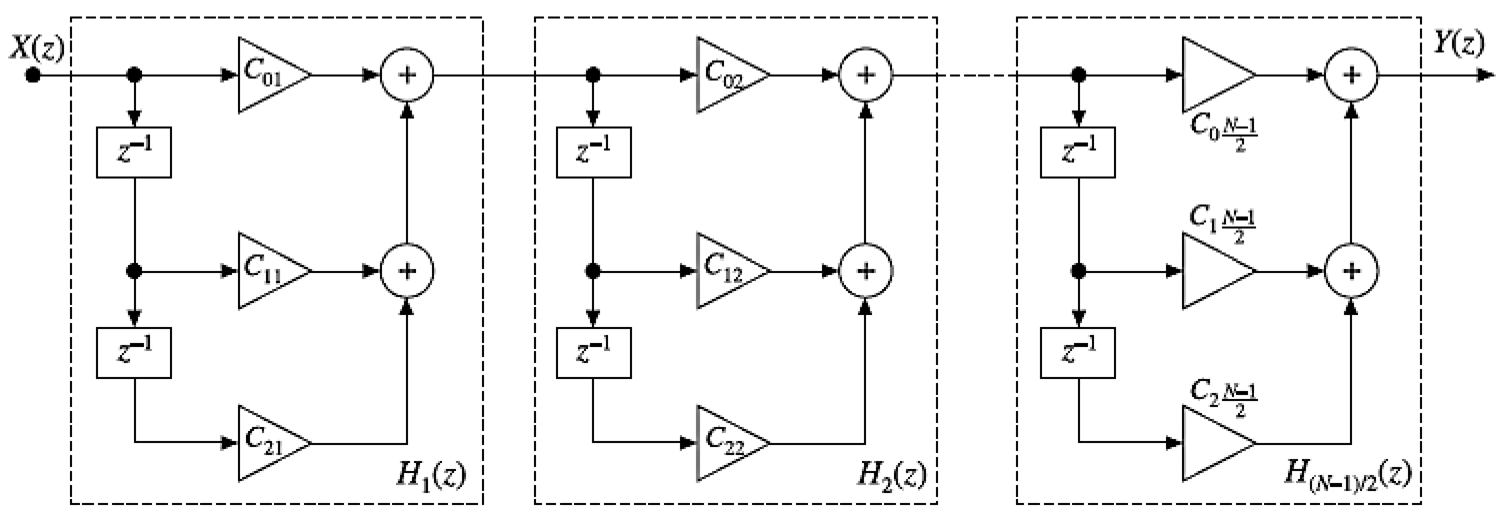


✓ The transfer function of FIR system, $H(z)$ is an $(N-1)^{th}$ order polynomial in z.

✓ When N is odd, then $(N-1)$ will be even and so $H(z)$ will have $(N-1)/2$ second order factors. When N is odd, then

$$H(z)=\sum_{k=0}^{N-1} b_k z^{-k} = \prod_{i=1}^{\frac{N-1}{2}} (C_{0i}+C_{1i}z^{-1}+C_{2i}z^{-2}) = H_1 H_2 \ldots H_{\frac{N-1}{2}}$$

✓ Each second order factor of the above equation for $H(z)$ can be realized in direct form and all the second order systems are connected in cascade to realize $H(z)$ as shown in Figure.

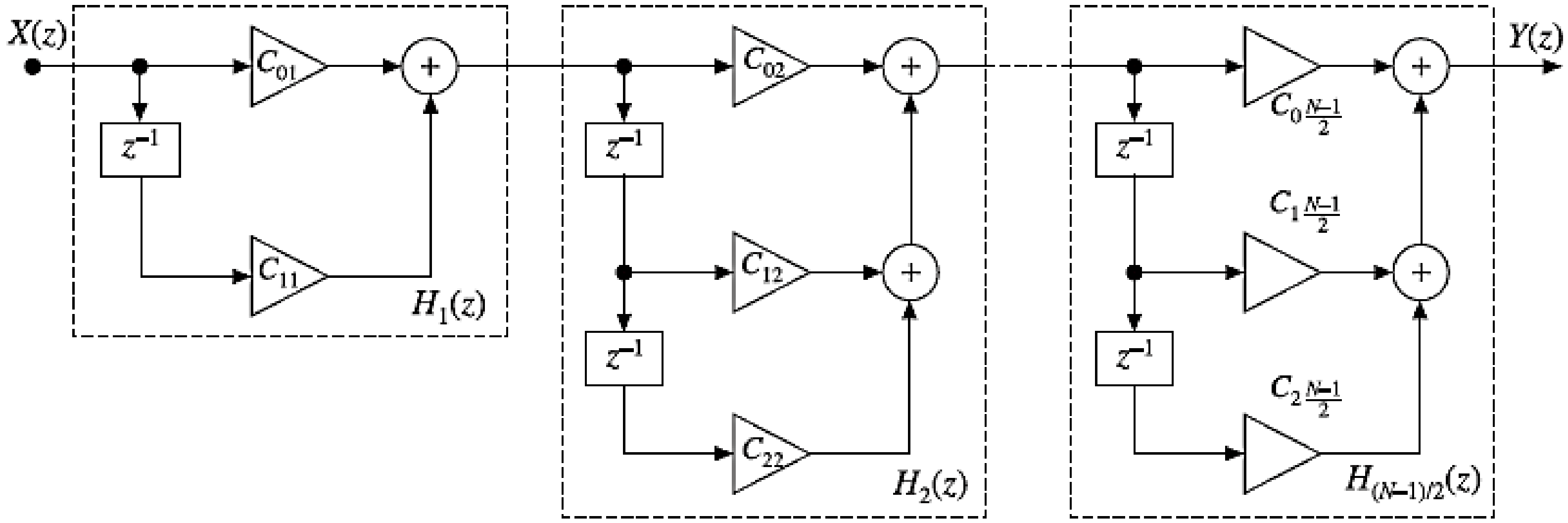


- ✓ When N is even, then $(N - 1)$ will be odd and so $H(z)$ will have one first order factor and $(N - 2)/2$ second order factors. When N is even, then

$$H(z) = \sum_{k=0}^{N-1} b_k z^{-k} = (C_{0i} + C_{1i} z^{-1}) \prod_{i=2}^{N/2} (C_{0i} + C_{1i} z^{-1} + C_{2i} z^{-2})$$

$$= H_1, H_2, \ldots, H_{N/2}$$

- ✓ In this case, the cascade structure will have one first order section and $(N - 2)/2$ second order sections. Each one of them can be realized in direct form and all of them connected in cascade as shown in Figure.



Example: Obtain direct form and cascade form realisation for the transfer function of an FIR system given by

$$H(z) = \left(1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2}\right)\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right)$$

Solution

Direct Form Realisation Expanding the transfer function $H(z)$, we have

$$H(z) = 1 - \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} + \frac{5}{64}z^{-3} - \frac{3}{16}z^{-4}$$

This function is realised in FIR direct form as shown in Fig. E9.13(a).

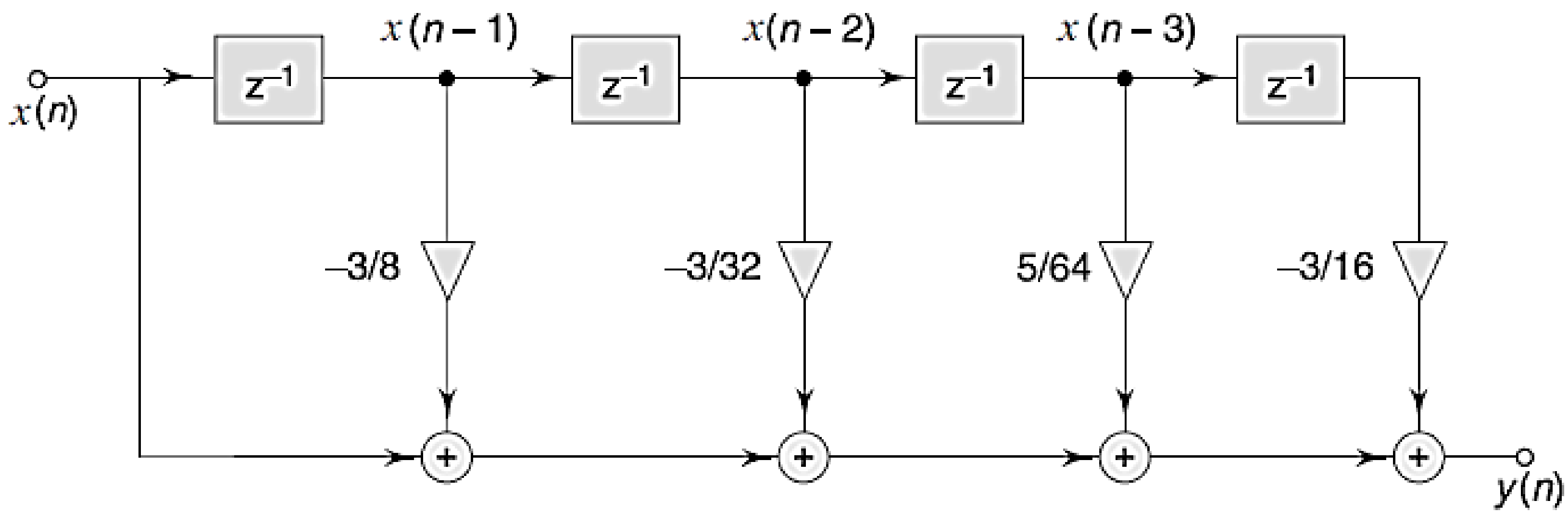


Fig. E9.13(a)

Cascade Realisation

$$H(z)=\left(1-\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}\right)\left(1-\frac{1}{8}z^{-1}-\frac{1}{2}z^{-2}\right) = H_1(z)\cdot H_2(z)$$

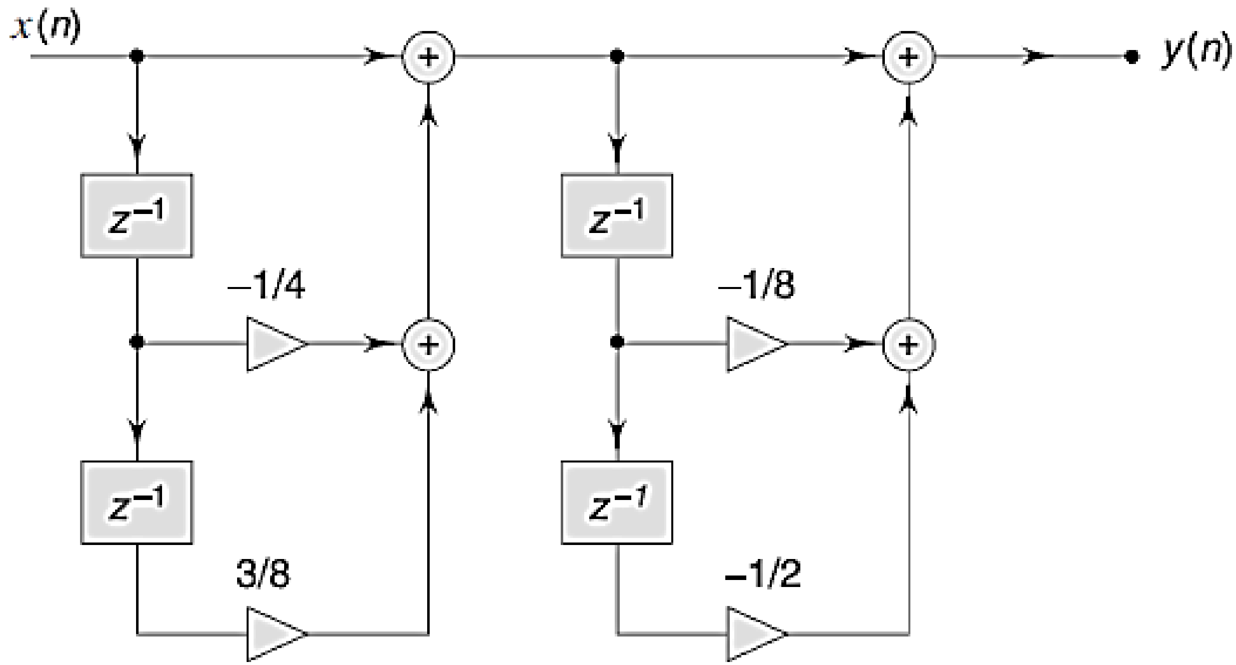
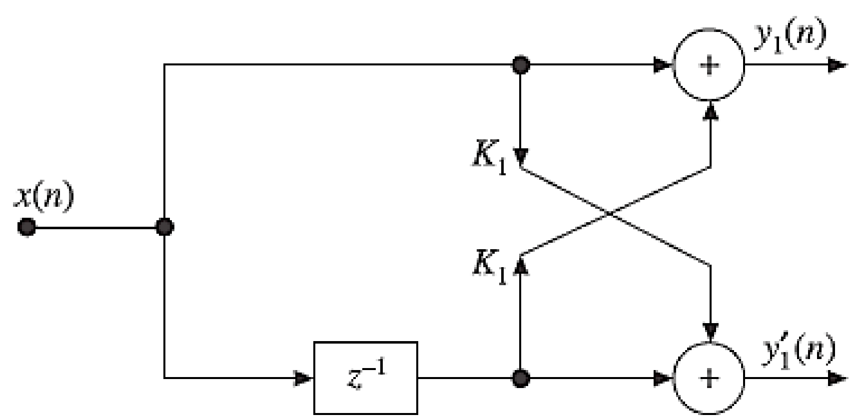


Fig. E9.13(b)

This function is realised in FIR cascade form as shown in Fig. E9.13(b).

Lattice Structure Realization

- ✓ The FIR structures discussed till now are used in general. However, the lattice structure is extensively used in digital speech processing.
- ✓ The lattice structure consists of two different paths through which the input $x(n)$ is processed. Hence, the lattice structure has two different output set-ups: $y(n)$ and $y'(n)$.
- ✓ $y(n)$ is the real output and $y'(n)$ is the supporting output, which offers support for obtaining the output for the next stage.
- ✓ A single stage lattice structure is shown in Figure, wherein K is called the reflection coefficient.

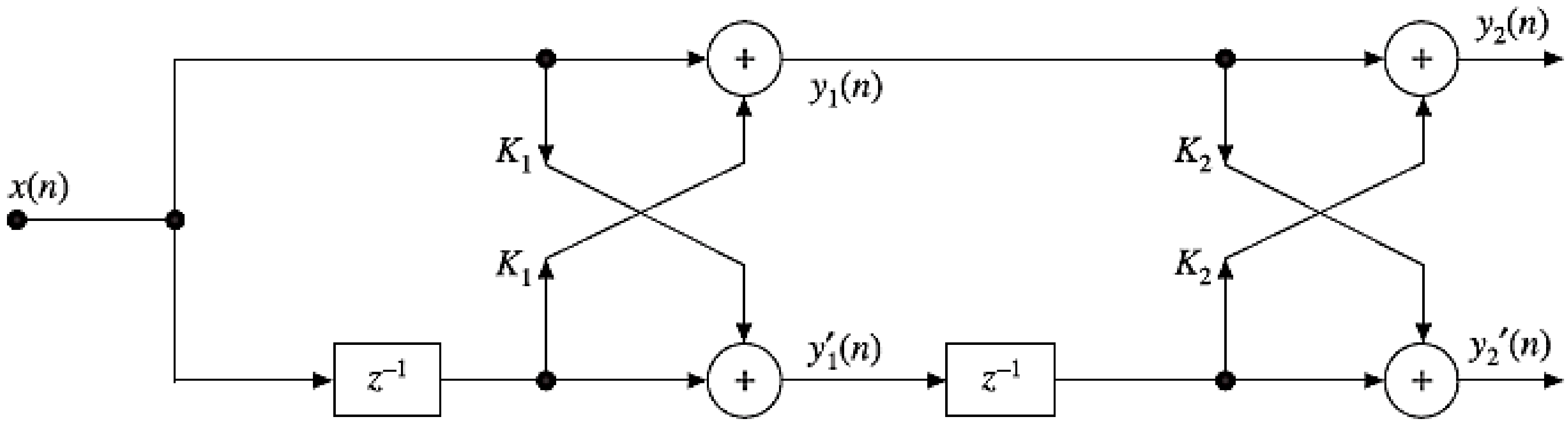


- ✓ Therefore, the output from the first stage of the lattice structure, which is the first order FIR system, is given below.

$$y_1(n) = x(n) + K_1 x(n-1)$$

$$y_1'(n) = K_1 x(n) + x(n-1)$$

- ✓ Similarly, the two-stage lattice structure is shown in Figure.



- ✓ The output from the second stage of Figure, which is the second order FIR system is given as follows:

$$y_2(n) = y_1(n) + K_2 y_1'(n-1)$$

$$y_2'(n) = K_2 y_1(n) + y_1'(n-1)$$

Substituting the values of $y_1(n)$ and $y'_1(n)$ in the above equations for $y_2(n)$ and $y'_2(n)$ we get

$$y_2(n) = [x(n) + K_1 x(n-1)] + K_2[K_1 x(n-1) + x(n-2)]$$

or

$$y_2(n) = x(n) + K_1[1 + K_2]x(n-1) + K_2 x(n-2)$$

$$y'_2(n) = K_2[x(n) + K_1 x(n-1)] + K_1 x(n-1) + x(n-2)]$$

or

$$y'_2(n) = K_2 x(n) + K_1[1 + K_2]x(n-1) + x(n-2)$$

Let $\alpha_2(0) = 1$, $\alpha_2(1) = K_1(1 + K_2)$ and $\alpha_2(2) = K_2$

Then the equation for $y_2(n)$ changes to

$$y_2(n) = \alpha_2(0)\,x(n) + \alpha_2(1)\,x(n-1) + \alpha_2(2)\,x(n-2)$$

or

$$y_2(n) = \sum_{k=0}^{2} \alpha_2(k)\,x(n-k)$$

Therefore, in general, the output of the *m*th order FIR system by using the lattice structure can be written as:

$$y_m(n) = \sum_{k=0}^{m} \alpha_m(k)x(n-k)$$

The above equation is the convolution sum. Using the convolution property it can be re-written as:

$$Y_m(z) = \alpha_m(z)\,X(z)$$

Hence, the transfer function of the FIR system can be obtained as

$$\alpha_m(z) = \frac{Y_m(z)}{X(z)}$$

The FIR filter having a system function given above is called the *forward prediction*.

Procedure to realize the lattice structure of FIR system

1. If the coefficient of the present input $x(n)$ is not unity, convert it to unity by taking common of the coefficient of the present input.
2. Find the order of the difference equation and compare the coefficients of the given difference equation with the coefficients of the same order lattice structure output involving the reflection coefficients K_1, K_2, K_3, ...
3. Assign the calculated values of K_1, K_2, K_3, ... and construct the structure.

In order to realize by using lattice structure, it is enough to find the values of reflection coefficients. Therefore, it is easily programmable. But the number of components used, specially adders and multipliers will increase.

EXAMPLE: Realize a system with $H(z) = 5 + 3z^{-1}$ by using lattice structure.

Solution: Given
$$H(z) = \frac{Y(z)}{X(z)} = 5 + 3z^{-1}$$

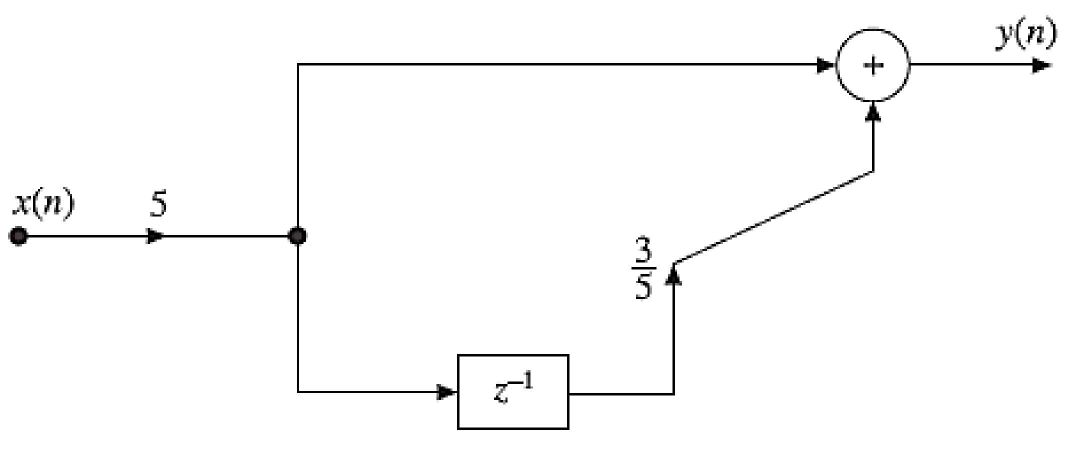
or
$$Y(z) = 5X(z) + 3z^{-1}X(z)$$

Taking the inverse Z-transform on both sides, we get

$$\begin{aligned} y(n) &= 5x(n) + 3x(n-1) \\ &= 5[x(n) + \frac{3}{5}x(n-1)] = 5p(n) \end{aligned}$$

This is of first order. So comparing $p(n) = x(n) + \frac{3}{5}x(n-1)$ with the standard equation for first order lattice structure, i.e. with $y(n) = x(n) + K_1x(n-1)$, we get $K_1 = \frac{3}{5}$

Therefore, the lattice structure realization of the given first order FIR system is as shown in Figure .



EXAMPLE: Determine the lattice coefficients corresponding to the FIR system with the system function $H(z) = 1 + \frac{7}{9}z^{-1} + \frac{3}{5}z^{-2}$ and realize it.

Solution: Given
$$H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{7}{9}z^{-1} + \frac{3}{5}z^{-2}$$

$\therefore$
$$Y(z) = X(z) + \frac{7}{9}z^{-1}X(z) + \frac{3}{5}z^{-2}X(z)$$

Taking inverse Z-transform on both sides, we get

$$y(n) = x(n) + \frac{7}{9}x(n-1) + \frac{3}{5}x(n-2)$$

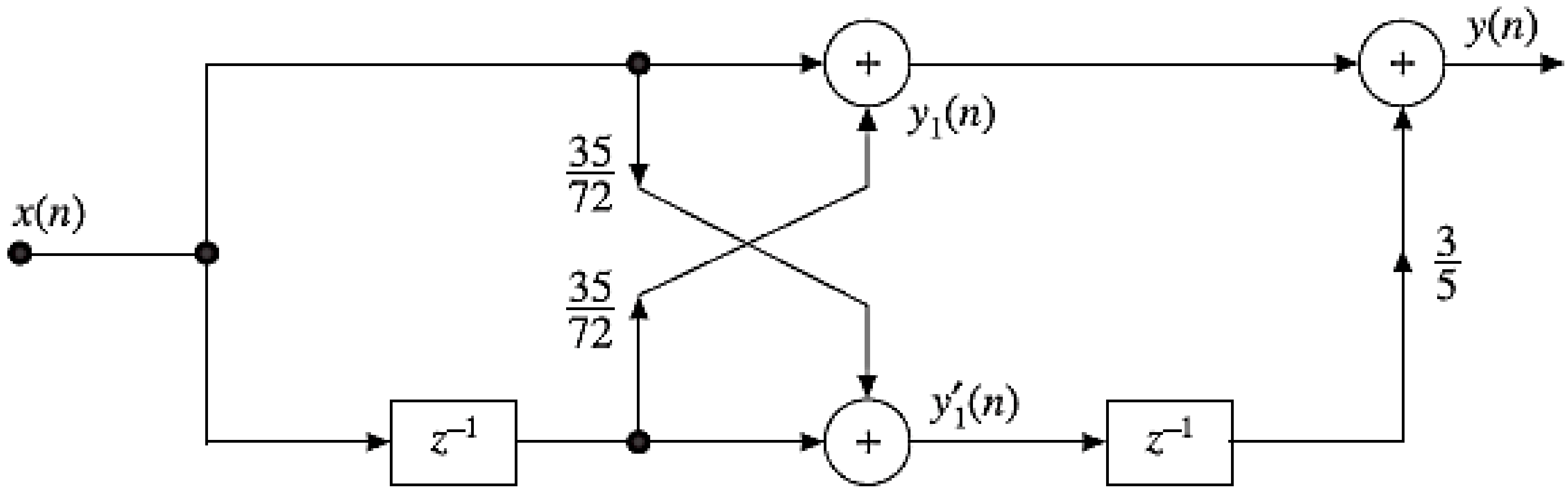
This corresponds to a second order system. Comparing this with the standard equation for second order lattice structure, i.e. with $y(n) = x(n) + K_1(1 + K_2)x(n-1) + K_2x(n-2)$, we get

$$K_2 = \frac{3}{5}$$

and
$$K_1(1 + K_2) = \frac{7}{9}$$

or
$$K_1 = \frac{7/9}{1 + (3/5)} = \frac{35}{72}$$

Therefore, the lattice structure realization of the given FIR system is shown in Figure .



$x(n)$
$\frac{35}{72}$
$\frac{35}{72}$
z^{-1}
$y_1(n)$
$y'_1(n)$
z^{-1}
$\frac{3}{5}$
$y(n)$

Linear Phase Realisation

- An FIR system is said to be linear phase, if it satisfies the condition

$$h(k) = \pm h(N - 1 - k)$$

- An FIR system which satisfies the relation $h(k) = h(N-1-k)$ is called a *symmetric FIR system* and an FIR system which satisfies the relation $h(k) = -h(N-1-k)$ is called an anti symmetric FIR system.
- Here we discuss symmetric FIR systems. The symmetric condition may be viewed as 1. Odd symmetry 2. Even symmetry
- The impulse response is symmetrical, i.e. $h(k) = h(N-1-k)$ means

$$h(0) = h(N-1);\ h(1) = h(N-2);\ \ldots;\ h\left(\frac{N}{2}-1\right) = h\left(\frac{N}{2}\right)$$

- By using this symmetry condition, it is possible to reduce the number of multipliers required for the realization of FIR system.

Odd Symmetry (N=Odd)

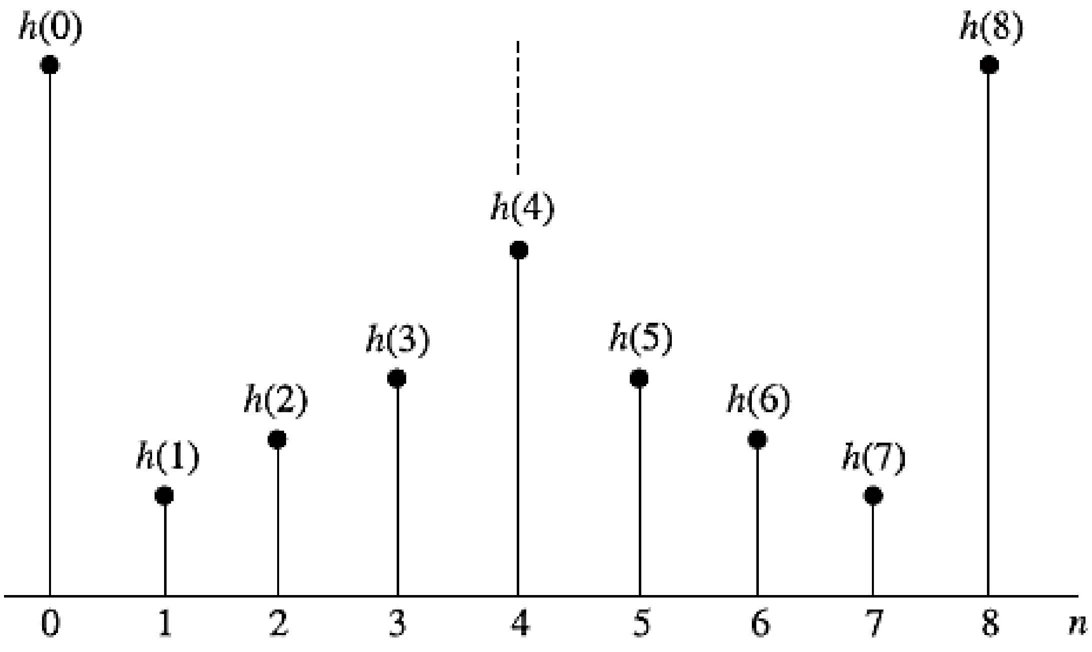
A function is said to be of odd symmetry if it has an odd number of samples and satisfies the condition $h(k) = h(N - 1 - k)$.

For example, if $N = 11$ (i.e. N is odd), then a linear phase FIR system will have

$$h(0) = h(9-1-0) = h(8),\ h(1) = h(9-1-1) = h(7),\ h(2) = h(9-1-2) = h(6)$$

$$h(3) = h(9-1-3) = h(5),\ h(4) = h(9-1-4) = h(4)$$

In case of odd symmetry, it is obvious that the central sample is the same for both LHS and RHS. Hence, in case of odd symmetry, the common or central sample will be lying at $(N - 1)/2$ The graphical representation of the above impulse response for $N = 11$ is shown in Figure .



The linear phase FIR system can be remodeled as

$$H(z) = \sum_{k=0}^{N-1} h(k)\, z^{-k} = \frac{Y(z)}{X(z)}$$

or $$Y(z) = [h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots + h(N-2)z^{-(N-2)} + h(N-1)z^{-(N-1)}]X(z)$$

For an odd symmetry linear phase FIR system, we have

$$h(0) = h(N-1),\ h(1) = h(N-2),\ \ldots,\ h\left(\frac{N-3}{2}\right) = h\left(\frac{N+1}{2}\right),\ h\left(\frac{N-1}{2}\right) = h\left(\frac{N-1}{2}\right)$$

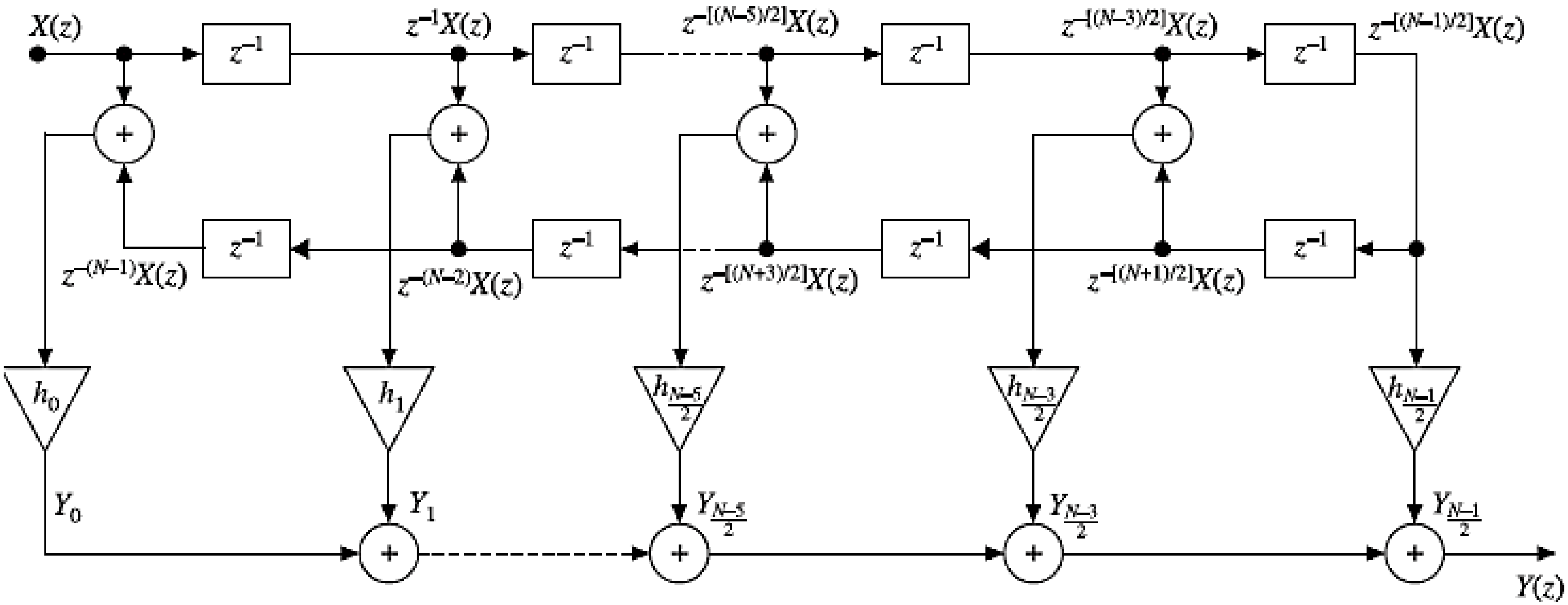
The impulse $h\left(\frac{N-1}{2}\right)$ will remain single.

Substituting these in the expression for $Y(z)$, we have the direct form structure of linear phase FIR system as

$$Y(z) = h_0[X(z) + z^{-(N-1)}X(z)] + h_1[z^{-1}X(z) + z^{-(N-2)}X(z)] + \cdots +$$

$$h_{\frac{N-5}{2}}\left[z^{-\frac{(N-5)}{2}}X(z) + z^{-\frac{(N+3)}{2}}X(z)\right] + h_{\frac{N-3}{2}}\left[z^{-\frac{(N-3)}{2}}X(z) + z^{-\frac{(N+1)}{2}}X(z)\right] + h_{\frac{N-1}{2}}z^{-\frac{(N-1)}{2}}X(z)$$

which is constructed as shown in Figure .



In Figure

$$Y_0 = h_0[X(z) + z^{-(N-1)}X(z)];\ Y_1 = h_1[z^{-1}X(z) + z^{-(N-2)}X(z)]$$

$$Y_{\frac{N-5}{2}} = h_{\frac{N-5}{2}}\left[z^{-\frac{(N-5)}{2}}X(z) + z^{-\frac{(N+3)}{2}}X(z)\right]$$

$$Y_{\frac{N-3}{2}} = h_{\frac{N-3}{2}}\left[z^{-\frac{(N-3)}{2}}X(z) + z^{-\frac{(N+1)}{2}}X(z)\right]$$

Even Symmetry (N=Even)

A function is said to have even symmetry if it has even number of samples and satisfies the condition $h(k) = h(N-1-k)$.

For example, if $N = 8$ (i.e. N is even), then a linear phase symmetric FIR system will have

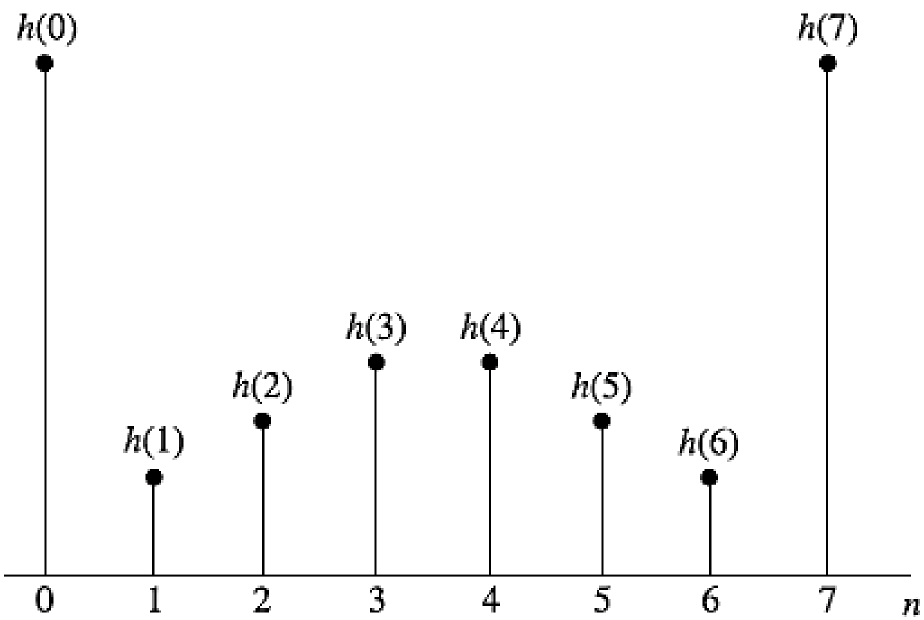
$$h(0) = h(8-1-0) = h(7),\ h(1) = h(8-1-1) = h(6),\ h(2) = h(8-1-2) = h(5)$$

and

$$h(3) = h(8-1-3) = h(4)$$

In the case of even symmetry, since there is no central sample, a virtual central sample is selected at $k = (N-1)/2$. In this case virtual sample is at $k = (8–1)/2 = 3.5$.

The graphical representation of the above impulse responses for $N = 8$ is shown in Figure .



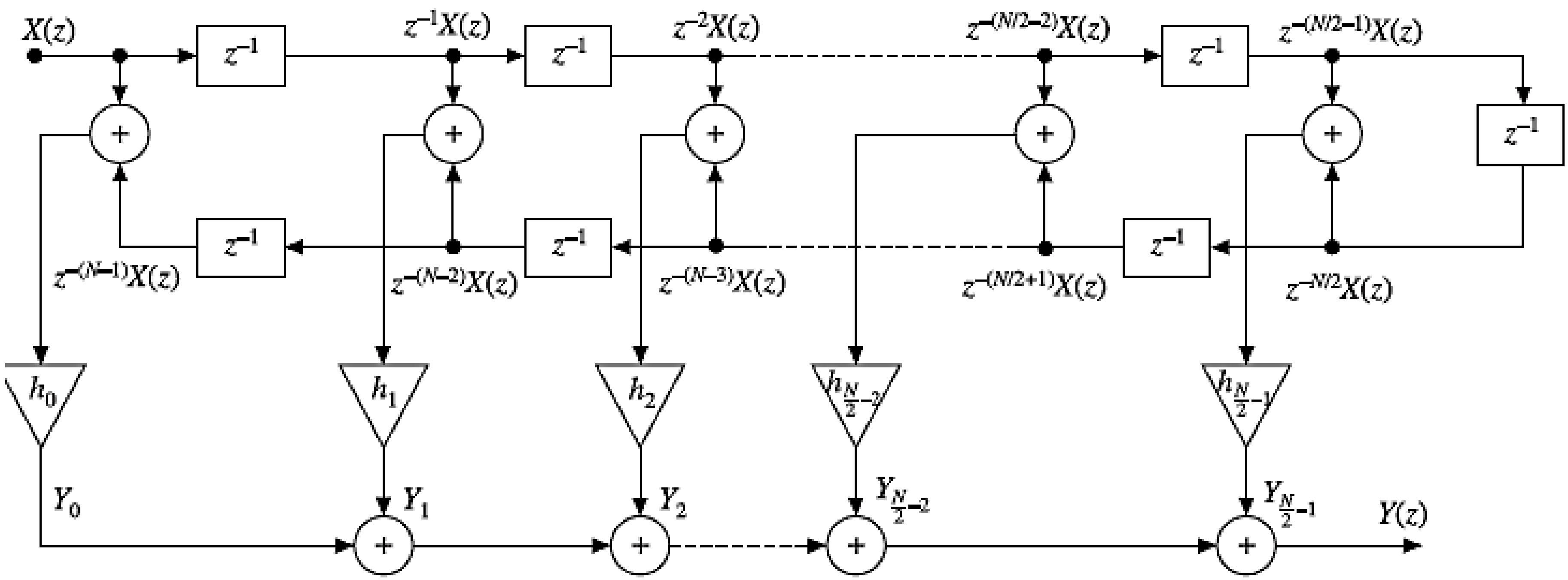
For an even symmetry linear phase FIR system, we have

$$h(0) = h(N-1),\ h(1) = h(N-2),\ \ldots,\ h\left(\frac{N}{2}-1\right) = h\left(\frac{N}{2}\right)$$

Substituting these in the expression for $Y(z)$, we have the direct form structure of linear phase FIR system as:

$$Y(z) = h_0[X(z) + z^{-(N-1)}X(z)] + h_1[z^{-1}X(z) + z^{-(N-2)}X(z)] + \cdots + h_{\frac{N}{2}-1}[z^{-\left(\frac{N}{2}-1\right)}X(z) + z^{-\frac{N}{2}}X(z)]$$

as shown in Figure .



In Figure

$$Y_0 = h_0[X(z) + z^{-(N-1)}X(z)]; \quad Y_1 = h_1[z^{-1}X(z) + z^{-(N-2)}X(z)]$$

$$Y_{\frac{N}{2}-2} = h_{\frac{N}{2}-2}\left[z^{-\left(\frac{N}{2}-2\right)}X(z) + z^{-\left(\frac{N}{2}+1\right)}X(z)\right]$$

$$Y_{\frac{N}{2}-1} = h_{\frac{N}{2}-1}\left[z^{-\left(\frac{N}{2}-1\right)}X(z) + z^{-\left(\frac{N}{2}\right)}X(z)\right]$$

$$Y_{\frac{N-1}{2}} = h_{\frac{N-1}{2}}\left[z^{-\left(\frac{N-1}{2}\right)}X(z)\right]$$

Example: Obtain FIR linear-phase and cascade realisations of the system function
$$H(z)=\left(1+\frac{1}{2}z^{-1}+z^{-2}\right)\left(1+\frac{1}{4}z^{-1}+z^{-2}\right)$$

Solution

Linear Phase Realisation Expanding $H(z)$, we have

$$H(z)=1+\frac{3}{4}z^{-1}+\frac{17}{8}z^{-2}+\frac{3}{4}z^{-3}+z^{-4}$$

Here $M = 5$ and the realisation is shown in Fig. E9.14(a).

Cascade Realisation

$$H(z)=\left(1+\frac{1}{2}z^{-1}+z^{-2}\right)\left(1+\frac{1}{4}z^{-1}+z^{-2}\right)$$

$H(z)$ has a product of two sections which have the linear phase symmetry property. The corresponding cascade realisation is shown in Fig. E9.14(b).

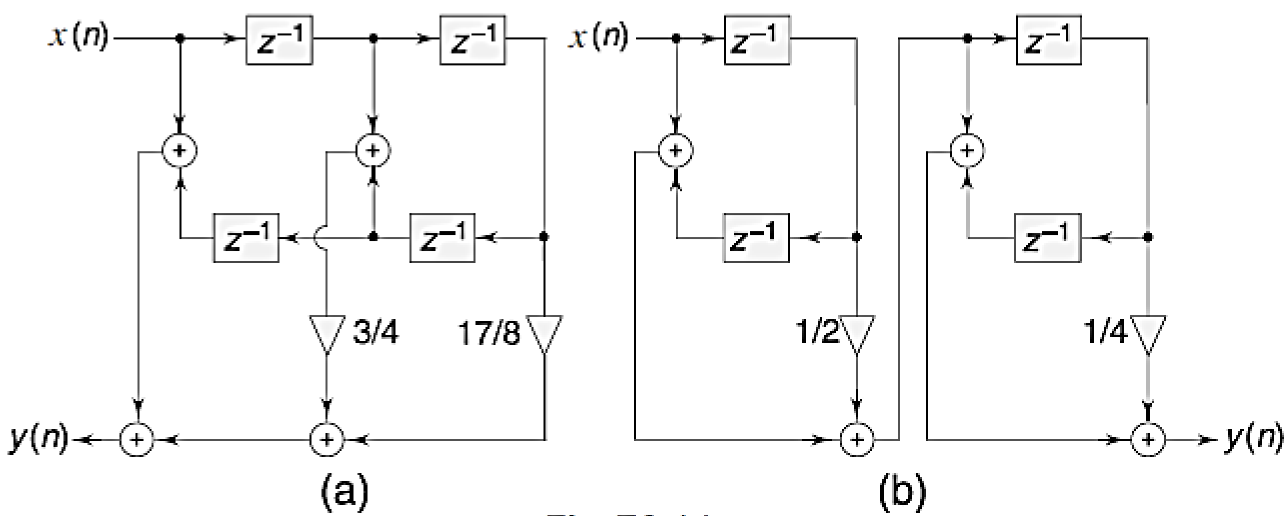


Fig. E9.14

Design of IIR Digital Filters

Introduction

- ✓ Filters are of two types—FIR and IIR.
- ✓ The type of filters which make use of feedback connection to get the desired filter implementation are known as recursive filters. Their impulse response is of infinite duration. So they are called IIR filters. The type of filters which do not employ any kind of feedback connection are known as non-recursive filters. Their impulse response is of finite duration. So they are called FIR filters.
- ✓ IIR filters are designed by considering all the infinite samples of the impulse response. The impulse response is obtained by taking inverse Fourier transform of ideal frequency response.

- ✓ An important step in the development of a digital filter is the determination of a realizable transfer function $H(z)$ approximating the given frequency response specifications.
- ✓ The process of deriving a transfer function is called *digital filter design*.
- ✓ In the most general sense, a digital filter is a linear shift-invariant discrete-time system that is realized using finite-precision arithmetic.
- ✓ The design of digital filters involves three basic steps:

  (1) the specification of the desired properties of the system;

  (2) the approximation of these specifications using a causal discrete-time system; and

  (3) the realization of the system using finite-precision arithmetic.

Design of IIR Digital Filters from Analog Filters

✓ The traditional approach to the design of IIR digital filters involves the transformation of an analog filter into a digital filter meeting prescribed specifications. This is a reasonable approach because:

1. The art of analog filter design is highly advanced and, since useful results can be achieved, it is advantageous to utilize the design procedures already developed for analog filters.
2. Many useful analog design methods have relatively simple closed-form design formulas. Therefore, digital filter design methods based on such analog design formulas are rather simple to implement.
3. In many applications it is of interest to use a digital filter to simulate the performance of an analog linear time-invariant filter.

The system function describing an analog filter may be written as:

$$H_a(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k\, s^k}{\sum_{k=0}^{N} a_k\, s^k}$$

- ✓ The impulse response of these filter coefficients is related to $H_a(s)$ by the Laplace transform

$$H_a(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st} dt$$

- ✓ The analog filter having the rational system function $H_a(s)$ given above can also be described by the linear constant coefficient differential equation.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- ✓ The above three equivalent characterizations of an analog filter leads to three alternative methods for transforming the analog filter into digital domain.
- ✓ The restriction on the design is that the filters should be realizable and stable.
- ✓ For stability and causality of analog filter, the analog transfer function should satisfy the following requirements:

1. The $H_a(s)$ should be a rational function of s, and the coefficients of s should be real.
2. The poles should lie on the left half of s-plane.
3. The number of zeros should be less than or equal to the number of poles.

- ✓ For stability and causality of digital filter, the digital transfer function should satisfy the following requirements:

1. The $H(z)$ should be a rational function of z and the coefficients of z should be real.
2. The poles should lie inside the unit circle in z-plane.
3. The number of zeros should be less than or equal to the number of poles.

✓ We know that the analog filter with transfer function $H_a(s)$ is stable if all its poles lie in the left half of the s-plane. Consequently for the conversion technique to be effective, it should possess the following desirable properties:

1. The imaginary axis in the s-plane should map into the unit circle in the z-plane. Thus, there will be a direct relationship between the two frequency variables in the two domains.
2. The left half of the s-plane should map into the interior of the unit circle centred at the origin in z-plane. Thus, a stable analog filter will be converted to a stable digital filter.

The following three methods are used for transforming the analog filter into digital domain:

1. Approximation of derivatives
2. Impulse Invariant Transformation
3. Bilinear Transformation

Approximation of Derivatives

The analog filter having the rational system function $H(s)$ can also be described by the linear constant coefficient differential equation.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- ✓ In this method of IIR filter design by approximation of derivatives, an analog filter is converted into a digital filter by approximating the above differential equation into an equivalent difference equation.

The backward difference formula is substituted for the derivative $\frac{dy(t)}{dt}$ at time $t = nT$. Thus,

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(n-1)T}{T}$$

or

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(n) - y(n-1)}{T}$$

where T is the sampling interval and $y(n) = y(nT)$.

The system function of an analog differentiator with an output $dy(t)/dt$ is $H(s) = s$, and the digital system which produces the output $[y(n) - y(n-1)]/T$ has the system function $H(z) = [1 - z^{-1}]/T$. Comparing these two, we can say that the frequency domain equivalent for the relationship $\left.\dfrac{dy(t)}{dt}\right|_{t=nT} = \dfrac{y(n)-y(n-1)}{T}$ is:

$$s = \frac{1-z^{-1}}{T}$$

Thus, this is the analog domain to digital domain transformation.

Also, the second derivative $\dfrac{d^2y(t)}{dt^2}$ can be replaced by the second backward difference:

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \left.\frac{d}{dt}\left[\frac{dy(t)}{dt}\right]\right|_{t=nT}$$

$$= \frac{[y(nT)-y(nT-T)]/T - [y(nT-T)-y(nT-2T)]/T}{T}$$

$$= \frac{y(n)-2y(n-1)+y(n-2)}{T^2}$$

The equivalent expression in frequency domain is:

$$s^2 = \frac{1 - 2z^{-1} + z^{-2}}{T^2}$$

or

$$s^2 = \left(\frac{1 - z^{-1}}{T} \right)^2$$

The ith derivative of function $y(t)$ results in the equivalent frequency domain relationship as:

$$s^i = \left(\frac{1 - z^{-1}}{T} \right)^i$$

As a result, the digital filter's system function $H(z)$ can be obtained from the analog filter's system function $H_a(s)$ by the method of approximation of the derivatives as:

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}}$$

The outcomes of the mapping of the z-plane from the s-plane are discussed below.

We have $$s = \frac{1 - z^{-1}}{T}, \quad \text{i.e.} \quad z = \frac{1}{1 - sT}$$

Substituting $s = j\Omega$ in the expression for z, we have

$$z = \frac{1}{1 - j\Omega T}$$

$$= \frac{1}{1 + \Omega^2 T^2} + j \frac{\Omega T}{1 + \Omega^2 T^2}$$

Varying Ω from $-\infty$ to ∞ the corresponding locus of points in the z-plane is a circle with radius 1/2 and with centre at $z = 1/2$, as shown in Figure.

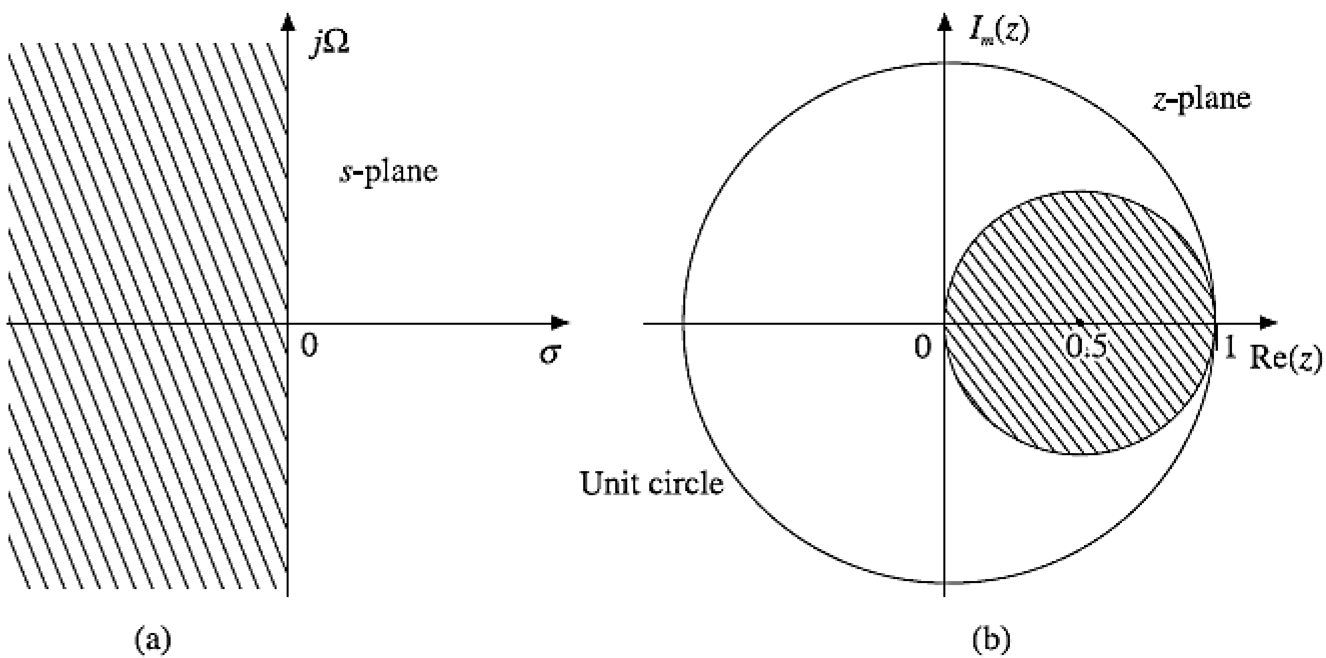


Figure Mapping of s-plane into z-plane by the backward difference method.

- ✓ It can be observed that the mapping of the equation $s = (1 - z^{-1})/T$, takes the left half plane of s-domain into the corresponding points inside the circle of radius 0.5 and centre at $z = 0.5$.
- ✓ Also the right half of the s-plane is mapped outside the unit circle. Because of this, this mapping results in a stable analog filter transformed into a stable digital filter. However, since the location of poles in the z-domain are confined to smaller frequencies, this design method can be used only for transforming analog low-pass filters and band pass filters which are having smaller resonant frequencies.
- ✓ This means that neither a high-pass filter nor a band-reject filter can be realized using this technique.
- ✓ The forward difference can be substituted for the derivative instead of the backward difference.

This provides

$$\frac{dy(t)}{dt} = \frac{y(nT+T) - y(nT)}{T}$$
$$= \frac{y(n+1) - y(n)}{T}$$

The transformation formula would be

$$s = \frac{z-1}{T}$$

or

$$z = 1 + sT$$

- ✓ The mapping of the equation $z = 1 + sT$ is shown in Figure. This results in a worse situation than the backward difference substitution for the derivative.
- ✓ When $s = j\Omega$, the mapping of these points in the s-domain results in a straight line in the z-domain with co-ordinates $(z_{real}, z_{imag}) = (1, \Omega T)$. As a result of this, stable analog filters do not always map into stable digital filters.

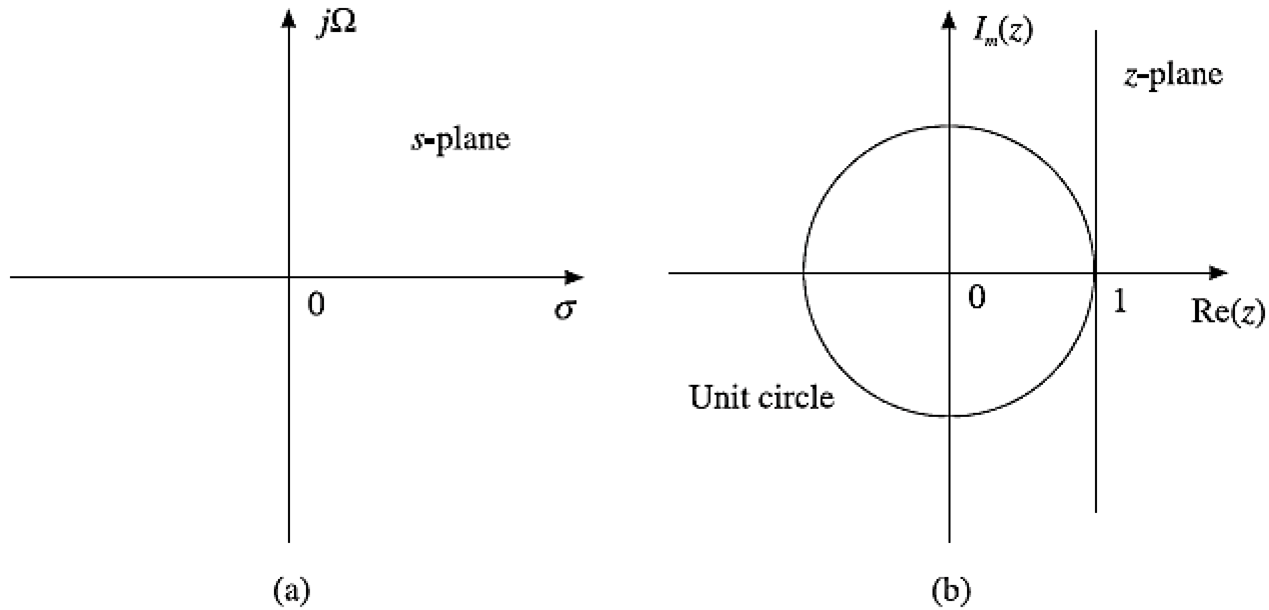


Figure Mapping of s-plane into z-plane by the forward difference method.

The limitations of the mapping methods discussed above can be overcome by using more complex substitution for the derivatives. An Nth order difference is proposed for the derivative, as shown

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{1}{T}\sum_{k=1}^{N} a_k \frac{y(nT+kT) - y(nT-kT)}{T}$$

Here $\{a_k\}$ are a set of parameters selected so as to optimize the approximation. The transformation from the s-plane to the z-plane will be

$$s = \frac{1}{T}\sum_{k=1}^{N} a_k (z^k - z^{-k})$$

EXAMPLE: Convert the analog low-pass filter specified by

$$H_a(s) = \frac{2}{s+3}$$

into a digital filter making use of the backward difference for the derivative.

Solution: We know that the mapping formula for the backward difference for the derivative is given by

$$s = \frac{1-z^{-1}}{T}$$

For the given analog filter function $H_a(s) = \dfrac{2}{s+3}$, the corresponding digital filter function is:

$$H(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{T}} = \frac{2}{\dfrac{(1-z^{-1})}{T}+3}$$

$$= \frac{2T}{1-z^{-1}+3T}$$

If $T = 1$ s, $$H(z) = \frac{2}{1-z^{-1}+3} = \frac{2}{4-z^{-1}}$$

Impulse Invariant Transformation

- ✓ In this technique, the desired impulse response of the digital filter is obtained by uniformly sampling the impulse response of the equivalent analog filter.
- ✓ The main idea behind this is to preserve the frequency response characteristics of the analog filter.
- ✓ For the digital filter to possess the frequency response characteristics of the corresponding analog filter, the sampling period T should be sufficiently small (or the sampling frequency should be sufficiently high) to minimize (or completely avoid) the effects of aliasing.

Let $h_a(t)$ = Impulse response of analog filter
T = Sampling period
$h(n)$ = Impulse response of digital filter

For impulse invariant transformation,

$$h(n) = h_a(t)|_{t=nT} = h_a(nT)$$

The Laplace transform of the analog filter impulse response $h_a(t)$ gives the transfer function of analog filter.

$$\therefore \qquad L[h_a(t)] = H_a(s)$$

The transformation technique can be well understood by first considering a simple distinct poles case for the analog filter's system function as shown below.

$$H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s - p_i}$$

The impulse response $h_a(t)$ of the analog filter is obtained by taking the inverse Laplace transform of the system function $H_a(s)$.

$$\therefore \qquad h_a(t) = \mathrm{L}^{-1}[H_a(s)] = \sum_{i=1}^{N} A_i e^{p_i t} u_a(t)$$

where $u_a(t)$ is the unit step function in the continuous-time case.

✓ The impulse response $h(n)$ of the equivalent digital filter is obtained by uniformly sampling $h_a(t)$, i.e.,

$$h(n) = h_a(nT) = \sum_{i=1}^{N} A_i e^{p_i nT} u_a(nT)$$

✓ The system function of the digital system of above expression can be obtained by taking z-transform, i.e.

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

Using the above equation for $h(n)$, we have

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{N} A_i e^{p_i nT} u_a(nT) \right] z^{-n}$$

Interchanging the order of summation, we have

$$H(z) = \sum_{i=1}^{N} \left[\sum_{n=0}^{\infty} A_i e^{p_i nT} u_a(nT) \right] z^{-n}$$

$$= \sum_{i=1}^{N} \frac{A_i}{1 - e^{p_i T} z^{-1}}$$

Comparing the above expressions for $H_a(s)$ and $H(z)$, we can say that the impulse invariant transformation is accomplished by the mapping.

$$\frac{1}{s-p_i} \xrightarrow[\text{(is tranformed to)}]{} \frac{1}{1-e^{p_iT}z^{-1}}$$

Relation between analog and digital poles

The above mapping shows that the analog pole at $s = p_i$ is mapped into a digital pole at $z = e^{p_cT}$. Therefore, the analog poles and the digital poles are related by the relation.

$$z = e^{sT}$$

The general characteristic of the mapping $z = e^{sT}$ can be obtained by substituting $s = \sigma + j\Omega$ and expressing the complex variable z in polar form as $z = re^{j\omega}$.

$$\therefore \qquad re^{j\omega} = e^{(\sigma+j\Omega)T} = e^{\sigma T}e^{j\Omega T}$$

That means

$$|z| = r = e^{\sigma T}$$

and

$$\angle z = \omega = \Omega T$$

So the relationship between analog frequency Ω and digital frequency ω is $\omega = \Omega T$ or $\Omega = \dfrac{\omega}{T}$.

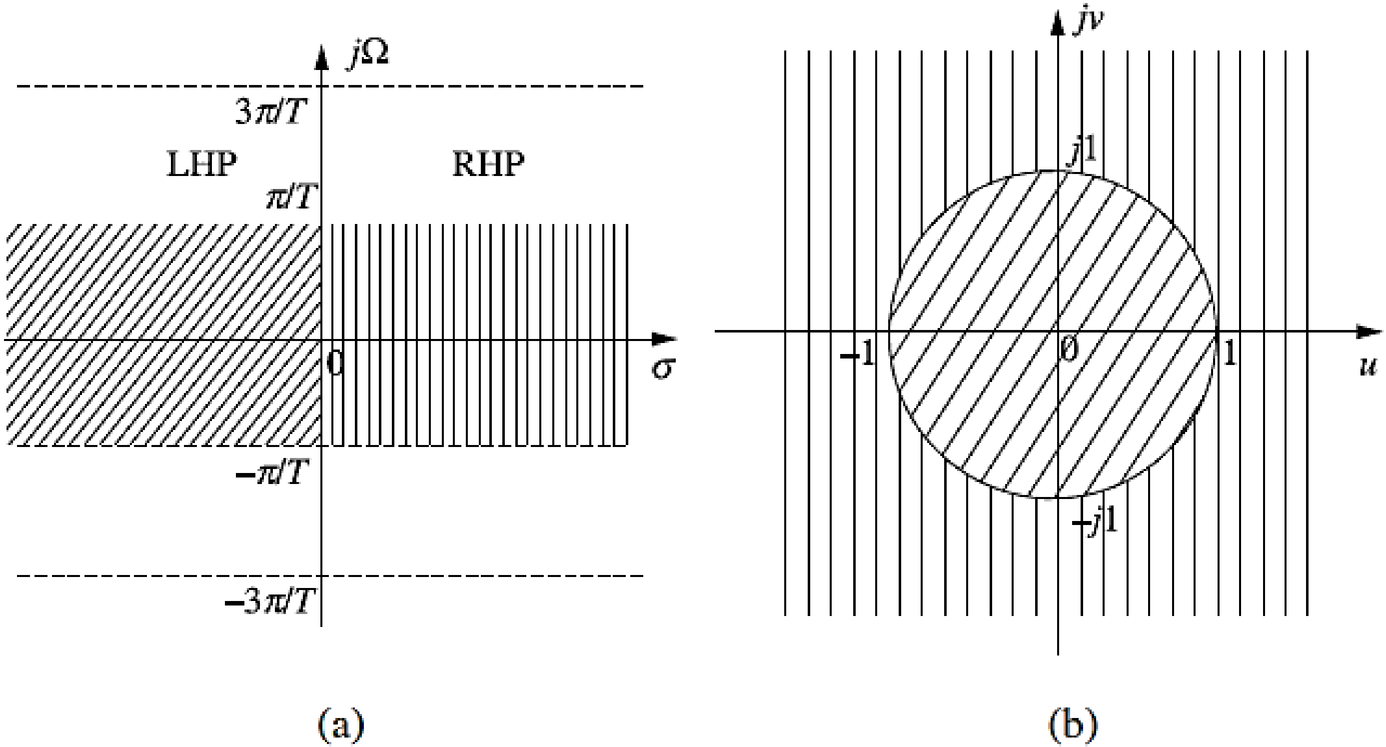


Figure Mapping of (a) s-plane into (b) z-plane by impulse invariant transformation.

- ✓ The stability of a filter (or system) is related to the location of the poles.
- ✓ For a stable analog filter the poles should lie on the left half of the s-plane. That means for a stable digital filter the poles should lie inside the unit circle in the z-plane.

- ✓ Some of the useful impulse invariant transformations are given below.
- ✓ The first one can be used when the analog real pole has a multiplicity of *m*.
- ✓ The second and third equations can be used when the analog poles are complex conjugate.

1. $$\frac{1}{(s+p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left(\frac{1}{1-e^{-sT}z^{-1}}\right); \quad s=p_i$$

2. $$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

3. $$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

EXAMPLE: For the analog transfer function

$$H_a(s) = \frac{2}{(s+1)(s+3)}$$

determine $H(z)$ if (a) $T = 1$ s and (b) $T = 0.5$ s using impulse invariant method.

Solution: Given, $H_a(s) = \dfrac{2}{(s+1)(s+3)}$

Using partial fractions, $H_a(s)$ can be expressed as:

$$H_a(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = (s+1)\,H_a(s)\big|_{s=-1} = \left.\frac{2}{s+3}\right|_{s=-1} = 1$$

$$B = (s+3)\,H_a(s)\big|_{s=-3} = \left.\frac{2}{s+1}\right|_{s=-3} = -1$$

$$\therefore \qquad H_a(s) = \frac{1}{s+1} - \frac{1}{s+3} = \frac{1}{s-(-1)} - \frac{1}{s-(-3)}$$

By impulse invariant transformation, we know that

$$\frac{A_i}{s-p_i} \xrightarrow[\text{(is transformed to)}]{} \frac{A_i}{1-e^{p_i T} z^{-1}}$$

Here $H_a(s)$ has two poles and $p_1 = -1$ and $p_2 = -3$.

Therefore, the system function of the digital filter is:

$$H(z) = \frac{1}{1 - e^{p_1 T} z^{-1}} - \frac{1}{1 - e^{p_2 T} z^{-1}}$$

$$= \frac{1}{1 - e^{-T} z^{-1}} - \frac{1}{1 - e^{-3T} z^{-1}}$$

(a) When $T = 1$ s

$$H(z) = \frac{1}{1 - e^{-1} z^{-1}} - \frac{1}{1 - e^{-3} z^{-1}}$$

$$= \frac{1}{1 - 0.3678 z^{-1}} - \frac{1}{1 - 0.0497 z^{-1}}$$

$$= \frac{(1 - 0.0497 z^{-1}) - (1 - 0.3678 z^{-1})}{(1 - 0.3678 z^{-1})(1 - 0.0497 z^{-1})}$$

$$= \frac{0.3181 z^{-1}}{1 - 0.4175 z^{-1} + 0.0182 z^{-2}}$$

(b) When $T = 0.5$ s

$$H(z) = \frac{1}{1-e^{-0.5}z^{-1}} - \frac{1}{1-e^{-3\times0.5}z^{-1}}$$

$$= \frac{1}{1-0.606\,z^{-1}} - \frac{1}{1-0.223\,z^{-1}}$$

$$= \frac{(1-0.223\,z^{-1})-(1-0.606\,z^{-1})}{(1-0.606\,z^{-1})(1-0.223\,z^{-1})}$$

$$= \frac{0.383\,z^{-1}}{1-0.829\,z^{-1}+0.135\,z^{-2}}$$

Bilinear Transformation

- ✓ In the previous sections, we have studied the IIR filter design using (a) approximation of derivatives method and (b) Impulse invariant transformation method.
- ✓ However the IIR filter design using these methods is appropriate only for the design of low-pass filters and band pass filters whose resonant frequencies are small. These techniques are not suitable for high-pass or band reject filters.
- ✓ The limitation is overcome in the mapping technique called the **bilinear transformation.**
- ✓ This transformation is a one-to-one mapping from the *s*-domain to the *z*-domain. That is, the bilinear transformation is a conformal mapping that transforms the imaginary axis of *s*-plane into the unit circle in the *z*-plane only once, thus avoiding aliasing of frequency components.

- ✓ In this mapping, all points in the left half of *s*-plane are mapped inside the unit circle in the *z*-plane, and all points in the right half of *s*-plane are mapped outside the unit circle in the *z*-plane. So the transformation of a stable analog filter results in a stable digital filter.
- ✓ The bilinear transformation can be obtained by using the trapezoidal formula for the numerical integration.
- ✓ Let the system function of the analog filter be $H_a(s) = \frac{b}{s+a}$

The differential equation describing the above analog filter can be obtained as:

$$H_a(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

or $$sY(s) + aY(s) = bX(s)$$

Taking inverse Laplace transform on both sides, we get

$$\frac{dy(t)}{dt} + a\,y(t) = bx(t)$$

Integrating the above equation between the limits $(nT - T)$ and nT, we have

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}\,dt + a\int_{nT-T}^{nT} y(t)\,dt = b\int_{nT-T}^{T} x(t)\,dt$$

The trapezoidal rule for numeric integration is expressed as:

$$\int_{nT-T}^{nT} a(t)\,dt = \frac{T}{2}[a(nT) + a(nT-T)]$$

Therefore, we get

$$y(nT) - y(nT-T) + a\frac{T}{2}y(nT) + a\frac{T}{2}y(nT-T) = b\frac{T}{2}x(nT) + b\frac{T}{2}x(nT-T)$$

Taking z-transform, we get

$$Y(z)[1-z^{-1}] + a\frac{T}{2}[1+z^{-1}]\,Y(z) = b\frac{T}{2}[1+z^{-1}]\,X(z)$$

Therefore, the system function of the digital filter is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{b}{\dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}} + a}$$

Comparing this with the analog filter system function $H_a(s)$ we get

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

Rearranging, we can get

$$z = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}$$

This is the relation between analog and digital poles in bilinear transformation. So to convert an analog filter function into an equivalent digital filter function, just put

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \quad \text{in } H_a(s)$$

The general characteristic of the mapping $z = e^{sT}$ may be obtained by putting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$ in the above equation for s.

Thus,
$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$$

or
$$s = \frac{2}{T}\frac{(re^{j\omega}-1)(re^{-j\omega}+1)}{(re^{j\omega}+1)(re^{-j\omega}+1)} = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

Since $s = \sigma + j\Omega$, we get

$$\sigma = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}\right]$$

and
$$\Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right]$$

From the above equation for σ, we observe that if $r < 1$ then $\sigma < 0$ and if $r > 1$, then $\sigma > 0$, and if $r = 1$, then $\sigma = 0$. Hence the left half of the s-plane maps into points inside the unit circle in the z-plane, the right half of the s-plane maps into points outside the unit circle in the z-plane and the imaginary axis of s-plane maps into the unit circle in the z-plane. This transformation results in a stable digital system.

Relation between analog and digital frequencies

On the imaginary axis of s-plane $\sigma = 0$ and correspondingly in the z-plane $r = 1$.

$$\therefore \qquad \Omega = \frac{2}{T}\left(\frac{2\sin\omega}{1 + 1 + 2\cos\omega}\right) = \frac{2}{T}\left(\frac{\sin\omega}{1 + \cos\omega}\right)$$

$$= \frac{2}{T}\left(\frac{2\sin\frac{\omega}{2}\,\cos\frac{\omega}{2}}{1 + 2\cos^2\omega/2 - 1}\right) = \frac{2}{T}\tan\frac{\omega}{2}$$

∴ The relation between analog and digital frequencies is:

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

or equivalently, we have $\omega = 2\tan^{-1} \frac{\Omega T}{2}$.

The above relation between analog and digital frequencies shows that the entire range in Ω is mapped only once into the range $-\pi \leq \omega \leq \pi$. The entire negative imaginary axis in the s-plane (from $\Omega = -\infty$ to 0) is mapped into the lower half of the unit circle in z-plane (from $\omega = -\pi$ to 0) and the entire positive imaginary axis in the s-plane (from $\Omega = \infty$ to 0) is mapped into the upper half of unit circle in z-plane (from $\omega = 0$ to $+\pi$).

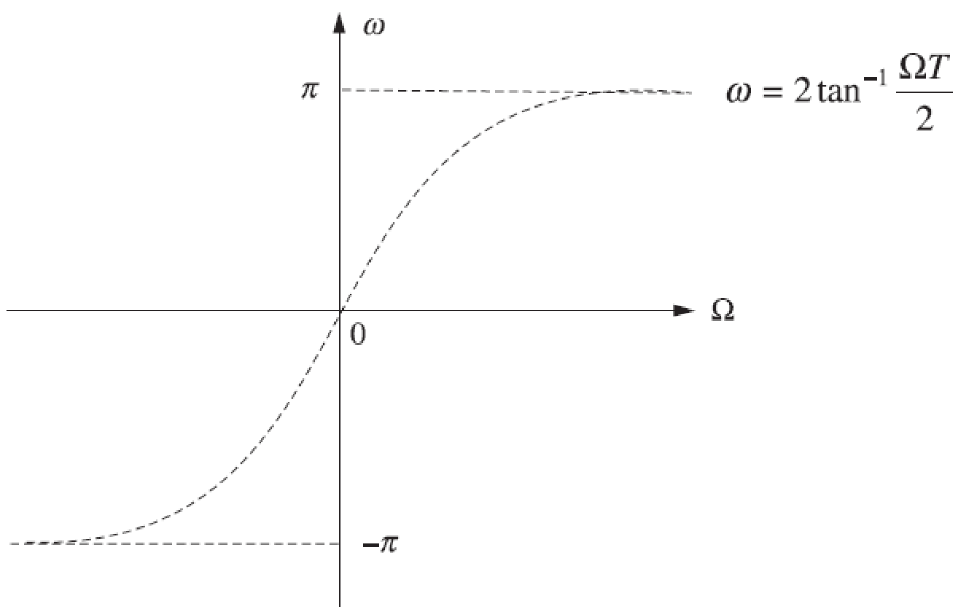
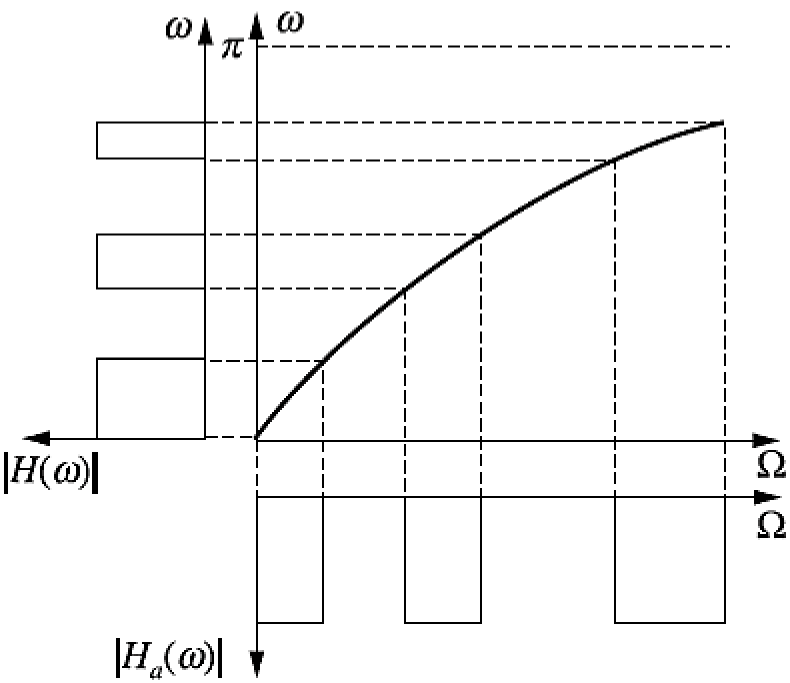


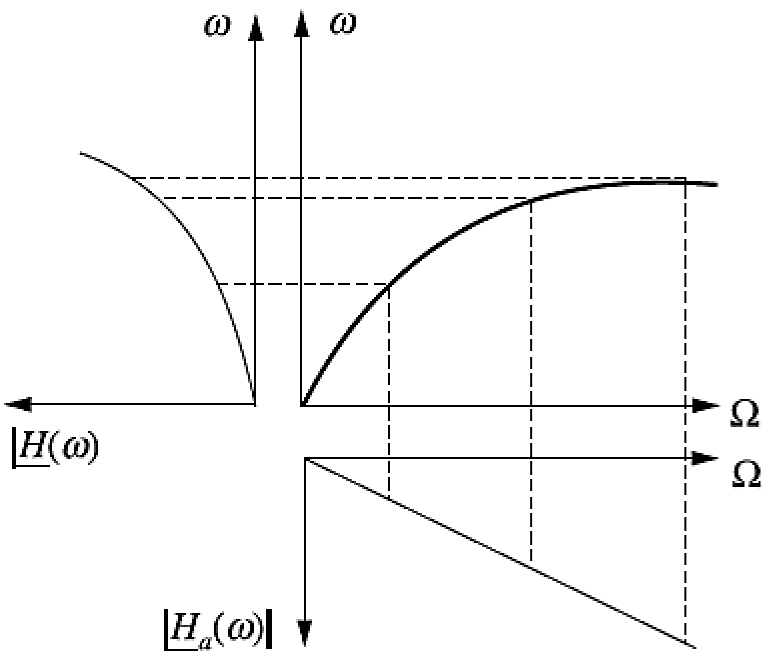
Figure Mapping between Ω and ω in bilinear transformation.

- ✓ But as seen in Figure, the mapping is non-linear and the lower frequencies in analog domain are expanded in the digital domain, whereas the higher frequencies are compressed.
- ✓ This is due to the nonlinearity of the arctangent function and usually known as frequency warping.

- ✓ The effect of warping on the magnitude response can be explained by considering an analog filter with a number of pass bands as shown in Figure (a).
- ✓ The corresponding digital filter will have same number of pass bands, but with disproportionate bandwidth, as shown in Figure (a).
- ✓ In designing digital filter using bilinear transformation, the effect of warping on amplitude response can be eliminated by prewarping the analog filter. In this method, the specified digital frequencies are converted to analog equivalent using the equation.



(a)



(b)

- ✓ This analog frequencies are called prewarp frequencies. Using the prewarp frequencies, the analog filter transfer function is designed, and then it is transformed to digital filter transfer function.
- ✓ This effect of warping on the phase response can be explained by considering an analog filter with linear phase response as shown in Figure (b). The phase response of corresponding digital filter will be nonlinear.
- ✓ From the earlier discussions, it can be stated that the bilinear transformation preserves the magnitude response of an analog filter only if the specification requires piecewise constant magnitude, but the phase response of the analog filter is not preserved.
- ✓ Therefore, the bilinear transformation can be used only to design digital filters with prescribed magnitude response with piecewise constant values. A linear phase analog filter cannot be transformed into a linear phase digital filter using the bilinear transformation.

EXAMPLE: Convert the following analog filter with transfer function

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

into a digital IIR filter by using bilinear transformation. The digital IIR filter is having a resonant frequency of $\omega_r = \pi/2$.

Solution: From the transfer function, we observe that $\Omega_c = 3$. The sampling period T can be determined using the equation:

$$\Omega_c = \frac{2}{T} \tan \frac{\omega_r}{2}$$

$$\therefore \qquad T = \frac{2}{\Omega_c} \tan \frac{\omega_r}{2} = \frac{2}{3} \tan \frac{\pi/2}{2} = 0.6666 \text{ s}$$

Using the bilinear transformation, the digital filter system function is:

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = H_a(s)\Big|_{s=3\frac{1-z^{-1}}{1+z^{-1}}}$$

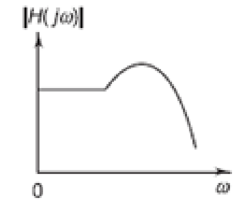
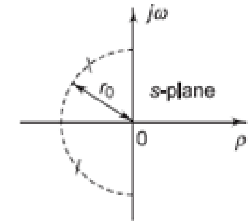
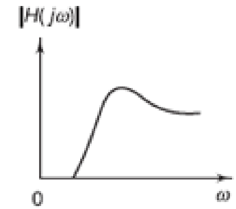
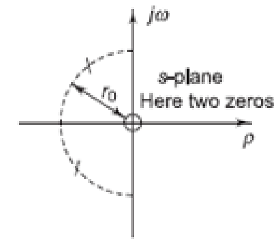
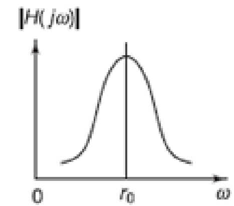
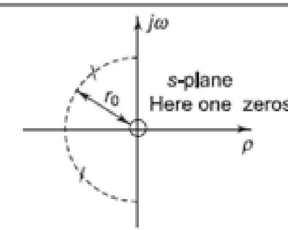
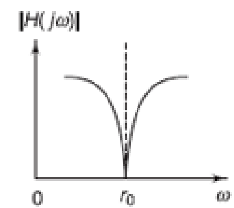
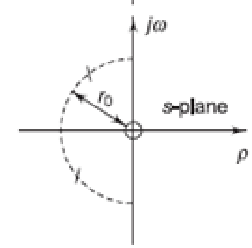
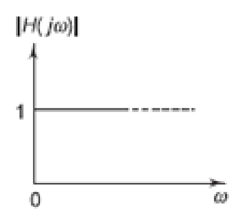
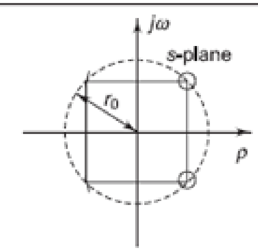
$\therefore$

$$H(z) = \left.\frac{s + 0.1}{(s + 0.1)^2 + 9}\right|_{s = 3\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{3\,\dfrac{1 - z^{-1}}{1 + z^{-1}} + 0.1}{\left[3\,\dfrac{1 - z^{-1}}{1 + z^{-1}} + 0.1\right]^2 + 9}$$

$$= \frac{\left[3(1 - z^{-1}) + 0.1(1 + z^{-1})\right][1 + z^{-1}]}{\left[3(1 - z^{-1}) + 0.1\,(1 + z^{-1})\right]^2 + 9(1 + z^{-1})^2}$$

$$= \frac{3.1 + 0.2\,z^{-1} - 2.9\,z^{-2}}{18.61 + 0.02\,z^{-1} + 17.41\,z^{-2}}$$

Name of the analog filter	*Its transfer function H(S)*	*Frequency response H(jω)*	*Pole-zero locations of |H(jω)|*
Low pass filter (LPF)	$\dfrac{r_0^2}{s^2+\left(\dfrac{r_0}{Q}\right)s+r_0^2}$		
High pass filter (HPF)	$\dfrac{s^2}{s^2+\left(\dfrac{r_0}{Q}\right)s+r_0^2}$		
Band Pass filter (BPF)	$\dfrac{(r_0/Q)s}{s^2+\left(\dfrac{r_0}{Q}\right)s+r_0^2}$		
Band stop filter (BSF)	$\dfrac{s^2+r_0^2}{s^2+\left(\dfrac{r_0}{Q}\right)s+r_0^2}$		
All pass filter (APF)	$\dfrac{s^2-(r_0/Q)s+r_0^2}{s^2+\left(\dfrac{r_0}{Q}\right)s+r_0^2}$		

Frequency Response of Analog and Digital IIR Filters

- ✓ The filters are frequency selective devices and so they are designed to pass the spectral content of the input signal in a specified band of frequencies.
- ✓ Hence, based on frequency response the filters are classified into four basic types. They are lowpass, high pass, band pass and band stop filters.
- ✓ The ideal magnitude response has sudden transition from pass band to stop band which is practically not realizable. Hence the ideal response is approximated using a filter approximation function.
- ✓ The approximation problem is solved to meet a specified tolerance in the pass band and stop band.

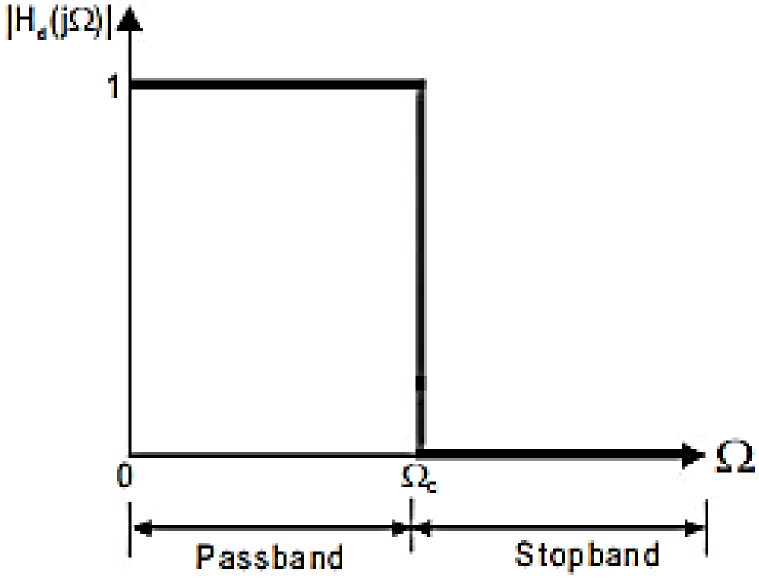


Fig : Normalized magnitude response of ideal analog lowpass filter.

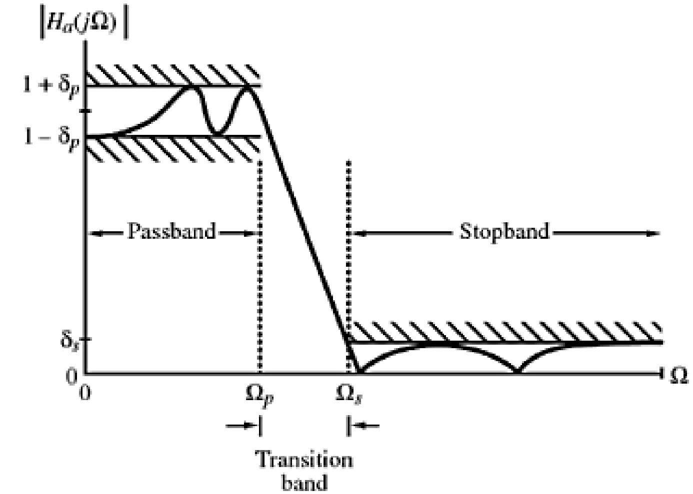


Figure: Typical magnitude specifications for an analog lowpass filter.

Ω_p - passband edge frequency.

Ω_s - stopband edge frequency.

δ_p - peak ripple in the passband.

δ_s - peak ripple in the stopband.

α_p- peak passband ripple.

α_s - minimum stopband attenuation.

$\frac{1}{\sqrt{1+\varepsilon^2}}$ – pass band ripple

$\frac{1}{\sqrt{(1+\lambda^2)}}$ or $\frac{1}{A}$ – maximum stop band ripple

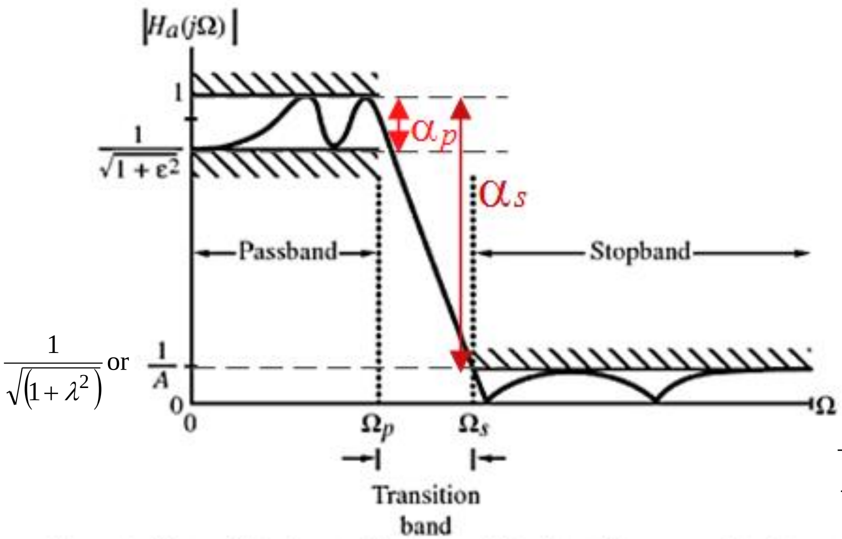


Figure: Normalized magnitude specifications for an analog lowpass filter.

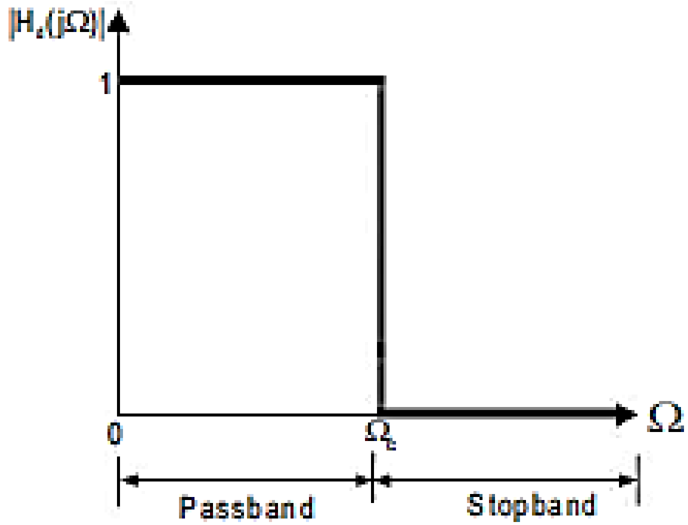


Fig a : Normalized magnitude response of ideal analog lowpass filter.

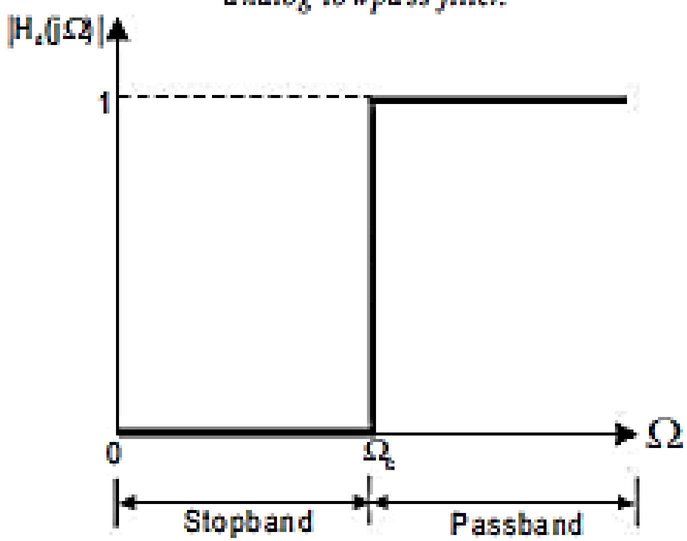


Fig b : Normalized magnitude response of ideal analog highpass filter.

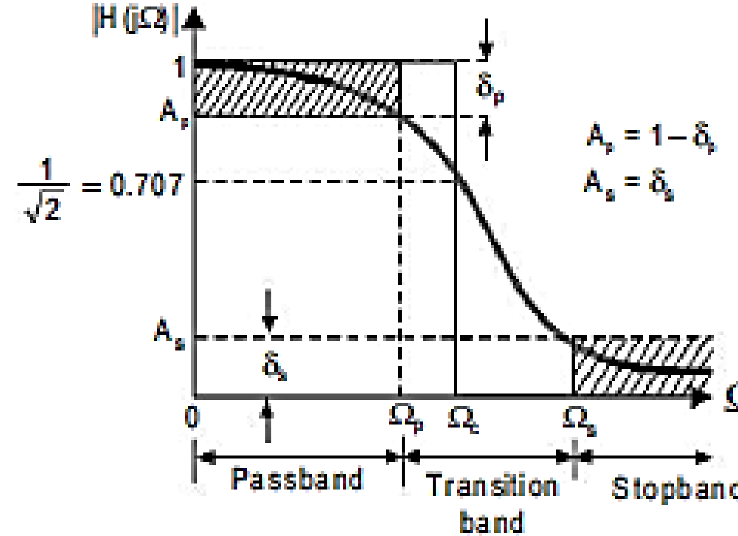


Fig e : Normalized magnitude response of practical analog low pass filter.

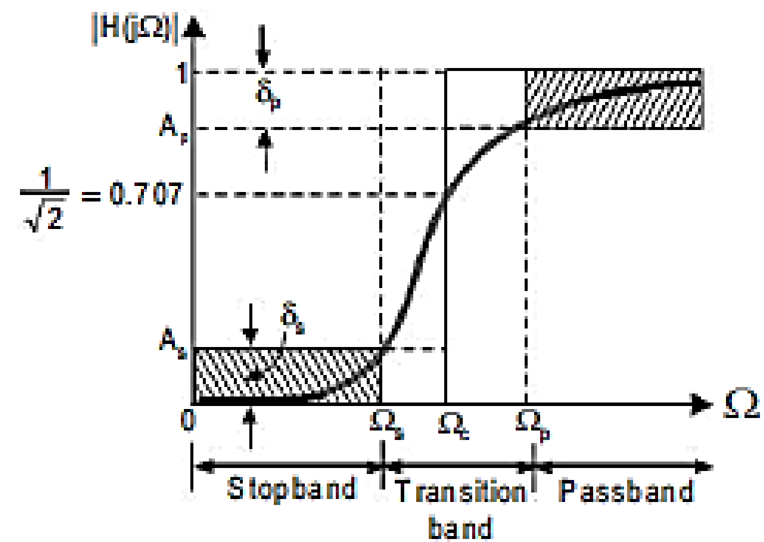


Fig f : Normalized magnitude response of practical analog highpass filter.

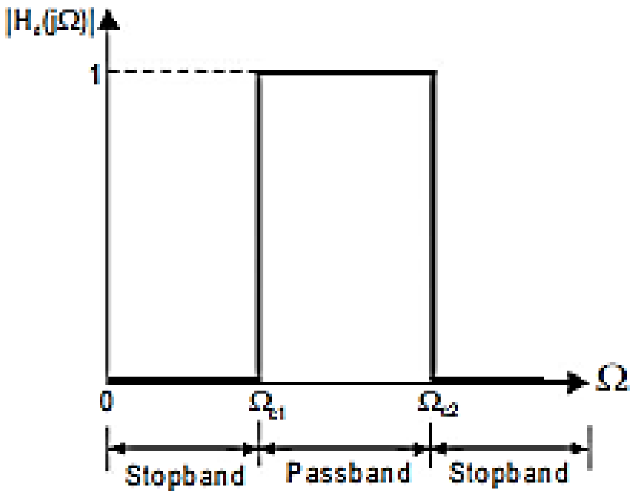


Fig c : Normalized magnitude response of ideal analog bandpass filter.

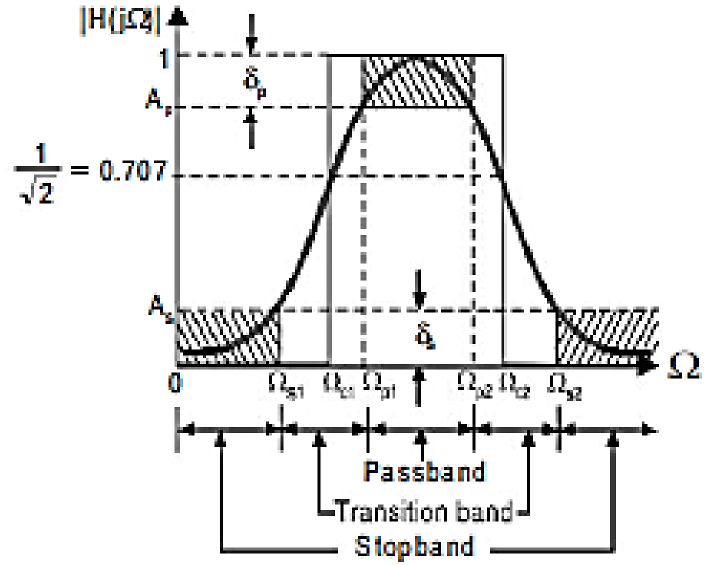


Fig g : Normalized magnitude response of practical analog bandpass filter.

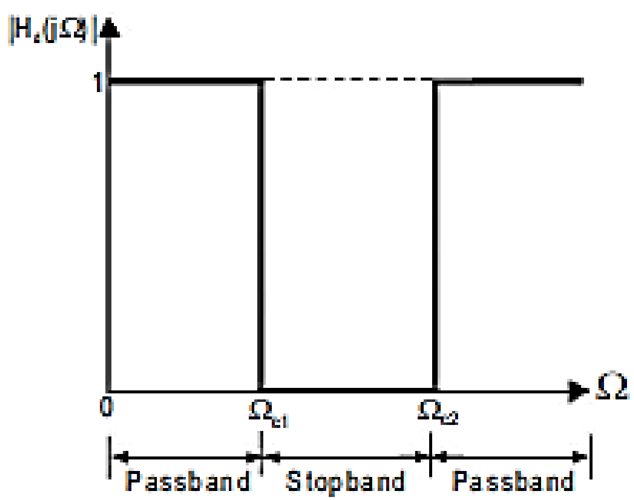


Fig d : Normalized magnitude response of ideal analog bandstop filter.

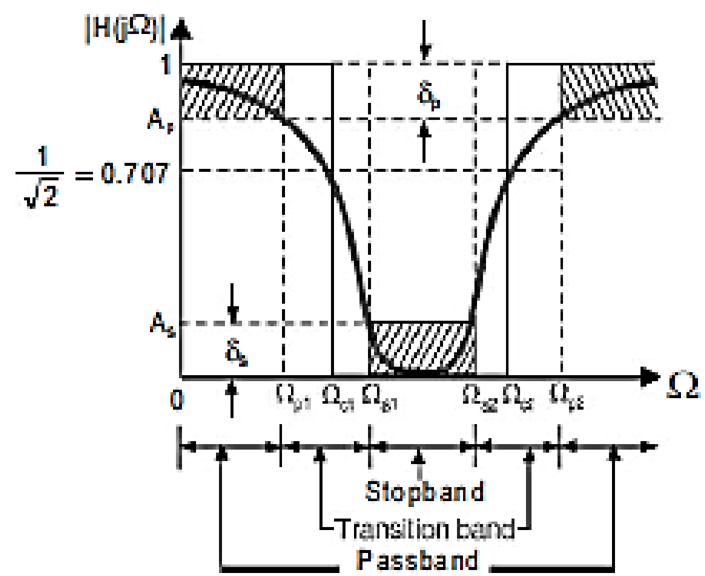


Fig h : Normalized magnitude response of practical analog bandstop filter.

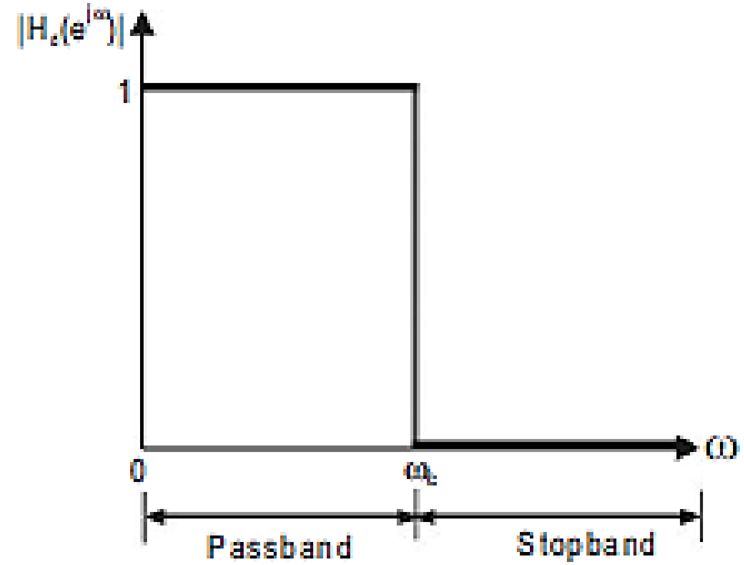


Fig a : Normalized magnitude response of ideal digital IIR lowpass filter.

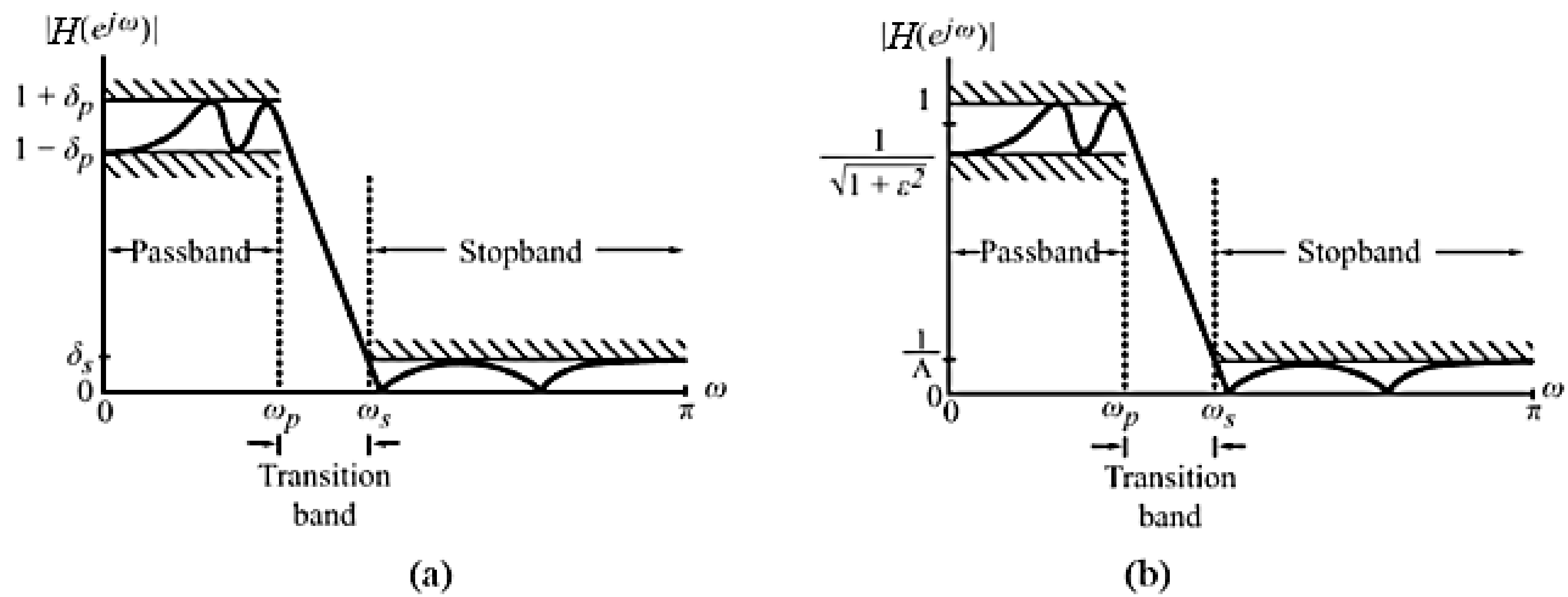


Fig: Digital lowpass filter (a) Typical magnitude response (b)normalized magnitude response

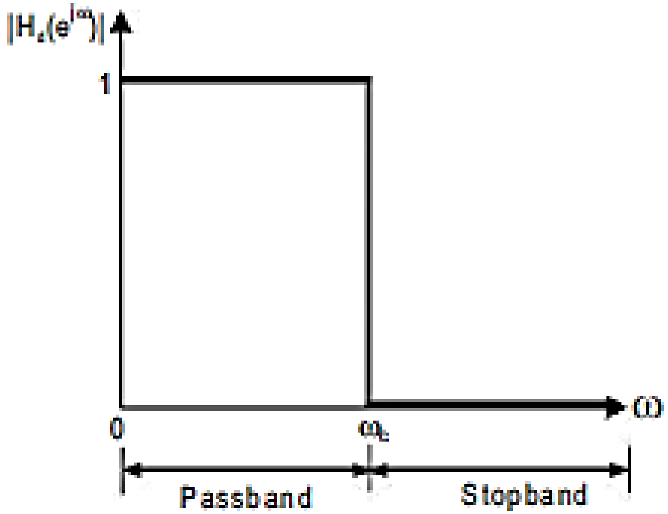


Fig a : Normalized magnitude response of ideal digital IIR lowpass filter.

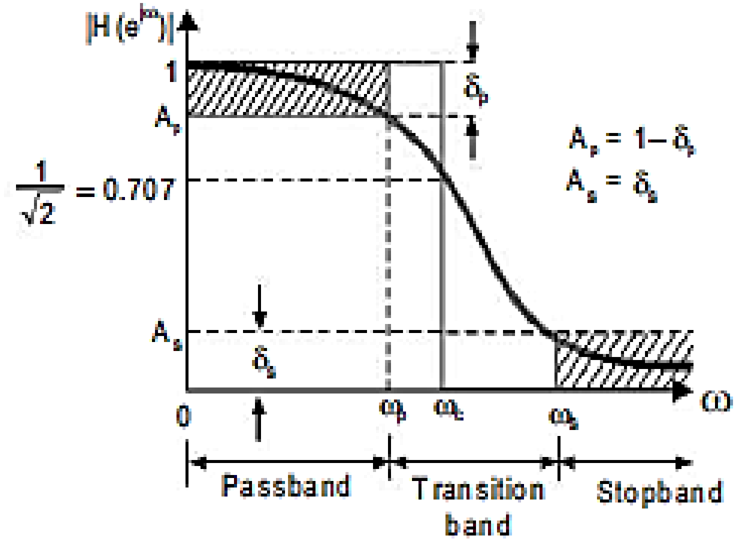


Fig e : Normalized magnitude response of practical digital IIR lowpass filter.

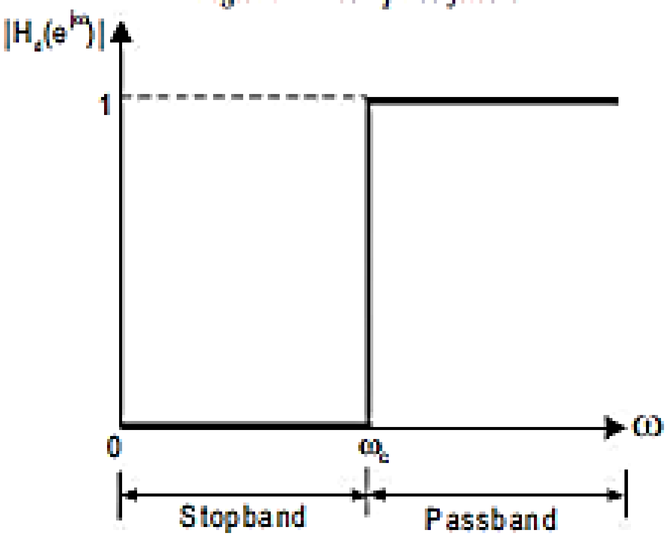


Fig b : Normalized magnitude response of ideal digital IIR highpass filter.

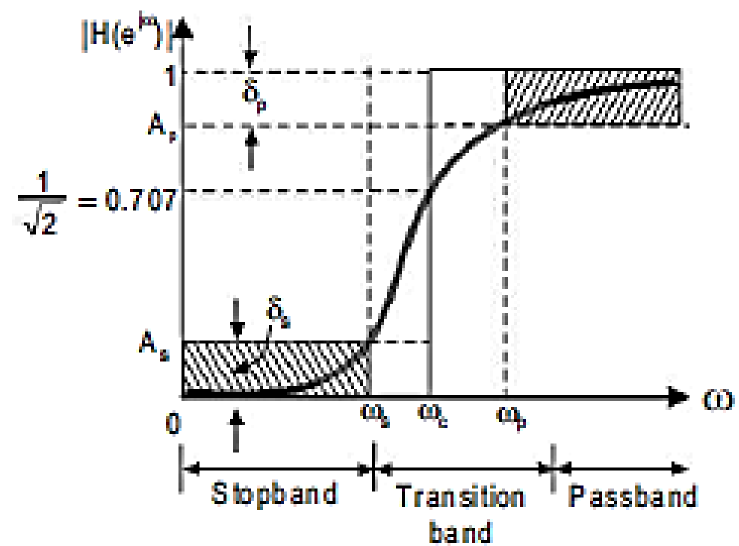


Fig f : Normalized magnitude response of practical digital IIR highpass filter.

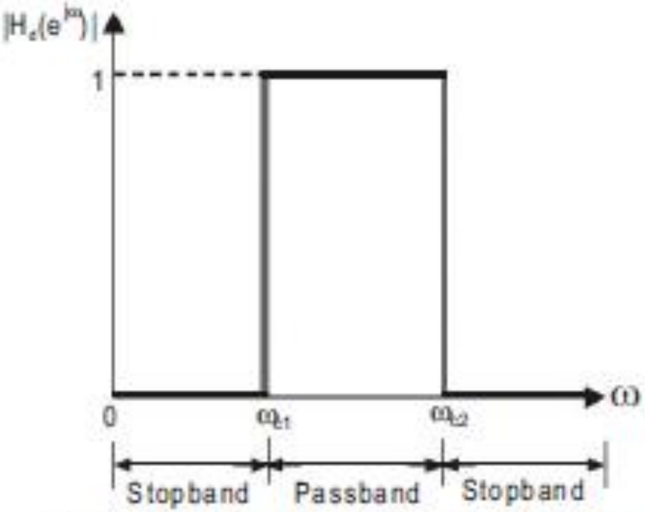


Fig c : Normalized magnitude response of ideal digital IIR bandpass filter.

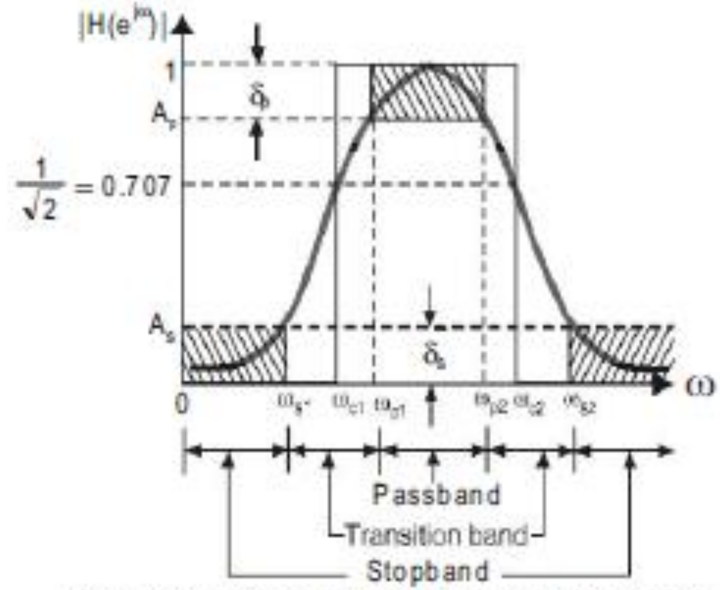


Fig g : Normalized magnitude response of practical digital IIR bandpass filter.

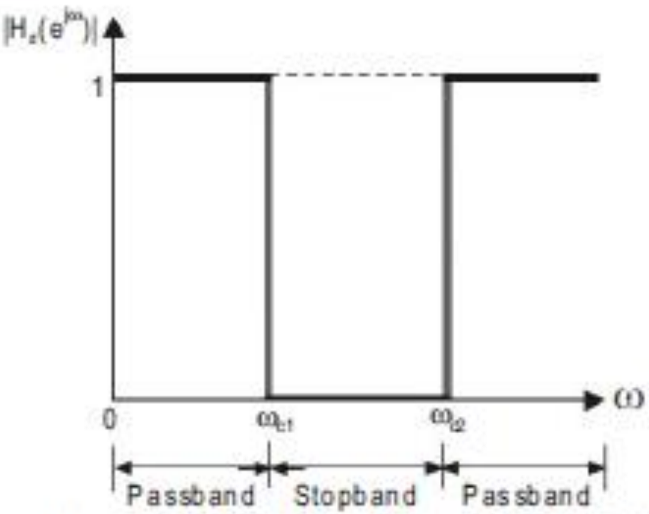


Fig d : Normalized magnitude response of ideal digital IIR bandstop filter.

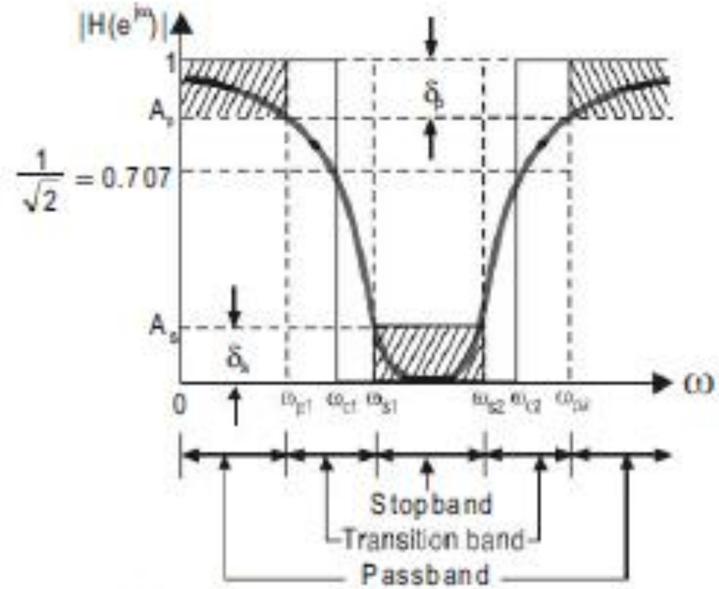


Fig h : Normalized magnitude response of practical digital IIR bandstop filter.

Design of IIR Digital filter using Analog filter approximations

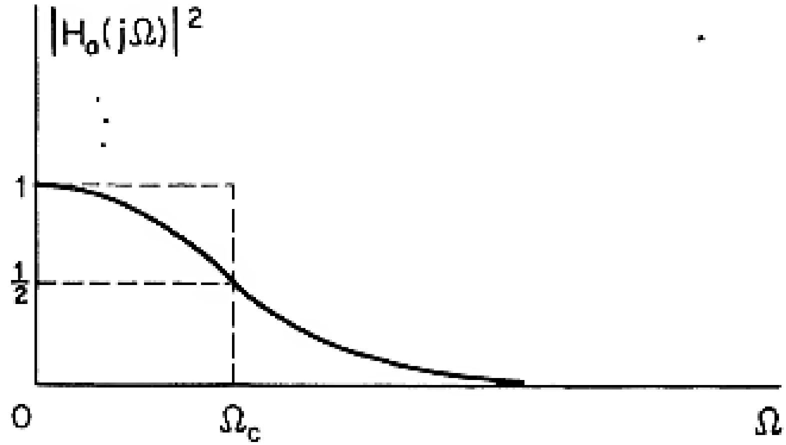
- ✓ The popular methods of designing IIR digital filter involves the design of equivalent analog filter and then converting the analog filter to digital filter.
- ✓ In this section we discuss examples of several analog lowpass approximation techniques, including Butterworth, and Chebyshev approximations.
- ✓ The discussion is organized as follows: First, we present the basic design formulas for a particular approximation method. Then, using the same lowpass filter specifications for each approximation method, we carry out the design of a digital filter using analog to digital transformation.

Design of Lowpass Digital Butterworth Filter

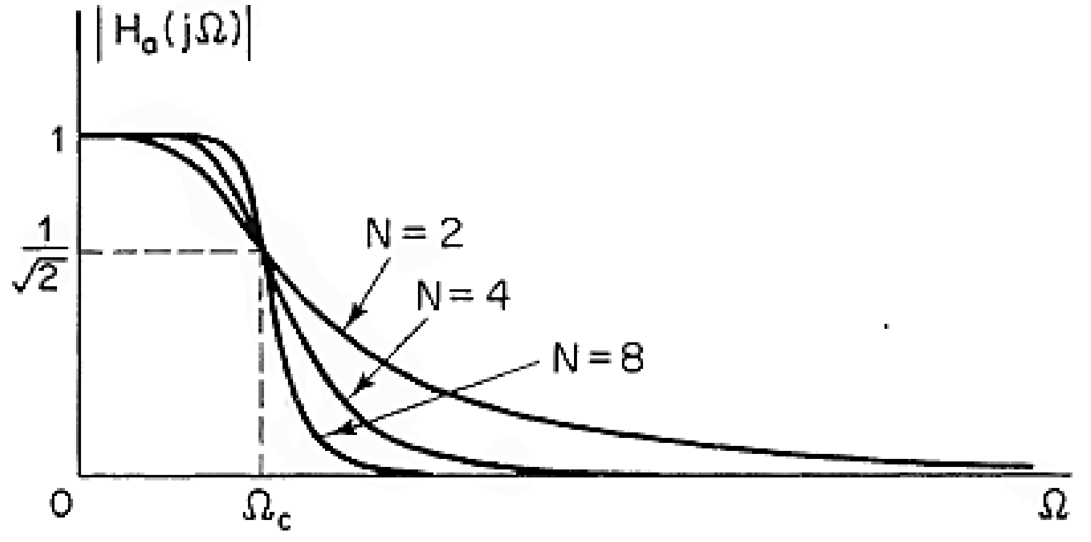
- ✓ The popular methods of designing IIR digital filter involves the design of equivalent analog filter and then converting the analog filter to digital filter.
- ✓ Hence to design a Butterworth IIR digital filter, first an analog Butterworth filter transfer function is determined using the given specifications.
- ✓ Then the analog filter transfer function is converted to a digital filter transfer function using either impulse invariant transformation or bilinear transformation.

Analog Butterworth Filter

- ✓ Butterworth filters are defined by the property that the magnitude response is maximally flat in the pass band.
- ✓ For an Nth-order lowpass filter, this means that the first $2N–1$ derivatives of the squared magnitude function are zero at $\Omega = 0$.
- ✓ Another property is that the approximation is monotonic in the pass band and the stop band. The squared magnitude function for an analog Butterworth filter is of the form $\left|H(j\Omega)\right|^2 = \dfrac{1}{1+\left(\dfrac{\Omega}{\Omega_C}\right)^{2N}}$ as sketched in Fig.



- ✓ As the parameter N in increases, the filter characteristics become sharper; that is, they remain closer to unity over more of the pass band and become close to zero more rapidly in the stop band, although the magnitude function at the cut-off frequency Ω_c will always be $1/\sqrt{2}=0.707$ which corresponds to –3 dB because of the nature of Eq.
- ✓ The dependence of the Butterworth filter characteristic on the parameter N is indicated. in Fig..



- ✓ The squared magnitude function of normalized Butterworth filter (i.e. Ω_c =1 rad/sec) is

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega)^{2N}}$$

- ✓ Now, let us derive the transfer function of a stable filter. For this purpose substitute $\Omega=s/j$ in the above equation, we can write the above equation as

$$|H(j\Omega)|^2 = H(\Omega^2)$$

$$= H\left(\left(\frac{s}{j}\right)^2\right) = H(-s^2)$$

$$= H(s)H(-s)$$

$$\Rightarrow H(s)H(-s) = \frac{1}{1+\left(\frac{s}{j}\right)^{2N}}$$

$$= \frac{1}{1+(-1)^N s^{2N}} = \frac{1}{1+\left(-s^2\right)^N}$$

- ✓ The above relation tell us that this function poles in the Left Half of s-plane as well as in the Right Half of s-plane, because of the presence of two factors H(s) and H($-s$).
- ✓ If the H(s) has roots in the LHP then H($-s$) has the corresponding roots in the RHP.
- ✓ We can obtain these roots by equating the denominator to zero.

That is, $1+\left(-s^2\right)^N=0$

For N odd, $s^{2N}=1=e^{j2\pi k}$

Now the roots can be found as

$$s_k=e^{j\pi k/N} \quad \text{for} \quad k=1,2,\cdots,2N$$

For N even, $s^{2N}=-1=e^{j(2k-1)\pi}$

Now the roots can be found as

$$s_k=e^{j\pi(2k-1)/2N} \quad \text{for} \quad k=1,2,\cdots,2N$$

For N = 3, $s^6 = 1$

We know for N odd the roots can be obtained as

$$s_1 = e^{j\pi/3} = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = 0.5 + j0.866$$

$$s_2 = e^{j2\pi/3} = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = -0.5 + j0.866$$

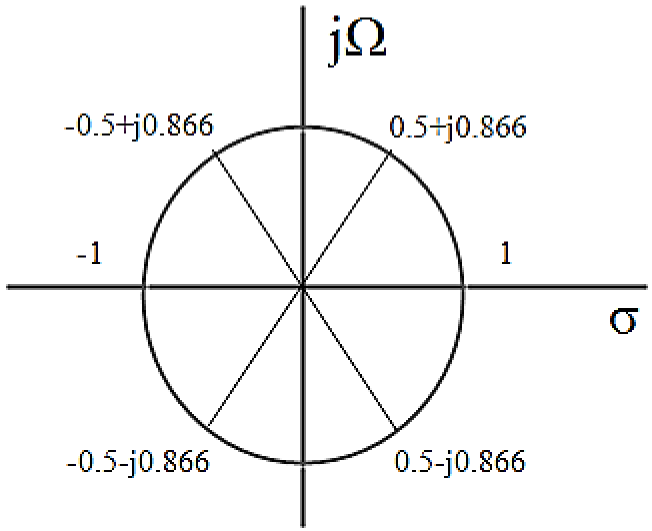
$$s_3 = e^{j\pi} = \cos\pi + j\sin\pi = -1$$

$$s_4 = e^{j4\pi/3} = \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} = -0.5 - j0.866$$

$$s_5 = e^{j5\pi/3} = \cos\frac{5\pi}{3} + j\sin\frac{5\pi}{3} = 0.5 - j0.866$$

$$s_6 = e^{j2\pi} = \cos 2\pi + j\sin 2\pi = 1$$

- ✓ All the poles can be located on the s-plane and is found that the angular separation between the poles is 360°/2N, which in this case is equal to 60° and all poles are lie on a unit circle.



- ✓ To ensure stability and considering only the poles that lie on LHP, we can write the denominator of the transfer function $H(s)$ as

$$(s+1)\left\{(s+0.5)^2+(0.866)^2\right\}=(s+1)(s^2+s+1)$$

- ✓ Therefore, the transfer function of a 3rd Butterworth filter for cut-off frequency Ω_c =1 rad/sec is

$$H(s)=\frac{1}{(s+1)(s^2+s+1)}$$

- ✓ As we are interested on the poles, which lies in LHP in s-plane, the same can be found by using the formula

$$s_k = e^{j\phi_k} \quad \text{where}$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad \text{for } k = 1,2,\ldots,N$$

- ✓ The following table gives the Butterworth polynomial for various values of N for Ω_c =1 rad/sec.

N	Factors of Polynomial
1	$(s+1)$
2	$(s^2+1.4142s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.7654s+1)(s^2+1.8478s+1)$
5	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$
6	$(s^2+0.5176s+1)(s^2+1.4142s+1)(s^2+1.9319s+1)$
7	$(s+1)(s^2+0.4450s+1)(s^2+1.2470s+1)(s^2+1.8019s+1)$
8	$(s^2+0.3902s+1)(s^2+1.1111s+1)(s^2+1.6629s+1)(s^2+1.9616s+1)$

- ✓ The equation gives us the poles locations of Butterworth for Ω_c =1 rad/sec are known as normalized poles.
- ✓ In general, the unnormalized poles are given by $s_k^{'} = \Omega_c s_k \Rightarrow s_k \to \frac{s_k^{'}}{\Omega_c}$
- ✓ The transfer function of such type of Butterworth filter can be obtained by substituting $s \to s/\Omega_c$ in the transfer function of Butterworth filter.
- ✓ We can now proceed to determine the order of equation given the filter specifications. The squared magnitude of Butterworth filter was restricted to –3 dB at Ω_c.
- ✓ Now let the maximum pass band attenuation in positive dB is α_p (< 3 dB) at pass band frequency Ω_p and α_s is the minimum stop band attenuation in positive dB at the stop band frequency Ω_s.
- ✓ Now the magnitude function can be written as $|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$

✓ Taking logarithm on both sides

$$20\log\left|H(j\Omega)\right| = 10\log 1 - 10\log\left(1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right)$$

✓ We know that at $\Omega = \Omega_p$ the attenuation is equal to α_p

$$\text{At } \Omega=\Omega_p;\ \left|H(j\Omega_p)\right|^2 = \frac{1}{1+\left(\frac{\Omega_p}{\Omega_C}\right)^{2N}} = \frac{1}{1+\varepsilon^2}$$

✓ Taking logarithm on both sides we have

$$20\log\left|H(j\Omega_p)\right| = 10\log 1 - 10\log\left(1+\varepsilon^2\right)$$

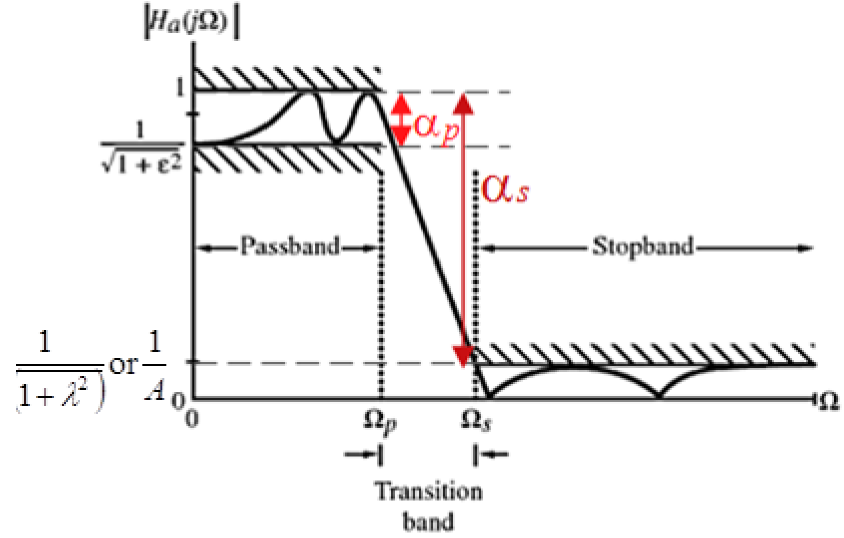
which gives $-\alpha_p = -10\log\left(1+\varepsilon^2\right)$

$$0.1\alpha_p = \log\left(1+\varepsilon^2\right)$$

Taking antilog on both side

$$1+\varepsilon^2 = 10^{0.1\alpha_p}$$

$$\varepsilon = \left(10^{0.1\alpha_p}-1\right)^{1/2}$$

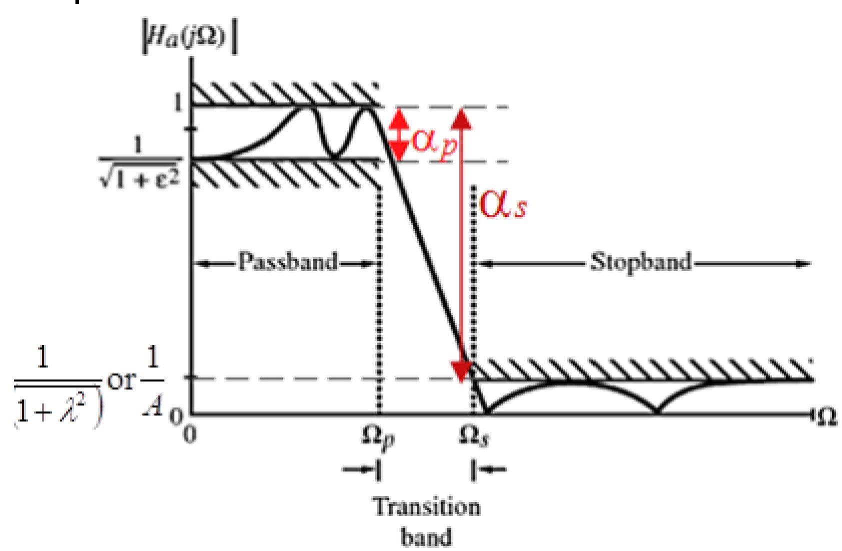


✓ Reference to the figure shows that at $\Omega = \Omega_s$ the minimum stop band attenuation is equal to α_s.

$$20\log\left|H(j\Omega_s)\right| = 10\log 1 - 10\log\left(1+\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right)$$

$$-\alpha_s = -10\log\left(1+\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right)$$

$$0.1\alpha_s = \log\left(1+\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right)$$



After simplification, we get

$$\varepsilon^2\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = 10^{0.1\alpha_s} - 1$$

✓ Substituting the ε in the above equation, we get

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}$$

✓ Since we are interested in finding the expression for N, taking logarithm for above equation

$$N\log\left(\frac{\Omega_s}{\Omega_p}\right) = \log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}$$

$$\therefore \qquad N = \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

✓ Since this expression normally does not result in an integer value, we therefore, round off to the next higher integer.

$$\therefore \qquad N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} \qquad \text{where} \quad \varepsilon = \left(10^{0.1\alpha_p}-1\right)^{1/2}$$

$$\lambda = \left(10^{0.1\alpha_s}-1\right)^{1/2}$$

$$\geq \frac{\log\left(\dfrac{\lambda}{\varepsilon}\right)}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{1/2N}} = \frac{\Omega_s}{\left(10^{0.1\alpha_s}-1\right)^{1/2N}}$$

Usually the specifications of the filter are given in terms of gain A or attenuation α at a passband or stopband frequency as given below:

$$A_1 \le |H(\omega)| \le 1, \qquad 0 \le \omega \le \omega_1$$

$$|H(\omega)| \le A_2, \qquad \omega_2 \le \omega \le \pi$$

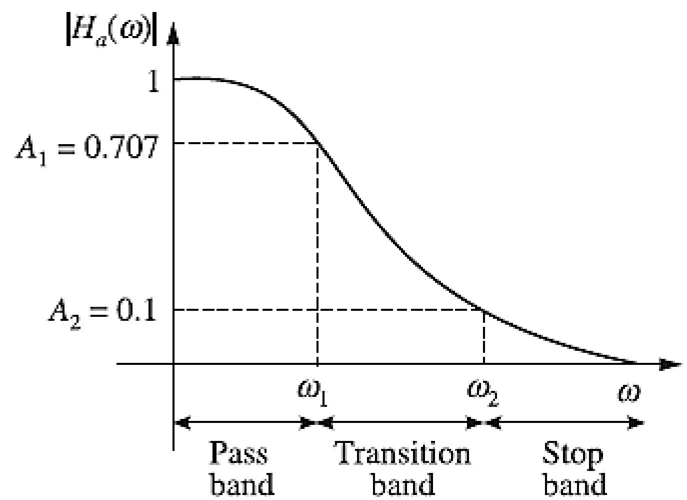
The order of the filter is determined as given below.

Let Ω_1 and Ω_2 be the analog filter edge frequencies corresponding to digital frequencies ω_1 and ω_2. The values of Ω_1 and Ω_2 are obtained using the bilinear transformation or impulse invariant transformation.

$$\therefore \qquad A_1^2 \le \frac{1}{1+\left(\dfrac{\Omega_1}{\Omega_c}\right)^{2N}} \le 1$$

and

$$\frac{1}{1+\left(\dfrac{\Omega_2}{\Omega_c}\right)^{2N}} \le A_2^2$$



These two equations can be written in the form

$$\left(\frac{\Omega_1}{\Omega_c}\right)^{2N} \le \frac{1}{A_1^2} - 1$$

and

$$\left(\frac{\Omega_2}{\Omega_c}\right)^{2N} \ge \frac{1}{A_2^2} - 1$$

Assuming equality we can obtain the filter order N and the 3 dB cutoff frequency Ω_c. Dividing the first equation by the second, we have

$$\left(\frac{\Omega_1}{\Omega_2}\right)^{2N} = \frac{\dfrac{1}{A_1^2} - 1}{\dfrac{1}{A_2^2} - 1}$$

From this equation, the order of the filter N is obtained approximately as:

$$N = \frac{1}{2} \frac{\log\left\{\left(\dfrac{1}{A_2^2} - 1\right) \Big/ \left(\dfrac{1}{A_1^2} - 1\right)\right\}}{\log \dfrac{\Omega_2}{\Omega_1}}$$

If N is not an integer, the value of N is chosen to be the next nearest integer. Also we can get

$$\Omega_c = \frac{\Omega_1}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$$

when parameters A_1 and A_2 are given in dB.

A_1 in dB is given by

$$A_1 \text{ dB} = -20 \log A_1$$

i.e.
$$\log A_1 = -\frac{A_1 \text{ dB}}{20}$$

or
$$A_1 = 10^{-\frac{A_1 \text{ dB}}{20}}$$

$\therefore$
$$\frac{1}{A_1^2} - 1 = \frac{1}{\left(10^{-\frac{A_1 \text{ dB}}{20}}\right)^2} - 1$$

i.e.
$$\frac{1}{A_1^2} - 1 = 10^{0.1 A_1 \text{ dB}} - 1$$

Similarly
$$\frac{1}{A_2^2} - 1 = 10^{0.1 A_2 \text{ dB}} - 1$$

$\therefore$
$$N = \frac{1}{2} \frac{\log\{[\frac{1}{A_2^2} - 1]/[\frac{1}{A_1^2} - 1]\}}{\log\left(\frac{\Omega_2}{\Omega_1}\right)} = \frac{1}{2} \frac{\log\left(\frac{10^{0.1 A_2 \text{dB}} - 1}{10^{0.1 A_1 \text{dB}} - 1}\right)}{\log\left(\frac{\Omega_2}{\Omega_1}\right)}$$

and Ω_c is given by

$$\Omega_c = \frac{\Omega_1}{(10^{0.1A_1\text{dB}} - 1)^{1/2N}} \qquad \text{or} \qquad \Omega_c = \frac{\Omega_2}{(10^{0.1A_2\text{dB}} - 1)^{1/2N}}$$

In fact,

$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_1}{\left[10^{0.1A_1\text{dB}} - 1\right]^{1/2N}} + \frac{\Omega_2}{\left[10^{0.1A_2\text{dB}} - 1\right]^{1/2N}}\right]$$

Design procedure for low-pass digital Butterworth IIR filter

The low-pass digital Butterworth filter is designed as per the following steps:

Let A_1 = Gain at a passband frequency ω_1
A_2 = Gain at a stopband frequency ω_2
Ω_1 = Analog frequency corresponding to ω_1
Ω_2 = Analog frequency corresponding to ω_2

Step 1 Choose the type of transformation, i.e., either bilinear or impulse invariant transformation.

Step 2 Calculate the ratio of analog edge frequencies Ω_2/Ω_1.

For bilinear transformation

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2}, \Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} \quad \therefore \frac{\Omega_2}{\Omega_1} = \frac{\tan \omega_2/2}{\tan \omega_1/2}$$

For impulse invariant transformation,

$$\Omega_1 = \frac{\omega_1}{T}, \quad \Omega_2 = \frac{\omega_2}{T} \quad \therefore \frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1}$$

Step 3 Decide the order N of the filter. The order N should be such that

$$N \geq \frac{1}{2}\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log\frac{\Omega_2}{\Omega_1}}$$

Choose N such that it is an integer just greater than or equal to the value obtained above.

Step 4 Calculate the analog cutoff frequency $\Omega_c = \dfrac{\Omega_1}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$

For bilinear transformation $\Omega_c = \dfrac{\dfrac{2}{T}\tan \omega_1/2}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$

For impulse invariant transformation $\Omega_c = \dfrac{\omega_1/T}{\left[\dfrac{1}{A_1^2} - 1\right]^{1/2N}}$

Step 5 Determine the transfer function of the analog filter.

Let $H_a(s)$ be the transfer function of the analog filter. When the order N is even, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k\Omega_c s + \Omega_c^2}$$

When the order N is odd, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k\Omega_c s + \Omega_c^2}$$

The coefficient b_k is given by

$$b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

For normalized case, $\Omega_c = 1$ rad/s

Step 6 Using the chosen transformation, transform the analog filter transfer function $H_a(s)$ to digital filter transfer function $H(z)$.

Step 7 Realize the digital filter transfer function $H(z)$ by a suitable structure.

EXAMPLE Design a digital Butterworth filter satisfying the following constraints:

$$0.8 \le |H(\omega)| \le 1; \qquad 0 \le \omega \le 0.2\pi$$

$$|H(\omega)| \le 0.2; \quad 0.32\pi \le \omega \le \pi$$

with $T = 1$ s. Apply impulse invariant transformation.

Solution: From the given specifications, we have

$$A_1 = 0.8 \qquad \omega_1 = 0.2\pi$$
$$A_2 = 0.2 \qquad \omega_2 = 0.32\pi \qquad \text{and } T = 1 \text{ s}$$

The Butterworth IIR digital filer is designed as per the following steps.

Step 1 Choice of the type of transformation
Here, the impulse invariant transformation is already specified.

Step 2 Determination of the ratio of analog filter's edge frequencies, Ω_2/Ω_1

$$\Omega_2 = \frac{\omega_2}{T} = \frac{0.32\pi}{1} = 0.32\pi$$

$$\Omega_1 = \frac{\omega_1}{T} = \frac{0.2\pi}{1} = 0.2\pi$$

$$\frac{\Omega_2}{\Omega_1} = \frac{0.32\pi}{0.2\pi} = 1.6$$

Step 3 Determination of the order of the filter N

$$N \geq \frac{1}{2} \frac{\log \left\{ \left[\frac{1}{A_2^2} - 1 \right] \Big/ \left[\frac{1}{A_1^2} - 1 \right] \right\}}{\log \frac{\Omega_2}{\Omega_1}}$$

$$\geq \frac{1}{2} \frac{\log \left\{ \left[\frac{1}{0.2^2} - 1 \right] \Big/ \left[\frac{1}{0.8^2} - 1 \right] \right\}}{\log 1.6}$$

$$\geq \frac{1}{2} \frac{\log \{24/0.5625\}}{\log 1.6} \geq 3.9931$$

So the order of the filter $N \geq 3.9931$. Choose $N = 4$.

Step 4 Determination of the analog cutoff frequency Ω_c (i.e., -3 dB frequency)

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1 \right]^{1/2N}} = \frac{0.2\pi}{\left[\frac{1}{0.8^2} - 1 \right]^{1/2\times 4}} = 0.675 \text{ rad/s}$$

Step 5 Determination of the transfer function of analog low-pass Butterworth filter. The transfer function of the low-pass filter for even values of N is:

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

where $$b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

Here $N = 4$; $\therefore$ $k = 1, 2$

When $k = 1$, $b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2 \times 4}\right] = 0.765$

When $k = 2$, $b_k = b_2 = 2\sin\left[\frac{(2 \times 2-1)\pi}{2 \times 4}\right] = 1.848$

$$\therefore \; H_a(s) = \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2} \times \frac{\Omega_c^2}{s^2 + b_2 \Omega_c s + \Omega_c^2}$$

$$= \frac{(0.675)^2}{s^2 + (0.765 \times 0.675)s + 0.675^2} \times \frac{(0.675)^2}{s^2 + (1.848 \times 0.675)\, s + (0.675)^2}$$

$$= \frac{0.2076}{(s^2 + 0.516s + 0.456)(s^2 + 1.247s + 0.456)}$$

Step 6 Determination of the digital filter transfer function $H(z)$

By partial fraction expansion, $H_a(s)$ can be expressed as:

$$H_a(s) = \frac{0.2076}{(s^2 + 0.516s + 0.456)\,(s^2 + 1.247s + 0.456)}$$

$$= \frac{As + B}{s^2 + 0.516s + 0.456} + \frac{Cs + D}{s^2 + 1.247s + 0.456}$$

On cross multiplying the above equation and simplifying, we get

$$0.2076 = (A + C)s^3 + (1.247A + B + 0.516C + D)s^2 + (0.456A + 1.247B + 0.456C$$

$$+ 0.516D)s + (0.456B + 0.456D)$$

On solving, we get

$$A = -0.622,\ B = -0.321,\ C = 0.622 \text{ and } D = 0.776$$

Therefore, $H_a(s)$ can be written as:

$$H_a(s) = \frac{-0.622s - 0.321}{s^2 + 0.516s + 0.456} + \frac{0.622s + 0.776}{s^2 + 1.247s + 0.456}$$

$$= \frac{-0.622(s + 0.516)}{(s^2 + 2 \times 0.258s + 0.258^2) + \left(\sqrt{0.456 - 0.258^2}\right)^2}$$

$$+ \frac{0.622(s + 1.248)}{(s^2 + 2 \times 0.624s + 0.624^2) + \left(\sqrt{0.456 - 0.624^2}\right)^2}$$

$$= \frac{-0.622\,(s+0.258+0.258)}{(s+0.258)^2+(0.624)^2} + \frac{0.622\,(s+0.624+0.624)}{(s+0.624)^2+(0.258)^2}$$

$$= -0.622\frac{s+0.258}{(s+0.258)^2+(0.624)^2} - 0.257\,\frac{0.624}{(s+0.258)^2+(0.624)^2}$$

$$+0.622\frac{s+0.624}{(s+0.624)^2+(0.258)^2} + 1.504\,\frac{0.258}{(s+0.624)^2+(0.258)^2}$$

The analog transfer function of the above equation can be transformed to digital transfer function using the following standard impulse invariant transformations.

$$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)\,z^{-1}}{1-2e^{-aT}(\cos bT)\,z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT)\,z^{-1}}{1-2e^{-aT}(\cos bT)\,z^{-1}+e^{-2aT}z^{-2}}$$

Taking T = 1 s, the above transformation can be applied to $H_a(s)$ to get $H(z)$.

$$H(z) = -0.622\frac{1-e^{-0.258}(\cos 0.624)z^{-1}}{1-2e^{-0.258}(\cos 0.624)z^{-1}+e^{-2\times 0.258}z^{-2}}$$

$$-0.257\frac{e^{-0.258}(\sin 0.624)z^{-1}}{1-2e^{-0.258}(\cos 0.624)z^{-1}+e^{-2\times 0.258}z^{-2}}$$

$$+0.622\frac{1-e^{-0.624}(\cos 0.258)z^{-1}}{1-2e^{-0.624}(\cos 0.258)z^{-1}+e^{-2\times 0.624}z^{-2}}$$

$$+1.504\frac{e^{-0.624}(\sin 0.258)z^{-1}}{1-2e^{-0.624}(\cos 0.258)z^{-1}+e^{-2\times 0.624}z^{-2}}$$

$$=\frac{-0.622+0.39z^{-1}}{1-1.254z^{-1}+0.597z^{-2}}+\frac{-0.116z^{-1}}{1-1.254z^{-1}+0.597z^{-2}}$$

$$+\frac{0.622-0.322z^{-1}}{1-1.036z^{-1}+0.287z^{-2}}+\frac{0.206z^{-1}}{1-1.036z^{-1}+0.287z^{-2}}$$

$$=\frac{-0.622+0.274z^{-1}}{1-1.254z^{-1}+0.597z^{-2}}+\frac{0.622-0.116z^{-1}}{1-1.036z^{-1}+0.287z^{-2}}$$

$$=\frac{0.0224z^{-1}+0.0544z^{-2}+0.0094z^{-3}}{1-2.29z^{-1}+2.1831z^{-2}-0.977z^{-3}+0.1713z^{-4}}$$

EXAMPLE Design a Butterworth digital filter using the bilinear transformation. The specifications of the desired low-pass filter are:

$$0.9 \le |H(\omega)| \le 1; \quad 0 \le \omega \le \frac{\pi}{2}$$

$$|H(\omega)| \le 0.2; \quad \frac{3\pi}{4} \le \omega \le \pi$$

with $T = 1$ s

Solution: The Butterworth digital filter is designed as per the following steps.

From the given specification, we have

$$A_1 = 0.9 \text{ and } \omega_1 = \frac{\pi}{2}$$

$$A_2 = 0.2 \text{ and } \omega_2 = \frac{3\pi}{4} \qquad \text{and } T = 1 \text{ s}$$

Step 1 Choice of the type of transformation

Here the bilinear transformation is already specified.

Step 2 Determination of the ratio of the analog filter's edge frequencies, Ω_2/Ω_1

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = \frac{2}{1}\tan\left[\frac{(3\pi/4)}{2}\right] = 2\tan\frac{3\pi}{8} = 4.828$$

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = \frac{2}{1}\tan\left[\frac{(\pi/2)}{2}\right] = 2\tan\frac{\pi}{4} = 2$$

$$\therefore \qquad \frac{\Omega_2}{\Omega_1} = \frac{4.828}{2} = 2.414$$

Step 3 Determination of the order of the filter N

$$N \geq \frac{1}{2}\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log\frac{\Omega_2}{\Omega_1}}$$

$$\geq \frac{1}{2}\frac{\log\left\{\left[\frac{1}{(0.2)^2}-1\right]\Big/\left[\frac{1}{(0.9)^2}-1\right]\right\}}{\log 1.207}$$

$$\geq \frac{1}{2}\frac{\log\{24/0.2345\}}{\log 2.414} \geq 2.626$$

Since $N \geq 2.626$, choose $N = 3$.

Step 4 Determination of the analog cutoff frequency Ω_c (i.e., –3 dB frequency)

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}} = \frac{2}{\left[\frac{1}{0.9^2}-1\right]^{1/2\times 3}} = 2.5467$$

Step 5 Determination of the transfer function of the analog Butterworth filter $H_a(s)$

For odd N, we have $H_a(s) = \dfrac{\Omega_c}{s+\Omega_c}\displaystyle\prod_{k=1}^{\frac{N-1}{2}}\dfrac{\Omega_c^2}{s^2+b_k\Omega_c s+\Omega_c^2}$

where $b_k = 2\sin\left[\dfrac{(2k-1)\pi}{2N}\right]$

For $N = 3$, we have

$$H_a(s) = \frac{\Omega_c}{s+\Omega_c}\;\frac{\Omega_c^2}{s^2 + b_1\Omega_c s + \Omega_c^2}$$

where $b_1 = 2\sin\left[\frac{(2\times 1-1)\pi}{2\times 3}\right] = 2\sin\frac{\pi}{6} = 1$

Therefore, $H_a(s) = \left(\frac{2.5467}{s+2.5467}\right)\left(\frac{(2.5467)^2}{s^2 + 1(2.5467)\,s + (2.5467)^2}\right)$

Step 6 Conversion of $H_a(s)$ into $H(z)$

Since bilinear transformation is to be used, the digital filter transfer function is:

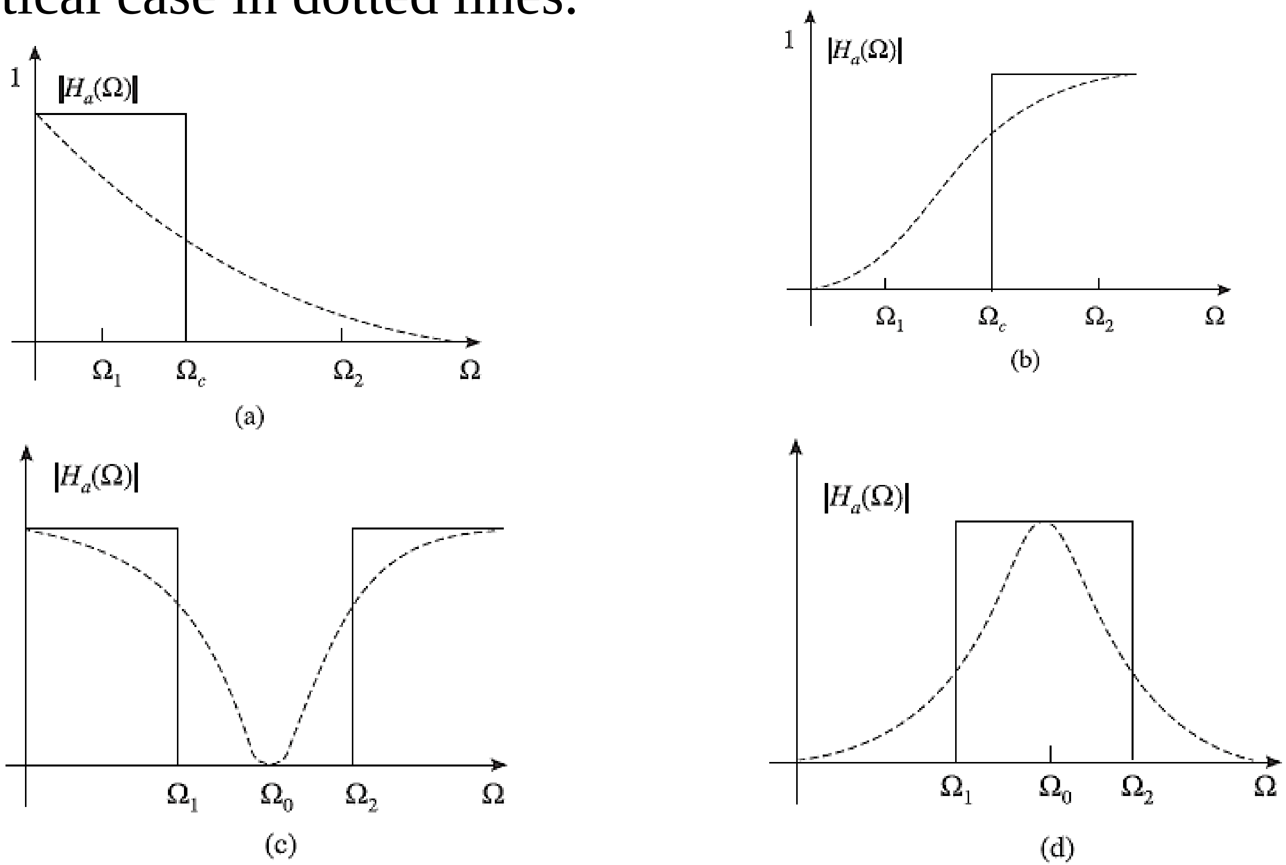
$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = H_a(s)\Big|_{s=2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \left(\frac{2.5467}{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2.5467}\right)\left[\frac{(2.5467)^2}{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 2.5467\left[2\,\frac{1-z^{-1}}{1+z^{-1}}\right] + (2.5467)^2}\right]$$

$$= \frac{0.2332\,(1+z^{-1})^3}{1 + 0.4394\,z^{-1} + 0.3845\,z^{-2} + 0.0416\,z^{-3}}$$

Frequency Transformation

- ✓ Basically there are four types of frequency selective filters, viz. low-pass, high-pass, band pass and band stop.
- ✓ In below Figure, the frequency response of the ideal case is shown in solid lines and practical case in dotted lines.



- ✓ In the design techniques discussed so far, we have considered only low-pass filters.
- ✓ This low-pass filter can be considered as a prototype filter and its system function $H_p(s)$ can be determined.
- ✓ The high-pass or band pass or band stop filters are designed by designing a low-pass filter and then transforming that low-pass transfer function into the required filter function by frequency transformation.
- ✓ Frequency transformation can be accomplished in two ways.

(1) Analog frequency transformation

(2) Digital frequency transformation

Analog frequency transformation

- ✓ In the analog frequency transformation, the analog system function $H_p(s)$ of the prototype filter is converted into another analog system function $H(s)$ of the desired filter (a low-pass filter with another cut off frequency or a high-pass filter or a band pass filter or a band stop filter).
- ✓ Then using any of the mapping techniques (impulse invariant transformation or bilinear transformation) this analog filter is converted into the digital filter with a system function $H(z)$.

- ✓ The frequency transformation formulae used to convert a prototype low-pass filter into a low-pass (with a different cut off frequency), high-pass, band pass or band stop are given in below Table.
- ✓ Here Ω_c is the cut off frequency of the low-pass prototype filter. Ω_c* cut off frequency of new low-pass filter or high-pass filter and Ω_1 and Ω_2 are the cut off frequencies of band pass or band stop filters.

Type	*Transformation*
Low-pass	$s \rightarrow \Omega_c \dfrac{s}{\Omega_c^*}$
High-pass	$s \rightarrow \Omega_c \dfrac{\Omega_c^*}{s}$
Band pass	$s \rightarrow \Omega_c \dfrac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}$
Band stop	$s \rightarrow \Omega_c \dfrac{s(\Omega_2 - \Omega_1)}{s^2 + \Omega_1 \Omega_2}$

Ω_0 is the centre frequency $\Omega_0 = \sqrt{\Omega_1 \Omega_2}$

$$\text{Quality factor } Q = \frac{\Omega_0}{\Omega_2 - \Omega_1}$$

EXAMPLE A Prototype low-pass filter has the system function $H_p(s) = \dfrac{1}{s^2 + 3s + 2}$.

Obtain a band pass filter with $\Omega_0 = 3$ rad/s and $Q = 12$.

Solution: We know that the centre frequency $\Omega_0 = \sqrt{\Omega_1 \Omega_2}$ and quality factor $Q = \dfrac{\Omega_0}{\Omega_2 - \Omega_1}$.

From Table , we have the low-pass to band pass transformation

$$s \to \Omega_c \frac{s^2 + \Omega_1\Omega_2}{s(\Omega_2 - \Omega_1)} = \Omega_c \frac{s^2 + \Omega_0^2}{s(\Omega_0/Q)}$$

$$s \to \Omega_c \frac{s^2 + 3^2}{s(3/12)} = 4\Omega_c \left(\frac{s^2 + 9}{s} \right)$$

Therefore, the transfer function of band pass filter is:

$$H(s) = H_p(s)\Big|_{s = 4\Omega_c\left(\frac{s^2+9}{s}\right)}$$

$$= \frac{1}{\left[4\Omega_c\left(\dfrac{s^2 + 9}{s}\right)\right]^2 + 3\left[4\Omega_c\left(\dfrac{s^2 + 9}{s}\right)\right] + 2}$$

$$= \frac{1}{16} \frac{s^2}{\Omega_c^2 s^4 + 0.75\Omega_c s^3 + (18\Omega_c^2 + 0.125)s^2 + 6.75\Omega_c s + 81\Omega_c^2}$$

Digital frequency transformation

- ✓ As in the analog domain, frequency transformation is possible in the digital domain also.
- ✓ The frequency transformation is done in the digital domain by replacing the variable z^{-1} by a function of z^{-1}, i.e., $f(z^{-1})$. This mapping must take into account the stability criterion.
- ✓ All the poles lying within the unit circle must map onto itself and the unit circle must also map onto itself.

✓ The below Table gives the formulae for the transformation of the prototype low pass digital filter into a digital low-pass, high-pass, band pass or band stop filters.

Type	*Transformation*	*Design parameter*
Low-pass	$z^{-1} \to \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \dfrac{\sin\left[(\omega_c - \omega_c^*)/2\right]}{\sin\left[(\omega_c + \omega_c^*)/2\right]}$
High-pass	$z^{-1} \to -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left[(\omega_c - \omega_c^*)/2\right]}{\cos\left[(\omega_c + \omega_c^*)/2\right]}$
Band pass	$z^{-1} \to -\dfrac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}$	$\alpha_1 = \dfrac{-2\alpha k}{(k+1)}$
		$\alpha_2 = \dfrac{(k-1)}{(k+1)}$
		$\alpha = \dfrac{\cos[(\omega_2 + \omega_1)/2]}{\cos[(\omega_2 - \omega_1)/2]}$
		$k = \cot\left(\dfrac{\omega_2 - \omega_1}{2}\right)\tan\left(\dfrac{\omega_c}{2}\right)$
Band stop	$z^{-1} \to \dfrac{z^{-2} - \alpha_1 z^{-1} + \alpha_2}{\alpha_2 z^{-2} - \alpha_1 z^{-1} + 1}$	$\alpha_1 = \dfrac{-2\alpha}{(k+1)}$
		$\alpha_2 = \dfrac{(1-k)}{(1+k)}$
		$\alpha = \dfrac{\cos[(\omega_2 + \omega_1)/2]}{\cos[(\omega_2 - \omega_1)/2]}$
		$k = \tan\left(\dfrac{\omega_2 - \omega_1}{2}\right)\tan\left(\dfrac{\omega_c}{2}\right)$

- ✓ The frequency transformation may be accomplished in any of the available two techniques, however, caution must be taken to which technique to use.
- ✓ For example, the impulse invariant transformation is not suitable for high-pass or band pass filters whose resonant frequencies are higher.
- ✓ In such a case, suppose a low-pass prototype filter is converted into a high-pass filter using the analog frequency transformation and transformed later to a digital filter using the impulse invariant technique. This will result in aliasing problems.
- ✓ However, if the same prototype low-pass filter is first transformed into digital filter using the impulse invariant technique and later converted into a high-pass filter using the digital frequency transformation, then it will not have any aliasing problem.
- ✓ Whenever the bilinear transformation is used, it is of no significance whether analog frequency transformation is used or digital frequency transformation.
- ✓ In this case, both analog and digital frequency transformation techniques will give the same result.

Design of Lowpass Digital Chebyshev Filter

- ✓ For designing a Chebyshev IIR digital filter, first an analog filter is designed using the given specifications. Then the analog filter transfer function is transformed to digital filter transfer function by using either impulse invariant transformation or bilinear transformation.
- ✓ The analog Chebyshev filter is designed by approximating the ideal frequency response using an error function.
- ✓ There are two types of Chebyshev approximations.
- ✓ In type-1 approximation, the error function is selected such that the magnitude response is equiripple in the pass band and monotonic in the stop band.
- ✓ In type-2 approximation, the error function is selected such that the magnitude function is monotonic in the pass band and equiripple in the stop band. The type-2 magnitude response is also called inverse Chebyshev response.

The magnitude response of type-1 Chebyshev low-pass filter is given by

$$\left|H_a(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 c_N^2\left(\dfrac{\Omega}{\Omega_c}\right)}$$

where ε is attenuation constant given by $\varepsilon = \left[\dfrac{1}{A_1^2} - 1\right]^{\frac{1}{2}}$

A_1 is the gain at the passband edge frequency ω_1 and $c_N\left(\dfrac{\Omega}{\Omega_c}\right)$ is the Chebyshev polynomial of the first kind of degree N given by

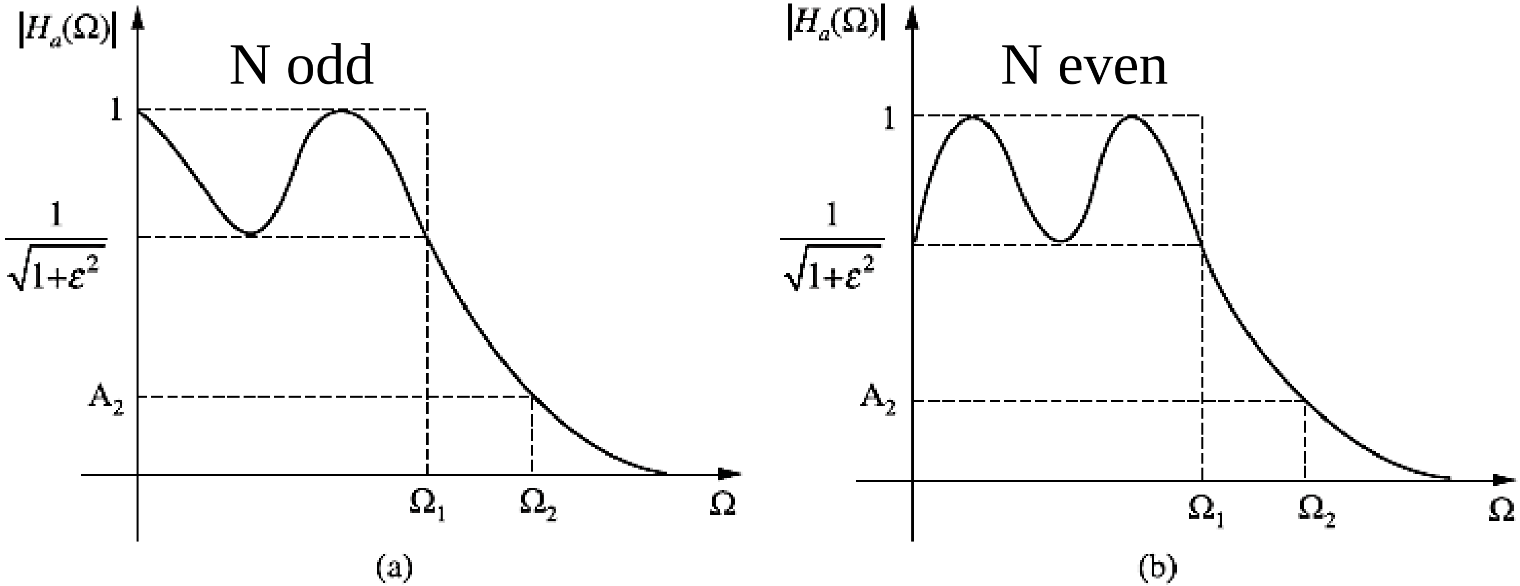
$$c_N(x) = \{\cos(N\cos^{-1} x), \quad \text{for } |x| \le 1$$

$$= \{\cos(N\cosh^{-1} x), \quad \text{for } |x| \ge 1$$

and Ω_c is the 3 dB cutoff frequency.

✓ The frequency response of Chebyshev filter depends on order *N*. The approximated response approaches the ideal response as the order *N* increases. The phase response of the Chebyshev filter is more nonlinear than that of the Butterworth filter for a given filter length *N*.

- ✓ The magnitude response of type-1 Chebyshev filter is shown in Figure.



- ✓ The design parameters of the Chebyshev filter are obtained by considering the low-pass filter with the desired specifications as given below.

$$A_1 \le |H(\omega)| \le 1 \qquad 0 \le \omega \le \omega_1$$

$$|H(\omega)| \le A_2 \qquad \omega_2 \le \omega \le \pi$$

- ✓ The corresponding analog magnitude response is to be obtained in the design process. We have

$$A_1^2 \le \frac{1}{1+\varepsilon^2 c_N^2(\Omega_1/\Omega_2)} \le 1$$

$$\frac{1}{1+\varepsilon^2 c_N^2(\Omega_1/\Omega_2)} \le A_2^2$$

Assuming $\Omega_c = \Omega_1$, we will have $c_N(\Omega_1/\Omega_c) = c_N(1) = 1$.

Therefore, from the above inequality involving A_1^2, we get

$$A_1^2 \le \frac{1}{1+\varepsilon^2}$$

Assuming equality in the above equation, the expression for ε is

$$\varepsilon = \left[\frac{1}{A_1^2} - 1\right]^{\frac{1}{2}}$$

The order of the analog filter, N can be determined from the inequality for A_2^2.

Assuming $\Omega_c = \Omega_1$,

$$c_N(\Omega_2/\Omega_1) \geq \frac{1}{\varepsilon}\left[\frac{1}{A_2^2} - 1\right]^{\frac{1}{2}}$$

Since $\Omega_2 > \Omega_1$,

$$\cosh[N\cosh^{-1}(\Omega_2/\Omega_1)] \geq \frac{1}{\varepsilon}\left[\frac{1}{A_2^2} - 1\right]^{\frac{1}{2}}$$

or

$$N \geq \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{A_2^2} - 1\right]^{\frac{1}{2}}\right\}}{\cosh^{-1}(\Omega_2/\Omega_1)}$$

Choose N to be the next nearest integer to the value given above. The values of Ω_2 and Ω_1 are determined from ω_1 and ω_2 using either impulse invariant transformation or bilinear transformation.

The transfer function of Chebyshev filters are usually written in the factored form as given below.

When N is even, $$H_a(s)=\prod_{k=1}^{\frac{N}{2}} \frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+c_k\Omega_c^2}$$

When N is odd, $$H_a(s)=\frac{B_0\Omega_c}{s+\Omega_c}\prod_{k=1}^{\frac{N-1}{2}} \frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+c_k\Omega_c^2}$$

where

$$b_k=2y_N\sin\left(\frac{(2k-1)\pi}{2N}\right)$$

$$c_k=y_N^2+\cos^2\left(\frac{(2k-1)\pi}{2N}\right)$$

$$c_0=y_N$$

$$y_N=\frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\varepsilon}\right]^{\frac{1}{N}}-\left[\left(\frac{1}{\varepsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\varepsilon}\right]^{\frac{-1}{N}}\right\}$$

For even values of N and unity dc gain filter, the parameter B_k are evaluated using the equation:

$$H_a(s)\Big|_{s=0} = \frac{1}{[1+\varepsilon^2]^{1/2}}$$

For odd values of N and unity dc gain filter, the parameter B_k are evaluated using the equation:

$$H_a(s)\Big|_{s=0} = 1$$

Poles of a normalized Chebyshev filter

The transfer function of the analog system can be obtained from the equation for the magnitude squared response as:

$$H_a(s)H_a(-s) = \frac{1}{1 + \varepsilon^2 c_N^2\left(\dfrac{s/j}{\Omega_c}\right)}$$

For the normalized transfer function, let us replace s/Ω_c by s_n.

$\therefore$

$$H_a(s_n)H_a(-s_n) = \frac{1}{1 + \varepsilon^2 c_N^2(-js_n)}$$

The normalized poles in the s-domain can be obtained by equating the denominator of the above equation to zero, i.e., $1 + \varepsilon^2 c_N^2(-js_n)$ to zero.

The solution to the above expression gives us the $2N$ poles of the filter given by

$$s_n = -\sin x \sinh y + j\cos x \cosh y = \sigma_n + j\Omega_n$$

where $n = 1, 2, ..., (N+1)/2$ for N odd

$= 1, 2, ..., N/2$ for N even

and

$$x = \frac{(2n-1)\pi}{2N} \qquad n = 1,2, ..., N$$

$$y = \pm\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right) \qquad n = 1, 2, ..., N$$

The unnormalized poles, s'_n can be obtained from the normalized poles as shown below.

$$s'_n = s_n\Omega_c$$

The normalized poles lie on an ellipse in s-plane. Since for a stable filter all the poles should lie in the left half of s-plane, only the N poles on the ellipse which are in the left half of s-plane are considered.

For N even, all the poles are complex and exist in conjugate pairs. For N odd, one pole is real and all other poles are complex and occur in conjugate pairs.

Design procedure for low-pass digital Chebyshev IIR filter

The low-pass Chebyshev IIR digital filter is designed following the steps given below.

Step 1 Choose the type of transformation.
(Bilinear or impulse invariant transformation)

Step 2 Calculate the attenuation constant ε.

$$\varepsilon = \left[\frac{1}{A_1^2} - 1\right]^{\frac{1}{2}}$$

Step 3 Calculate the ratio of analog edge frequencies Ω_2/Ω_1.
For bilinear transformation,

$$\frac{\Omega_2}{\Omega_1} = \frac{\frac{2}{T}\tan\frac{\omega_2}{2}}{\frac{2}{T}\tan\frac{\omega_1}{2}} = \frac{\tan\frac{\omega_2}{2}}{\tan\frac{\omega_1}{2}}$$

For impulse invariant transformation,

$$\frac{\Omega_2}{\Omega_1} = \frac{\omega_2/T}{\omega_1/T} = \frac{\omega_2}{\omega_1}$$

Step 4 Decide the order of the filter N such that

$$N \geq \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{A_2^2} - 1\right]\right\}}{\cosh^{-1}\left\{\frac{\Omega_2}{\Omega_1}\right\}}$$

Step 5 Calculate the analog cutoff frequency Ω_c.
For bilinear transformation,

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}} = \frac{\frac{2}{T}\tan\frac{\omega_1}{2}}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$$

For impulse invariant transformation

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}} = \frac{\omega_1/T}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$$

Step 6 Determine the analog transfer function $H_a(s)$ of the filter. When the order N is even, $H_a(s)$ is given by

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

When the order N is odd, $H_a(s)$ is given by

$$H_a(s) = \frac{B_0 \Omega_c}{s + c_0 \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}$$

where $b_k = 2 y_N \sin\left(\dfrac{(2k-1)\pi}{2N}\right)$

$$c_k = y_N^2 + \cos^2 \frac{(2k-1)\pi}{2N}$$

$$c_0 = y_N$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{-1}{N}}\right\}$$

For even values of N and unity dc gain filter, find B_k's such that

$$H_a(0) = \frac{1}{(1+\varepsilon^2)^{1/2}}$$

For odd values of N and unity dc gain filter, find B_k's such that

$$\prod_{k=0}^{\frac{N-1}{2}} \frac{B_k}{c_k} = 1$$

(It is normal practice to take $B_0 = B_1 = B_2 = \ldots = B_k$)

Step 7 Using the chosen transformation, transform $H_a(s)$ to $H(z)$, where $H(z)$ is the transfer function of the digital filter.

[The high-pass, band pass and band stop filters are obtained from low-pass filter design by frequency transformation].

EXAMPLE Design a Chebyshev IIR digital low-pass filter to satisfy the constraints.

$$0.707 \le |H(\omega)| \le 1, \qquad 0 \le \omega \le 0.2\pi$$

$$|H(\omega)| \le 0.1, \qquad 0.5\pi \le \omega \le \pi$$

using bilinear transformation and assuming $T = 1$ s.

Solution: Given

$$A_1 = 0.707, \qquad \omega_1 = 0.2\pi$$

$$A_2 = 0.1, \qquad \omega_2 = 0.5\pi$$

$T = 1$ s and bilinear transformation is to be used. The low-pass Chebyshev IIR digital filter is designed as follows:

Step 1 Type of transformation
Here bilinear transformation is to be used.

Step 2 Attenuation constant ε

$$\varepsilon = \left[\frac{1}{A_1^2} - 1\right]^{\frac{1}{2}} = \left[\frac{1}{0.707^2} - 1\right]^{\frac{1}{2}} = 1$$

Step 3 Ratio of analog edge frequencies, Ω_2/Ω_1.
Since bilinear transformation is to be used,

$$\frac{\Omega_2}{\Omega_1} = \frac{\frac{2}{T}\tan\frac{\omega_2}{2}}{\frac{2}{T}\tan\frac{\omega_1}{2}} = \frac{\tan\frac{0.5\pi}{2}}{\tan\frac{0.2\pi}{2}} = \frac{2}{0.6498} = 3.0779$$

Step 4 Order of the filter N

$$N \geq \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{A_2^2}-1\right]^{\frac{1}{2}}\right\}}{\cosh^{-1}\left\{\frac{\Omega_2}{\Omega_1}\right\}} \geq \frac{\cosh^{-1}\left\{\frac{1}{1}\left[\frac{1}{0.1^2}-1\right]^{0.5}\right\}}{\cosh^{-1}\{3.0779\}} \geq 1.669 \approx 2.$$

Step 5 Analog cutoff frequency Ω_c

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}} = \frac{\frac{2}{T}\tan\frac{\omega_1}{2}}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}} = \frac{0.6498}{\left[\frac{1}{0.7077}-1\right]^{\frac{1}{4}}} = 0.6498$$

Step 6 Analog filter transfer function $H_a(s)$

$$H_a(s) = \prod_{k=1}^{N/2} \frac{B_k\Omega_c^2}{s^2 + b_k\Omega_c s + c_k\Omega_c^2} = \frac{B_1\Omega_c^2}{s^2 + b_1\Omega_c s + c_1\Omega_c^2}$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\varepsilon}\right]^{\frac{-1}{N}}\right\}$$

$$= \frac{1}{2}\left\{\left[\left(\frac{1}{1^2}+1\right)^{\frac{1}{2}}+\frac{1}{1}\right]^{\frac{1}{2}} - \left[\left(\frac{1}{1^2}+1\right)^{\frac{1}{2}}+\frac{1}{1}\right]^{\frac{-1}{2}}\right\}$$

$$= \frac{1}{2}\left\{[2.414]^{\frac{1}{2}} - [2.414]^{\frac{-1}{2}}\right\} = 0.455$$

$$b_1 = 2y_N \sin\left[\frac{(2k-1)\pi}{2N}\right] = 2\times 0.455\ \sin\left[\frac{(2\times 1-1)\pi}{2\times 2}\right] = 0.6435$$

$$c_1 = y_N^2 + \cos^2\left[\frac{(2k-1)\pi}{2N}\right] = (0.455)^2 + \cos^2\left[\frac{(2\times 1-1)\pi}{2\times 2}\right] = 0.707$$

For N even,

$$\prod_{k=1}^{\frac{N}{2}} \frac{B_k}{c_k} = \frac{A}{(1+\varepsilon^2)^{0.5}} = 0.707$$

That is $B_1 = c_1 \times 0.707 = 0.707 \times 0.707 = 0.5$.
Therefore, the system function is:

$$H_a(s) = \frac{0.5(0.6498)^2}{s^2 + (0.6435)(0.6498)s + (0.707)(0.6498)^2}$$

On simplifying, we get

$$H_a(s) = \frac{0.2111}{s^2 + 0.4181s + 0.2985}$$

Step 7 Digital filter transfer function $H(z)$

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{0.2111}{s^2 + 0.4181s + 0.2985}\Bigg|_{s=2\,\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{0.2111}{\left[2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 0.4181\left[2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)\right] + 0.2985}$$

$$= \frac{0.2111(1+z^{-1})^2}{5.1347 - 7.403z^{-1} + 3.463z^{-2}}$$

$$= \frac{0.0411(1+z^{-1})^2}{1 - 1.441z^{-1} + 0.6744z^{-2}}$$

Design of FIR Digital Filters

Introduction

- ✓ A filter is a frequency selective system. Digital filters are classified as finite duration unit impulse response (FIR) filters or infinite duration unit impulse response (IIR) filters, depending on the form of the unit impulse response of the system.
- ✓ In the FIR system, the impulse response sequence is of finite duration, i.e., it has a finite number of non-zero terms. The FIR filters are usually implemented using non-recursive structures (no feedback-only zeros).
- ✓ The response of the FIR filter depends only on the present and past input samples.

The following are the main advantages of FIR filters over IIR filters:

1. FIR filters are always stable.
2. FIR filters with exactly linear phase can easily be designed.
3. FIR filters can be realized in both recursive and non-recursive structures.
4. FIR filters are free of limit cycle oscillations, when implemented on a finite word length digital system.
5. Excellent design methods are available for various kinds of FIR filters.

The disadvantages of FIR filters are as follows:

1. The implementation of narrow transition band FIR filters is very costly, as it requires considerably more arithmetic operations and hardware components such as multipliers, adders and delay elements.
2. Memory requirement and execution time are very high.

- ✓ FIR filters are employed in filtering problems where linear phase characteristics within the pass band of the filter is required. If this is not required, either an FIR or an IIR filter may be employed.
- ✓ An IIR filter has lesser number of side lobes in the stop band than an FIR filter with the same number of parameters.
- ✓ For this reason if some phase distortion is tolerable, an IIR filter is preferable.
- ✓ Also, the implementation of an IIR filter involves fewer parameters, less memory requirements and lower computational complexity.

Characteristics of FIR Filters with Linear Phase

✓ The transfer function of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

where $h(n)$ is the impulse response of the filter. The frequency response [Fourier transform of $h(n)$] is given by

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

which is periodic in frequency with period 2π, i.e.,

$$H(\omega) = H(\omega + 2k\pi), \qquad k = 0, 1, 2, \ldots$$

Since $H(\omega)$ is complex it can be expressed as

$$H(\omega) = \pm |H(\omega)| e^{j\theta(\omega)}$$

where $|H(\omega)|$ is the magnitude response and $\theta(\omega)$ is the phase response.

We define the phase delay τ_p and group delay τ_g of a filter as:

$$\tau_p = -\frac{\theta(\omega)}{\omega} \text{ and } \tau_g = -\frac{d\theta(\omega)}{d\omega}$$

For FIR filters with linear phase, we can define

$$\theta(\omega) = -\alpha\omega \qquad -\pi \leq \omega \leq \pi$$

where α is constant phase delay in samples.

$$\tau_g = -\frac{d\theta(\omega)}{d\omega} = -\frac{d}{d\omega}(-\alpha\omega) = \alpha \text{ and } \tau_p = -\frac{\theta(\omega)}{\omega} = \frac{\alpha\omega}{\omega} = \alpha$$

i.e. $\tau_p = \tau_g = \alpha$ which means that α is independent of frequency.

We have

$$\sum_{n=0}^{N-1} h(n)e^{-j\omega n} = \pm\left|H(\omega)\right|e^{j\theta(\omega)}$$

i.e. $$\sum_{n=0}^{N-1} h(n)[\cos\omega n - j\sin\omega n] = \pm\left|H(\omega)\right|[\cos\theta(\omega) + j\sin\theta(\omega)]$$

This gives us

$$\sum_{n=0}^{N-1} h(n) \cos \omega n = \pm |H(\omega)| \cos \theta(\omega)$$

and

$$-\sum_{n=0}^{N-1} h(n) \sin \omega n = \pm |H(\omega)| \sin \theta(\omega)$$

Therefore,

$$\frac{\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} = \frac{\sin \theta(\omega)}{\cos \theta(\omega)} = \frac{\sin \alpha\omega}{\cos \alpha\omega}$$

i.e.

$$\sum_{n=0}^{N-1} h(n) [\sin \omega n \cos \alpha\omega - \cos \omega n \sin \alpha\omega] = 0$$

i.e.

$$\sum_{n=0}^{N-1} h(n) \sin (\alpha - n)\omega = 0$$

This will be zero when

$$h(n) = h(N-1-n) \text{ and } \alpha = \frac{N-1}{2}, \qquad \text{for } 0 \le n \le N-1$$

This shows that FIR filters will have constant phase and group delays when the impulse response is symmetrical about $\alpha = (N-1)/2$.

The impulse response satisfying the symmetry condition $h(n) = h(N - 1 - n)$ for odd and even values of N is shown in Figure . When $N = 9$, the centre of symmetry of the sequence occurs at the fourth sample and when $N = 8$, the filter delay is $3\frac{1}{2}$ samples.

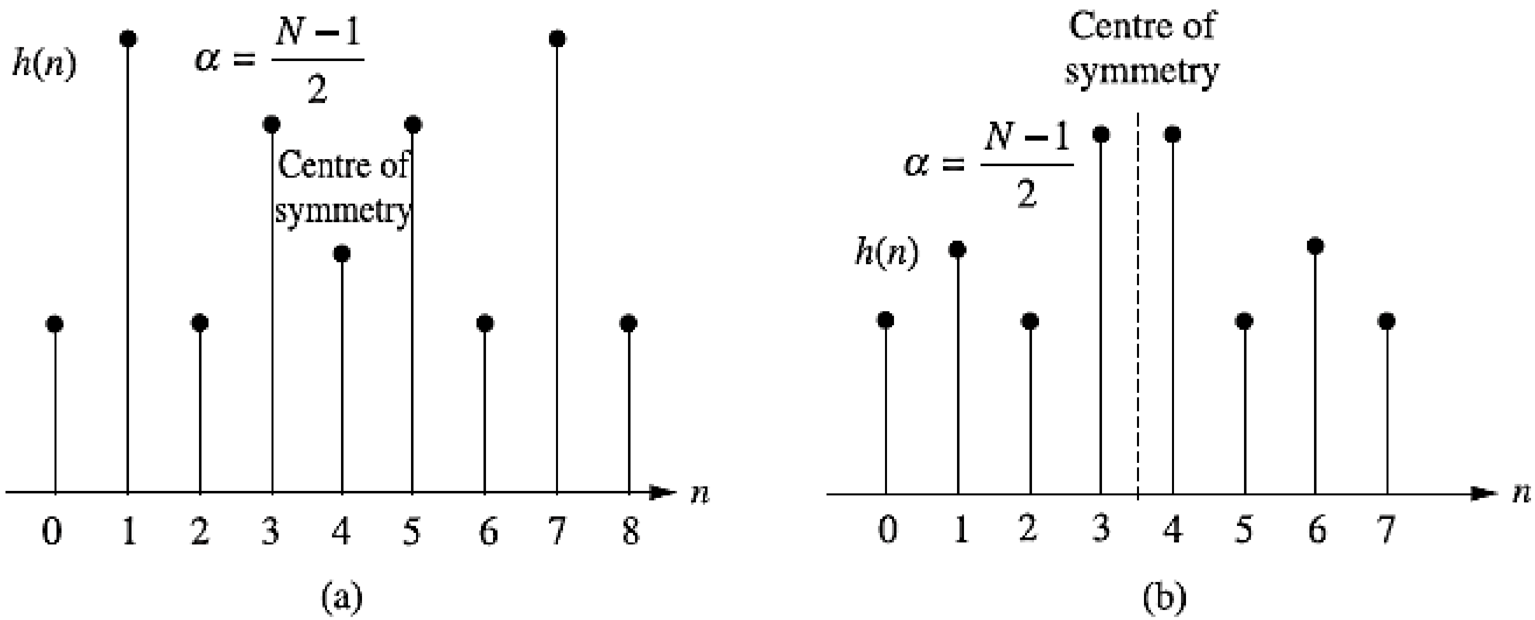


Figure Impulse response sequence of symmetrical sequences for (a) N odd (b) N even.

If only constant group delay is required and not the phase delay, we can write

$$\theta(\omega) = \beta - \alpha\omega$$

Now, we have

$$H(\omega) = \pm|H(\omega)|e^{j(\beta-\alpha\omega)}$$

i.e.
$$\sum_{n=0}^{N-1} h(n)e^{-j\omega n} = \pm|H(\omega)|e^{j(\beta-\alpha\omega)}$$

i.e. $$\sum_{n=0}^{N-1} h(n)[\cos \omega n - j \sin \omega n] = \pm |H(\omega)| [\cos (\beta - \alpha\omega) + j \sin (\beta - \alpha\omega)]$$

This gives

$$\sum_{n=0}^{N-1} h(n) \cos \omega n = \pm |H(\omega)| \cos (\beta - \alpha\omega)$$

and $$-\sum_{n=0}^{N-1} h(n) \sin \omega n = \pm |H(\omega)| \sin (\beta - \alpha\omega)$$

$\therefore$ $$-\frac{\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} = \frac{\sin (\beta - \alpha\omega)}{\cos (\beta - \alpha\omega)}$$

Cross multiplying and rearranging, we get

$$\sum_{n=0}^{N-1} h(n) [\cos \omega n \sin (\beta - \alpha\omega) + \sin \omega n \cos (\beta - \alpha\omega)] = 0$$

i.e. $$\sum_{n=0}^{N-1} h(n) \sin [\beta - (\alpha - n)\omega] = 0$$

If $\beta = \dfrac{\pi}{2}$, the above equation can be written as:

$$\sum_{n=0}^{N-1} h(n) \cos(\alpha - n)\,\omega = 0$$

This equation will be satisfied when

$$h(n) = -h(N-1-n) \text{ and } \alpha = \frac{N-1}{2}$$

This shows that FIR filters have constant group delay τ_g and not constant phase delay when the impulse response is antisymmetrical about $\alpha = (N-1)/2$.

The impulse response satisfying the antisymmetry condition is shown in Figure . When $N = 9$, the centre of antisymmetry occurs at fourth sample and when $N = 8$, the centre of antisymmetry occurs at $3\frac{1}{2}$ samples. From Figure 9.2, we find that $h[(N-1)/2] = 0$ for antisymmetric odd sequence.

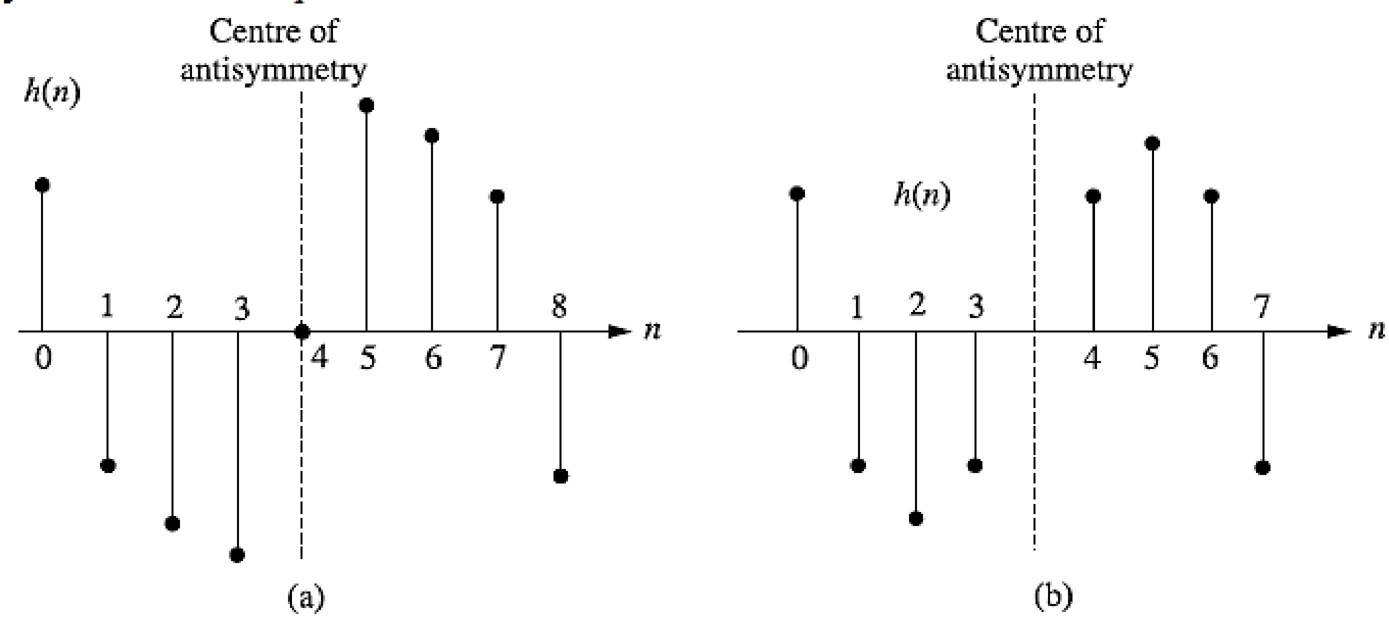


Figure Impulse response sequence of antisymmetric sequences for (a) N odd (b) N even.

Frequency Response of linear phase FIR filters

- ✓ The frequency response of the filter is the Fourier transform of its impulse response. If $h(n)$ is the impulse response of the system, then the frequency response of the system is denoted by $H(e^{j\omega})$ or $H(\omega)$.
- ✓ $H(\omega)$ is a complex function of frequency and so it can be expressed as magnitude function $|H(\omega)|$ and phase function $\angle H(\omega)$.
- ✓ Depending on the value of N (odd or even) and the type of symmetry of the filter impulse response sequence (symmetric or antisymmetric), there are following four possible types of impulse response for linear phase FIR filters.

    1. Symmetrical impulse response when N is odd.
    2. Symmetrical impulse response when N is even.
    3. Antisymmetric impulse response when N is odd.
    4. Antisymmetric impulse response when N is even.

Frequency Response of linear phase FIR filters when impulse response is Symmetrical and N is odd

Let $h(n)$ be the impulse response of the system. The frequency response of the system $H(\omega)$ is given as:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

Since the impulse response of the FIR filter has only N samples, the limits of summation can be changed to $n = 0$ to $N - 1$.

$$\therefore \qquad H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

When N is odd number, the symmetrical impulse response will have the centre of symmetry at $n = (N - 1)/2$. Hence $H(\omega)$ is expressed as:

$$H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=(N+1)/2}^{N-1} h(n)\, e^{-j\omega n}$$

Let $m = N - 1 - n$, $\qquad \therefore\ n = N - 1 - m$

When $n = \dfrac{N+1}{2}$, $\qquad m = (N-1) - \left(\dfrac{N+1}{2}\right) = \dfrac{N-3}{2}$

When $n = N - 1$, $\qquad m = (N - 1) - (N - 1) = 0$

Therefore,

$$H(\omega)=\sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n}+h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)}+\sum_{m=0}^{(N-3)/2} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Replacing m by n, we get

$$H(\omega)=\sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n}+h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)}+\sum_{n=0}^{(N-3)/2} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

For symmetrical impulse response, $h(n) = h(N-1-n)$.

Hence

$$H(\omega)=\sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n}+h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)}+\sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega(N-1)-j\omega(-n)}$$

$$=e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right)+\sum_{n=0}^{(N-3)/2} h(n)\left[e^{-j\omega n+j\omega\left(\frac{N-1}{2}\right)}+e^{-j\omega(N-1)+j\omega\frac{N-1}{2}-j\omega(-n)}\right]\right\}$$

$$=e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right)+\sum_{n=0}^{(N-3)/2} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)}+e^{-j\omega\left[(N-1)-\frac{N-1}{2}-n\right]}\right]\right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left[(N-1)-\frac{N-1}{2}-n\right]} \right] \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{(N-3)/2} h(n)\, 2\cos\left[\left(\frac{N-1}{2} - n\right)\omega\right] \right\}$$

Let $k = \frac{N-1}{2} - n$, $\quad\quad \therefore\ n = \frac{N-1}{2} - k$

When $n = 0$, $\quad\quad k = \frac{N-1}{2}$

When $n = \frac{N-3}{2}$, $\quad\quad k = \frac{N-1}{2} - \frac{N-3}{2} = 1$

$$\therefore \quad H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{k=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - k\right) \cos \omega k \right\}$$

Replacing k by n, we get

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right)\cos \omega n\right\}$$

The above equation for $H(\omega)$ is the frequency response of linear phase FIR filter when impulse response is symmetrical and N is odd.

The magnitude function of $H(\omega)$ is given by

$$|H(\omega)| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right)\cos \omega n$$

The phase function of $H(\omega)$ is given by

$$\angle H(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega\alpha \qquad \text{where } \alpha = \frac{N-1}{2}$$

Figure (a) shows a symmetrical impulse response when $N = 9$ and Figure (b) shows the corresponding magnitude function of frequency response. From these figures it can be observed that the magnitude function of $H(\omega)$ is symmetric with $\omega = \pi$, when the impulse response is symmetric and N is odd number.

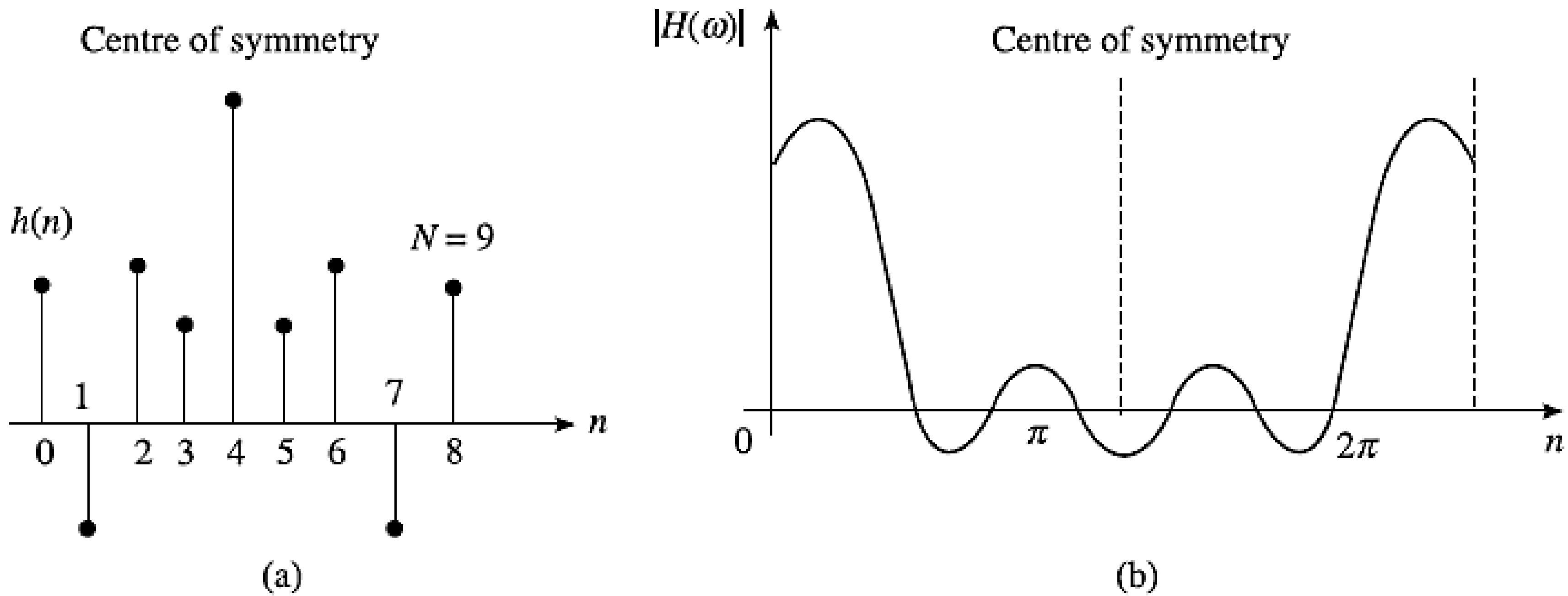


Figure (a) Symmetrical impulse response, $N = 9$ (b) Magnitude function of $H(\omega)$.

Frequency Response of linear phase FIR filters when impulse response is Symmetrical and N is even

The Frequency response of FIR filter, with impulse response $h(n)$ of length N is:

$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

For symmetrical impulse response with even number of samples (i.e. when N is even), the centre of symmetry lies between $n = (N/2) - 1$ and $n = N/2$. Hence $H(\omega)$ is expressed as:

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n}$$

Let $m = N - 1 - n$, $\quad\therefore\quad n = N - 1 - m$

When $n = \dfrac{N}{2}$, $\quad m = N - 1 - \dfrac{N}{2} = \dfrac{N}{2} - 1$

When $n = N - 1$, $\quad m = N - 1 - (N - 1) = 0$

Therefore, the above equation for $H(\omega)$ can be written as:

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{m=0}^{(N/2)-1} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Replacing m by n, we get

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N/2)-1} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

By the symmetry condition, $h(N - 1 - n) = h(n)$

Hence $H(\omega)$ can be written as:

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega(N-1)-j\omega(-n)}$$

$$= e^{-j\omega\frac{N-1}{2}} \left\{ \sum_{n=0}^{(N/2)-1} h(n) \left[e^{-j\omega n + j\omega\frac{N-1}{2}} + e^{-j\omega(-n) - j\omega(N-1) + j\omega\frac{N-1}{2}} \right] \right\}$$

$$= e^{-j\omega\frac{N-1}{2}} \left\{ \sum_{n=0}^{(N/2)-1} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right\}$$

$$= e^{-j\omega\frac{N-1}{2}} \left\{ \sum_{n=0}^{(N/2)-1} h(n)\, 2\cos\left(\omega\left(\frac{N-1}{2} - n \right) \right) \right\}$$

Let $k = \dfrac{N}{2} - n$, $\quad\therefore\quad n = \dfrac{N}{2} - k$

When $n = 0$, $\quad k = \dfrac{N}{2}$

When $n = \dfrac{N}{2} - 1$, $\quad k = \dfrac{N}{2} - \left(\dfrac{N}{2} - 1\right) = 1$

Therefore, the above expression for $H(\omega)$ becomes

$$H(\omega) = e^{-j\omega\frac{N-1}{2}}\left\{\sum_{k=1}^{N/2} 2h\left(\frac{N}{2} - k\right)\cos\omega\left(k - \frac{1}{2}\right)\right\}$$

On replacing k by n, we get

$$H(\omega) = e^{-j\omega\frac{N-1}{2}}\left\{\sum_{n=1}^{N/2} 2h\left(\frac{N}{2} - n\right)\cos\omega\left(n - \frac{1}{2}\right)\right\}$$

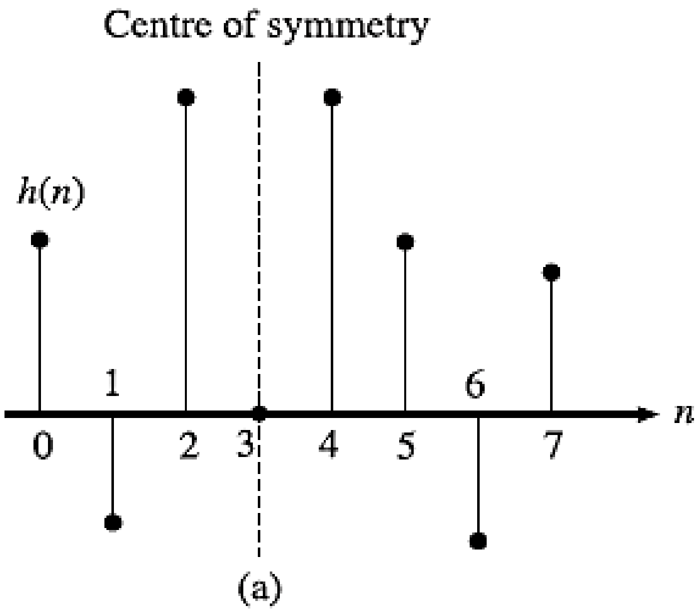
This is the expression for frequency response of linear phase FIR filter when impulse response is symmetrical and N is even. The magnitude function of $H(\omega)$ is given by

$$|H(\omega)| = \left[\sum_{n=1}^{N/2} 2h\left(\frac{N}{2} - n\right)\cos\omega\left(n - \frac{1}{2}\right)\right]$$

The phase function of $H(\omega)$ is given by

$$\angle H(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega\alpha \qquad \text{where } \alpha = \frac{N-1}{2}$$

Figure (a) shows a symmetrical impulse response when N = 8, and Figure (b) shows the corresponding magnitude function of frequency response. From these figures it can be observed that the magnitude function of $H(\omega)$ is antisymmetric with $\omega = \pi$, when impulse response is symmetric and N is even number.



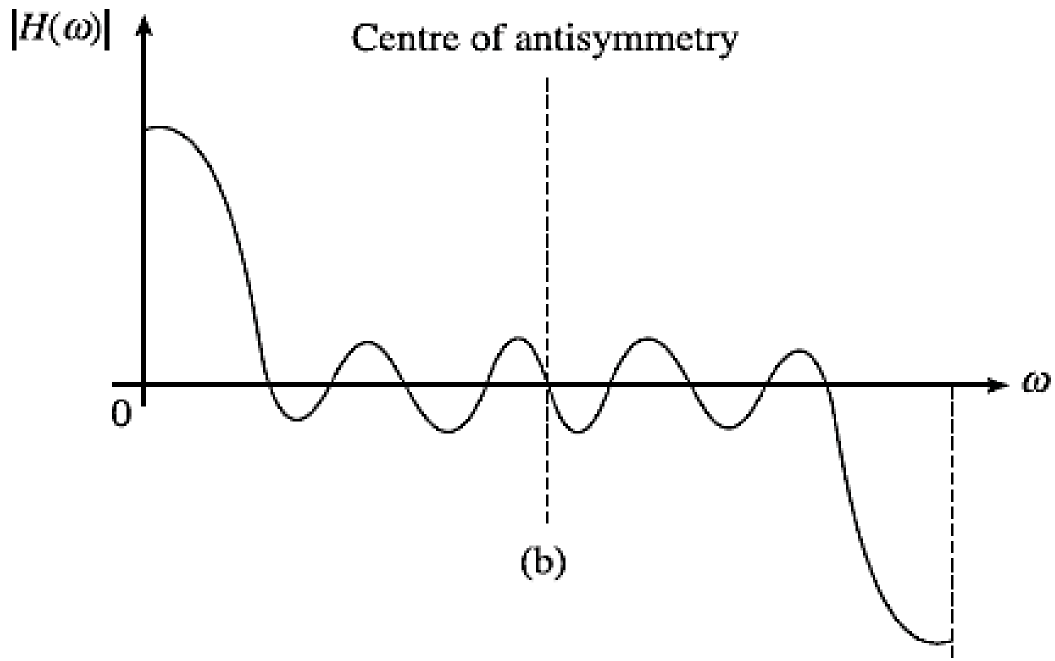


Figure (a) Symmetrical impulse response, N=8, (b) Magnitude function of $H(\omega)$.

Frequency Response of linear phase FIR filters when impulse response is Antisymmetric and N is odd

The frequency response of linear phase FIR filter with impulse response $h(n)$ of length N is:

$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

The impulse response is antisymmetric with centre of antisymmetry at $n = (N - 1)/2$. Also $h[(N - 1)/2] = 0$. Hence $H(\omega)$ can be expressed as:

$$H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=(N+1)/2}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + \sum_{n=(N+1)/2}^{N-1} h(n)\, e^{-j\omega n}$$

Let $m = N - 1 - n$, $\quad\therefore\; n = N - 1 - m$

When $n = \dfrac{N+1}{2}$, $\quad m = N - 1 - \left(\dfrac{N+1}{2}\right) = \dfrac{N-3}{2}$

When $n = N - 1$, $\quad m = N - 1 - (N - 1) = 0$

$$\therefore \quad H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + \sum_{m=0}^{(N-3)/2} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

On replacing m by n, we get

$$H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N-3)/2} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

For antisymmetric impulse response, $h(N - 1 - n) = -h(n)$. Hence, the above equation for $H(\omega)$ can be written as:

$$H(\omega) = \sum_{n=0}^{(N-3)/2} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N-3)/2} -h(n)\, e^{-j\omega(-n)-j\omega(N-1)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{(N-3)/2} h(n) \left[e^{-j\omega n + j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(-n) - j\omega(N-1) + j\omega\left(\frac{N-1}{2}\right)} \right] \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{(N-3)/2} h(n) \left[e^{j\omega\left(\frac{N-1}{2} - n\right)} - e^{-j\omega\left(\frac{N-1}{2} - n\right)} \right] \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{(N-3)/2} h(n)\, 2j \sin\left(\omega\left(\frac{N-1}{2} - n \right) \right) \right\}$$

Since $j = e^{j\pi/2}$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N-3)/2} 2\,h(n)\,e^{j\frac{\pi}{2}}\,\sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right]$$

$$= e^{j\left(\frac{\pi}{2}-\omega\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N-3)/2} 2h(n)\,\sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right]$$

Let $k = \dfrac{N-1}{2} - n,$ $\quad\therefore\ n = \dfrac{N-1}{2} - k$

When $n = 0,$ $\quad k = \dfrac{N-1}{2}$

When $n = \dfrac{N-3}{2},$ $\quad k = \dfrac{N-1}{2} - \dfrac{N-3}{2} = 1$

$$\therefore \quad H(\omega) = e^{j\left(\frac{\pi}{2}-\omega\left(\frac{N-1}{2}\right)\right)}\left[\sum_{k=1}^{(N-1)/2} 2h\left(\frac{N-1}{2}-k\right)\sin\omega k\right]$$

Replacing k by n, we get

$$H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right)\right)}\left[\sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right)\sin \omega n\right]$$

This is the equation for frequency response of linear phase FIR filter when impulse response is antisymmetric and n odd. The magnitude function is given by

$$|H(\omega)| = \sum_{n=1}^{(N-1)/2} 2h\left(\frac{N-1}{2} - n\right)\sin \omega n$$

The phase function is given by

$$\angle H(\omega) = \frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right) = \beta - \alpha\omega$$

where $$\beta = \frac{\pi}{2} \text{ and } \alpha = \frac{N-1}{2}$$

Figure (a) shows an antisymmetric impulse response when $N = 9$, and Figure (b) shows the corresponding magnitude function of frequency response. From these figures, it can be observed that the magnitude function is antisymmetic with $\omega = \pi$, when the impulse response is antisymmetric and N is odd.

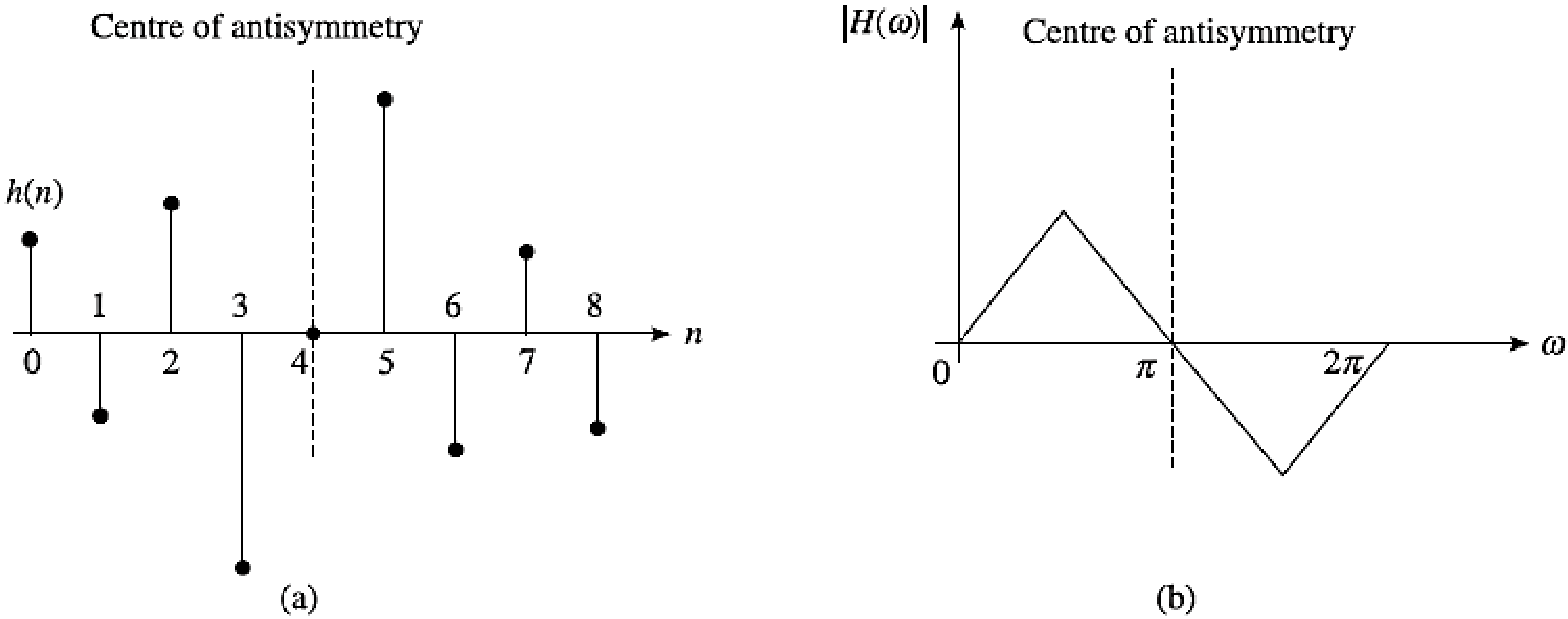


Figure (a) Antisymmetric impulse response for $N = 9$, (b) Magnitude function of $H(\omega)$.

Frequency Response of linear phase FIR filters when impulse response is Antisymmetric and N is even

The frequency response of linear phase FIR filter with impulse response $h(n)$ of length N is:

$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

The impulse response $h(n)$ is antisymmetric with centre of antisymmetry in between $n = (N/2) - 1$ and $n = (N/2)$. Hence $H(\omega)$ can be expressed as:

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n}$$

Let $m = N - 1 - n,\qquad \therefore\ n = N - 1 - m$

When $n = \dfrac{N}{2},\qquad m = N - 1 - \dfrac{N}{2} = \dfrac{N}{2} - 1$

When $n = N - 1,\qquad m = N - 1 - (N - 1) = 0$

$$\therefore\quad H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{m=0}^{(N/2)-1} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Replacing m by n, we have

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N/2)-1} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

For antisymmetric impulse response, $h(N-1-n) = -h(n)$. Hence the above equation for $H(\omega)$ can be written as:

$$H(\omega) = \sum_{n=0}^{(N/2)-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{(N/2)-1} -h(n)\, e^{-j\omega(-n)-j\omega(N-1)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N/2)-1} h(n)\left[e^{-j\omega n + j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(-n)-j\omega(N-1)+j\omega\left(\frac{N-1}{2}\right)}\right]\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N/2)-1} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N/2)-1} h(n)\, 2j\sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right]$$

Replacing j by $e^{j(\pi/2)}$, we have

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N/2)-1} 2h(n)\, e^{j(\pi/2)} \sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right]$$

$$= e^{j\left(\frac{\pi}{2}-\omega\frac{N-1}{2}\right)}\left[\sum_{n=0}^{(N/2)-1} 2h(n)\sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right]$$

Let $k = \frac{N}{2} - n$, $\quad \therefore \quad n = \frac{N}{2} - k$

When $n = 0$, $\quad k = \frac{N}{2}$

When $n = \frac{N}{2} - 1$, $\quad k = \frac{N}{2} - \left(\frac{N}{2} - 1\right) = 1$

$$\therefore \qquad H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\frac{N-1}{2}\right)}\left[\sum_{k=1}^{N/2} 2h\left(\frac{N}{2} - k\right)\sin\left(\omega\left(k - \frac{1}{2}\right)\right)\right]$$

Replacing k by n, we get

$$H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\frac{N-1}{2}\right)}\left[\sum_{n=1}^{N/2} 2h\left(\frac{N}{2} - n\right)\sin\left(\omega\left(n - \frac{1}{2}\right)\right)\right]$$

This is the equation for the frequency response of linear phase FIR filter when impulse response is antisymmetric and N is even.

The magnitude function is given by

$$|H(\omega)| = \sum_{n=1}^{N/2} 2h\left(\frac{N}{2} - n\right)\sin\left(\omega\left(n - \frac{1}{2}\right)\right)$$

The phase function is given by

$$\angle H(\omega) = \frac{\pi}{2} - \omega\,\frac{N-1}{2} = \beta - \alpha\omega$$

where $\beta = \frac{\pi}{2}$ and $\alpha = \frac{N-1}{2}$.

Figure (a) shows an antisymmetric impulse response when $N = 8$, and Figure (b) shows its corresponding magnitude function of frequency response. From Figure 9.6, it can be observed that the magnitude function of $H(\omega)$ is symmetric with $\omega = \pi$ when the impulse response is antisymmetric and N is even number.

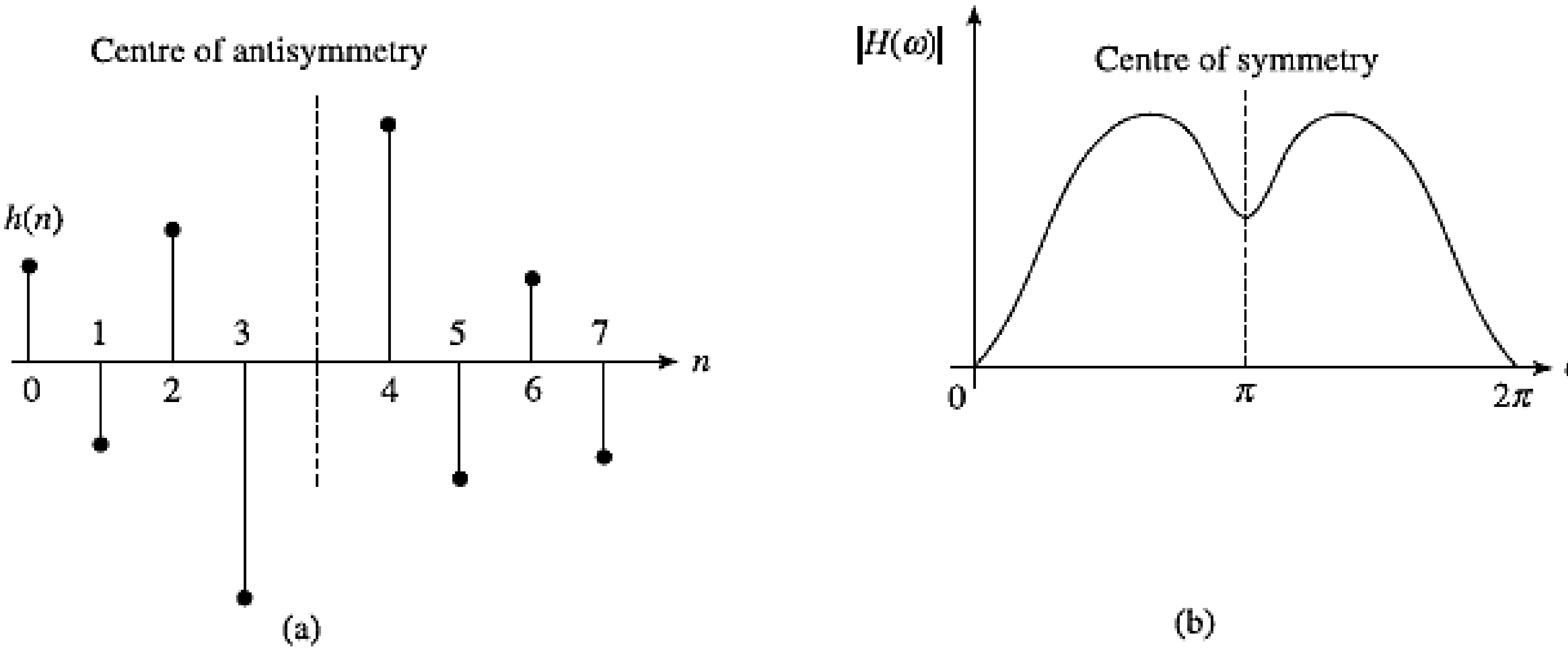


Figure (a) Antisymmetrical impulse response for $N = 8$, (b) Magnitude function of $H(\omega)$.

✓ The various steps in designing FIR filters are as follows:

1. Choose an ideal(desired) frequency response, $H_d(\omega)$.
2. Take inverse Fourier transform of $H_d(\omega)$ to get $h_d(n)$ or sample $H_d(\omega)$ at finite number of points (N-points) to get $H(k)$.
3. If $h_d(n)$ is determined, then convert the infinite duration $h_d(n)$ to a finite duration $h(n)$ (usually $h(n)$ is an N-point sequence) or if $H(k)$ is determined, then take N-point inverse DFT to get $h(n)$.
4. Take Z-transform of h(n) to get H(z), where H(z) is the transfer function of the digital filter.
5. Choose a suitable structure and realize the filter.

Design Techniques for Linear Phase FIR Filters

- ✓ The well known methods of designing FIR filters are as follows:

    1. Fourier series method
    2. Window method
    3. Frequency sampling method
    4. Optimum filter design

- ✓ In Fourier series method, the desired frequency response $H_d(\omega)$ is converted to a Fourier series representation by replacing by $2\pi fT$, where T is the sampling time. Then using this expression, the Fourier coefficients are evaluated by taking inverse Fourier transform of $H_d(\omega)$, which is the desired impulse response of the filter $h_d(n)$.
- ✓ The Z-transform of $h_d(n)$ gives $H_d(z)$ which is the transfer function of the desired filter. The $H_d(z)$ obtained from $H_d(n)$ will be a transfer function of unrealizable non causal digital filter of infinite duration.
- ✓ A finite duration impulse response $h(n)$ can be obtained by truncating the infinite duration impulse response $h_d(n)$ to N-samples.
- ✓ Now, take Z-transform of $h(n)$ to get $H(z)$. This $H(z)$ corresponds to a non-causal filter. So multiply this $H(z)$ by $z^{-(N-1)/2}$ to get the transfer function of realizable causal filter of finite duration.

- ✓ In window method, we begin with the desired frequency response specification $H_d(\omega)$ and determine the corresponding unit sample response $h_d(n)$. The $h_d(n)$ is given by the inverse Fourier transform of $H_d(\omega)$.
- ✓ The unit sample response $h_d(n)$ will be an infinite sequence and must be truncated at some point, say, at $n = N - 1$ to yield an FIR filter of length N.
- ✓ The truncation is achieved by multiplying $h_d(n)$ by a window sequence $w(n)$. The resultant sequence will be of length N and can be denoted by $h(n)$. The Z-transform of $h(n)$ will give the filter transfer function $H(z)$.
- ✓ There have been many windows proposed like Rectangular window, Triangular window, Hanning window, Hamming window, Blackman wndow and Kaiser window that approximate the desired characteristics.

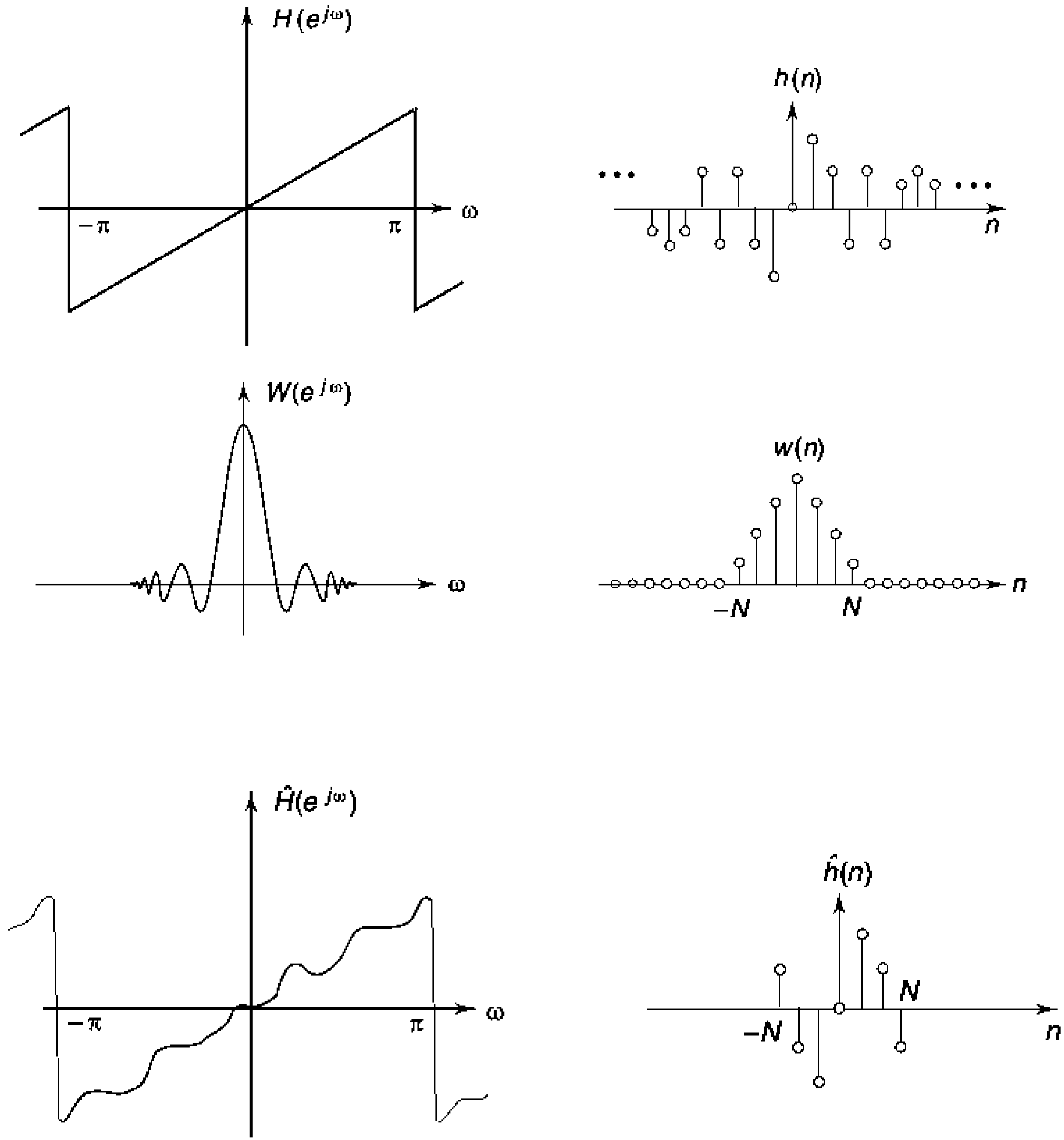
- ✓ In frequency sampling method of filter design, we begin with the desired frequency response specification $H_d(\omega)$, and it is sampled at N-points to generate a sequence $H(k)$ which corresponds to the DFT coefficients.
- ✓ The N-point IDFT of the sequence $H(k)$ gives the impulse response of the filter $h(n)$. The Z-transform of $h(n)$ gives the transfer function $H(z)$ of the filter.

- ✓ In optimum filter design method, the weighted approximation error between the desired frequency response and the actual frequency response is spread evenly across the pass band and evenly across the stop band of the filter.
- ✓ This results in the reduction of maximum error.
- ✓ The resulting filter have ripples in both the pass band and the stop band. This concept of design is called optimum equiripple design criterion.

Design of FIR Filters Using Windows

✓ The procedure for designing FIR filter using windows is:

1. Choose the desired frequency response of the filter $H_d(\omega)$.
2. Take inverse Fourier transform of $H_d(\omega)$ to obtain the desired impulse response $h_d(n)$.
3. Choose a window sequence $w(n)$ and multiply $h_d(n)$ by $w(n)$ to convert the infinite duration impulse response to a finite duration impulse response $h(n)$.
4. The transfer function $H(z)$ of the filter is obtained by taking Z-transform of $h(n)$.



H(e^{jω})
ω
−π
π
h(n)
n
W(e^{jω})
ω
w(n)
−N
N
n
Ĥ(e^{jω})
ω
−π
π
ĥ(n)
N
−N
n

✓ The different types of window sequences discussed in this are,

1. Rectangular window, $w_{\mathrm{R}}(n)$
2. Bartlett or Triangular window, $w_{\mathrm{T}}(n)$
3. Hanning window, $w_{\mathrm{Hn}}(n)$
4. Hamming window, $w_{\mathrm{H}}(n)$
5. Blackman window, $w_{\mathrm{B}}(n)$
6. Kaiser window, $w_{\mathrm{K}}(n)$

Rectangular Window:

✓ The weighting function (window function) for an N-point rectangular window is given by

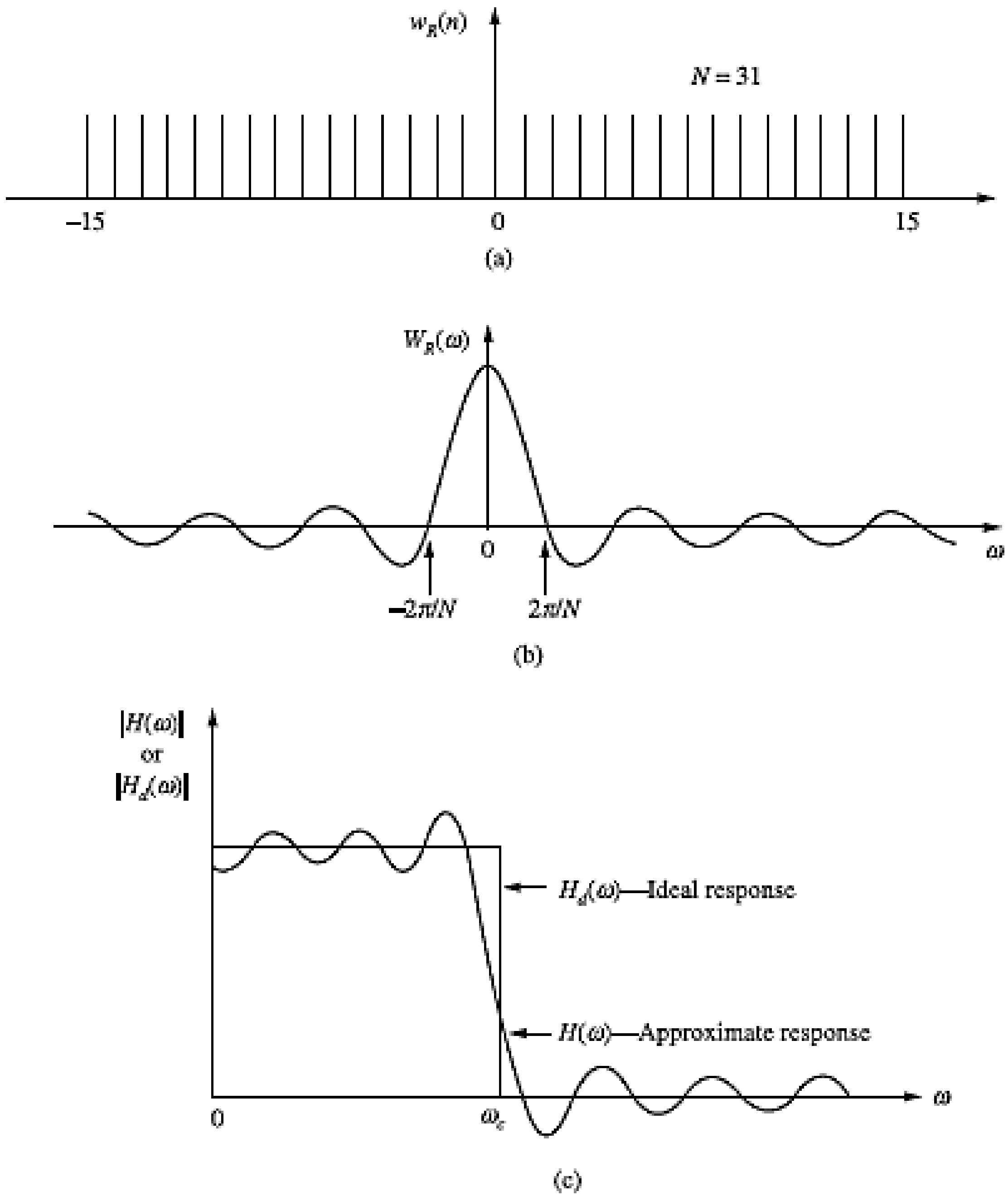
$$w_R(n) = \begin{cases} 1, & -\dfrac{(N-1)}{2} \le n \le \left(\dfrac{N-1}{2}\right) \\ 0, & \text{elsewhere} \end{cases} \quad \text{or} \quad w_R(n) = \begin{cases} 1, & 0 \le n \le (N-1) \\ 0, & \text{elsewhere} \end{cases}$$

✓ The spectrum (frequency response) of rectangular window $W_R(\omega)$ is given by the Fourier transform of $w_R(n)$.

$$W_R(\omega) = \sum_{n=-(N-1)/2}^{(N-1)/2} e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega\left(n-\frac{N-1}{2}\right)} = \sum_{n=0}^{N-1} e^{-j\omega n} e^{j\omega\frac{N-1}{2}} = e^{j\omega\frac{N-1}{2}} \sum_{n=0}^{N-1} e^{-j\omega n} = e^{j\omega\left(\frac{N-1}{2}\right)}\left[\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right]$$

$$= e^{j\frac{\omega N}{2}} e^{-j\frac{\omega}{2}} \frac{e^{-j\frac{\omega N}{2}}\left[e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}\right]}{e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]} = \frac{e^{j\frac{\omega N}{2}} - e^{-j\frac{\omega N}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} = \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

✓ The frequency spectrum for N = 31 is shown in Figure.



- ✓ The spectrum $W_R(\omega)$ has two features that are important. They are the width of the main lobe and the side lobe amplitude.
- ✓ The frequency response is real and its zero occurs when $= 2k\pi/N$ where k is an integer.
- ✓ The response for between $-2\pi/N$ and $2\pi/N$ is called the main lobe and the other lobes are called side lobes.
- ✓ For rectangular window the width of main lobe is $4\pi/N$. The first side lobe will be 13 dB down the peak of the main lobe and the roll off will be at 20 dB/decade.
- ✓ As the window is made longer, the main lobe becomes narrower and higher, and the side lobes become more concentrated around $\omega = 0$, but the amplitude of side lobes is unaffected.
- ✓ So increase in length does not reduce the amplitude of ripples, but increases the frequency when rectangular window is used.

- ✓ If we design a low-pass filter using rectangular window, we find that the frequency response differs from the desired frequency response in many ways.
- ✓ It does not follow quick transitions in the desired response. The desired response of a low-pass filter changes abruptly from pass band to stop band, but the actual frequency response changes slowly.
- ✓ This region of gradual change is called filter's transition region, which is due to the convolution of the desired response with the window response's main lobe.
- ✓ The width of the transition region depends on the width of the main lobe. As the filter length N increases, the main lobe becomes narrower decreasing the width of the transition region.

- ✓ The convolution of the desired response and the window response's side lobes gives rise to the ripples in both the pass band and stop band.
- ✓ The amplitude of the ripples is dictated by the amplitude of the side lobes. This effect, where maximum ripple occurs just before and just after the transition band, is known as **Gibb's phenomenon**.
- ✓ The Gibbs phenomenon can be reduced by using a less abrupt truncation of filter coefficients. This can be achieved by using a window function that tapers smoothly towards zero at both ends.

Bartlett or Triangular Window:

- ✓ The triangular window has been chosen such that it has tapered sequences from the middle on either side. The window function $w_T(n)$ is defined as

$$w_T(n) = \begin{cases} 1 - \dfrac{2|n|}{N-1}, & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0 \qquad , & \text{otherwise} \end{cases}$$

or

$$w_T(n) = \begin{cases} 1 - \dfrac{2|n-(N-1)/2|}{N-1}, & 0 \le n \le N-1 \\ 0 \qquad , & \text{otherwise} \end{cases}$$

- ✓ In magnitude response of triangular window, the side lobe level is smaller than that of the rectangular window being reduced from –13 dB to –25 dB.
- ✓ However, the main lobe width is now $8\pi/N$ or twice that of the rectangular window.

- ✓ The triangular window produces a smooth magnitude response in both pass band and stop band, but it has the following disadvantages when compared to magnitude response obtained by using rectangular window:

    1. The transition region is more.
    2. The attenuation in stop band is less.

- ✓ Because of these characteristics, the triangular window is not usually a good choice.

Raised Cosine Windows:

- ✓ The raised cosine window multiplies the central Fourier coefficients by approximately unity and smoothly truncates the Fourier coefficients toward the ends of the filter.
- ✓ The smoother ends and broader middle section produces less distortion of $h_d(n)$ around $n = 0$. It is also called generalized Hamming window.
- ✓ The window sequence is of the form:

$$w_H(n) = \begin{cases} \alpha + (1-\alpha)\cos\left(\dfrac{2\pi n}{N-1}\right), & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0, & \text{elsewhere} \end{cases}$$

Hanning Window:

- ✓ The Hanning window function is given by

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\left(\dfrac{2\pi n}{N-1}\right), & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0 & , \quad \text{otherwise} \end{cases}$$

or

$$w_{Hn}(n) = \begin{cases} 0.5 - 0.5\cos\dfrac{2n\pi}{N-1}, & \text{for } 0 \le n \le N-1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

- ✓ The width of main lobe is $8\pi/N$, i.e., twice that of rectangular window which results in doubling of the transition region of the filter. The peak of the first side lobe is –32 dB relative to the maximum value. This results in smaller ripples in both pass band and stop band of the low-pass filter designed using Hanning window. The minimum stop band attenuation of the filter is 44 dB.
- ✓ At higher frequencies the stop band attenuation is even greater. When compared to triangular window, the main lobe width is same, but the magnitude of the side lobe is reduced, hence the Hanning window is preferable to triangular window.

Hamming Window:

- ✓ The Hamming window function is given by

$$w_H(n) = \begin{cases} 0.54 + 0.46 \cos\left(\dfrac{2\pi n}{N-1}\right), & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0 \quad , & \text{otherwise} \end{cases}$$

or

$$w_H(n) = \begin{cases} 0.54 - 0.46 \cos\left(\dfrac{2n\pi}{N-1}\right), & 0 \le n \le N-1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

- ✓ In the magnitude response for $N = 31$, the magnitude of the first side lobe is down about 41 dB from the main lobe peak, an improvement of 10 dB relative to the Hanning window.
- ✓ But this improvement is achieved at the expense of the side lobe magnitudes at higher frequencies, which are almost constant with frequency.

- ✓ The width of the main lobe is $8\pi/N$. In the magnitude response of low-pass filter designed using Hamming window, the first side lobe peak is –51 dB, which is –7 dB lesser with respect to the Hanning window filter.
- ✓ However, at higher frequencies, the stop band attenuation is low when compared to that of Hanning window.
- ✓ Because the Hamming window generates lesser oscillations in the side lobes than the Hanning window for the same main lobe width, the Hamming window is generally preferred.

Blackman Window:

- ✓ The Blackman window function is another type of cosine window and given by the equation

$$w_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{N-1} + 0.08\cos\dfrac{4\pi n}{N-1}, & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \left(\dfrac{N-1}{2}\right) \\ 0 & , \text{ otherwise} \end{cases}$$

or $$w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2n\pi}{N-1} + 0.08\cos\dfrac{4n\pi}{N-1}, & 0 \le n \le N-1 \\ 0 & , \text{ otherwise} \end{cases}$$

- ✓ In the magnitude response, the width of the main lobe is $12\pi/N$, which is highest among windows. The peak of the first side lobe is at –58 dB and the side lobe magnitude decreases with frequency. This desirable feature is achieved at the expense of increased main lobe width.
- ✓ However, the main lobe width can be reduced by increasing the value of N. The side lobe attenuation of a low-pass filter using Blackman window is –78 dB.

The below Table gives the important frequency domain characteristics of some window functions.

Type of window	*Approximate transition width of main lobe*	*Minimum stop band attenuation* (dB)	*Peak of first side lobe* (dB)
Rectangular	$4\pi/N$	–21	–13
Bartlett	$8\pi/N$	–25	–25
Hanning	$8\pi/N$	– 44	–31
Hamming	$8\pi/N$	–51	– 41
Blackmann	$12\pi/N$	–78	–58

EXAMPLE A low-pass filter is to be designed with the following desired frequency response:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega}, & -\dfrac{\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ 0, & \dfrac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

Determine the filter coefficients $h(n)$ if the window function is defined as:

$$w(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Also determine the frequency response $H(e^{j\omega})$ of the designed filter.

Solution: For the given filter with

$$H_d(\omega) = \begin{cases} e^{-j2\omega}, & -\dfrac{\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ 0, & \dfrac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j2\omega}\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j\omega(n-2)}\, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(n-2)}}{j(n-2)} \right]_{-\pi/4}^{\pi/4} = \frac{1}{\pi(n-2)} \left[\frac{e^{j(n-2)\frac{\pi}{4}} - e^{-j(n-2)\frac{\pi}{4}}}{2j} \right]$$

$$= \frac{1}{\pi(n-2)} \sin\,(n-2)\,\frac{\pi}{4}, \quad n \neq 2.$$

For $n = 2$, the filter coefficient can be obtained by applying L'Hospital rule to the above expression. Thus,

$$h_d(2) = \underset{n \to 2}{\mathrm{Lt}}\ \frac{1}{\pi} \frac{\sin\,(n-2)\,\frac{\pi}{4}}{(n-2)} = \frac{1}{\pi} \cdot \frac{\pi}{4} = \frac{1}{4}$$

Since it is a linear phase filter, the other filter coefficients are given by

$$h_d(0) = \frac{1}{\pi(0-2)} \sin\,(0-2)\,\frac{\pi}{4} = \frac{1}{2\pi}$$

$$h_d(1) = \frac{1}{\pi(1-2)} \sin (1-2)\frac{\pi}{4} = \frac{1}{\sqrt{2}\pi}$$

$$h_d(3) = \frac{1}{\pi(3-2)} \sin (3-2)\frac{\pi}{4} = \frac{1}{\sqrt{2}\pi}$$

$$h_d(4) = \frac{1}{\pi(4-2)} \sin (4-2)\frac{\pi}{4} = \frac{1}{2\pi}$$

The filter coefficients of the filter using rectangular window would be then

$$h(n) = h_d(n) \cdot w(n) = h_d(n)$$

Therefore, $h(0) = \dfrac{1}{2\pi} = h(4)$, $h(1) = \dfrac{1}{\sqrt{2}\pi} = h(3)$, and $h(2) = \dfrac{1}{4}$

are the coefficients of the designed digital filter.

The realizable digital filter is:

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4}$$

$$= \frac{1}{2\pi} + \frac{1}{\sqrt{2}\pi} z^{-1} + 0.25\, z^{-2} + \frac{1}{\sqrt{2}\pi} z^{-3} + \frac{1}{2\pi} z^{-4}$$

$$= z^{-2}\left[0.25 + \frac{1}{\sqrt{2}\pi}(z + z^{-1}) + \frac{1}{2\pi}(z^2 + z^{-2})\right]$$

The frequency response $H(\omega)$ of the digital filter is given by

$$H(\omega) = \sum_{n=0}^{4} h(n) e^{-j\omega n}$$

$$= h(0) + h(1)\, e^{-j\omega} + h(2)\, e^{-j2\omega} + h(3)\, e^{-j3\omega} + h(4)\, e^{-j4\omega}$$

$$= e^{-j2\omega} [h(0)\, e^{j2\omega} + h(1)\, e^{j\omega} + h(2) + h(3)\, e^{-j\omega} + h(4)\, e^{-j2\omega}]$$

$$= e^{-j2\omega} [h(2) + h(1)\, (e^{j\omega} + e^{-j\omega}) + h(0)\, (e^{j2\omega} + e^{-j2\omega})]$$

$$= e^{-j2\omega} \left[\frac{1}{4} + \frac{\sqrt{2}}{\pi} \cos\omega + \frac{1}{\pi} \cos 2\omega \right]$$

EXAMPLE Design a filter with

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & -\dfrac{\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ 0 \quad , & \dfrac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

using a Hamming window with $N = 7$.

Solution: For the given filter with

$$H_d(\omega) = \begin{cases} e^{-j3\omega}, & -\dfrac{\pi}{4} \le \omega \le \dfrac{\pi}{4} \\ 0 \quad , & \dfrac{\pi}{4} \le |\omega| \le \pi \end{cases}$$

the filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j3\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j\omega(n-3)}\, d\omega = \frac{1}{2\pi} \left[\frac{e^{j\omega(n-3)}}{j(n-3)} \right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{\pi(n-3)} \left[\frac{e^{j\pi(n-3)/4} - e^{-j\pi(n-3)/4}}{2j} \right]$$

$$= \frac{\sin \pi(n-3)/4}{\pi(n-3)}, \qquad n \ne 3$$

For $n = 3$, the filter coefficient can be obtained by applying L'Hospital's rule to the above expression. Thus,

$$h_d(3) = \underset{n \to 3}{\mathrm{Lt}} \frac{\sin \frac{1}{4}(n-3)\pi}{\sin(n-3)\pi} = \frac{1}{4}$$

The other filter coefficients are given by

$$h_d(0) = \frac{\sin \pi(0-3)/4}{\pi(0-3)} = \frac{0.707}{3\pi}, \qquad h_d(1) = \frac{\sin \pi(1-3)/4}{\pi(1-3)} = \frac{1}{2\pi}$$

$$h_d(2) = \frac{\sin \pi(2-3)/4}{\pi(2-3)} = \frac{0.707}{\pi}, \qquad h_d(4) = \frac{\sin \pi(4-3)/4}{\pi(4-3)} = \frac{0.707}{\pi}$$

$$h_d(5) = \frac{\sin \pi(5-3)/4}{\pi(5-3)} = \frac{1}{2\pi}, \qquad h_d(6) = \frac{\sin \pi(6-3)/4}{\pi(6-3)} = \frac{0.707}{3\pi}$$

So the filter coefficients are:

$$h_d(0) = \frac{0.707}{3\pi}, \quad h_d(1) = \frac{1}{2\pi}, \quad h_d(2) = \frac{0.707}{\pi}, \quad h_d(3) = \frac{1}{4},$$

$$h_d(4) = \frac{0.707}{\pi}, \quad h_d(5) = \frac{1}{2\pi}, \quad h_d(6) = \frac{0.707}{3\pi}$$

The Hamming window function of a causal filter is:

$$w(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2\pi n}{N-1}, & 0 \le n \le N-1 \\ 0 \qquad\qquad\qquad\qquad\qquad , & \text{otherwise} \end{cases}$$

Therefore, with $N = 7$

$$w(0) = 0.54 - 0.46 \cos 0 = 0.08, \qquad w(1) = 0.54 - 0.46\cos \frac{2\pi \times 1}{7-1} = 0.31$$

$$w(2) = 0.54 - 0.46\cos \frac{2\pi \times 2}{7-1} = 0.77, \qquad w(3) = 0.54 - 0.46\cos \frac{2\pi \times 3}{7-1} = 1$$

$$w(4) = 0.54 - 0.46\cos \frac{2\pi \times 4}{7-1} = 0.77, \qquad w(5) = 0.54 - 0.46\cos \frac{2\pi \times 5}{7-1} = 0.31$$

$$w(6) = 0.54 - 0.46\cos \frac{2\pi \times 6}{7-1} = 0.08$$

The filter coefficients of the resultant filter are:

$$h(n) = h_d(n)\, \omega(n), \qquad n = 0, 1, 2, 3, 4, 5, 6$$

Therefore,

$$h(0) = h_d(0)\, w(0) = \frac{0.707}{3\pi} \times 0.08 = 0.006, \qquad h(1) = h_d(1)\, w(1) = \frac{1}{2\pi} \times 0.31 = 0.049$$

$$h(2) = h_d(2)\, w(2) = \frac{0.707}{\pi} \times 0.77 = 0.173, \qquad h(3) = h_d(3)\, w(3) = \frac{1}{4} \times 1 = \frac{1}{4} = 0.25$$

$$h(4) = h_d(4)\, w(4) = \frac{0.707}{\pi} \times 0.77 = 0.173, \qquad h(5) = h_d(5)\, w(5) = \frac{1}{2\pi} \times 0.31 = 0.049$$

$$h(6) = h_d(6)\, w(6) = \frac{0.707}{3\pi} \times 0.08 = 0.006$$

The frequency response of a causal filter is given by

$$H(\omega) = \sum_{n=0}^{6} h(n) e^{-j\omega n}$$

$$= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} + h(4)e^{-j4\omega} + h(5)e^{-j5\omega} + h(6)e^{-j6\omega}$$

$$= e^{-j3\omega}\{h(3) + [h(0)\, e^{j3\omega} + h(6)\, e^{-j3\omega}] + [h(1)\, e^{j2\omega} + h(5)\, e^{-j2\omega}] + [h(2)\, e^{j\omega} + h(4)\, e^{-j\omega}]\}$$

$$= e^{-j3\omega}\,[h(3) + 2h(0)\cos 3\omega + 2h(1)\cos 2\omega + 2h(2)\cos\omega]$$

$$= e^{-j3\omega}\,[0.25 + 0.012\cos 3\omega + 0.098\cos 2\omega + 0.346\cos\omega]$$

The transfer function of the digital FIR low-pass filter is:

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

$$= h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4} + h(5) z^{-5} + h(6) z^{-6}$$

$$= z^{-3}\left[h(3) + h(2)\,(z^{-1} + z) + h(1)\,(z^{-2} + z^{2}) + h(0)\,(z^{-3} + z^{3}) \right]$$

$$= z^{-3}\left[0.25 + 0.173\,[z + z^{-1}] + 0.049\,[z^{2} + z^{-2}] + 0.006\,[z^{3} + z^{-3}] \right]$$

$$= 0.006 + 0.049\, z^{-1} + 0.173 z^{-2} + 0.25\, z^{-3} + 0.173\, z^{-4} + 0.049\, z^{-5} + 0.006^{-6}$$

Kaiser Window:

- ✓ From the frequency domain characteristics of the window functions, it can be seen that a trade off exists between the main lobe width and side lobe amplitude. The main lobe width is inversely proportional to N. As the length of the filter is increased, the width of the main lobe becomes narrower and narrower, and the transition band is reduced considerably.
- ✓ However, the minimum stop band attenuation is independent of N and is a function of the selected window. Thus, in order to achieve prescribed minimum stop band attenuation and pass band ripple, the designer must find a window with an appropriate side lobe level and then choose N to achieve the prescribed transition width.
- ✓ In this process, the designer may often have to settle for a window with undesirable design specifications.

- ✓ A desirable property of the window function is that the function is of finite duration in the time domain and that the Fourier transform has maximum energy in the main lobe or a given peak side lobe amplitude.
- ✓ A simple approximation to these functions have been developed by Kaiser in terms of zeroth order modified Bessel functions of the first kind. In a Kaiser window, the side lobe level can be controlled with respect to the main lobe peak by varying a parameter α.
- ✓ The width of the main lobe can be varied by adjusting the length of the filter. The Kaiser window function is given by

$$w_k(n) = \begin{cases} \dfrac{I_0(\beta)}{I_0(\alpha)}, & \text{for } |n| \leq \dfrac{N-1}{2} \\ 0 & \text{, otherwise} \end{cases}$$

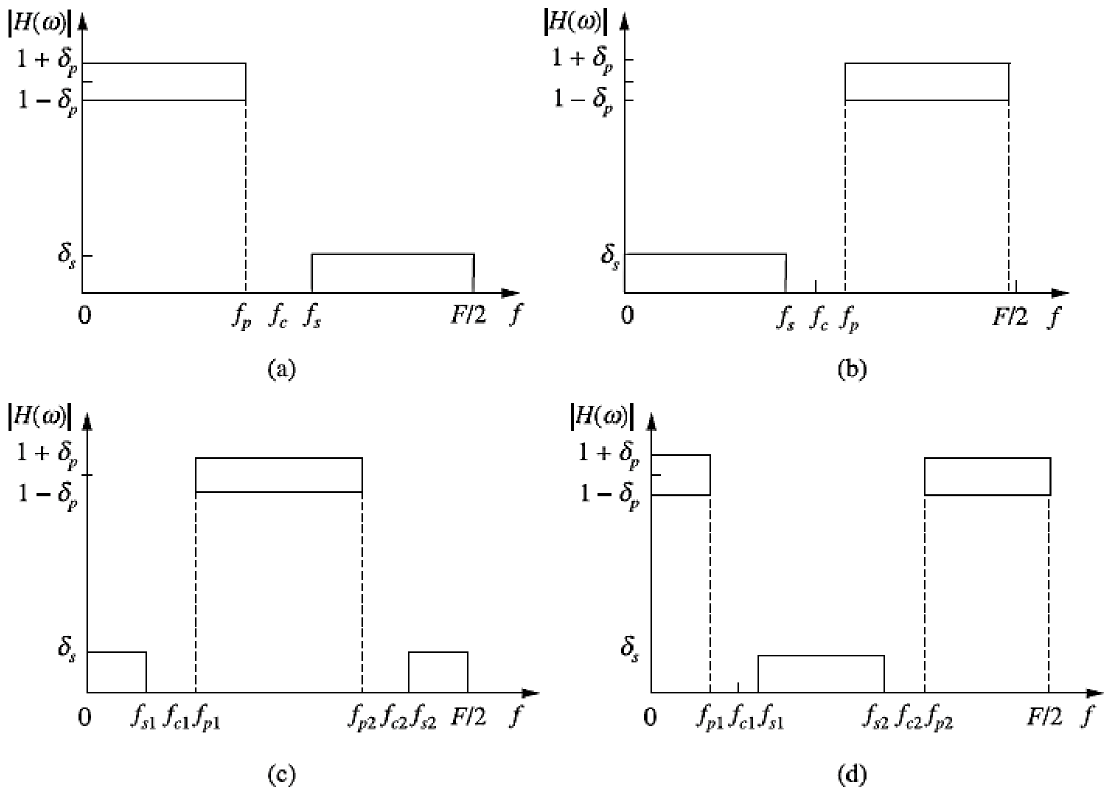
where α is an independent variable determined by Kaiser. The parameter β is expressed by

$$\beta = \alpha\left[1 - \left(\frac{2n}{N-1}\right)^2\right]^{\frac{1}{2}}$$

✓ The modified Bessel function of the first kind, $I_0(x)$ can be computed from its power series expansion given by

$$I_0(x) = 1 + \sum_{k=1}^{\infty} \left[\frac{1}{k!} \left(\frac{x}{2} \right)^k \right]^2$$

$$= 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^3} + \cdots$$



$$\alpha_p = 20 \log_{10} \frac{1 + \delta_p}{1 - \delta_p} \text{ dB}$$

$$\alpha_s = -20 \log_{10} \alpha_s \text{ dB}$$

$$\Delta F = f_s - f_p$$

Figure Idealized frequency responses: (a) low-pass filter, (b) high-pass filter, (c) band pass filter and (d) band stop filter.

Design specifications

1. Filter type; low-pass, high-pass, band pass or band stop
2. Pass band and stop band frequencies in hertz
   For low-pass/high-pass: f_p and f_s
   For band pass/band stop: $f_{p1}, f_{p2}, f_{s1}, f_{s2}$
3. Pass band ripple and minimum stop band attenuation in positive dB; α'_p and α'_s
4. Sampling frequency in hertz: F
5. Filter order N-odd

Design procedure

1. Determine $h_d(n)$ for an ideal frequency response $H(\omega)$
2. Choose δ according to equations

$$\alpha_p = 20 \log_{10} \frac{1+\delta_p}{1-\delta_p} \text{ dB and } \alpha_s = -20 \log \delta_s \text{ dB}$$

$$\text{and } \alpha_p \le \alpha'_p \text{ and } \alpha_s \ge \alpha'_s$$

where the actual design parameter can be determined from

$$\delta = \min(\delta_p, \delta_s)$$

$$\text{where } \delta_s = 10\,e^{-0.05\alpha'_s} \text{ and } \delta_p = \frac{10\,e^{0.05\alpha'_p} - 1}{10\,e^{0.05\alpha'_p} + 1}$$

3. Calculate α_s using the formula

$$\alpha_s = -20 \log_{10} \delta_s$$

4. Determine the parameter α from the following equation for

$$\alpha = \begin{cases} 0, & \text{for } \alpha_s < 21 \\ 0.5842\,(\alpha_s - 21)^{0.4} + 0.07886\,(\alpha_s + 21), & \text{for } 21 < \alpha_s \leq 50 \\ 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \end{cases}$$

5. Determine the parameter D from the following Kaiser's design equation

$$D = \begin{cases} 0.9222, & \text{for } \alpha_s \leq 21 \\ \dfrac{\alpha_s - 7.95}{14.36}, & \text{for } \alpha_s > 21 \end{cases}$$

6. Choose the filter order for the lowest odd value of N

$$N \geq \frac{\omega_{sf} D}{B} + 1$$

7. Compute the window sequence using equation

$$w_k(n) = \begin{cases} \dfrac{I_0\left[\alpha\sqrt{1 - \left(\dfrac{2n}{N-1}\right)^2}\right]}{I_0(\alpha)}, & \text{for } |n| \leq \dfrac{N-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

8. Compute the modified impulse response using

$$h(n) = w_k(n) h_d(n)$$

9. The transfer function is given by

$$H(z) = z^{-\left(\frac{N-1}{2}\right)}\left[h(0) + 2\sum_{n=1}^{(N-1)/2} h(n)\,(z^{n} + z^{-n})\right]$$

10. The magnitude response can be obtained using

$$\bar{H}(\omega) = \sum_{n=0}^{(N-1)/2} a(n)\cos\omega n$$

where

$$a(0) = h\left(\frac{N-1}{2}\right)$$

$$a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

The different windows parameters are compared in Table.

Name of window	*Window sequence*
Rectangular window	$w_R(n)=\begin{cases}1, & -\dfrac{(N-1)}{2}\le n\le\dfrac{N-1}{2}\\ 0, & \text{otherwise}\end{cases}$ $w_R(n)=\begin{cases}1, & 0\le n\le N-1\\ 0, & \text{otherwise}\end{cases}$
Triangular window	$w_T(n)=\begin{cases}1-\dfrac{2\lvert n\rvert}{N-1}, & -\dfrac{(N-1)}{2}\le n\le\dfrac{(N-1)}{2}\\ 0, & \text{otherwise}\end{cases}$ (or) $w_T(n)=\begin{cases}1-\dfrac{2\lvert n-(N-1)/2\rvert}{N-1}, & 0\le n\le N-1\\ 0, & \text{otherwise}\end{cases}$
Hanning window	$w_{hn}(n)=\begin{cases}0.5+0.5\cos\dfrac{2n\pi}{N-1}, & -\dfrac{(N-1)}{2}\le n\le\dfrac{(N-1)}{2}\\ 0, & \text{otherwise}\end{cases}$ (or) $w_{hn}(n)=\begin{cases}0.5-0.5\cos\dfrac{2n\pi}{N-1}, & 0\le n\le N-1\\ 0, & \text{otherwise}\end{cases}$

Hamming window	$w_H(n) = \begin{cases} 0.54 + 0.46 \cos \dfrac{2n\pi}{N-1}, & -\dfrac{(N-1)}{2} \le n \le \dfrac{N-1}{2} \\ 0 & , \text{ otherwise} \end{cases}$ (or) $w_H(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2n\pi}{N-1}, & 0 \le n \le N-1 \\ 0 & , \text{ otherwise} \end{cases}$
Blackman window	$w_B(n) = \begin{cases} 0.42 + 0.5 \cos \dfrac{2n\pi}{N-1} + 0.08 \cos \dfrac{4n\pi}{N-1}, & -\dfrac{(N-1)}{2} \le 0 \le \left(\dfrac{N-1}{2}\right) \\ 0 & , \text{otherwise} \end{cases}$ (or) $w_B(n) = \begin{cases} 0.42 - 0.5 \cos \dfrac{2n\pi}{N-1} + 0.08 \cos \dfrac{4n\pi}{N-1}, & 0 \le n \le N-1 \\ 0 & , \text{otherwise} \end{cases}$
Kaiser	$w_k(n) = \begin{cases} \dfrac{I_0\left[\alpha\sqrt{1-\left(\dfrac{2n}{N-1}\right)^2}\right]}{I_0(\alpha)}, & -\dfrac{(N-1)}{2} \le n \le \dfrac{(N-1)}{2} \\ 0 & , \text{ otherwise} \end{cases}$ (or) $w_k(n) = \begin{cases} \dfrac{I_0\left[\alpha\sqrt{\left(\dfrac{N-1}{2}\right)^2 - \left(n-\dfrac{N-1}{2}\right)^2}\right]}{I_0\left(\alpha\dfrac{N-1}{2}\right)}, & 0 \le n \le \dfrac{(N-1)}{2} \\ 0 & , \text{ otherwise} \end{cases}$

Comparison of Different Windows

Sl. No.	Rectangular Window	Hanning Window	Hamming Window	Blackman Window	Kaiser Window
1	The width of the main lobe is $4\pi/M$.	The width of the main lobe is $8\pi/M$.	The width of the main lobe is $8\pi/M$.	The width of the main lobe in window spectrum is $12\pi/M$.	The width of the main lobe in window spectrum depends on α and M.
2	The maximum magnitude of the side lobe in window spectrum is –13 dB.	The maximum magnitude of the side lobe in window spectrum is –31 dB.	The maximum magnitude of the side lobe in window spectrum is –41dB.	The maximum magnitude of the side lobe in window spectrum is –58 dB.	The maximum magnitude of the side lobe with respect to peak of main lobe is variable using the parameter α.
3	The magnitude of the side lobe slightly decreases with increasing ω.	The magnitude of the side lobe decreases with increase in ω.	The magnitude of the side lobe remains constant. Here, the increased side-lobe attenuation is achieved at the expense of constant attenuation at high frequencies.	The magnitude of the side lobe decreases rapidly with increasing ω.	The magnitude of the side lobe decreases with increasing ω.
4	The minimum stopband attenuation is 22 dB.	The minimum stopband attenuation is 44 dB.	The minimum stopband attenuation is 51 dB.	The minimum stopband attenuation is 78 dB.	The minimum stopband attenuation is variable and depends on the value of α.

- ✓ Looking at the parameters for rectangular and triangular window, it can be noted that the triangular window has a transition width twice that of rectangular window.
- ✓ However, the attenuation in stop band for triangular window is less. Therefore, it is not very popular for FIR filter design.
- ✓ The Hanning and Hamming windows have same transition width. But the Hamming window is most widely used because it generates less ringing in the side lobes.
- ✓ The Blackman window reduces the side lobe level at the cost of increase in transition width.
- ✓ The Kaiser window is superior to other windows because for given specifications its transition width is always small. By varying the parameter α, the desired side lobe level and main lobe peak can be achieved.
- ✓ Further the main lobe width can be varied by varying the length N. That is why Kaiser window is the favourite window for many digital filter designers.

✓ The window design for FIR filter has certain advantages and disadvantages.

Advantages

1. The filter coefficients can be obtained with minimum computation effort.
2. The window functions are readily available in closed-form expression.
3. The ripples in both stop band and pass band are almost completely eliminated.

Disadvantages

1. It is not always possible to obtain a closed form expression for the Fourier series coefficients $h(n)$.
2. Windows provide little flexibility in design.
3. It is somewhat difficult to determine, in advance, the type of window and duration N required to meet a given prescribed frequency specification.

Design of FIR Filters by Frequency Sampling Technique

- ✓ In this method, the ideal frequency response is sampled at sufficient number of points (i.e. *N*-points). These samples are the DFT coefficients of the impulse response of the filter. Hence the impulse response of the filter is determined by taking IDFT.

Let $H_d(\omega)$ = Ideal (desired) frequency response

$\tilde{H}(k)$ = The DFT sequence obtained by sampling $H_d(\omega)$

$h(n)$ = Impulse response of FIR filter

The impulse response $h(n)$ is obtained by taking IDFT of $\tilde{H}(k)$. For practical realizability, the samples of impulse response should be real. This can happen if all the complex terms appear in complex conjugate pairs. This suggests that the terms can be matched by comparing the exponentials. The terms $\tilde{H}(k)\, e^{j(2\pi nk/N)}$ should be matched by the term that has the exponential $e^{-j(2\pi nk)/N}$ as a factor.

- ✓ Two design techniques are available, viz., type-I design and type-II design.
- ✓ In the type-I design, the set of frequency samples includes the sample at frequency $\omega = 0$. In some cases, it may be desirable to omit the sample at $\omega = 0$ and use some other set of samples. Such a design procedure is referred to as the type-II design.

Procedure for type-I design

1. Choose the ideal (desired) frequency response $H_d(\omega)$.
2. Sample $H_d(\omega)$ at N-points by taking $\omega = \omega_k = \dfrac{2\pi k}{N}$, where $k = 0, 1, 2, 3, ..., (N-1)$ to generate the sequence $\tilde{H}(k)$.

$$\therefore \qquad \tilde{H}(k) = H_d(\omega)\big|_{\omega=(2\pi k)/N}; \quad \text{for } k = 0, 1, 2, \ldots, (N-1)$$

3. Compute the N samples of $h(n)$ using the following equations:

$$\text{When } N \text{ is odd, } h(n) = \frac{1}{N}\left[\tilde{H}(0) + 2\sum_{k=1}^{(N-1)/2} \operatorname{Re}\left(\tilde{H}(k)\, e^{j\frac{2\pi nk}{N}}\right)\right]$$

$$\text{When } N \text{ is even, } h(n) = \frac{1}{N}\left[\tilde{H}(0) + 2\sum_{k=1}^{\left(\frac{N}{2}-1\right)} \left(\tilde{H}(k)\, e^{j\frac{2\pi nk}{N}}\right)\right]$$

where 'Re' stands for 'real part of'.

4. Take Z-transform of the impulse response $h(n)$ to get the transfer function $H(z)$.

$$\therefore \qquad H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

Procedure for type-II design

1. Choose the ideal (desired) frequency response $H_d(\omega)$.

2. Sample $H_d(\omega)$ at N-points by taking $\omega = \omega_k = \dfrac{2\pi(2k+1)}{2N}$, where $k = 0, 1, 2, \ldots, (N-1)$ to generate the sequence $\tilde{H}(k)$.

$$\therefore \qquad \tilde{H}(k) = H_d(\omega)\big|_{\omega = \frac{2\pi(2k+1)}{2N}}; \quad \text{for } k = 0, 1, 2, \ldots, (N-1)$$

3. Compute the N samples of $h(n)$ using the following equations:

When N is odd, $h(n) = \dfrac{2}{N} \displaystyle\sum_{k=0}^{(N-3)/2} \text{Re}\left[\tilde{H}(k)\, e^{jn\pi(2k+1)/N}\right]$

When N is even, $h(n) = \dfrac{2}{N} \displaystyle\sum_{k=0}^{\left(\frac{N}{2}-1\right)} \text{Re}\left[\tilde{H}(k)\, e^{jn\pi(2k+1)/N}\right]$

4. Take Z-transform of the impulse response $h(n)$ to get the transfer function $H(z)$.

$$\therefore \qquad H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

The procedure for FIR filter design by frequency sampling method is:

1. Choose the desired frequency response $H_d(\omega)$.

2. Take N samples of $H_d(\omega)$ to generate the sequence $H(k)$.

3. Take inverse DFT of $H(k)$ to get the impulse response $h(n)$.

4. The transfer function $H(z)$ of the filter is obtained by taking Z-transform of the impulse response $h(n)$.

EXAMPLE Design a linear phase FIR filter of length $N = 11$ which has a symmetric unit sample response and a frequency response that satisfies the conditions:

$$H\left(\frac{2\pi k}{11}\right) = \begin{cases} 1\ , & \text{for } k = 0, 1, 2 \\ 0.5\ , & \text{for } k = 3 \\ 0\ , & \text{for } k = 4, 5 \end{cases}$$

Solution: For linear-phase FIR filter, the phase function, $\theta(\omega) = -\alpha\omega$, where $\alpha = (N-1)/2$.

Here $N = 11$, $\quad \therefore \quad \alpha = (11 - 1)/2 = 5.$

Also, here $\omega = \omega_k = \dfrac{2\pi k}{N} = \dfrac{2\pi k}{11}$.

Hence we can go for type-I design. In this problem, the samples of the magnitude response of the ideal (desired) filter are directly given for various values of k. Therefore,

$$\bar{H}(k) = H_d(\omega)\,|_{\omega=\omega_k} = \begin{cases} 1\, e^{-j\alpha\omega_k} & , \quad k = 0, 1, 2 \\ 0.5\, e^{-j\alpha\omega_k} & , \quad k = 3 \\ 0 & , \quad k = 4, 5 \end{cases}$$

where, $\omega_k = \dfrac{2\pi k}{11}$

When $k = 0$, $\quad \bar{H}(0) = e^{-j\alpha\omega_0} = e^{-j5\times\left(\frac{2\pi\times 0}{11}\right)} = 1$

When $k = 1$, $\quad \bar{H}(1) = e^{-j\alpha\omega_1} = e^{-j5\times\left(\frac{2\pi\times 1}{11}\right)} = e^{-j\left(\frac{10\pi}{11}\right)}$

When $k = 2$, $\quad \bar{H}(2) = e^{-j\alpha\omega_2} = e^{-j5\times\left(\frac{2\pi\times 2}{11}\right)} = e^{-j\left(\frac{20\pi}{11}\right)}$

When $k = 4$, $\quad \bar{H}(4) = 0$

When $k = 5$, $\quad \bar{H}(5) = 0$

The samples of impulse response are given by

$$h(n) = \frac{1}{N}\left\{\tilde{H}(0) + 2\sum_{k=1}^{(N-1)/2} \operatorname{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/N}\right]\right\}$$

$$= \frac{1}{11}\left\{\tilde{H}(0) + 2\sum_{k=1}^{5} \operatorname{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/N}\right]\right\}$$

$$= \frac{1}{11}\left\{\tilde{H}(0) + 2\sum_{k=1}^{2} \operatorname{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/11}\right] + 2\operatorname{Re}\left[\tilde{H}(3)\, e^{j2\pi n3/11}\right]\right\}$$

$$= \frac{1}{11}\left\{1 + 2\sum_{k=1}^{2} \operatorname{Re}\left[e^{-j5\left(\frac{2\pi k}{11}\right)} \times e^{j\left(\frac{2\pi nk}{11}\right)}\right] + 2\operatorname{Re}\left[0.5^{-j5\left(\frac{2\pi\times 3}{11}\right)} \times e^{j\left(\frac{2\pi n3}{11}\right)}\right]\right\}$$

$$= \frac{1}{11}\left\{1 + 2\sum_{k=1}^{2} \operatorname{Re}\left[e^{j\frac{2\pi k}{11}(n-5)}\right] + 2\operatorname{Re}\left[0.5\, e^{j\frac{6\pi}{11}(n-5)}\right]\right\}$$

$$= \frac{1}{11} + \frac{2}{11}\cos\frac{2\pi}{11}(n-5) + \frac{2}{11}\cos\frac{4\pi}{11}(n-5) + \cos\frac{6\pi}{11}(n-5)$$

$$n=0,\ h(0) = \frac{1}{11} + \frac{2}{11}\cos\left(\frac{-10\pi}{11}\right) + \frac{2}{11}\cos\left(\frac{-20\pi}{11}\right) + \cos\left(\frac{-30\pi}{11}\right) = -0.5854$$

$$n=1,\ h(1) = \frac{1}{11} + \frac{2}{11}\cos\left(\frac{-8\pi}{11}\right) + \frac{2}{11}\cos\left(\frac{-16\pi}{11}\right) + \cos\left(\frac{-24\pi}{11}\right) = 0.787$$

$$n=2,\ h(2) = \frac{1}{11} + \frac{2}{11}\cos\left(\frac{-6\pi}{11}\right) + \frac{2}{11}\cos\left(\frac{-12\pi}{11}\right) + \cos\left(\frac{-18\pi}{11}\right) = 0.3059$$

$$n=3,\ h(3)=\frac{1}{11}+\frac{2}{11}\cos\left(\frac{-4\pi}{11}\right)+\frac{2}{11}\cos\left(\frac{-8\pi}{11}\right)+\cos\left(\frac{-12\pi}{11}\right)=-0.9120$$

$$n=4,\ h(4)=\frac{1}{11}+\frac{2}{11}\cos\left(\frac{-2\pi}{11}\right)+\frac{2}{11}\cos\left(\frac{-4\pi}{11}\right)+\cos\left(\frac{-6\pi}{11}\right)=0.1770$$

$$n=5,\ h(5)=\frac{1}{11}+\frac{2}{11}\cos 0+\frac{2}{11}\cos 0+\cos 0=\frac{5}{11}+1=1.4545$$

For linear-phase FIR filters, the condition $h(N-1-n) = h(n)$ will be satisfied when $\alpha = (N-1)/2$.

When $n = 6$, $h(6) = h(11-1-6) = h(4) = 0.1770$

When $n = 7$, $h(7) = h(11-1-7) = h(3) = -0.9120$

When $n = 8$, $h(8) = h(11-1-8) = h(2) = 0.3059$

When $n = 9$, $h(9) = h(11-1-9) = h(1) = 0.787$

When $n = 10$, $h(10) = h(11-1-10) = h(0) = -0.5854$

The transfer function of the filter $H(z)$ is given by Z-transform of $h(n)$. Therefore,

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{10} h(n) z^{-n}$$

$$= \sum_{n=0}^{4} h(n) z^{-n} + h(5) z^{-5} + \sum_{n=6}^{10} h(n) z^{-n}$$

$$= \sum_{n=0}^{4} h(n) z^{-n} + h(5) z^{-5} + \sum_{n=0}^{4} h(10-n) z^{-(10-n)}$$

$$= \sum_{n=0}^{4} h(n) \left[z^{-n} + z^{-(10-n)} \right] + h(5) z^{-5}$$

$$= h(0)\left[1 + z^{-10}\right] + h(1)\left[z^{-1} + z^{-9}\right] + h(2)\left[z^{-2} + z^{-8}\right] + h(3)\left[z^{-3} + z^{-7}\right]$$

$$+ h(4)\left[z^{-4} + z^{-6}\right] + h(5) z^{-5}$$

$$= -0.5854\left[1 + z^{-10}\right] + 0.787\left[z^{-1} + z^{-9}\right] + 0.3059\left[z^{-2} + z^{-8}\right] - 0.9120\left[z^{-3} + z^{-7}\right]$$

$$+ 0.1770\left[z^{-4} + z^{-6}\right] + 1.4545 z^{-5}$$

Thank You