* Eigen value and Eigen vector :-

$\begin{bmatrix} L_A \ \mathbb{R}^n \to \mathbb{R}^n \\ L_A(x) = Ax \end{bmatrix}$

Let $T: V \to V$ be a linear transformation.
A non-zero vector $v \in V$ is called eigen vector of T if there exists $\lambda \in \mathbb{F}$ such that $T(v) = \lambda v$. ; $v \neq 0$.
λ is called eigen value of T.

* $A \in M_{n \times n}(\mathbb{F})$, A non-zero vector $v \in \mathbb{F}^n$ is called eigen value of A if $Av = \lambda v$, for some $\lambda \in \mathbb{F}$.

* For any L.T., is eigen value exists ? ✓ yes

Eg. ① $I: \mathbb{R}^2 \to \mathbb{R}^2$, $I(a,b) = (a,b)$

$$I(a,b) = 1 \cdot (a,b)$$

$a=1$, $b=0$; $I(1,0) = 1 \cdot (1,0)$, $v = (1,0)$

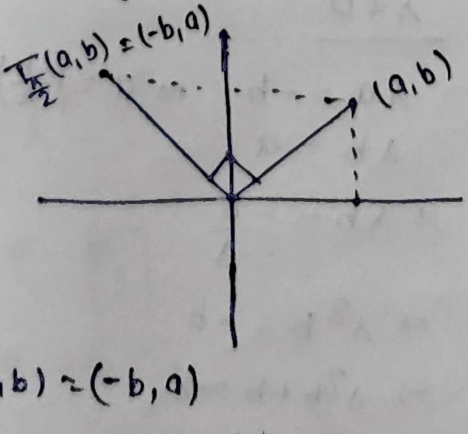
↳ 1 is an eigen value of I ; Corresponding to eigen vector $v = (1,0)$.

Any non-zero vector $v \in \mathbb{R}^2$ is an eigen vector corresponding to eigen value 1.

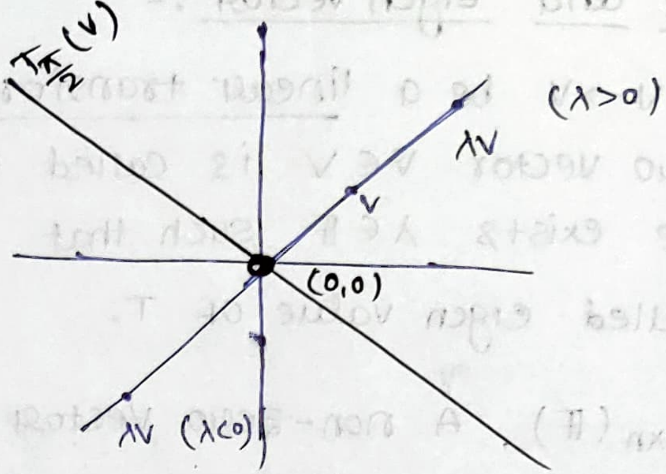
② $T(v) = \lambda v$

$\boxed{v \neq 0}$

$T_{\pi/2}$: rotation of vector by angle $\pi/2$.



$T_{\frac{\pi}{2}}(a,b) = (-b,a)$



$T_{\frac{\pi}{2}}(v) \neq \lambda v$

for: $v \neq 0$

(because it is meeting only at origin.)

so, ($T_{\frac{\pi}{2}}$ doesnot have an eigen vector, eigen value.)

↳ $T(a,b) = (-b, a)$

Suppose λ is an eigen value of T with eigen vector $v(a,b) \neq 0$

$\Rightarrow (-b,a) = \lambda(a,b)$

$\Rightarrow \boxed{\begin{aligned} \lambda a &= -b \\ \lambda b &= a \end{aligned}}$

Case-① $\lambda = 0 \Rightarrow (a,b) = (0,0)$

but eigen vector $(a,b) \neq (0,0)$

$\Rightarrow\Leftarrow$ (contradiction)

Case-② $\lambda \neq 0$

$\lambda a = -b \Rightarrow a = \left(\frac{-b}{\lambda}\right)$

$\lambda b = a$

$\Rightarrow \lambda b = -\frac{b}{\lambda}$

$\Rightarrow \lambda^2 b = -b$

$\Rightarrow \lambda^2 b + b = 0$

$\Rightarrow (\lambda^2 + 1) b = 0$

$\Rightarrow$ Case-ⓐ

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\boxed{\lambda = \pm i}$ but $\lambda \in \mathbb{R}$

$\Rightarrow\Leftarrow$ (contradiction)

$\Rightarrow$ Case-ⓑ

$b = 0, a = 0$

$(a,b) = (0,0)$

$\Rightarrow\Leftarrow$ (contradiction.)

* $A \in M_{n\times n}(\mathbb{F})$ $\qquad$ $I \in M_{n\times n}(\mathbb{F})$

Characteristic polynomial : $\det(A-\lambda I) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0$

$$\det(A-\lambda I) = 0$$

eigen value of $A = \{\lambda \in \mathbb{F} \,/\, \det(A-\lambda I) = 0\}$.

Eg. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$. find eigen value eigen vector.

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 2 & 3-\lambda \end{bmatrix} = (A-\lambda I)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1, 1, 4$$

for $\lambda = 1$

$Av = \lambda v$ $\qquad$ $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

(or) $Av = 1v$

(or) $(A-\lambda I)v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$(A - 1\cdot I)v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y + z = 0 \Rightarrow \boxed{y = -z}$

$y + z = 0$

$2y + 2z = 0$

$\Rightarrow y + z = 0$

So: eigen vector corresponding to $\lambda = 1$ is.

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Big/ \begin{array}{l} x, y \in \mathbb{R}. \\ \& \; x ; y \neq 0. \end{array} \right\}.$$

$= \text{span of two vectors } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \& \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$

For $\lambda = 4$.

$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$-3x + y + z = 0 \Rightarrow 3x = y + z \Rightarrow$

$-2y + z = 0$

$2y - z = 0 \Rightarrow \left(y = \frac{z}{2}\right)$ (or) $z = 2y$

from ① $x = \dfrac{(y + 2y)}{3} = y$

$\therefore x = y$

and $z = 2y$

$$\left\{ \begin{array}{l} \text{So eigen vector} \\ \text{corresponding to } \lambda = 4 \end{array} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x \\ 2x \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \Big/ \begin{array}{l} x \in \mathbb{R} \\ x \neq 0 \end{array} \right.$$

$\hookrightarrow$ span of one vector $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

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Theorem: The eigen vectors corresponding to distinct eigen values are L.I.

Pf: Let $T: V \to V$ be a L.T. and $\lambda_1, \lambda_2 \dots \lambda_K$ be distinct eigen values with $V_1, V_2 \dots V_K$ be corresponding eigen vectors.

To prove $\{V_1, V_2 \dots V_K\}$ is L.I.

Proof is done by Induction method. ??

To prove

$K=1$. $\{V_1\}$ is L.I.

$(\alpha_1 V_1 + \dots \alpha_{K-1} V_{K-1} = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots \alpha_{K-1} = 0\}$.

Since $V_1 \neq 0$, $\{V_1\}$ is L.I. $\{V_1$ is an eigen vector$\}$

Assume that $\{V_1, V_2 \dots V_{K-1}\}$ is L.I., to prove $\{V_1, V_2 \dots V_K\}$ is L.I.

$$\alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_K V_K = 0. \quad \text{—— (1)}$$

apply $(T - \lambda_K I)$ on both side

$$(T - \lambda_K I)(\alpha_1 V_1 + \alpha_2 V_2 + \dots \alpha_{K-1} V_{K-1} + \alpha_K V_K) = (T - \lambda_K I) 0$$

$$T(\alpha_1 V_1 + \alpha_2 V_2 + \dots \alpha_K V_K) - \lambda_K I(\alpha_1 V_1 + \alpha_2 V_2 + \dots + \alpha_K V_K) = T(0) - \lambda_K I(0).$$

$I(V) = V$

$$\Rightarrow \alpha_1 T(V_1) + \alpha_2 T(V_2) + \dots \alpha_K T(V_K) - [\lambda_K \alpha_1 I(V_1) + \lambda_K \alpha_2 I(V_2) + \dots \lambda_K \alpha_K I(V_K)] = 0 - 0 = 0$$

$T(V_1) = \lambda_1 V_1 \cdot \dots \; T(V_K) = \lambda_K V_K$

$\Rightarrow$ ~~$\alpha_1 \lambda_1 V_1 + \alpha$~~

$$\Rightarrow (\alpha_1 \lambda_1 V_1 + \alpha_2 \lambda_2 V_2 + \dots \cancel{\alpha_K \lambda_K V_K}) - (\lambda_K \alpha_1 V_1 + \alpha_2 \lambda_K V_2 + \dots \cancel{\lambda_K \alpha_K V_K}) = 0$$

$\Rightarrow \alpha_1(\lambda_1 - \lambda_K)v_1 + \alpha_2(\lambda_2 - \lambda_K)v_2 + \cdots \alpha_{K-1}(\lambda_{K-1} - \lambda_K)v_{K-1}$

since $\{v_1 \ldots v_K\}$ is L.I.

$\Rightarrow \underbrace{\alpha_1(\lambda_1 - \lambda_K)}_{\neq 0} = 0 = \alpha_2\underbrace{(\lambda_2 - \lambda_K)}_{\neq 0} = \cdots\cdots = \alpha_{K-1}\underbrace{(\lambda_{K-1} - \lambda_K)}_{\neq 0}.$

W.K.T. $\lambda_1, \lambda_2 \ldots \lambda_K$ are distinct

$$\lambda_i \neq \lambda_K \;;\; i = 1, 2, \ldots K-1$$

$$\lambda_i - \lambda_K \neq 0$$

Since $\lambda_i - \lambda_K \neq 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_{K-1} = 0$

so: eq^n (1) becomes

$$\alpha_K v_K = 0 \;;\; \alpha_K = 0 \;(\because v_K \neq 0).$$

$\therefore \alpha_1 = \alpha_2 = \cdots = \alpha_K = 0 \Rightarrow \{v_1, v_2 \ldots v_K\}$ is L.I.

* Caylay Hamilton Theorem:-

$\boxed{f(t) = \det(A - tI).}$

Let $A \in M_{n \times n}(\mathbb{F})$, and $f(t)$ be the characteristic polynomial of A. Then —

$$f(A) = 0$$

Eg. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$,

$f(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)$
$= \lambda^2 - 3\lambda + 2$

$f(\lambda) = \lambda^2 - 3\lambda + 2 = 0$

$f(A) = A^2 - 3A + 2I = 0.$

If A is invertible ; $A^{-1}(A^2 - 3A + 2I) = A^{-1}0$

$A - 3I + 2A^{-1} = 0$

$$\boxed{A^{-1} = \frac{3I - A}{2}}$$

* Similar Matrices $(B \sim A)$

Let $A, B \in M_{n \times n}(\mathbb{F})$, B is similar to A

If there exist an invertible matrix P.

such that $B = P^{-1}AP$.

(or) $\boxed{PB = AP}$

Eg:- $I \sim P^{-1}IP = P^{-1}P = I$

$\Rightarrow I \sim I$

and also $0 \sim 0$ $(0 \sim P^{-1}0P = 0$

• properties :-

1. $B \sim A \Rightarrow \det(B) = \det(A)$

Pf. $B \sim A \Rightarrow B = P^{-1}AP$

$|B| = |P^{-1}AP|$

$= |P^{-1}|.|A|.|P|$ $\left(|A^{-1}| = \frac{1}{|A|}\right)$

$= \frac{1}{\cancel{|P|}}.|A|.\cancel{|P|} = |A| \Rightarrow |B| = |A|$

(2.) Let $B \sim A$. then eigen values of A and B are equal.

$\boxed{\det(A - \lambda I) = \det(B - \lambda I)}$

$\because$ $B \sim A$. $B = P^{-1}AP$.

$\Rightarrow \det(B - \lambda I) = 0$ | ~~$\det(P^{-1}AP)$~~ -

$\det(A - I)$

~~$\Rightarrow \det(B)$~~

~~$\det(B) = \det(P$~~

$\det(B - \lambda I) = \det(P^{-1}AP - \lambda I)$

$I = P^{-1}P$

$= \det(P^{-1}AP - \lambda P^{-1}P)$

~~$= \det[(P^{-1}A - \lambda P^{-1})P]$~~

$= \det(P^{-1}AP - P^{-1}\lambda P)$

$= \det[P^{-1}(AP - \lambda P)]$

$= \det[P^{-1}(A - \lambda I)P]$

$= \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P)$

$= \frac{1}{\cancel{\det(P)}} \cdot \det(A - \lambda I) \cdot \cancel{\det(P)}$

$\boxed{\det(B - \lambda I) = \det(A - \lambda I)}$

$\therefore$ eigen values of A and B are equal.

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* Recall

$B \sim A$, $\exists$ a invertible matrix P. such that $\boxed{B = P^{-1}AP}$

$PB = AP$

* $\det(A) = \det(B)$
* eigen values of A and B are equal.

(3) x is an eigen vector of A corresponding to eigen value λ, and $B \sim A$. then $P^{-1}x$ is an eigen vector of B corresponding to same eigen value λ.

Given

$Ax = \lambda x$ for $x \neq 0$ — (1)

$B = P^{-1}AP$

$B \cdot P^{-1}x = \lambda P^{-1}x$; $P^{-1}x \neq 0$

$BP^{-1}x = P^{-1}AP \cdot P^{-1}x \quad (PP^{-1} = I)$

$= P^{-1}AIx$

$= P^{-1}Ax$

$= P^{-1}\lambda x \quad$ (from (1))

$\Rightarrow BP^{-1}x = \lambda P^{-1}x$.

Note:. $\boxed{x \neq 0} \Rightarrow \boxed{P^{-1}x \neq 0}$

let $P^{-1}x = 0$

$P \cdot P^{-1}x = P \cdot 0$

$Ix = 0$

$x = 0$

Which is contradict to $\underline{0}$.

* Eigen vector can be different for similar matrices.

④ $B \sim A \Rightarrow B^k \sim A^k$; $k \geq 2$

$\Rightarrow B^2 \sim A^2$

$\because B \sim A$

$\Rightarrow B = P^{-1}AP$

$B^2 = P^{-1}AP \cdot P^{-1}AP \quad (PP^{-1} = I)$

$= P^{-1}AA \cdot P$

$B^2 = P^{-1}A^2P$

$\Rightarrow \underline{\underline{B^2 \sim A^2}}$

$\underline{\text{ly}}$ for $B^3 \sim A^3$ and $\underline{\underline{B^k \sim A^k}}$.

*** Theorem :-** Eigen values of A and A^T are same.

Pf Char. polynomial of A = characteristic polynomial of B

$\Rightarrow |A - \lambda I| = |A^T - \lambda I|$

$\det(A^T - \lambda I) = \det[(A^T - \lambda I)^T]$

$= \det[(-\lambda I^T + A^{T^T})]$

$= \det[-\lambda I + A]$ $\quad \left\{ \because \det A = \det A^T \right.$

$\Rightarrow \det(A^T - \lambda I) = \det(A - \lambda I)$.

*** Complex matrices :-**

$A \in M_{n \times n} \; (\mathbb{C})$

$\bar{A} = \bar{a}_{ij}$, $A = a_{ij}$

$z = a + ib$
$\bar{z} = a - ib$

$A, B \in M_{n \times n} (\mathbb{C})$

$(\because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ & $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$;

① $\overline{(A+B)} = \bar{A} + \bar{B}$

② $\overline{(\alpha A)} = \bar{\alpha} \cdot \bar{A}$ $\quad \alpha \in \mathbb{C}$

③ $(\bar{A})^T = \overline{(A^T)}$

④ $\overline{(AB)} = \bar{A} \cdot \bar{B}$.

Note:- If λ is an eigen value of A then $\bar{\lambda}$ is an eigen value of $\bar{A}$.

$$Ax = \lambda x \qquad x \neq 0$$

$$\overline{Ax} = \overline{\lambda x}$$

$$\bar{A}.\bar{x} = \bar{\lambda}.\bar{x}\ , \quad \bar{x} \neq 0$$

$\Rightarrow$ $\bar{\lambda}$ is an eigen value of $\bar{A}$.

Define: $A^* = (\bar{A})^T$

$A = [a_{ij}]$

$A^T = [a_{ji}]$

$\bar{A}^T = [\bar{a}_{ji}]$

then
1. $(A^*)^* = A$
2. $(A+B)^* = A^* + B^*$
3. $(\alpha A)^* = \bar{\alpha} . A^*$
4. $(AB)^* = B^* A^*$

① $\Rightarrow$ $(A^*)^* = A$

$$(A^*)^* = (\bar{A}^T)^*$$
$$= (\overline{\bar{A}^T})^T$$
$$= (\bar{\bar{A}}^T)^T$$
$$= (A^T)^T$$
$$= A$$

④ $(AB)^* = (\overline{AB})^T$
$$= (\bar{A}.\bar{B})^T$$
$$= (\bar{A}.\bar{B})^T$$
$$= \bar{B}^T . \bar{A}^T$$
$$= B^* . A^*$$

* Harmition matrix : $A \in M_{n\times n}(\mathbb{C})$

- A is a Harmition if $A = A^* = \bar{A}^T$ $(A = \bar{A}^T)$
- A is a (Anti) skew-harmition if $A = -A^* = -(\bar{A})^T$

$A = -(\bar{A})^T$

$A \in M_{n\times n}(\mathbb{R})$ $(A = \bar{A})$

A is symmetric if $A = A^T$
A is Anti-symmetric if $A = -A^T$.

Eg. $A = \begin{bmatrix} 3 & i \\ -i & -2 \end{bmatrix}$ is a harmition matrix. Varify.

$$\bar{A} = \begin{bmatrix} 3 & -i \\ i & -2 \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 3 & i \\ -i & -2 \end{bmatrix} = A$$

$\Rightarrow$ $\bar{A}^T = A$

Theorem: If A is hermition then eigen values of A is Real. $(A = \bar{A}^T)$

Pf. Let λ be an eigen value of A, $Ax = \lambda x$, $x \neq 0$

$(\lambda = \bar{\lambda})$.

$$Ax = \lambda x$$

$$\bar{x}^T A x = \bar{x}^T \lambda x$$

$$\boxed{\bar{x}^T A x = \lambda \bar{x}^T x} \quad \text{— (a)}$$

$$\boxed{\bar{x}^T \bar{A}^T x = \bar{x}^T \cdot \lambda x} = \lambda \bar{x}^T \cdot x$$

(b)

$\Rightarrow$ ($\because \bar{\lambda}$ is an eigen value of $\bar{A}^T$)

$a, b \in \mathbb{R}$

$z = a + ib = \overline{a+ib} = a \in \mathbb{R}$

$z \in \mathbb{R}$

$z\bar{z} = |z|^2$

$\boxed{z = \bar{z}}$

$a + ib = a - ib$

$2ib = 0$

$b = 0$

since $A = \bar{A}^T$

$\bar{x}^T A x = \bar{x}^T (\bar{A}^T) x$

$\lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x$ (from a & b)

$(\lambda - \bar{\lambda}) \bar{x}^T x = 0$ — (2)

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\bar{x}^T = (\bar{x}_1 \ \bar{x}_2 \ \bar{x}_3)$

$\bar{x}^T x = (\bar{x}_1 \ \bar{x}_2 \ \bar{x}_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\bar{x}^T x = |x_1|^2 + |x_2|^2 + |x_3|^3 \neq 0$

(2) $\Rightarrow$

$(\lambda - \bar{\lambda}) \bar{x}^T x = 0$

since : $x \neq 0 \Rightarrow \bar{x}^T x \neq 0$

$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\bar{x}^T x \neq 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$\lambda = \bar{\lambda}$

$\therefore \lambda$ is real.

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Note:-

① λ is an eigen value of A, λ^k is an eigen value of A^k.

$$Ax = \lambda x \quad x \neq 0$$

$$A^2x = A \cdot Ax = A(\lambda x)$$
$$= \lambda \cdot (Ax)$$
$$= \lambda \cdot \lambda x$$
$$= \lambda^2 x, \quad x \neq 0.$$

λ^2 is an eigen value of A^2.

② if λ is an eigen value of A, then $\lambda + k$ is an eigen value of $(A+kI)$; $k \in \mathbb{C}$

Pf

$$(A+kI)_{n\times n}\, x = Ax + kIx = Ax + k \cdot x$$
$$= \lambda x + kx$$
$$= (\lambda + k)x \qquad x \neq 0$$

		Eigen values
Hermitian	$A = \bar{A}^T$	Real
Skew-Hermitian	$A = -\bar{A}^T$	
Unitary	$A \times \bar{A}^T = I$	

(If A is skew-hermitian)

Theorem:- Let $A \in M_{n\times n}(\mathbb{C})$ then the eigen values of A is zero (or) purely Imaginary.

Pf $A = -\bar{A}^T$

Suppose λ is an eigen value of A, $\boxed{Ax = \lambda x}$ $x \neq 0$

$\bar{x}^T A x = \bar{x}^T(-\bar{A}^T)x$ — ①

$\bar{A}x = \bar{\lambda}x$

$(-\bar{A}^T)x = (-\bar{\lambda})x$ — ②

Put in ①

$\bar{x}^T(\lambda x) = \bar{x}^T(-\bar{\lambda})x$

$\lambda \bar{x}^T . x = -\bar{\lambda}(\bar{x}^T)x$

$\Rightarrow (\lambda+\bar{\lambda})(\bar{x}^T x) = 0$ $\qquad \{ \bar{x}^T x = |x_1|^2 + \dots |x_n|^2 \}$

$x = \begin{pmatrix} x \\ \vdots \\ x_n \end{pmatrix}$

Since, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, $\bar{x}^T x \neq 0$

$\therefore \lambda + \bar{\lambda} = 0.$

let $\lambda = a+ib$, $\bar{\lambda} = a-ib$

$\Rightarrow \lambda+\bar{\lambda} = 0$

$a+ib+a-bi = 0$

$2a = 0$

$\Rightarrow \boxed{a=0}$

$\therefore \boxed{\lambda = ib}$

* Diagonalization :- $A \in M_{n\times n}(F)$.

A is diagonilizeble if ~~A ~ B~~. A ~ D.

Where D is diagonal mateix. $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$.

i.e; A is diagonalizable if $\exists$ an invertible mateix P Such that $A = P^{-1}DP$.

i.e, $AP = DP$

↳ Importance of diagonalization

① $A = P^{-1}DP$

$A^{2023} = ?$

$\because A = P^{-1}AP$

$A^2 = A\cdot A = P^{-1}DP\,P^{-1}DP$

$= P^{-1}D^2P.$

$D^2 = \begin{bmatrix} a^2 & 0 & 0 & - \\ 0 & b^2 & 0 & - \\ 0 & 0 & c^2 & - \\ - & - & - & - \end{bmatrix}$

$\therefore A^{2023} = P^{-1}D^{2023}P = P^{-1}\begin{pmatrix} d_1^{2023} & 0 & 0 & - \\ 0 & d_2^{2023} & 0 & -- \\ \vdots & & & \\ 0 & 0 & -- & d_n^{2023} \end{pmatrix}P$

(2) $Rank(A) = Rank(P^{-1}DP) = Rank(D) \Rightarrow$ No. of non-zero diagonal entries.

* **Theorem**:

$A = P^{-1}DP$

$A \in M_{n \times n}(\mathbb{C})$ is diagonilizable $\Leftrightarrow$ A has n L.I. eigen vectors

Proof: A is diagonilizable, $\Rightarrow AP = PA$

$P = [v_1 \dots v_n]$; $A[v_1, v_2 \dots v_n] = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & - & 0 \\ 0 & 0 & \dots & d_n \end{bmatrix}$

$v_1, v_2 \dots v_n \Rightarrow$ column matrix

$\Rightarrow [Av_1, Av_2 \dots Av_n] = [d_1 v_1, d_2 v_2 \dots d_n v_n]$

$\Rightarrow Av_i = d_i v_i$; $i = 1, 2 \dots n$.

since $P = [v_1 \dots v_n]$ is invertible, $v_i \neq 0$.

$v_i = 1, 2, \dots n$.

$\Rightarrow \{v_1 \dots v_n\}$ are L.I.

$Av_i = d_i v_i$, $v_i \neq 0$, $i = 1, 2, \dots n$

Which implies A has n L.I. eigen vectors, with eigen values $d_1, d_2 \dots d_n$.

Given

$\Leftarrow$ A has 'n' L.I. eigen vectors to prove,

$Av_i = d_i v_i$: $v_i \neq 0$, take $P = [v_1 \dots v_n]$

$AP = A[v_1 \dots v_n] = [Av_1, Av_2 \dots Av_n]$

$= (d_1 v_1 \dots d_n v_n)$

$= \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} [v_1 \dots v_n]$

$\Rightarrow \boxed{AP = DP}$

$A \in M_{n\times n}(\mathbb{C}) \Rightarrow$ ① find eigen values ② find eigen vectors

$\lambda_1, \ldots, \lambda_n$ $\qquad v_1, \ldots, v_n$

then ③ $P = [v_1 \ldots v_n]$ ④ $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

→ set of e.vectors.

P is invertible $\Rightarrow$ A is diagonilizable

$\boxed{A = P^{-1} D P}$

Eg. Check if $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$ is diagonilizable.

$A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Sol^n $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 6 \\ 0 & -1-\lambda \end{vmatrix} = 0$

$(2-\lambda)(-1-\lambda) = 0$

$\lambda = 2, -1$

eigen values

$v_i = \begin{bmatrix} -1-\lambda \\ 0 \end{bmatrix}$

$v_1 = \begin{bmatrix} -1-2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$AX = \lambda X$

$(A - \lambda I)x = 0$ $\qquad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 6 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$6x_2 = 0 \Rightarrow x_2 = 0$

$-3x_2 = 0 \Rightarrow x_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$AX = \lambda X$ (for $\lambda = 2$)

$\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\cancel{2x_1} + 6x_2 = \cancel{2x_1}$ $\qquad -x_2 = 2x_2$

$6x_2 = 0$ $\qquad -3x_2 = 0$

$x_2 = 0$ $\qquad x_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to v_1$

for $\lambda = -1$

$\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$2x_1 + 6x_2 = -x_1$

$3x_1 + 6x_2 = 0$

$-x_2 = -x_2$

$v_2 \to = x_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$3x_1 + 6x_2 = 0$

$x_1 = -2x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -\frac{1}{2}x_2 \end{bmatrix}$

$\begin{bmatrix} -2x_1 \\ x_2 \end{bmatrix}$

<u>eigen value</u> eigen vector corresponding to $\lambda = 2 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 2, -1$ eigen " " " $\lambda = -1 \Rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$|P| = (1-0) = 1 \neq 0$

so P is invertible $\Rightarrow$ A is diagonalisible.

$A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$ PA = DP

$\begin{vmatrix} 2-\lambda & 6 \\ 0 & -1-\lambda \end{vmatrix} = 0$

$\lambda = 2, -1$

For $\lambda = 2$

$(A - \lambda I) x = 0$

$\begin{bmatrix} 0 & 6 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$6x_2 = 0$

$-3x_2 = 0$

$\boxed{x_2 = 0}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix},$

v_1

For $\lambda = -1$

$\begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3x_1 + 6x_2 = 0$

$x_1 = -2x_2$

$\frac{x_1}{2} = \frac{x_2}{-1}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

v_2

$x_2 \neq 0$

$\therefore P = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$|P| = -1 \neq 0$

$\therefore P$ is invertible

$\Rightarrow$ A is diagonalisable

$A = P^{-1} D P$

$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$P^{-1} = -1 \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$P^{-1} D P = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}.$

$$A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(1-\lambda)(2-\lambda)-2\right] - 1(-4+2\lambda) - 1(-2) = 0$$

$$(3-\lambda)(2-3\lambda+\lambda^2-2) - 1(-4+2\lambda) + 2 = 0$$

$$(3-\lambda)(0-3\lambda+\lambda^2) + 4 - 2\lambda + 2 = 0$$

$$-9\lambda + 3\lambda^2 + 3\lambda^2 - \lambda^3 + 6 - 2\lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - \underset{11\lambda}{\cancel{9\lambda}} + 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1, 3, 2$$

For $\lambda = 1$

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_2 - x_3 = 0 \Rightarrow \cancel{2x_3 - x_3 - x_3 = 0}$$

$$-2x_1 + 2x_3 = 0$$

$$\boxed{x_3 = x_1}$$

$$x_2 + x_3 = 0$$

$$x_2 = -x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

v_1

For $\lambda = 3$

$$\begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 - x_3 = 0$$

$$x_2 = x_3$$

$$-2x_1 - 2x_2 + 2x_3 = 0$$

$$x_2 - x_3 = 0$$

$$\Rightarrow -2x_1 - 2x_3 + 2x_3 = 0$$

$$x_1 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

v_2

for $\lambda = 2$

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 - x_3 = 0$

$-2x_1 - x_2 + 2x_3 = 0$

$x_2 = 0$

$\Rightarrow \quad x_1 = x_3$

$$V_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad P = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$|P| = 1(1-0) + 1(-1-1)$

$= 1 - 2 = -1 \neq 0$

$\therefore$ P is invertible

$\Rightarrow$ A is diagonalizable.

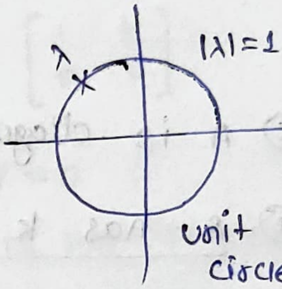
Theorem

$$U = \{\lambda \in \mathbb{C} \,/\, |\lambda| = 1\}$$

$A \in M_{n\times n}(\mathbb{C})$

If A is unitary then eigen values of A are unit module U.

$$A\bar{A}^T = I \Rightarrow \bar{A}^T = A^{-1}$$



PF: Let λ be an eigen value of A

$$\boxed{Ax = \lambda x,} \quad x \neq 0,$$

$$(\bar{A}\bar{x})^T = (\bar{\lambda}\bar{x})^T,$$

$$\bar{x}^T \cdot \bar{A}^T = \bar{\lambda}\bar{x}^T$$

$$\bar{x}^T \cdot A^{-1} = \bar{\lambda}\bar{x}^T \quad (\because \bar{A}^T = A^{-1})$$

$$(\bar{x}^T \cdot A^{-1}) A x = \bar{\lambda}\bar{x}^T A x$$

$$\bar{x}^T I x = \bar{\lambda}\bar{x}^T \lambda x$$

$$\bar{x}^T x = \bar{\lambda}\lambda \bar{x}^T x$$

$$\bar{x}^T x = \bar{\lambda}\lambda \bar{x}^T x$$

$$\bar{x}^T x(1 - |\lambda|^2) = 0 \quad (\because \bar{x}^T x \neq 0)$$

$$\Rightarrow 1 - |\lambda|^2 = 0$$

$$\Rightarrow |\lambda|^2 = 1 \Rightarrow |\lambda| = 1$$

↳ If A is unitary, then $\det(A) = ?$

$$A\bar{A}^T = I$$

$$\det(A\bar{A}^T) = \det(I)$$

$\det \bar{c} = (\overline{\det c})$

$\det c = \det c^T$

$$\det(A) \cdot \det(\bar{A}^T) = 1$$

$$\det(A) \cdot (\overline{\det A}) = 1$$

$$|\det(A)|^2 = 1$$

$$\Rightarrow |\det(A)| = 1$$

Note: ① If A is unitary $\Rightarrow$ A is invertible ($\because |\det A| = 1$).

② A is unitary $\Rightarrow$ $\bar{A}, A^T, \bar{A}^T, A^{-1}$ are unitary.

Given, $A\bar{A}^T = I$, Let $B = A^T$, to prove B is unitary

i.e: $\underline{B\bar{B}^T = I}$.

④ A is diagonalizable $\Leftrightarrow$ A has n L.I. eigen vectors.

④ $\underline{A \text{ has } k \text{ distinct eigen } \cancel{\text{vectors}} \text{ values}}$ $\Rightarrow$ $\{v_1 - - v_k\}$ is L.I.

(or) $\underline{\text{eigen vectors}}$

$\underline{v_1 - - - v_k}$

④ $A_{n\times n}$ has n distinct eigen values $\Rightarrow$ A is diagonizable.

$\hookrightarrow$ $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$

$\because$ $A_{2\times2}$ has two distinct eigen values

then A is diagoonlizable

($\because$ by theorem ③).

* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(or) $(1-\lambda)(1-\lambda) = 0$

$\lambda = 1, 1$

$A = P^{-1}DP$

$I = I^{-1}II$

$I = I$

$\Rightarrow A = I$ is diagoonilizable

$(A - \lambda I)x = 0$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0.x_1 + 0.x_2 = 0$

$\Rightarrow$ e.vectors $= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{matrix} x_1, x_2 \in \mathbb{R} \\ x_1 = x_2 \neq 0 \end{matrix} \right\}$

$= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\because$ P is invertible $\Rightarrow$ A is diagonalizable.

$*\hookrightarrow \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

eigen value $\lambda^2 = 0 \Rightarrow \lambda = 0$

$\hookrightarrow 0, 0$

$(A - 0I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0x_1 + 1 \cdot x_2 = 0$

$\Rightarrow \boxed{x_2 = 0}$

$\therefore P = \begin{array}{cc} v_1 & v_2 \\ \end{array}$ $\begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}$

$\det P = 0$

$\therefore$ P is not invertible

$\Rightarrow$ P doesnot have two L.I.

$\Rightarrow$ A is not diagonalizable.

Eigen vector $= \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\middle|\, x_1 \neq 0 \right\}$

Note:- $A_{n \times n}$ is nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$.

$*$ A is nilpotent then A is not diagonizable.

Proof: Let A is diagonilizable ; $A \neq 0$

$A = P^{-1} D P$ — ①

$A^k = P^{-1} D^k P$

$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ 0 & 0 & \cdots & d_n \end{bmatrix}$

$\because$ ~~$A^k = 0 \Rightarrow D = 0$~~

$\Rightarrow$ ~~A~~ . $0 = P^{-1} D^k P$

$\Rightarrow D^k = 0 \Rightarrow d_i = 0$

from ① $\quad A = P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} P$

$\Rightarrow A = 0 \iff$ this is contradiction to $A \neq 0$

$\Rightarrow$ A is not diagonizable.