LPP (primal) $\longleftrightarrow$ LPP (Dual)

Canonical form: A LPP written in the following form is said to be LPP in canonical form.

$$\text{Max } Z = C^T X \quad \text{s.t.} \quad AX \le b \;;\; X \ge 0$$

OR

$$\text{Min } Z = C^T X \quad \text{s.t.} \quad AX \ge b \;;\; X \ge 0$$

Rules for constructing Dual

Primal problem / constraint		Dual problem objective / constraint	
Max	$\le$	Min	$\ge$
Min	$\ge$	Max	$\le$

Primal $\Longleftrightarrow$ **Dual**

$$\begin{array}{lcl} \text{Max } Z = C^T X & & \text{Min } Z^* = b^T Y \\ \text{s.t. } AX \le b & \Longleftrightarrow & \text{s.t. } A^T Y \ge C \\ X \ge 0 & & Y \ge 0 \end{array}$$

Note: Dual of a dual is primal.

Pb1. Write the dual of the LPP

$$\text{Max } Z = x_1 + 2x_2 - x_3$$

$$\text{s.t.} \quad 4x_1 + x_2 - x_3 \le 6 \qquad y_1$$

$$x_1 - 2x_2 \ge 5 \quad (-1) \quad y_2$$

$$x_1, x_2 \ge 0$$

Sol:- **Dual**

$$\text{Min } z = 6y_1 - 5y_2$$

$$4y_1 - y_2 \ge 1$$

$$y_1 + 2y_2 \ge 2$$

$$-y_1 \ge -1 \quad : \quad y_1, y_2, y_3 \ge 0.$$

Pb2

$$\text{Min } Z = x_1 + 4x_2 + 2x_3$$

s.t.

y_1: $x_1 - 4x_2 + x_3 \geq 5$

y_2: (-1) $2x_1 - x_3 \leq 6$

y_3: $x_1 + x_2 + x_3 \geq 1$

$x_1, x_2, x_3 \geq 0$;

Dual

$$\text{Max } Z^* = 5y_1 - 6y_2 + y_3$$

s.t.

$y_1 - 2y_2 + y_3 \leq 1$

$-4y_1 + y_3 \leq 4$

$y_1 + y_2 + y_3 \leq 2$

$y_1, y_2, y_3 \geq 0$

Pb3

$$\text{Max } Z = 6x_1 + 4x_2$$

s.t. $x_1 + 2x_2 \leq 4$

$x_1 - 6x_2 \geq 7$

$x_1, x_2 \geq 0$

Dual

$$\text{Min } Z^* = 4y_1 - 7y_2$$

s.t.

$y_1 - y_2 \geq 6$

$2y_1 + 6y_2 \geq 4$

$y_1, y_2 \geq 0$

OR

$$\text{Max}(-Z^*) = -4y_1 + 7y_2$$

s.t.

$-y_1 + y_2 \leq -6$

$-2y_1 - 6y_2 \leq -4$

$y_1, y_2 \geq 0$

Pb4 Write dual of

$$\text{Min } Z = x_1 + 4x_2 - x_3$$

s.t. $x_1 - 2x_2 + 2x_3 \leq 6$

$2x_1 + 4x_2 - x_3 \geq 5$

$x_1 + x_2 = 4$

$x_1, x_2, x_3 \geq 0$

$\longrightarrow x_1 + x_2 \geq 4$

$x_1 + x_2 \leq 4$

Sol

$$\text{Min } Z = x_1 + 4x_2 - x_3$$

s.t.

$x_1 - 2x_2 + 2x_3 \leq 6$ (-1) y_1

$2x_1 + 4x_2 - x_3 \geq 5$ y_2

$x_1 + x_2 \geq 4$ y_3'

$x_1 + x_2 \leq 4$ (-1) y_3''

$\Downarrow$ Dual

$$\text{Max } Z^* = -6y_1 + 5y_2 + 4y_3' - 4y_3''$$

s.t.

$-y_1 + 2y_2 + y_3' - y_3'' \leq 1$

$2y_1 + 4y_2 + y_3' - y_3'' \leq 4$

$-2y_1 - y_2 \leq -1$

$y_1, y_2, y_3', y_3'' \geq 0$

OR

Dual

$$\text{Max } Z^* = -6y_1 + 5y_2 + 4y_3$$

s.t.

$-y_1 + 2y_2 + y_3 \leq 1$

$2y_1 + 4y_2 + y_3 \leq 4$

$-2y_1 - y_2 \leq -1$

$y_1, y_2 \geq 0$; $y_3 = y_3' - y_3''$

$y_3 \longrightarrow$ unrestricted.

Pb 5.

$$\text{Min } Z = x_1 + 2x_2$$

s.t.

$$4x_1 - 5x_2 = 4 \qquad y_1$$
$$x_1 + x_2 \geq 7 \qquad y_2$$
$$x_1, x_2 \geq 0$$

Dual

$$\text{Max } Z^* = 4y_1 + 7y_2$$

s.t.

$$4y_1 + y_2 \leq 1$$
$$-5y_1 + y_2 \leq 2$$

y_1 – unrest. ; $y_2 \geq 0$

Pb 6.

$$\text{Min } Z = x_1 + 2x_2$$

s.t.

$$4x_1 - 5x_2 \geq 4$$
$$x_1 + x_2 \geq 7$$

x_1 – unrest.

$x_2 \geq 0$

Dual

$$\text{Max } Z^* = 4y_1 + 7y_2$$

s.t.

$$4y_1 + y_2 = 1$$
$$-5y_1 + y_2 \leq 2$$
$$y_1, y_2 \geq 0$$

Pb 7.

$$\text{Max } Z = 6x_1 - 5x_2 + 7x_3 + x_4$$

y_1 s.t. $2x_1 + 4x_2 - x_3 + x_4 \leq 4$

$+ 6x_3 + 7x_4 \geq 5$

y_2 (-1) $x_1 - x_2 \qquad + 5x_4 = 6$

y_3 $2x_1 + 2x_2 + 4x_3 \qquad = 7$

y_4 $x_1 + 8x_2 + x_3$

x_1, x_4 – unrest.

$x_2, x_3 \geq 0$

Dual

$$\text{Min } Z^* = 4y_1 - 5y_2 + 6y_3 + 7y_4$$

s.t.

$$2y_1 - y_2 + 2y_3 + y_4 = 6$$
$$4y_1 + y_2 + 2y_3 + 8y_4 \geq -5$$
$$-y_1 - 6y_2 + 4y_3 + y_4 \geq 7$$
$$y_1 - 7y_2 + 5y_3 = 1$$

$y_1 ; y_2 \geq 0$

$y_3 ; y_4$ — unrest.

Scanned with CamScanner

Weak Duality Theorem :-

Thm : Consider the LPP (primal) Max $Z = C^T X$

s.t. $AX \le b$; $X \ge 0$ and

Dual Min $Z^* = b^T Y$ s.t. $A^T Y \ge C$; $Y \ge 0$

Let X be the feasible solution of the primal and Y be the feasible solution of the dual.

Then

$$C^T X \le b^T Y$$

(Max) or $(Z \le Z^*)$ (Min.)

Cor : Let $\bar{x}$ be feasible the primal and $\bar{y}$ be feasible for the dual. Also, let $C^T \bar{x} = b^T \bar{y}$.

Then $\bar{x}$ is optimal to the primal and $\bar{y}$ is optimal to the dual.

Strong Duality Theorem

i) Let $\bar{x}$ be an optimal solⁿ of the primal. Then there exist a $\bar{y}$ which is optimal to the dual. Also $C^T \bar{x} = b^T \bar{y}$.

ii) Let y^* be an optimal solution of the dual. Then $\exists$ x^* which is optimal to the primal. Also $b^T y^* = C^T x^*$.

How to find solution of Dual problem from Primal problem.

Dual optimal solution - Formula

Primal LPP

$\text{Max } Z = C^T X$

s.t. $Ax = b$

$X \geq 0$

$\longrightarrow$

Dual LPP

$\text{Min } Z^* = b^T Y$

s.t. $A^T Y \geq C$

Y - unrest. in sign.

Cost of B.V → C_B from the optimal table

B (Basis Matrix) → find → B^{-1} from the optimal table

$$\boxed{Y_B^T = C_B^T B^{-1}}$$

Y_B → Dual variables (Dual optimal sol^n.)

Pb 1. Write the dual of the following LPP (primal)

$\text{Max } Z = 4x_1 + 3x_2$

s.t. $x_1 + x_2 \overset{+S_1}{\leq} 8$

$2x_1 + x_2 \overset{+S_2}{\leq} 10$

$x_1, x_2 \geq 0$

And find the solution of dual by solving the primal.

Sol

starting B.V: S_1, S_2

First simplex Table / last simplex Table:

B.V	x_1	x_2	S_1	S_2	X_B
Z	-4	-3	0	0	0
S_1	1	1	1	0	8
S_2	2	1	0	1	10
Z	0	0	2	1	26
$3\ x_2$	0	1	2	-1	6
$4\ x_1$	1	0	-1	1	2

$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ " B^{-1}

$B.V = (x_2, x_1)$

$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad B^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$\boxed{\text{Max } Z = 26 \text{ at } x_1 = 2;\ x_2 = 6}$

Dual

$\text{Min } Z^* = 8y_1 + 10y_2$

s.t. $y_1 + 2y_2 \geq 4$

$y_1 + y_2 \geq 3$

$y_1, y_2 \geq 0$

$Y_B^T = (y_1, y_2) = C_B^T B^{-1}$

$= [3\ \ 4]\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = [2\ \ 1]$

$$\boxed{\begin{aligned} \text{Min } Z^* &= 8 \times 2 + 10 \times 1 \\ &= 26 \\ \text{at } y_1 &= 2;\ y_2 = 1 \end{aligned}}$$

optimal.

Pb2. Write the dual of the following LPP and also find the solution of dual by solving the primal. (See - Case 3 of Big-M Method)

$$\text{Max } Z = -2x_1 - x_2$$

s.t.
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3 \quad ; \quad x_1, x_2 \geq 0$$

Sol^n

St. form

$$\text{Max } Z = -2x_1 - x_2 + 0\cdot S_1 + 0\cdot S_2$$

s.t.

y_1: $3x_1 + x_2 = 3$

y_2: $4x_1 + 3x_2 - S_1 = 6$

y_3: $x_1 + 2x_2 + S_2 = 3$

Dual

$$\text{Min } Z^* = 3y_1 + 6y_2 + 3y_3$$

s.t.
$$3y_1 + 4y_2 + y_3 \geq -2$$
$$y_1 + 3y_2 + 2y_3 \geq -1$$
$$-y_2 \geq 0 \Rightarrow y_2 \leq 0$$
$$y_3 \geq 0, \; y_1 \text{ - unrest.}$$

$\downarrow$ Add artificial variable.

$$\text{Max } Z = -2x_1 - x_2 + 0\cdot S_1 + 0\cdot S_2 - Ma_1 - Ma_2$$

s.t.
$$3x_1 + x_2 + a_1 = 3$$
$$4x_1 + 3x_2 - S_1 + a_2 = 6$$
$$x_1 + 2x_2 + S_2 = 3$$
$$x_1, x_2, S_1, S_2, a_1, a_2 \geq 0$$

Initial B.V

first table

	c_j	-2	-1	0	M	M	0	
	B.V	x_1	x_2	S_1	a_1	a_2	S_2	X_B
	Z	-7M+2	-4M+1	M	0	0	0	-9M
-M	a_1	3	1	0	1	0	0	3
-M	a_2	4	3	-1	0	1	0	6
0	S_2	1	2	0	0	0	1	3

Last table

	B.V	x_1	x_2	S_1	a_1	a_2	S_2	X_B
	Z	0	0	1/5	M-2/5	M-1/5	0	-12/5
-2	x_1	1	0	1/5	3/5	-1/5	0	3/5
-1	x_2	0	1	-3/5	-4/5	3/5	0	6/5
0	S_2	0	0	1	1	-1	1	0

B^{-1} (columns a_1, a_2, S_2 of the last table)

$$\text{B.V.} = (x_1, x_2, S_2)$$

$$B = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\begin{array}{cccc} & a_1 & a_2 & S_2 \\ B^{-1} = & \left(\begin{matrix} 3/5 \\ -4/5 \\ 1 \end{matrix}\right. & \begin{matrix} -1/5 \\ 3/5 \\ -1 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right) \end{array}$$

$$C_B^T = [-2 \quad -1 \quad 0]$$

$$Y_B^T = C_B^T B^{-1}$$
$$= \left(-2/5 \quad -1/5 \quad 0\right)$$

$$\text{Min } Z^* = 3\left(-\tfrac{2}{5}\right) + 6\left(-\tfrac{1}{5}\right) + 3\cdot 0$$
$$= -\frac{12}{5}$$

at $y_1 = -2/5$; $y_2 = -1/5$; $y_3 = 0$.

Existence Theorem :-

(finite optimal solution)

1. If primal and dual both have feasible solutions then both have optimal solution (same optimum value).
2. If primal/(dual) has unbounded solutions then the dual (primal) has no feasible solution or has infeasible solⁿ.
3. If primal (dual) has no feasible solution then the dual (primal) has unbounded or infeasible solution.

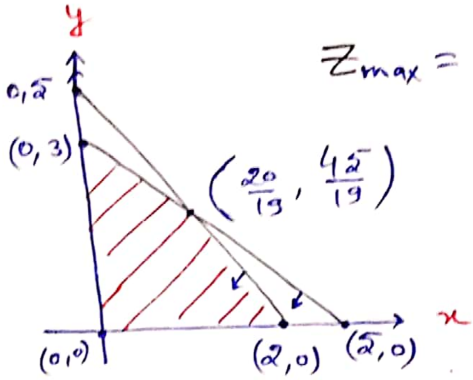
Pb1. Primal LPP :-

$$\text{Max } Z = 5x_1 + 3x_2$$

s.t. $3x_1 + 5x_2 \leq 15 \leftarrow w_1$

$5x_1 + 2x_2 \leq 10 \leftarrow w_2$

$x_1, x_2 \geq 0$



$Z_{max} = \frac{235}{19}$ at $(\frac{20}{19}, \frac{45}{19})$

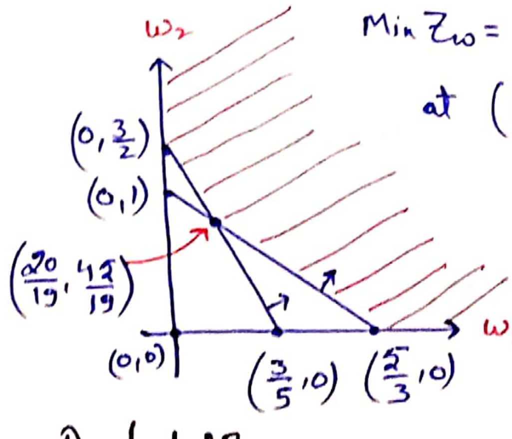
Dual

$$\text{Min } Z_w = 15w_1 + 10w_2$$

s.t. $3w_1 + 5w_2 \geq 5$

$5w_1 + 2w_2 \geq 3$

$w_1, w_2 \geq 0$



$\text{Min } Z_w = \frac{235}{19}$ at $(\frac{20}{19}, \frac{45}{19})$

Pb2.

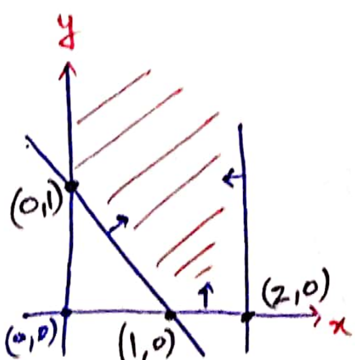
Primal LPP

$\text{Max } Z = x_1 + 4x_2$

s.t. $-x_1 - x_2 \leq -1$

$x_1 \leq 2$

$x_1, x_2 \geq 0$



$\text{Max } Z \to \infty$ (unbounded solⁿ)

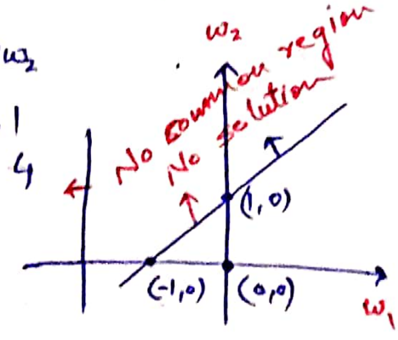
Dual LPP

$\text{Min } Z_w = -w_1 + 2w_2$

s.t. $-w_1 + w_2 \geq 1$

$-w_1 \geq 4$

$w_1, w_2 \geq 0$



No feasible solution / Infeasible Solⁿ

Pb3.

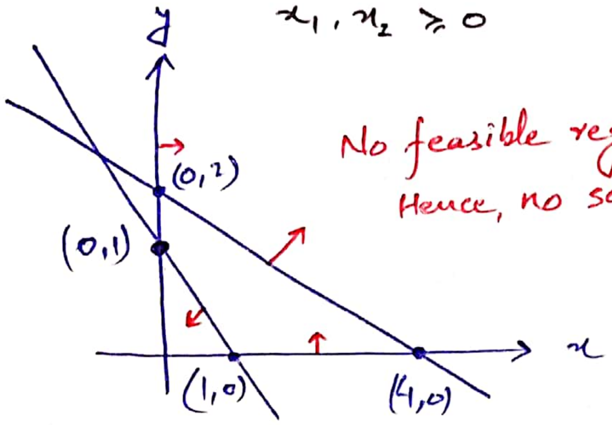
Primal LPP

$$\text{Max } Z_x = 4x_1 + 3x_2$$

s.t. $x_1 + x_2 \le 1 \leftarrow w_1$

$x_1 + 2x_2 \ge 4 \leftarrow w_2$

$x_1, x_2 \ge 0$



No feasible region

Hence, no solution

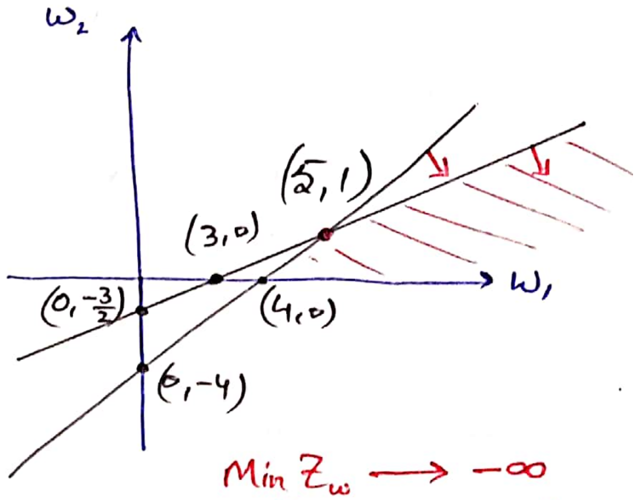
Dual LPP

$$\text{Min } Z_w = w_1 - 4w_2$$

s.t. $w_1 - w_2 \ge 4$

$w_1 - 2w_2 \ge 3$

$w_1, w_2 \ge 0$



Min $Z_w \longrightarrow -\infty$

(unbounded sol^n)

Pb4.

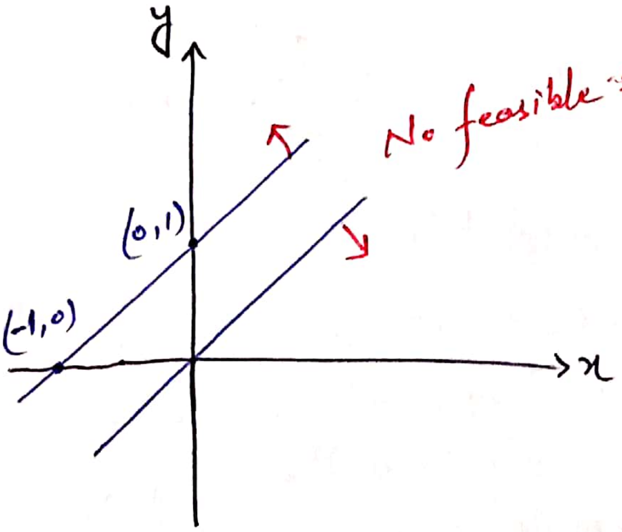
Primal LPP

$$\text{Max } Z_x = 3x_1 + 4x_2$$

s.t. $x_1 - x_2 \le -1 \leftarrow w_1$

$-x_1 + x_2 \le 0 \leftarrow w_2$

$x_1, x_2 \ge 0$



No feasible sol^n

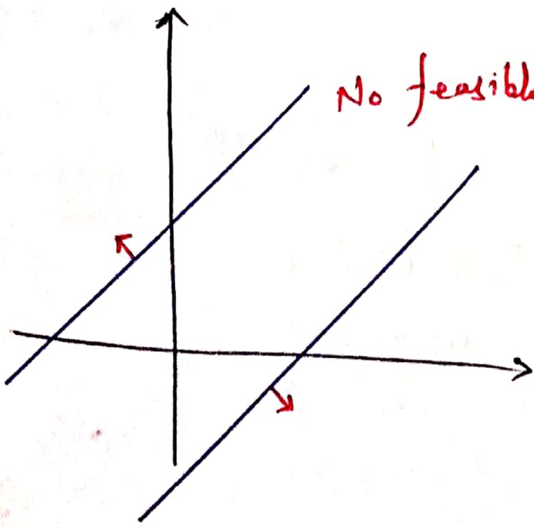
Dual LPP

$$\text{Min } Z_w = -w_1$$

s.t. $w_1 - w_2 \ge 3$

$-w_1 + w_2 \ge 4$

$w_1, w_2 \ge 0$



No feasible sol^n

Dual Simplex Method :-

L.P.P $\begin{cases} \text{Max/Min } Z = C^T X \\ AX \leq, =, \geq b \\ X \geq 0 \end{cases}$ $\xrightarrow{\text{Simplex Method}}$

- LPP must be in std. form
- R.H.S of constraints ≥ 0

optimality is achieved in next iteration

B.V	x_1 x_2 . . . s_1, s_2 . .	X_B
	$z_j - c_j$	
s_1	Scalars	$X_{B_1} \geq 0$
s_2		$X_{B_2} \geq 0$

feasibility is already maintained

optimality Criteria

$z_j - c_j \geq 0 \ \forall j$ (Max.)

$z_j - c_j \leq 0 \ \forall j$ (Min.)

Dual Simplex Method.

$z_j - c_j \geq 0$ (Max.) } Already Maintained.

$z_j - c_j \leq 0$ (Min.)

B.V.		X_B
	Current $z_j - c_j$ is optimal	
		X_{B_1}
		X_{B_2}
		$\vdots$

feasibility is disturbed.

Algorithm

1. Convert all the inequalities of '$\geq$' type into '$\leq$', and write the problem in maximization form with the constraints as equality by introducing only slack variables.
2. In general it is not possible to find a starting basic feasible solution to the L.P.P with all $z_j - c_j \geq 0$.

3. – To find the outgoing variable first

$$x_{Br} = \min \{ x_{Bi} \; ; \; x_{Bi} < 0 \}$$

i.e. the most –ve value.

– if atleast one α_{rj} is –ve then we find incoming variable using.

$$\text{Min} \left\{ \left| \frac{z_j - c_j}{\alpha_{ij}} \right| \; ; \; \alpha_{ij} < 0 \right\}$$

4. If all $\alpha_{ij} \geq 0$ in any row corresponding to a –ve x_{Bi} then the LPP has no feasible solⁿ.

	x_1	x_2	x_3	x_4	X_B
x_4	1	0	2	3	–4

Note: Dual Simplex Method cannot give unbounded solution.

5. When we get the set of basic feasible solution we stop the process and ~~these~~ this solution is our optimal solution.

Pb! Use dual simplex method to solve the given LPP.

$$\text{Max } Z = -2x_1 - x_2$$

s.t. $2x_1 - x_2 - x_3 \geq 3$

$x_1 - x_2 + x_3 \geq 2$; $x_1, x_2, x_3 \geq 0$

Sol:

$\text{Max } Z = -2x_1 - x_2$

s.t. $-2x_1 + x_2 + x_3 \leq -3$

$-x_1 + x_2 - x_3 \leq -2$

std. form

$\text{Max } Z = -2x_1 - x_2 + 0 \cdot x_3 + 0 \cdot S_1 + 0 \cdot S_2$

s.t.

$-2x_1 + x_2 + x_3 + S_1 = -3$

$-x_1 + x_2 - x_3 + S_2 = -2$

C_B	B.V	x_1 (−2)	x_2 (−1)	x_3 (0)	S_1 (0)	S_2 (0)	X_B
	Z	2	1	0	0	0	0
0	S_1	$\boxed{-2}$	1	1	1	0	−3
0	S_2	−1	1	−1	0	1	−2
		0	2	1	1	0	−3
	x_1	1	−1/2	−1/2	−1/2	0	3/2
	S_2	0	1/2	$\boxed{-3/2}$	−1/2	1	−1/2
		0	5/3	0	2/3	2/3	−10/3
	x_1	1	−2/3	0	−2/3	−1/3	5/3
	x_3	0	−1/3	1	1/3	−2/3	1/3

Current table satisfies the optimality criteria $z_j - c_j \geq 0 \ \forall j$

Remove (most −ve value)

feasibility is distubbuted.

First : Remove/leaving var

Second : Enter. variable

$\text{Min}\left\{ \left| \frac{z_j - c_j}{\alpha_{ij}} \right| ; \alpha_{ij} < 0 \right\}$

$= \text{Min}\left\{ \left| \frac{2}{-2} \right| \right\}$

$= 1$

$\text{Min} = \left\{ \left| \frac{1}{-3/2} \right| ; \left| \frac{1}{-1/2} \right| \right\}$

$= \frac{2}{3}$

feasible. (stop).

$$\text{Max } Z = -10/3$$

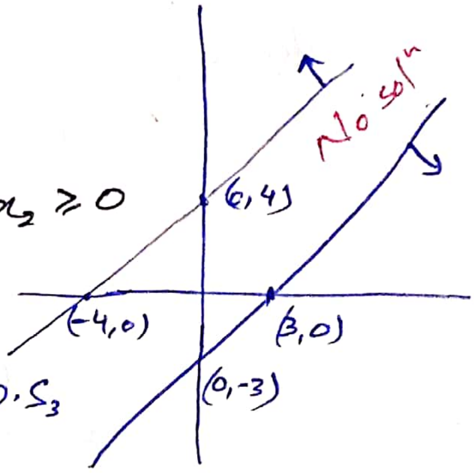
at $x_1 = 5/3$; $x_2 = 0$; $x_3 = 1/3$

Pb 2. Use dual-simplex method to solve LPP:

$$\text{Max } Z = -x_1$$

s.t. $x_1 - x_2 \geq 3$

$-x_1 + x_2 \geq 4$; $x_1, x_2 \geq 0$



Sol^n:

Std form:

$$\text{Max } Z = -x_1 + 0 \cdot x_2 + 0 \cdot S_1 + 0 \cdot S_3$$

s.t. $-x_1 + x_2 + S_1 = -3$

$x_1 - x_2 + S_2 = -4$; $x_1, x_2, S_1, S_2 \geq 0$

B.V	x_1	x_2	S_1	S_2	X_B
	1	0	0	0	0
S_1	-1	1	1	0	-3
S_2	1	$\boxed{-1}$	0	1	-4
	1	0	0	0	
S_1	0	0	1	1	-7
x_2	-1	1	0	-1	4

The current LPP has Infeasible sol^n since current table is optimal but not feasible.

leaving variable but no entering variable.

$$\text{Max } Z = -x_1 + 0 \cdot x_2 - Ma_1 - Ma_2$$

s.t. $x_1 - x_2 - S_1 + a_1 = 3$

$-x_1 + x_2 - S_2 + a_2 = 4$

B.V	x_1	x_2	S_1	S_2	a_1	a_2

Pb 3 Solve LPP ; using Dual Simplex Method.

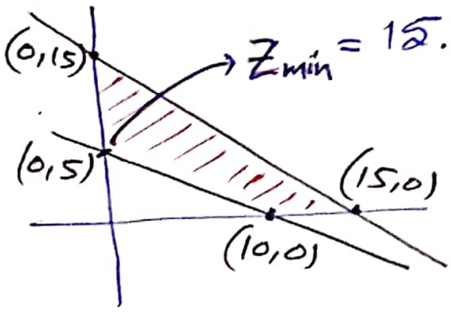
$$\text{Min } Z = 2x_1 + 3x_2$$
$$\text{s.t. } 2x_1 + 2x_2 \leq 30$$
$$x_1 + 2x_2 \geq 10$$
$$x_1, x_2 \geq 0$$

OR

$$\text{Min } Z = 2x_1 + 3x_2$$
s.t.
$$2x_1 + 2x_2 + S_1 = 30$$
$$x_1 + 2x_2 - S_2 = 10$$
$$x_1, x_2, S_1, S_2 \geq 0$$

OR Std. form

$$\text{Min } Z = 2x_1 + 3x_2$$
s.t.
$$2x_1 + 2x_2 + S_1 = 30$$
$$-x_1 - 2x_2 + S_2 = -10$$
$$x_1, x_2, S_1, S_2 \geq 0$$



B.V	x_1	x_2	S_1	S_2	X_B
Z	-2	-3	0	0	0
S_1	2	2	1	0	30
S_2	-1	$\boxed{-2}$	0	1	-10 ←
	$-1/2$	0	0	$-3/2$	15
S_1	1	0	1	1	20
x_2	$1/2$	1	0	$1/2$	5

$$\text{Min}\left\{\left|\frac{-2}{-1}\right|, \left|\frac{-3}{-2}\right|\right\} = \frac{3}{2}$$

$z_j - c_j \leq 0 \;\forall j$ (optimality)

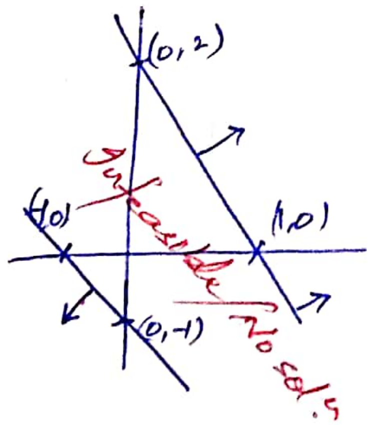
} feasibility criteria satisfied.

$$\boxed{Z_{min} = 15 \,;\; \text{at } x_1 = 0 \,;\; x_2 = 5}$$

Pb 4.

$$\text{Min. } Z = x_1 + x_2$$
$$\text{s.t. } 2x_1 + x_2 \geq 2$$
$$-x_1 - x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

$$\text{Min } Z = x_1 + x_2$$
s.t.
$$-2x_1 - x_2 + S_1 = -2$$
$$x_1 + x_2 + S_2 = -1$$
$$x_1, x_2, S_1, S_2 \geq 0$$



B.V	x_1	x_2	S_1	S_2	X_B
Z					
S_1	-2	-1	1	0	-2 ← leaving variable
S_2	1	1	0	1	-1 ← but corresponding to this all $\alpha_{ij} \geq 0$ (Step 4)

Hence infeasible sol^n.

Scanned with CamScanner