25/08/23

* vector space :-

$(V, +, \cdot)$

addition

$+ \quad V \times V \to V$

$(u, v) \to u+v \in V$

(Scalar multiplication) $\cdot \quad F \times V \to V$

$(\alpha, v) \to \alpha \cdot v \in V$

$V \times W = \{ (v, w) / v \in V, w \in W \}$

$F = \mathbb{R}$ or $\mathbb{C}$
Real ↓ Complex ↓

$u\ x$
$v\ y$
w

α, β

$F = \mathbb{R}$ or $\mathbb{C}$

$(V, +, \cdot)$ over $F = \mathbb{R}$ (or) $\mathbb{C}$ is called vector space; $+, \cdot$ are well defined: (i) $u+v \in V$
(ii) $\alpha \cdot u \in V$

$\cdot : (\alpha, u) \to \alpha \cdot u \in V$

$+ : (u, v) \to u+v \in V$

if it is satisfying the followings

(i) ~~u+v =~~

(VS1) $u+v = v+u$; $\forall\ u, v \in V$ (commutativity property)

(VS2) $(u+v)+w = u+(v+w)$; $\forall\ u, v, w \in V$. (associativity)

(VS3) there exist a zero vector 0.
$u+0 = u$. $\forall\ u \in V$ [additive identity]

(VS4) $\forall\ u \in V$, $\exists\ v \in V$, such that $u+v=0$
[additive inverse]

(VS5) $1 \cdot u = u \quad \forall\ u \in V$

(VS6) $(\alpha\beta) \cdot u = \alpha \cdot (\beta u)$ $\forall\ u \in V$, $\alpha, \beta \in F$ (associativity)

(VS7) $\alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$; $\forall\ u, v \in V$, $\alpha \in F$ (distributivity)

(VS8) $(\alpha+\beta) \cdot u = \alpha u + \beta \cdot u$; $\forall\ u \in V$, $\forall\ \alpha, \beta \in F$.
(distributivity)

Eg. ① $(\mathbb{R}^n, +, \cdot)$ is a v.s. over $\mathbb{R}$

$a = (a_1, a_2 \dots a_n)$, $b = (b_1, \dots b_n)$

$a + b = (a_1 + b_1, a_2 + b_2, \dots a_n + b_n)$

$\alpha \cdot a = (\alpha a_1, \alpha a_2 \dots \alpha a_n)$

② $M_{m\times n}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n} \middle/ \begin{array}{l} a_{ij} \in \mathbb{R} \\ 1 \le i \le m \\ 1 \le j \le n \end{array} \right\}$

↓

set of all $m\times n$ matrices

$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} & b_{1n} \\ b_{21} & b_{22} & b_{2n} \\ b_{m1} & b_{m2} & b_{mn} \end{bmatrix}$

$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{1n} + b_{1n} \\ \vdots & & \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix}$

$\alpha \cdot A = \begin{bmatrix} \alpha a_1 & \alpha a_2 & \alpha a_n \\ & & \\ \alpha a_{m1} & \alpha a_{m2} & \alpha a_{mn} \end{bmatrix}$

$M_{m\times n}(\mathbb{R})$ is a vector space (v.s) over $\mathbb{R}$.

3.) $P_n(\mathbb{R}) = \{ a_0 + a_1 x + \dots a_n x^n / a_i \in \mathbb{R} \}$

↓

set of all polynomials of degree $\le n$.

$u = f(x) = a_0 + a_1 x + \dots + a_n x^n$

$v = g(x) = b_0 + b_1 x + \dots + b_n x^n$

$f(x)+g(x) = (a_0+b_0)+(a_1+b_1)x + \cdots (a_n+b_n)x^n$

$\alpha \cdot f(x) = \alpha \cdot a_0 + \alpha \cdot a_1 x + \cdots \alpha \cdot a_n x^n$

$P_n(\mathbb{R})$ is a V.S over $\mathbb{R}$.

Note :- ① $\mathbb{R}$ is a vector space over $\mathbb{R}$

② Is $(\mathbb{R},+,\cdot)$ a vector space over $\mathbb{C}$?

$\cancel{V \in R}$, $\cancel{F \in}$

$V = \mathbb{R}$ $\quad F = \mathbb{C}$

$\alpha \in \mathbb{C}$, $u \in \mathbb{R}$, $\alpha \cdot u \in \mathbb{C}$

$\therefore (\mathbb{R},+,\cdot)$ is Not a vector space over $\mathbb{C}$

③ $(\mathbb{C},+,\cdot)$ is a V.S. over $\mathbb{C}$

④ $(\mathbb{C},+,\cdot)$ is a V.S. over $\mathbb{R}$

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Check $(\mathbb{R}^2,+,\cdot)$ over $\mathbb{R}$

$(a_1,a_2)+(b_1,b_2) = (a_1+b_1, a_2-b_2)$ ×

$\alpha(a_1,a_2) = (\alpha a_1, a_2)$ ✓

① $u+v = v+u$

$u+v = (a_1+b_1, a_2-b_2)$

$v+u = (b_1+a_1, b_2-a_2)$

$u+v \neq v+u$

③ $(u+v)+w \neq u+(v+w)$ → fails (take c as w)

(VS 2)

Let $c=(c_1,c_2)$

$(u+v)+w = (a_1+b_1, a_2-b_2)+(c_1+c_2)$

$= (a_1+b_1+c_1, a_2-b_2+c_2)$

$u+(v+w)$

$(a_1+b_1)+(a_2+c_1, b_2-c_2)$

$\neq (a_1+a_2+c_1, \ldots)$

(VS8) fails.

$(\alpha+\beta)\cdot u \neq \alpha\cdot u + \beta u$ $\quad (\alpha = 1, \beta = 1, u = (1,1))$

L.H.S $= (\alpha+\beta)\cdot u$
$= 2\cdot(1,1)$
$= (2\cdot 1, 1)$
$= (2,1)$

R.H.S $= \alpha\cdot u + \beta\cdot u$
$= 1\cdot(1,1) + 1\cdot(1,1)$
$= (1,1) + (1,1)$
$= (2,0)$

L.H.S $\neq$ R.H.S.

* properties :-

Cancellation Law

① If $x+z = y+z$ then $x = y$

Let $x, y, z \in V$

$z \in V$
$\exists\, v$ such that
$z + v = 0$ (VS4)

$x = x + 0$ (VS3)
$= x + (z+v)$ (VS4)
$= (x+z) + v$ (VS2)
$= (y+z) + v$ ($\because$ Given)

$x = y + (z+v)$
$= y + 0$

$\boxed{x = y}$

② The additive identity '0' is unique.

Let $0, 0'$ be two additive identity.

let $x \in V$, $\quad x + 0 = x$ (VS3)
$x + 0' = x$

$x+0 = x+0'$ $\qquad$ $x+z = y+z$

$0 = 0'$ (by cancellation Law) $\qquad$ $\Rightarrow (x=y)$

Exercise

③ Additive Inverse is unique.

④ $0 \cdot u = 0 \quad \forall\, u \in V$ $\qquad$ $[1 \cdot u = u \;\forall\, u \in V]$

$0 \cdot u = (0+0) \cdot u$

$0 + 0 \cdot u = 0 \cdot u + 0 \cdot u$ (VS8)

$0 = 0 \cdot u$ (by cancellation Law.)

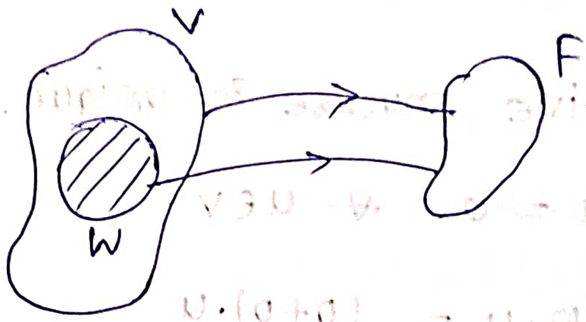
Exercise

① $(-\alpha) \cdot u = -(\alpha u) = \alpha(-u). \quad \forall\, \alpha \in F$

② $\alpha \cdot 0 = 0 \quad \forall\, \alpha \in F$

IS W a v. space over F ?

$V = (\mathbb{R}^2, +, \cdot)$



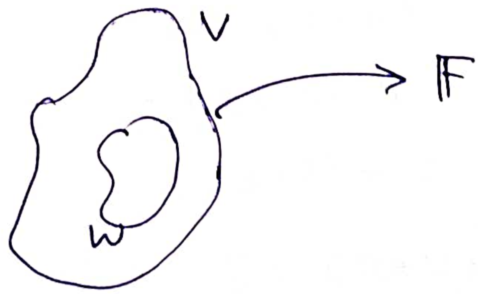
$W = \{(1,0), (0,1)\} \subseteq \mathbb{R}^2 = V$

$u + v \in W$, $(1,1) \notin W$; So W is not a vector space over $\mathbb{F}$.

* **Subspace :-**

Let V be a vector space over $\mathbb{F}$, and $W \subseteq V$. W is called subspace if W is a vector space over $\mathbb{F}$ under the same operation addition & scalar multiplication defined on V.

* Any subset $W \subseteq V$ may not be a V.S. over $\mathbb{F}$.



(VS1) $u + v = v + u$, $u, v \in W \subseteq V$.

(VS2) $u + (v + w) = (u + v) + w$, $u, v, w \in W \subseteq V$

X (VS3) $0 \notin W$ (0 may not be in W)

X (VS4) $z + v = 0 \Rightarrow v \notin W$ $(5,-5)$ may not be in W

(VS5), (VS6), (VS7), (VS8). holds for W.

$u + v \notin W$
$\alpha \cdot u \notin W$

only we have to verify (VS3), (VS4) and $\left.\begin{matrix} u+v \notin W \\ \alpha \cdot u \notin W \end{matrix}\right\}$.

Theorem

let $W \subseteq V$

Let V be a vector space over $\mathbb{F}$

W is a subspace of V. $\overset{\text{iff}}{\Longleftrightarrow}$ the following three condition holds

① $0 \in W$

② $u+v \in W, \ \forall\, u,v \in W$

③ $\alpha \cdot u \in W, \ \forall\, u \in W, \alpha \in F$

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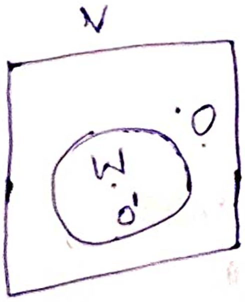
proof:

$\Rightarrow$ Given W is a subspace of V.

(W is a v.s. over $\mathbb{F}$)

Since by definition of W is a v.s.

(2), (3) holds.



Let $x \in W$

$x + 0' = x$ ($\because 0'$ is a additive identity of W)

$x \in W \subseteq V$, $x + 0 = x$ ($\because 0$ is a additive identity of V)

$x + 0' = x + 0$ $\quad 0 \in V$

$0' = 0 \in W$ ($\because 0' \in W$)

$\therefore$ ① $0 \in W$ holds.

Assume $W \subseteq V$, $0 \in W$, $u+v \in W$, $\alpha u \in W \ \forall\, u,v \in W, \alpha \in F$

to prove, W is a subspace of V.

(VS3) $\to x + 0 = x$

(VS4) $\to x + z = 0$, $z \in W$

Closed under $(+, \cdot)$

W is closed under $+, \cdot$ ($\because$ Assumption).

$0 \in W \Rightarrow u + 0 = u \ \forall\, u \in W$

$-(\alpha \cdot x) = -\alpha x$

Let $x \in W$, $(-1) \cdot x = -(1 \cdot x)$

$= (-x) \in W$

$x + (-x) = 0$ $\quad (\because)$ $-x$ is additive inverse of x.

$\therefore$ W is a subspace of V.

* vector space :-

$$D = \left\{ y(x) \,/\, a_0 + a_1 \frac{d\,y(x)}{dx} + a_2 \frac{d^2 y}{dx^2} + \cdots + a_n \frac{d^n y(x)}{dx^n} = 0 \right\}$$

$a_i \in \mathbb{R}$

D is a v.s. over $\mathbb{R}$.

Eg: of subspace

$W \subseteq V = M_{n \times n}(\mathbb{R})$ — set of all $n \times n$ matrices whose entries from $\mathbb{R}$

$W = \{ A_{n \times n} \,/\, A = A^T \}$ is a subspace of V over $\mathbb{R}$.

$\downarrow$

set of all $n \times n$ symmetric matrices.

(1) $0 \in W$.

(2) $A, B \in W$, $A + B \in W$.

(3) $\alpha \in \mathbb{F}$, $A \in W$, $\alpha A \in W$

Proof ① $0 = 0^T \Rightarrow 0 \in W$ holds

② To prove $A+B \in W$ *

$(A+B)^T = A^T + B^T$ $\quad$ $A, B \in W \Rightarrow A = A^T$, $B = B^T$

$= A + B$

$\Rightarrow (A+B)^T = A+B$

$\Rightarrow A+B \in W$

③ $(\alpha \cdot A)^T = \alpha A^T = \alpha A$ $\quad$ ($\because A \in W \Rightarrow A^T = A$)

$\Rightarrow \alpha A \in W$

$\therefore$ W is a subspace of V.

Ex ⑤ ②

$W_1 = \{A \,/\, A = -A^T\}$ set of all (skew) anti-symmetric matrices.

W_1 is a ~~v.space~~ s. space of $M_{m \times n}(\mathbb{R})$

Sol ① $0 = -0^T \Rightarrow 0 \in W$ holds.

② to prove $A+B \in W$ $\quad$ $A, B \in W$ $\Rightarrow A = -A^T$, $B = -B^T$

$\Rightarrow -(A+B)^T$

$\Rightarrow -\{A^T + B^T\}$

$= -A^T - B^T$

$= A + B \in W.$

② $(-\alpha A)^T = -\alpha A^T$

$= \alpha \cdot (-A^T)$

$= \alpha A \in W$

$\therefore W_1$ is a subspace of V.

↳ $A_{n\times n}$ is invertible (non-singular) if det of $A \neq 0$

$A_{n\times n}$ is singular if $\det(A) = 0$

3) $W_2 = \{ A / \det A \neq 0 \}$ set of all invertible matrices.

IS W_2 a subspace of $M_{n\times n}(\mathbb{R})$.

⇒ ① $0 \notin W_2$: since $\det(0) = 0$

② $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\det A = 1$, A is invertible

$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\det(B) = 1$, B is invertible

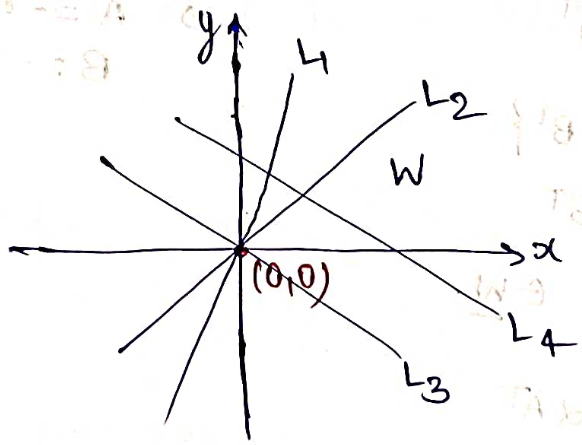
$A, B \in W_2$, $A + B = 0 \notin W_2$

∴ W_2 is not a subspace of $M_{n\times n}(\mathbb{R})$.

↳

$V = (\mathbb{R}^2, +, \cdot)$ over $\mathbb{R}$

$0 = (0,0)$

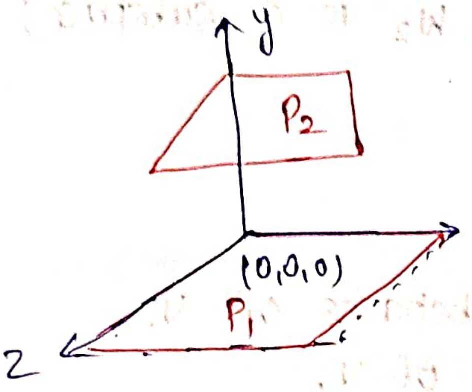


$0 \in W$

→ L_1, L_2, L_3 → is a subspace. (line passing through origin)

L_4 is not a subspace ∵ it is not passing through origin.

↳ $(\mathbb{R}^3, +, \cdot)$ over $\mathbb{R}$



⇒ P_1 plane passing through origin is a subspace of $\mathbb{R}^3$

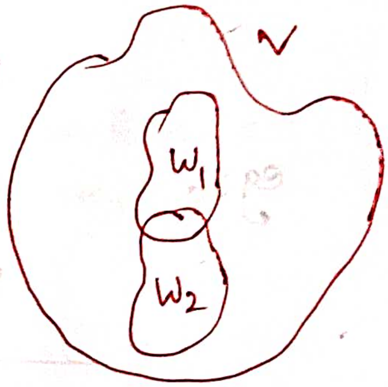
P_2 is not a subspace
(∵ it is not passing through origin.)

* Theorem:

⇒ W_1, W_2 are subspaces of V over $\mathbb{F}$

$W_1 \cap W_2$ is a subspace or not ?

$W_1 \cup W_2$ is a " " ?



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Proof :

To prove $W_1 \cap W_2$ is a subspace

(1) $0 \in W_1 \cap W_2$

(∵ Both W_1 and W_2 are subspaces.

⇒ $0 \in W_1$

and $0 \in W_2$

⇓

$0 \in W_1 \cap W_2$.)

$\mathbb{F}$

(ii) $u, v \in W_1 \cap W_2$

⇒ $u \in W_1$ and $u \in W_2$

$v \in W_1$ and $v \in W_2$

$u + v \in W_1$ (∵ W_1 is a subspace)

$u + v \in W_2$ (∵ W_2 is a subspace)

⇒ $u + v \in W_1 \cap W_2$

(iii) $u \in W_1 \cap W_2$. $\alpha \in F \Rightarrow \alpha u \in W_1 \cap W_2$

$\hookrightarrow u \in W_1$, $u \in W_2$.

$\alpha \cdot u \in W_1$, $\alpha \cdot u \in W_2$. ($\because W_1, W_2$ is a subspace)

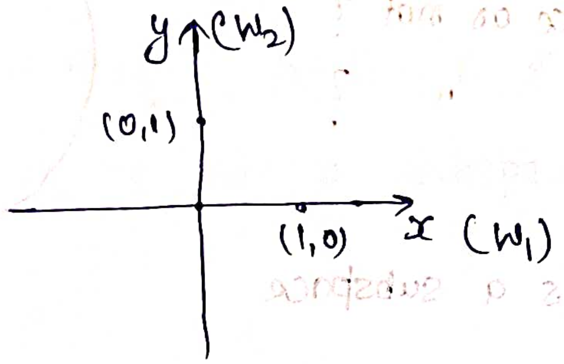
$\Rightarrow \alpha u \in W_1 \cap W_2$

* Theorem :- Let W_1, W_2 be two subspaces of V. $W_1 \cup W_2$ is a subspace of V.

Prove (or) disprove.

Let W_1 and W_2 be two subspaces of V.
Let $W_1 \cup W_2$ is not a subspace of V.

eg:-



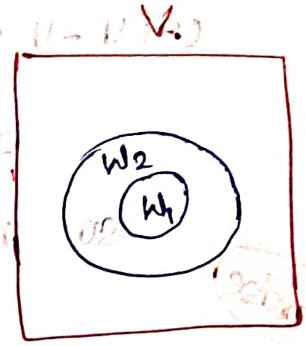
In general, $W_1 \cup W_2$ is not a subspace.
$V = (\mathbb{R}^2, +, \cdot)$ is a v.s. over $\mathbb{R}$.

x-axis $\leftarrow W_1 = \{(a,0) / a \in \mathbb{R}\}$

y-axis $\leftarrow W_2 = \{(0,b) / b \in \mathbb{R}\}$

$u = (1,0) \in W_1$
$v = (0,1) \in W_2$
$u+v = (1,1) \notin W_1 \cup W_2$

Theorem :- Let W_1, W_2 be two subspaces of V.
then $W_1 \cup W_2$ is a subspace of V
if and only if $\Rightarrow$ $W_1 \subseteq W_2$
(or)
$W_2 \subseteq W_1$



Pf $\Rightarrow$ Assume $W_1 \subseteq W_2$ (or) $W_2 \subseteq W_1$

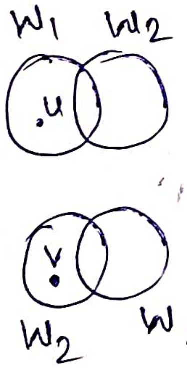
$W_1 \cup W_2 = W_2$ (or) W_1

$\because$ W_1, W_2 are subspaces.

$\Rightarrow$ $W_1 \cup W_2$ is a subspace.

$\Rightarrow$ Given $W_1 \cup W_2$ is a subspace, To prove $W_1 \subseteq W_2$ (or) $W_2 \subseteq W_1$.

Suppose $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$



There exist $u \in W_1$ but $u \notin W_2$

and $v \in W_2$ but $v \notin W_1$

$u \in W_1 \cup W_2$, $v \in W_1 \cup W_2$

$\Rightarrow$ $u+v \in W_1 \cup W_2$ (W_1, W_2 is a subspace)

Case-(i) $u+v \in W_1$

$u \in W_1 \Rightarrow -u \in W_1$ ($\because W_1$ is a subspace)

$-u+u+v = 0+v = v \in W_1$ (W_1 is a subspace)

but $v \notin W_1$. $\Rightarrow\Leftarrow$ (contradiction)

$\therefore$ our assumption is wrong.

Case-(ii) $u+u \in W_2$

$v \in W_2 \Rightarrow -v \in W_2$ ($\because W_2$ is a subspace)

$u+v-v = u+0 = u \in W_2$

but $u \notin W_2 \Rightarrow \Leftarrow$ (contradiction)

so our assumption is wrong.

Exercise

Theorem Let W_1, W_2 be two subspaces of V.

$$W_1 \oplus W_2 = \left\{ u+v \,/\, \begin{array}{l} u \in W_1 \\ v \in W_2 \end{array} \right\}$$

is a subspace.

Verify ① $0 \in W_1 + W_2$

② $U_1 = u_1+v_1$, $U_2 = u_2+v_2 \Rightarrow u_1+U_2 \in W_1+W_2$

③ $\alpha \cdot u_1 \in W_1+W_2$.

$\Rightarrow$

$\because$ W_1 and W_2 are subspaces

$\Rightarrow$ $0 \in W_1$

& $0 \in W_2$

~~Now~~, $\therefore 0 \in (W_1+W_2)$

$U_1 = u_1+v_1 \in W_1$

$U_2 = U_2+V_2 \in W_2$

$u_1+u_2 = \{u_1+u_2, v_1+v_2\} \in W_1+W_2$

$\in W_1$ $\in W_2$

$\alpha \cdot (u_1) = \alpha(u_1+v_1)$

$= (\alpha u_1 + \alpha v_1) \in W_1+W_2$

Def^n:- Linear Combination :- Let V be a V.S. over $\mathbb{F}$.

Let $S \subseteq V$ $(S \neq \phi)$ $\{v_1, v_2 \ldots v_n\}$ is called Linear combination of vectors in S.

Non-empty set.

iff $\exists\ \alpha_1 \ldots \alpha_n \in \mathbb{F}$

such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n$

(2) **Span (S) :-** span(S) is set of all Linear combination of S

$$\text{span}(S) = \left\{\alpha_1 v_1 + \cdots + \alpha_n v_n \;\middle|\; \begin{array}{l} n \in \mathbb{N} \\ \alpha_i \in \mathbb{F} \\ v_i \in R \\ \subseteq V \end{array}\right\}.$$

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Eg. ①

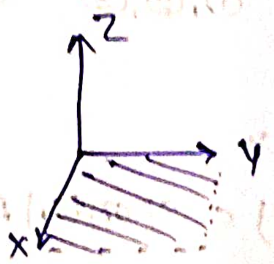
$$\text{span}(S) = \{\alpha_1 v_1, \alpha_1 v_1 + \alpha_2 v_2, \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots \alpha_n v_n + \cdots + \alpha_{10} v_{10})$$

Eg ②

$$S = \{\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}\} \subseteq R3$$

$$\text{span}(S) = \{\alpha_1(1,0,0) + \alpha_2(0,1,0)\}$$

$$= \{(\alpha_1, \alpha_2, 0) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$



* **Theorem :-** $S \subseteq V$ $\quad V \to F$

span(S) is a ~~vector~~ Sub space (or) Not ?

Pf.

$$\text{span}(S) = \left\{ \alpha_1 v_1 + \dots \alpha_n v_n \;\middle|\; \begin{array}{l} v_i \in S \\ \alpha_i \in F \end{array} \right\}$$

$0 \in \text{span}(S)$
$u+v \in \text{span}(S)$
$\alpha \cdot u \in \text{span}(S)$

① Choose, $\alpha_i = 0$

$$\Rightarrow 0.v_1 + 0v_2 + \dots 0.v_n = 0 + \dots 0 = 0$$
$$\in \text{span}(S)$$

② $u = \alpha_1 v_1 + \dots + \alpha_n v_n$ Let

$v = \beta_1 v_1 + \dots \beta_m v_m$ $\quad n > m$

$$u+v = \{(\alpha_1+\beta_1)v_1 + (\alpha_2+\beta_2)v_2 + \dots + (\alpha_m+\beta_m)v_m + \alpha_{m+1}v_{m+1} + \dots \alpha_n v_n\}$$

$\Rightarrow u+v \in \text{span}(S)$.

③ Let $u = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \in \text{span}(S)$

and $\alpha \in F$

$\alpha \cdot u = (\alpha\alpha_1)v_1 + \dots + (\alpha\alpha_n)v_n$

$\Rightarrow \alpha u \in \text{span}(S)$.

so from ①,②,③, span(S) is a subspace.

EXC. Let $S_1, S_2 \subseteq V$.

$\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$

$A \subseteq B$
$a \in A \Rightarrow a \in B$

Eg. $S = \{1, x, x^2, x^3\} \subseteq P(\mathbb{R})$

$P(\mathbb{R})$ → set of all polynomials over $\mathbb{R}$.

$\text{span}(S) = \{\alpha_1 \cdot 1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \,/\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}\}$.

↓

set of all the polynomials whose degree ≤ 3.

* Linearly independent

Let V be a v.s over $\mathbb{F}$.

$S \subseteq V$,

S is linearly independent

If $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

S is Linearly dependent

If $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0 \Rightarrow$ at least one $\alpha_i \neq 0$

Note :- S is L.D.

$\alpha_1 v_1 + \cdots + \alpha_k v_k + \cdots + \alpha_n v_n = 0 \Rightarrow \boxed{\alpha_k \neq 0}$

$\mathbb{F} = \mathbb{R}$ (or) $\mathbb{C}$

$\alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 - \cdots - \alpha_{k-1} v_{k-1} - \alpha_{k+1} v_{k+1} - \cdots - \alpha_n v_n$

multiply by α_k^{-1} on both side

$\alpha_k^{-1} \cdot \alpha_k v_k = -\alpha_k^{-1}\alpha_1 v_1 - \alpha_k^{-1}\alpha_2 v_2 + \cdots \alpha_k^{-1}\alpha_n v_n$

$\alpha_k^{-1} \cdot \alpha_k = 1$

$\Rightarrow \boxed{v_k = -\alpha_k^{-1}\alpha_1 v_1 - \cdots - \alpha_k^{-1}\alpha_n v_n}$

Eg: ① $S = \{(1,0),(0,1)\} \subseteq \mathbb{R}^2$ over $\mathbb{R}$

$\alpha_1 v_1 + \alpha_2 v_2 = \vec{0} \Rightarrow \alpha_1 = \alpha_2 = 0$

$\alpha_1(1,0) + \alpha_2(0,1) = (0,0)$

~~$\alpha_1 + \alpha_2$~~ $(\alpha_1, \alpha_2) = (0,0)$

$\Rightarrow \alpha_1 = 0$
$\alpha_2 = 0$

this means S is linearly independent

(2) $S = \{ (1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1) \}$

$\quad\quad\quad\quad v_1 \quad\quad\quad v_2 \quad\quad\quad v_3 \quad\quad\quad v_4$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$

$\Rightarrow \alpha_1(1,0,0,-1) + \alpha_2(0,1,0,-1) + \alpha_3(0,0,1,-1) + \alpha_4(0,0,0,1)$

$= (\alpha_1, \alpha_2, \alpha_3, -\alpha_1-\alpha_2-\alpha_3+\alpha_4) = (0,0,0,0)$

$\Rightarrow \alpha_1 = 0$

$\alpha_2 = 0$

$\alpha_3 = 0$

$\&$ $-\alpha_1-\alpha_2-\alpha_3+\alpha_4 = 0 \Rightarrow \alpha_4 = \alpha_1+\alpha_2+\alpha_3$

$\Rightarrow -0-0-0+\alpha_4 = 0 \quad \Rightarrow \alpha_4 = 0+0+0$

$\Rightarrow \alpha_4 = 0 \quad \Rightarrow \alpha_4 = 0$

so; S is linearly independent.

(3) $S = \{(1,0,-1), (2,-1,1), (3,-1,0)\} \subseteq \mathbb{R}^3$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \bar{0}$

$\alpha_1(1,0,-1) + \alpha_2(2,-1,1) + \alpha_3(3,-1,0) = (0,0,0)$

$(\alpha_1+2\alpha_2+3\alpha_3, -\alpha_2-\alpha_3, -\alpha_1+\alpha_2) = 0$

$\alpha_1+2\alpha_2+3\alpha_3 = 0 \quad\quad -\alpha_2-\alpha_3 = 0 \quad\quad -\alpha_1+\alpha_2 = 0$

$\Rightarrow \alpha_2 = -\alpha_3 \quad\quad \Rightarrow \alpha_2 = \alpha_1$

$\Rightarrow \alpha_1 + 2\alpha_1 + 3(-\alpha_2)$.

$\Rightarrow 3\alpha_1 - 3\alpha_1 = 0 \Rightarrow 0 = 0$

$\Rightarrow \cancel{\alpha_1 = 0}$ ~~So S is L.I.~~

$\Rightarrow \cancel{\alpha_2 = 0}$

$\Rightarrow \cancel{\alpha_3 = 0}$

$(\alpha_1, \alpha_2, \alpha_3)$ is a sol^n.

so $\{\alpha_2, \alpha_2, -\alpha_2 \mid \alpha_2 \in \mathbb{R}\}$

∵ α_2 may be any value from $\mathbb{R}$ but the sol^n should always be zero.

So S is Linearly dependent.

Since $V_3 = V_1 + V_2$

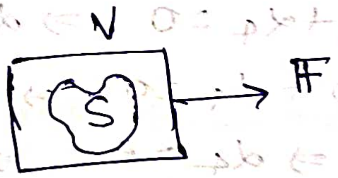
$\Rightarrow$ S · S is Linearly dependent

Date: 15/09/2023

Recall

① $S \subseteq V$

S is called L.I set



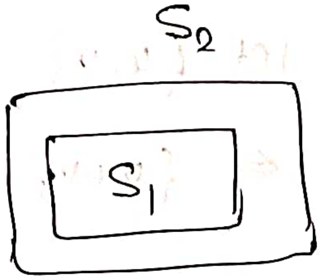
$v_i \in S$, $\alpha_i \in \mathbb{F}$ $\quad \alpha_1 v_1 + \dots + \alpha_n v_n = 0 \Rightarrow \alpha_i = 0 \ \forall \ i = 1, 2, \dots$

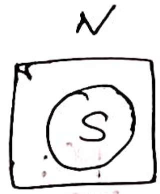
② $\alpha_1 v_1 + \dots + \alpha_n v_n = 0 \Rightarrow \alpha_i \neq 0$ for some i

then S is L.D.

S is L.I. if one vector can not be expressed as a Linear $comb^n$. of other vectors in S.

Theorem :-





① Let V be a v.s. over $\mathbb{F}$.

$S_1 \subseteq S_2$

If S_2 is Linearly independent then S_1 is also L.I.

Pf: If S_2 is L.I then S_1 is L.I.

$\Rightarrow$ to prove S_1 is L.I

(i.e) $\alpha_1 v_1 + \dots \alpha_n v_n = 0 \Rightarrow \alpha_i = 0 \ \forall i = 1, 2, \dots n.$

$v_i \in S_1$; $\alpha_i \in \mathbb{F}$

$v_i \in S_1 \subseteq S_2 \Rightarrow v_i \in S_2$

Given that S_2 is L.I. and $v_i \in S_2$

$\therefore \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n = 0 \Rightarrow \alpha_i = 0 \ \forall i = 1, 2, \dots n$

$\therefore S_1$ is L.I.

Theorem

Let V be a v.s. over $\mathbb{F}$.

$S_1 \subseteq S_2$

② If S_1 is L.D then S_2 is also L.D.

Pf: Given S_1 is L.D, To prove S_2 is L.D

- suppose, S_2 is Linearly independent

$S_1 \subseteq S_2$, by pre-theorem① S_1 is L.I.

but S_1 is L.D. (which is contradiction)

$\therefore \Rightarrow S_2$ is L.D.

problem (1) let $\{u, v\}$ is L.I.

$\Leftrightarrow \{u+v, u-v\}$ is L.I.

Pf:

let $\{u,v\} \to$ is L.I. To prove $\{u+v, u-v\}$ is L.I.

$\Rightarrow$ ~~$\alpha_i u_i = 0 \Rightarrow \alpha_i = 0$~~

~~and $\alpha_j v_j = 0 \Rightarrow \alpha_j = 0$~~

~~then for $u+v$~~

~~$\alpha_i u_i + \alpha_j v_j = 0$~~

~~$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_n u_n + \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_m v_m = 0$~~

~~$\because \alpha_i = 0 \ \forall i = 1 \dots n$ and $\alpha_j = 0 \ \forall j = 1, \dots m$~~

~~so~~

let u, v is L.I. to prove $\{u+v, u-v\}$ is L.I.

~~$\alpha_i (u_i v)$ is L.I.~~,

$\alpha_1(u+v) + \alpha_2(u-v) = 0$

$(\alpha_1+\alpha_2) u + (\alpha_1-\alpha_2) v = 0$

$\Rightarrow \alpha_1+\alpha_2 = 0 \ \& \ \alpha_1 - \alpha_2 = 0$

$\Rightarrow \boxed{\alpha_1 = \alpha_2 = 0}$ $\because (u,v)$ is L.I.

let $\{u+v, u-v\}$ is L.I to prove $\{u, v\}$ is L.I

$\alpha_1 u + \alpha_2 v = 0 \Rightarrow \alpha_1 = 0$ & $\alpha_2 = 0$

~~$\{u+v, u-v\}$~~

let $\alpha_1 u + \alpha_2 v = 0$

$u\left[\frac{\alpha_1}{2} + \frac{\alpha_1}{2} + \frac{\alpha_2}{2} - \frac{\alpha_2}{2}\right] + v\left[\frac{\alpha_2}{2} + \frac{\alpha_2}{2} + \frac{\alpha_1}{2} - \frac{\alpha_1}{2}\right] = 0$

$\cancel{u\left(\frac{\alpha_1}{2}\right) + v\left(\frac{\alpha_1}{2}\right) + u\left(\frac{-\alpha_2}{2}\right) + v\left(\frac{\alpha_2}{2}\right)}$

$\Rightarrow \left(\frac{\alpha_1+\alpha_2}{2}\right)(u+v) + \left(\frac{\alpha_1-\alpha_2}{2}\right)(u-v) = 0$

since $\{u+v, u-v\}$ L.I

$\Rightarrow \frac{\alpha_1+\alpha_2}{2} = 0$ and $\frac{\alpha_1-\alpha_2}{2} = 0$

$\Rightarrow \underline{\alpha_1 = \alpha_2 = 0}$

Exercise

(2) $\{u, v, w\}$ is L.I. $\Leftrightarrow \{u+v, v+w, u+w\}$ is L.I.

Pf: ~~let $\{u+v, v+w, u+w\}$ is L.I~~

~~to prove $\{u, v, w\}$ is L.I.~~

let u, v, w is L.I.

$\alpha_1(u+v) + \alpha_2(v+w) + \alpha_3(u+w) = 0$

$(\alpha_1+\alpha_3)u + (\alpha_1+\alpha_2)v + (\alpha_2+\alpha_3)w = 0$

$\Rightarrow \alpha_1+\alpha_3 = 0 \quad \alpha_1+\alpha_2 = 0 \quad \alpha_2+\alpha_3 = 0$

$\Rightarrow \boxed{\alpha_1 = \alpha_2 = \alpha_3 = 0}$ ($\because$ u, v, w is L.I.)

- $AX = b$

A is invertible, $\det A \neq 0$

$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ $\{v_1, v_2, v_3\}$ is L.I.

$\Leftrightarrow \det A \neq 0$

If $\{v_1, v_2, v_3\}$ is L.D

$\Leftrightarrow \det(A) = 0$

Note:- (1) $S = \{0 \neq u\} \subseteq V \Rightarrow S$ is L.I.

To prove

$\alpha \cdot u = 0 \Rightarrow \alpha = 0$

suppose $\alpha \neq 0 \Rightarrow \alpha^{-1}$ exist

$\alpha^{-1} \alpha u = \alpha^{-1} \cdot 0$

$\Rightarrow 1 \cdot u = 0$

$\Rightarrow u = 0$ So our assumption is wrong. $(\because u \neq 0)$

$\therefore \alpha = 0$

(2) $S = \{0, u_1, \dots u_n\}$ is ~~L.I.~~ (or) L.D.

$\Rightarrow \alpha_1 \cdot 0 + \alpha_2 u_1 + \dots \alpha_n u_n = 0$

$\Rightarrow \alpha_2 = \dots \alpha_n = 0$

but α_1 may (or) may not be 0. $(\alpha_1 \neq 0)$

So it is L.D.

$\Rightarrow S = \{0, u_1, \dots u_n\}$ is L.D.

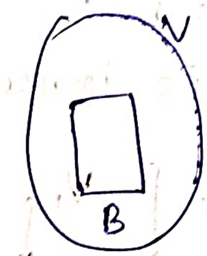
any vector containing 0 will be L.D.

Basis : Let V be a v.s. over $\mathbb{F}$

$B \subseteq V$. B is called basis

if 1) B is L.I.

2) $\boxed{\text{Span}(B) = V.}$ $\longrightarrow$ $\boxed{\begin{array}{l} v \in V \\ v = \alpha_1 v_1 + \dots \alpha_n v_n \\ v_i \in B \end{array}}$



$B \subseteq V$

Eg. ① $\mathbb{R}^2$ is a v.s. over $\mathbb{R}$.

$B = \{(1,0), (0,1)\}$ is a Basis.

$\mathbb{R}^2 = \{(a,b) \,/\, a, b \in \mathbb{R}\}$

(1) B is L.I.

let $\alpha, \beta \in \mathbb{R}$ $\quad \alpha(1,0) + \beta(0,1) = (\alpha, \beta)$

span $B = \mathbb{R}^2$

$\therefore$ B is a Basis of $\mathbb{R}^2$

Eg ① $\mathbb{R}^n$ is a v.s. over $\mathbb{R}$

$B = \{(1,0,0 \dots 0).(0,1,0 \dots 0) \dots (0,0 \dots 1)\}$ is a basis.

(n tuple)

② $V = M_{n \times n}(\mathbb{R})$ over $\mathbb{R}$

Basis $B = \left\{ B_{ij} \,/\, \text{only non zero entry is the ith row, jth column} \right\}$

$i \le 1, j \le n.$

For 2×2 matrices.

Basis (B) $= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$v_1 \quad v_2 \quad v_3 \quad v_4$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0 \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \alpha_4 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \Rightarrow$ B is L.I.

(3.) $V = P_n(\mathbb{R}) = \{a_0 + a_1 x + \cdots + a_n x^n \,/\, a_i \in \mathbb{R},\ i = 0, 1, 2 \ldots n\}$

Basis $B = \{1, x, x^2 \ldots x^n\}$

1) B is L.I., $\alpha_0 \cdot 1 + \alpha_1 \cdot x + \cdots + \alpha_n x^n = 0$

$\Rightarrow 0 \cdot 1 + 0 \cdot x + \cdots + 0 \cdot x^n = 0$

$\Rightarrow \alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$

$\Rightarrow$ B is a L.I.

ii) span B = V

<u>Theorem</u>:- let $B \subseteq V$ $B = \{v_1, v_2 \ldots v_n\}$

B is a basis of V $\Leftrightarrow$ $\forall$ $v \in V$ can be uniquely written as linear combination of vector in B.

<u>Pf</u>: B is a basis [i.e., B is LI, span B = V]

Suppose span B = V, $\forall$ $v \in V$, can expressed as L.C. of elements in B.

suppose $v \in V$,

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_1 v_1 + \cdots + \beta_n v_n$$

$$(\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_n - \beta_n) v_n = \bar{0}$$

since B is L.I. $\Rightarrow \alpha_i - \beta_i = 0$

$$\Rightarrow \boxed{\alpha_i = \beta_i} \quad \forall i$$

Given $\forall v \in V$, v can be written as Linear Combination of vectors in B.

i.e., $\boxed{\text{span } B = V.}$

To prove B is a Basis

to prove B is L.I.

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0 = 0 \cdot v_1 + \cdots + 0 \cdot v_n$$

$\Rightarrow \alpha_i = 0 \ \forall i$ ($\because$ unique way L.C.)

$\therefore$ B is L.I.

$V = \mathbb{R}$ over $\mathbb{R}$

Basis $B = \{1\}$, there exist many basis for V? $\hookrightarrow$ yes

B is L.I.

$$\alpha \cdot 1 = 0 \Rightarrow \alpha = 0$$

$$\text{span}(B) = \{\alpha \cdot 1 \mid \alpha \in \mathbb{R}\} = \{\alpha \mid \alpha \in \mathbb{R}\} = \mathbb{R}$$

$B = \{2\}$ is a basis

$$\alpha \cdot 2 = 0 \Rightarrow \alpha = 0$$

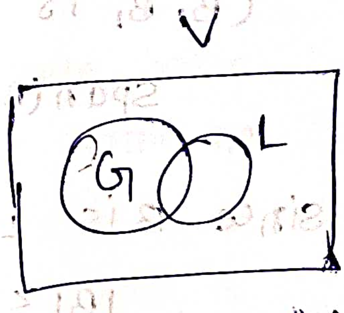
$$\text{span}(B) = \{\alpha \cdot 2 \mid \alpha \in \mathbb{R}\} = \mathbb{R}.$$

$|B|$ — no. of elements on B.

Note:- V is a v.s. over $\mathbb{R}$.

B, B_1 is a basis for V. $|B| = |B_1|$

Theorem Let V be a v.s. over $\mathbb{F}$, generated by a set G containing exactly 'n' vectors and L be a L.I. set of V containing 'm' vectors then $m \le n$.



Span $G = V$, L is L.I

$|G| = n$, $|L| = m$

$|\text{L.I. set}| \le |\text{Spanning set of } V|$

Theorem ②

B, B_1 is a basis for V $\Rightarrow |B| = |B_1|$

[Every basis of V contains same no. of elements.]

Pf. Given B, B_1 are basis of V.

(B, B_1 is L.I,

$\text{span}(B) = \text{span } B_1 = V$)

since B is L.I., $\text{span } B_1 = V$.

$|B| \leq |B_1|$ — ⓐ (∵ theorem ①)

since B_1 is L.I., $\text{span } B = V$

$|B_1| \leq |B|$ — ⓑ (∵ theorem ①)

from ⓐ and ⓑ

$|B| = |B_1|$

Date
21/09/23

Theorem:

1) Let V be a vector space and $B = \{u_1, u_2 \dots u_n\}$ be a subset of V. Then B is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of B.

that is, can be expressed as in the form -

$V = a_1u_1 + a_2u_2 + \dots + a_nu_n$. for unique scalars $a_1, a_2 \dots a_n$.

2) Let V be a vector space having a finite basis. Then every basis for V contains the same no. of vectors. eg. B, B_1

$$|B| = |B_1|$$

Dimension of V: is (number of elements in basis)

Defn:- A vector space is called finite dimensional if it has a basis consisting of a finite no. of vectors.

The unique number of vectors in each ~~vector~~ basis for V is called the dimension of V and is denoted by $\dim(V)$.

↳ A vector space that is not finite-dimensional is called infinite-dimensional.

Example

1.) The vector space $\{0\}$ has dimension zero. $V = \{0\}$, $\{0\} \leq V$, $V \subseteq V$

2.) The vector space F^n has dimension n.

$\mathbb{R}^n$ over $\mathbb{R}$. Basis $= \{e_1, e_2 \dots e_n\}$ where

nth place

$e_n = \{0, 0, 0 \dots 1, 0, 0 \dots\}$

3.) The vector space $M_{m \times n}(\mathbb{F})$ has dimension $(m \cdot n)$.

4.) The vector space $P^n(F)$ has dimension $n+1$.

$B = \{1, x, x^2, x^3, \ldots, x^n\}$. $|B| = n+1$.

5.) over the field of complex numbers, the vector space of complex numbers has dimension 1.
(A Basis is $\{1\}$)

$\mathbb{C}$ over $\mathbb{C}$. $B = \{1\}$ | span $B = \mathbb{C}$
$\alpha \cdot 1 = \alpha \in \mathbb{C}$.

dim $B = 1$.

6.) over the field of real numbers, the vector space of complex numbers has dimension 2. (A basis is $(1, i)$).

$V = \mathbb{C}$ over $\mathbb{R}$

$B = \{1, i\}$

$\boxed{\text{Span } B = \mathbb{C}}$

$\alpha \cdot 1 + \beta \cdot i = \alpha + i\beta \in \mathbb{C}$

$\alpha \cdot 1 + \beta \cdot i = 0 = 0 + 0i$

$\alpha, \beta \in \mathbb{R}$.

$\Rightarrow \boxed{\alpha = \beta = 0} \Rightarrow B$ is L.I.

* **Theorem:-** Let V be a v.s. with dimension n.

dim (a) Any finite generating set for V contains at least n vectors, and a generating set of V that contains exactly n vectors is a basis for V.

(b) Any linearly independent subset of V that contains exactly n vectors is a basis for V.

(c) Every Linearly ~~sub~~ independent subset of V can be extended to a basis for V.

(a)

dim $V = n$
span $(S) = V$
$|S| = n$
$\Rightarrow$ (S is a L.I.)
and $|S| = n$
$\Rightarrow S$ is a basis.

dim $V = n$, $S = \{v_1, v_2, v_3\}$ Check S' is L.I.?

$S = \{v_1, v_2\}$ is L.I.
$\Rightarrow \{v_1, v_2, \ldots, v_n\}$ is a basis.

* Dimension of subspace:-

Let W be a subspace of a finite dimensional vector space V. Then W is finite dimensional and $\dim(W) \le \dim(V)$ moreover, if $\dim(W) = \dim(V)$, the $V = W$.

Proof $\dim V = n$, $W \le V$, W is a subspace.

① $W = \{0\}$. $\dim W = 0 \le n \geq \dim V$.

② $W \neq \{0\}$, then $0 \neq v_1 \in W$,

$\{v_1\}$ is L.I.

since by prev. theorem, we can extend $\{v_1, v_2, v_3 \dots v_n\}$ is L.I. & $W \le V$

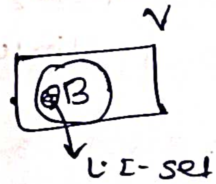
$\dim W \le \dim V$. $\quad k \le n$

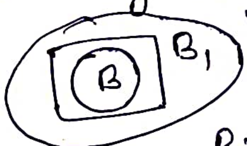
$\dim(V) = n$.

Theorem: ① B is a basis for V. $(\dim V = n)$

② B is a maximal L.I. set

③ B is minimal generating set.





B_1 - Generating set.

Problem $V = \mathbb{R}^5 \xrightarrow{\text{over } \mathbb{R}}$ $\mathbb{F} = \mathbb{R}$. set $\mathbb{R}$

Let $W = \{ (a_1, a_2, a_3, a_4, a_5) \} \in F^5 : a_1 + a_3 + a_5 = 0$

$a_2 = a_4 \}$.

find $\dim(W)$

$0 \in W$; $0 = (0,0,0,0,0)$

$a, b \in W \Rightarrow a + b \in W$

$\alpha \cdot a \in W \;\; \forall \alpha \in \mathbb{R}, a \in W$ $\Big\} \Rightarrow W$ is a subspace.

$W = \{ (a_1, a_2, a_3, a_2, -(a_1) - (a_3)) \}$ $\quad a_2 = a_4$

$a_1 + a_3 + a_5 = 0$

$= \{ \underbrace{a_1 (1,0,0,0,-1)}_{v_1} + a_2 \underbrace{(0,1,0,1,0)}_{v_2}$

$+ a_3 \underbrace{(0,0,1,0,-1)}_{v_3} \; (a_1, a_2, a_3 \in \mathbb{R} \}$

$W = \text{span}\{v_1, v_2, v_3\} = \text{span}(S)$ — ②

To prove $\{v_1, v_2, v_3\}$ is L.I.

$a_1(v_1) + a_2v_2 + a_3v_3 = (0,0,0,0,0)$

$(a_1, a_2, a_3, a_2, -a_1, -a_3) = (0,0,0,0,0)$

$a_1 = 0 = a_2 = a_3$

$\Rightarrow$ S is L.I. — ①

from ① & ② S is a basis for W.

$\dim W = |S| = 3$.

<u>Theorem</u> prove that if W_1 & W_2 are finite-dimensional subspace of a vector space V, then the subspace $W_1 + W_2$ is finite dimensional and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

<u>Hint:</u>

~~Step~~

<u>Proof:</u>

$W_1 + W_2 = \{u + v \,/\, u \in W_1 ; v \in W_2\}$

$W_1 \subseteq W_1 + W_2$, $W_2 \subseteq W_1 + W_2$

$W_1 \cap W_2 \subseteq W_1$ & $W_1 \cap W_2 \subseteq W_2$

fix the basis for $W_1 \cap W_2$.

$\{u_1, u_2 \ldots u_R\}$ is a basis for $W_1 \cap W_2 \subseteq W_1$.

$\{u_1, u_2 \ldots u_R, v_1, v_2 \ldots v_m\}$ is a basis for W_1.

since $W_1 \cap W_2 \subseteq W_2$

$\{u_1, u_2 \ldots u_R, w_1, w_2 \ldots w_p\}$ is a basis for W_2.

$\dim W_1 \cap W_2 = R$, $\dim W_1 = R + m$, $\dim W_2 = R + P$.

$\{u_1, u_2 \ldots u_R, v_1 \ldots v_m, w_1 \ldots w_p\}$ is a basis for $W_1 + W_2$.

$\dim W_1 + W_2 = R + m + p$.

$\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = R + m + R + p - R \Rightarrow \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

$= (R+m) + (R+p) - R$

$= R + m + p$.

Date 02/09/23

* Theorem :-

1.) Let V be a vector space and $B = \{u_1, u_2, \ldots u_n\}$ be a subset of V. The B is a basis for V if and only if each $v \in V$ can be uniquely expressed as a Linear combination of vector of B, i.e; can be expressed in the form of $v = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n$. for unique scalars $(a_1, a_2 \ldots a_n)$

2.) Let V be a vector space having a finite basis. Then every basis for V contains the same no. of vectors.

B, B_1

$\Rightarrow |B| = |B_1|$.

* Dimension of V: dimension of V is number of elements in Basis.

Def^n: A vector space is called finite dimensional if it has a basis consisting of a finite no. of vectors. The unique no. of vectors in each basis for V is called the dimension of V and is denoted by dim(V).

↳ A vector space that is not finite-dimensional is called infinite-dimensional.

Example:

1.) The vector space $\{0\}$ has dimension zero.

2.) The vector space F^n has dimension n.

3.)

Ques $V = M_{2\times 2}(\mathbb{R})$ over $\mathbb{R}$: $W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \middle/ a, b, c \in \mathbb{R} \right\}$

$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \middle/ a, b \in \mathbb{R} \right\}$. find dim W_1, dim W_2, dim $W_1 + W_2$.

Sol^n^. $W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{v_1} + b\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{v_2} + c\underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{v_3} \middle/ a, b, c \in \mathbb{R} \right\}$

$\text{span}\{v_1, v_2, v_3\} = W_1$ — ②

$av_1 + bv_2 + cv_3 = \begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow a = b = c = 0$

$\{v_1, v_2, v_3\}$ is L.I. — ①

Basis for $W_1 = \{v_1, v_2, v_3\}$, $\boxed{\text{dimension} = 3}$

$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} = a\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{u_1} + b\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{u_2} \middle/ a, b \in \mathbb{R} \right\}$

$\left.\begin{array}{l} W_2 = \text{span}\{u_1, u_2\} \\ \text{verify } \{u_1, u_2\} \text{ is L.I.} \end{array}\right\} \Rightarrow \{u_1, u_2\}$ is a basis for W_2. ✓

$W_1 \cap W_2$; $\begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} 0 & e \\ -e & f \end{pmatrix}$

$\Rightarrow a = 0 = f, \quad b = e, \quad c = -e$

$\begin{pmatrix} 0 & e \\ -e & f \end{pmatrix} \in W_1 \cap W_2$

$W_1 \cap W_2 = \left\{ \begin{pmatrix} 0 & e \\ -e & f \end{pmatrix} \middle/ e \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

$\{v_1\}$ is L.I. $\Rightarrow \{v_1\}$ is a basis for $W_1 \cap W_2$.

$\dim(W_1 \cap W_2) = 1$

$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$= 3 + 2 - 1$

$= 4$.

* Linear Transformation:-

Let V and W be vector space (over $\mathbb{F}$). We call a function $T: V \to W$ a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have

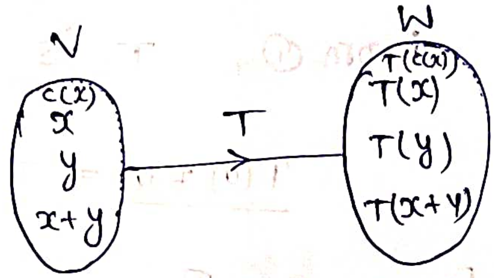
a.) $T(x+y) = T(x) + T(y)$ and

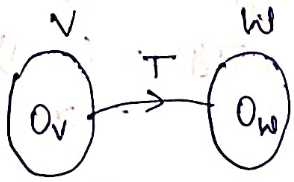
b.) $T(cx) = c \cdot T(x)$

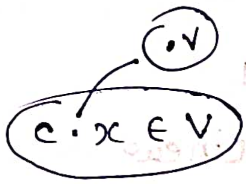
, if the underlying field $\mathbb{F}$ is the field of rational numbers, then (a) implies.

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$T: V \to W$







$T(cx) = c.T(x) \in W$

$c \in \mathbb{F}$

W

* properties:-

If T is linear, then

1.) $T(0) = 0$.

2.) T is linear if and only if $T(cx+y) = cT(x) + T(y)$ $\forall\, x, y \in V$ and $c \in \mathbb{F}$

3.) if T is Linear, then $T(x-y) = T(x) - T(y)$ $\forall\, x, y \in V$.

Pf: 1) $0_w + T(\cancel{0_v}) = T(0_v + 0_v) = T(0_v) + T(\cancel{0_v})$ ($\because T$ is linear)

$$\boxed{0_w = T(0_v)}$$

2) Combination of (a) & (b) is the defn of L.T.

3) $T(x+cy) = T(x) + c.T(y)$

$c = -1, \ T(x-y) = T(x) - T(y) \ \forall x, y \in V.$

from 1, T is Linear $\Rightarrow T(0) = 0$.

$\underline{T(0) \neq 0 \Rightarrow T \text{ is not linear}}$

Eg. $T: \mathbb{R} \to \mathbb{R}$

$T(x) = x+1$,

$T(0) = 0+1 = 1$

T is not a Linear trans.

$T(x+y) = x+y+1$

$T(x) + T(y) = (x+1) + (y+1)$

$= x+y+2$

$T(x+y) \neq T(x) + T(y)$

$\Rightarrow$ T is not a Linear transformation.

Example:

1) Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (2a_1 + a_2, a_1)$ to show that T is linear.

$T(x+y) = T(x) + T(y)$ & $T(cx) = c.T(x)$

$x, y \in \mathbb{R}^2$, $x = (b_1, b_2)$, $y = (c_1, c_2)$

$T(x) = T(b_1, b_2) = (2b_1 + b_2, b_1)$

$T(y) = T(c_1, c_2) = (2c_1 + c_2, c_1)$

$T(x+y) = T(b_1+c_1, b_2+c_2) = (2(b_1+c_1) + b_2+c_2, b_1+c_1)$

$T(x) + T(y) = (2(b_1+c_1) + b_2+c_2, b_1+c_1)$

$\Rightarrow T(x+y) = T(x) + T(y)$ — (a)

Sly $T(cx) = T(cb_1, cb_2) = (2cb_1 + cb_2, cb_2) = c(2b_1 + b_2, b_2)$

$cT(x) = c \cdot T(b_1, b_2) = c(2b_1 + b_2, b_2)$

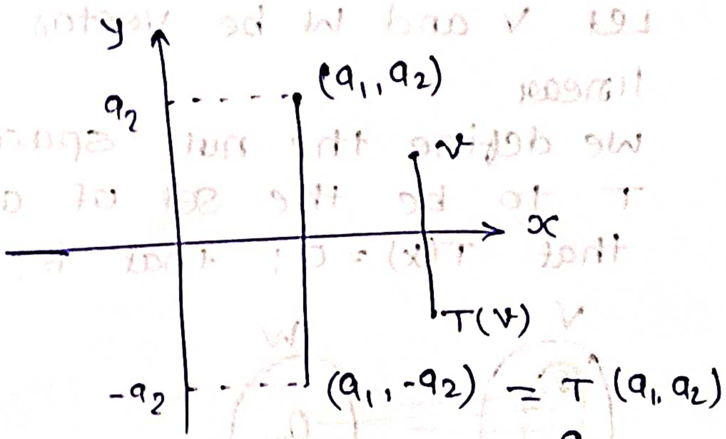
$\Rightarrow T(cx) = cT(x)$ — (b)

both a and b varify.

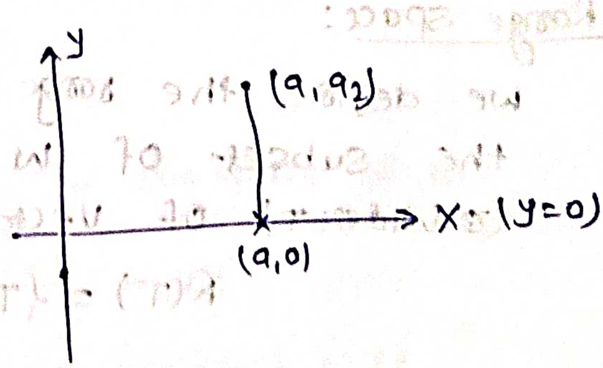
so T is Linear.

(2) Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, -a_2)$

T is called the Reflection about the x-axis.



(3) Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, 0)$. T is called the projection on the x-axis.



(4) Define $T: M_{m \times n}(F) \to M_{n \times m}(F)$ by $T(A) = A^T$.

$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B) \quad \forall A, B \in V$

$T(\alpha A) = (\alpha A)^T = \alpha A^T = \alpha A^T = \alpha T(A)$

5) Define $T: P^n(R) \to P_{n-1}(R)$ by $T(f(x)) = f'(x)$

$f : x + x^2$

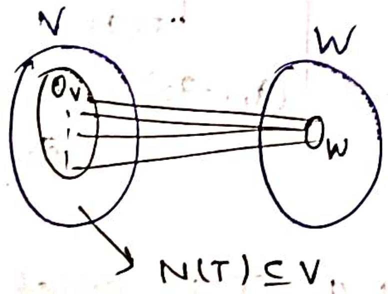
$f' : 1 + 2x$

$$T(f+g) = (f+g)' = f' + g' = T(f) + T(g)$$

$$T(\alpha f) = (\alpha f)' = \alpha f' = \alpha T(f)$$

* Null space:

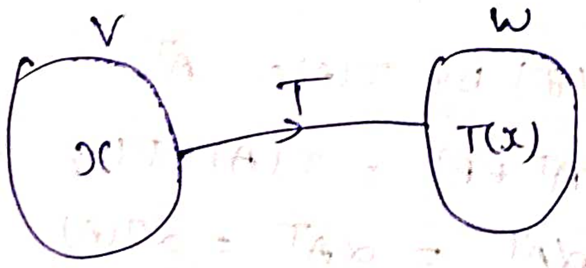
Let V and W be vector spaces, and $T: V \to W$ be linear.
We define the null space (or Kernel) $N(T)$ of T to be the set of all vector x in V such that $T(x) = 0$: that is, $N(T) = \{x \in V : T(x) = 0\}$.



* Range space:

we define the range (or image) $R(T)$ of T to be the subset of W consisting of all images (under T) of vectors in V: that is,

$$R(T) = \{T(x) : x \in V\}.$$



* Theorem :- Let $T : V \to W$ be linear then $R(T), N(T)$ are subspaces of W & V.

Pf: To prove $N(T)$ is a subspace of V

① $T(0) = 0 \Rightarrow 0 \in N(T)$

② Given $u, v \in N(T)$.

$$\left.\begin{array}{l} T(u) = 0 \\ T(v) = 0 \end{array}\right\} \text{To prove } u+v \in N(T) \text{ i.e; } T(u+v) = 0$$

$$T(u+v) = T(u) + T(v)$$
$$= 0 + 0$$
$$= 0$$

$\Rightarrow u+v \in N(T)$.

③ $u \in N(T) \Rightarrow \alpha u \in N(T)$, where $\alpha \in \mathbb{F}$

$T(\alpha u) = \alpha \cdot T(u) = \alpha \cdot 0 = 0$

$\Rightarrow \alpha u \in N(T)$

from ①, ② & ③,

$N(T)$ is a subspace of V.

Now;

$R(T) = \{T(u) / u \in V\} \subseteq W$ is a subspace of W.

① $0 \in R(T)$, $T(0) = 0 \in R(T)$

② Let $u, v \in R(T)$ to prove $u+v \in R(T)$

$u, v \in R(T)$ i.e; $\exists\, x, y \in V$.

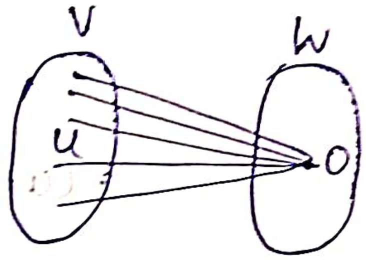
such that $T(x) = u$
$T(y) = v$

$u+v = T(x) + T(y) = T(x+y)$

$\Rightarrow u+v \in R(T)$.

③ $\alpha u \in R(T)$

Eg: ① <u>zero transformation</u> :- $T: V \to W$

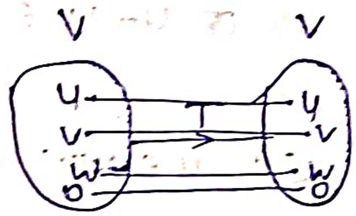


$T(u) = 0 \ \forall \ u \in V$

$R(T), N(T)$

$R(T) = \{0\}, \quad N(T) = V.$

② <u>identity</u>: $T(u) = u, \quad T: V \to V$



$R(T) = V \quad , N(T) = \{0\}.$

③ $T: \mathbb{R}^3 \to \mathbb{R}^2$

$T(a_1, a_2, a_3) = (\underset{\in \mathbb{R}}{a_1 - a_2}, \underset{\in \mathbb{R}}{2a_3})$

$N(T), R(T).$??

$\hookrightarrow R(T) = \mathbb{R}^2$

$N(T) = \{u \in \mathbb{R}^3 \,/\, T(u) = (0,0)\}$

$= \{(a_1, a_2, a_3) \,/\, T(a_1, a_2, a_3) = (0,0)\}.$

$(a_1 - a_2, 2a_3) = (0,0)$

$a_1 = a_2$

$a_3 = 0$

$\text{span}\{(1,1,0)\} = \{(a_1, a_1, 0) \,/\, a_1 = a_2, a_3 = 0\}.$

$= a_1(1,1,0).$

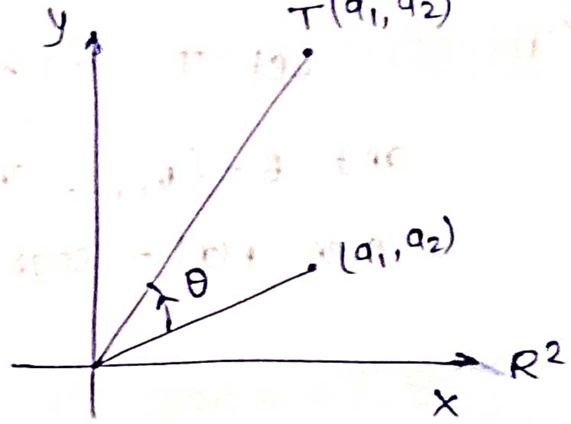
(4)

$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$\theta \to$ fixed. $\quad 0^\circ \le \theta \le 360^\circ$

$T_\theta (a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta,$

$a_1 \sin\theta + a_2 \cos\theta)$



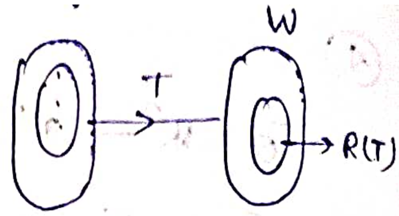
verify

T_θ is Linear

$\downarrow$

(rotation of vector by θ angle.)

Exc: **Theorem**: Let $T: V \to W$ be linear and $B = \{v_1, \dots v_k\}$ is a basis of V. then $R(T) = \text{span}\{T(v_1), \dots T(v_k)\}$.



Def^n

$$\dim(N(T)) = \text{Nullity}(T)$$

$$\dim(R(T)) = \text{rank}(T).$$

- **Rank - Nullity theorem (or) dimension theorem** :-

Let $T: V \to W$ be linear transformation.

If V is finite dimensional, then

$$\dim V = \dim N(T) + \dim R(T)$$

$$\dim V = \text{Nullity}(T) + \text{rank}(T)$$

$V = P(\mathbb{R})$ is set of all polynomials whose coefficients in $\mathbb{R}$.

V is a v.s. over $\mathbb{R}$.

Basis

$B = \{1, x, x^2 \ldots x^n \ldots\}$

② span $B = P(\mathbb{R})$

$\Rightarrow$ V is infinite dimensional v.s.

$\alpha_1 \cdot 1 + \cdots + \alpha_n x^n = 0$

$\Rightarrow \alpha_i = 0 \ \forall i$

$\therefore$ Any linear combination of vectors in B is L.I. $\Rightarrow$ ① B is L.I.

② span $B = P(\mathbb{R})$.

Recall:

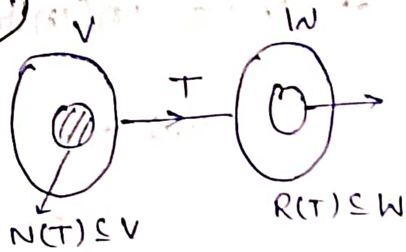
* Rank – Nullity theorem :-

Let $T: V \to W$ be linear and V is finite dimensional.

then, $\boxed{\dim V = \dim R(T) + \dim N(T)}$ { W can be infinite (or) finite dimensional. }

proof:

Since V is finite dimensional, $\dim V = n$.



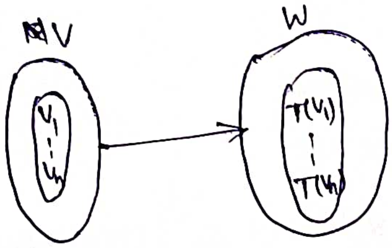
We know that,

N(T) is a subspace.

$B = \{v_1, v_2 \ldots v_k\}$ is a basis for N(T).

$\{v_1, v_2 \ldots v_k\}$ is L.I. $\Rightarrow \{v_1, v_2 \ldots v_k, v_{k+1}, \ldots v_n\}$ is a basis for V.

$B_1 = \{v_1, v_2 \ldots v_k, v_{k+1}, \ldots v_n\}$ is a basis for V.



{ $v \in N(T)$ $\Rightarrow T(v) = 0$ }

$R(T) = \text{span}\{\underset{0}{T(v_1)} \ldots \underset{0}{T(v_k)}, T(v_{k+1}) \ldots T(v_n)\}$.

Since $v_1 \ldots v_k \in N(T)$

$\Rightarrow T(v_1) = T(v_2) = \ldots = T(v_k) = 0$

$R(T) = \text{span}\{T(v_{k+1}), \ldots T(v_n)\}$

verify $\{T(v_{k+1}) \ldots T(v_n)\}$ is L.I.

$B_2 = \{T(v_{k+1}) \ldots T(v_n)\}$ is a basis for R(T).

$\Rightarrow \because \dim V = n$

& $\dim R(T) = n-k$

& $\dim N(T) = k$

$\Rightarrow \dim V = \dim R(T) + \dim N(T)$

$\Rightarrow n = (n-k) + k$

$\Rightarrow n = n$ verify.

→ Consider the L.T., $L_A : \mathbb{R}^n \to \mathbb{R}^n$

$A \in M_{n\times n}(\mathbb{R})$, $L_A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = A_{n\times n}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$

verify L_A is Linear.

↳ $\dim(\mathbb{R}^n) = \dim R(L_A) + \dim N(L_A)$.

$n = \text{rank}(A) + \dim N(L_A)$.

$\dim N(L_A) = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Big/ Ax = 0 \right\}$ = Solution set of homogeneous eqn $Ax = 0$.

$$\boxed{\dim N(L_A) = n - \text{Rank}(A).}$$

$T : \mathbb{R}^3 \to \mathbb{R}^2$

(1.) Eg: $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$\dim \mathbb{R}^3 = \dim R(T) + \dim N(T)$

$N(T) = \{(a_1, a_1, 0) = a_1(1,1,0) / a_1 \in \mathbb{R}\}$.

$\{(1,1,0)\}$ is a basis for $N(T)$

$\Rightarrow \underline{3 = 2+1}$.

P:01 $T: \underline{\mathbb{R}^3} \to \underline{\mathbb{R}^3}$, $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$

find basis, dim of $R(T)$ & $N(T)$.

~~$R(T) = \{(x+2y-z, y+z, x+y-2z)\}$~~

$R(T) = \{x(1,0,1) + y(2,1,1) + z(-1,1,-2) / x,y,z \in \mathbb{R}\}$.

$= \text{Span}\{(1,0,1), (2,1,1), (-1,1,-2)\}$.

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{pmatrix}$$

$R_2 = R_2 - 2R_1 \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & -2 \end{pmatrix}$

$R_3 = R_1 + R_3 \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$

$R_3 = R_3 - R_2 \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\therefore$ Basis for $R(T) = \{(1,0,1), (0,1,-1)\}$

$\boxed{\dim(R(T)) = 2.}$

Now;

$N(T) = \{(x,y,z) \mid T(x,y,z) = (0,0,0)\}$

soln of $x + 2y - z = 0 \Rightarrow x = -(2y - z) \Rightarrow x = -(-2z - z)$

$y + z = 0 \Rightarrow y = -z$ $\quad x = 3z$

$x + y + 2z = 0$

$x + y + 2z = 0$

$3z - z + 2z = 0$

$5z - z = 0$

$x + 2y - z = 0 \quad$ — ①

$y + z = 0 \quad$ — ②

$x + y + 2z = 0 \quad$ — ③

$\Rightarrow$ ① − ② = ③

$x + 2y - z = 0$

$y + z = 0$

$\Rightarrow \boxed{y = -z}$

so, $x + 2(-z) - z = 0$

$x - 3z = 0$

$\boxed{x = 3z}$

$\Rightarrow N(T) = \{(x, y, z) = (3z, -z, z) = z(3, -1, 1) \mid z \in \mathbb{R}\}$.

$= \text{span}\{(3, -1, 1)\}$

$\{(3, -1, 1)\}$ is L.I.

so Basis of $N(T) = \{(3, -1, 1)\}$

so; $\boxed{\dim N(T) = 1}$