Constrained optimization problem with inequality constraints. Let f and $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, \ldots, m$ be continuously differentiable functions. Consider the problem

$$\text{minimize/maximize } f(\mathbf{x}) \qquad \text{(CI)}$$
$$\text{subject to } g_i(\mathbf{x}) \leq 0 \text{ for } i = 1, 2, \ldots, m.$$

The inequality constraints in (CI) can be transformed to equality constraints by adding nonnegative slack variables, y_i^2, as

$$g_i(\mathbf{x}) + y_i^2 = 0 \text{ for } i = 1, 2, \ldots, m.$$

Now the problem can be solved by the method of Lagrange's multipliers. We can solve the constrained optimization problems with inequality constraints using Kuhn-Tucker conditions under certain circumstances.

Convex/concave functions. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if

$$f[\lambda\,\mathbf{x} + (1-\lambda)\,\mathbf{y}] \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \quad \lambda \in (0, 1).$$

A function f is concave if and only if $-f$ is convex.

Condition for convex/concave functions. Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable.

- If the Hessian matrix of f is positive semi-definite, then f is convex.
- If the Hessian matrix of f is negative semi-definite, then f is concave.

Condition for a matrix to be positive semi-definite or negative semi-definite. Let A be a symmetric matrix of order n, and $A_1, A_2, \ldots, A_n$ be its leading principal minors.

- If $A_1 \geq 0,\ A_2 \geq 0, \ldots,\ A_n \geq 0$, then A is positive semi-definite.
- If $A_1 \leq 0,\ A_2 \geq 0,\ A_3 \leq 0, \ldots$, then A is negative semi-definite.
- If all eigenvalues of A are non-negative, then A is positive semi-definite.
- If all eigenvalues of A are non-positive, then A is negative semi-definite.

Quadratic form. A function of the form $\sum_{i,j=1}^{n} a_{ij} x_i x_j$ is said to be in quadratic form in $x_1, x_2, \ldots, x_n$. For example, $x_1^2 + x_1 x_2$, $x_1^2 + 2x_3^2 + 3x_2 x_3$ are quadratic forms. If $A = (a_{ij})$ and $\mathbf{x} = (x_1, \ldots, x_n)^T$, then $\mathbf{x}^T A\,\mathbf{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j$. Matrices corresponding to $x_1^2 + x_1 x_2$ and $x_1^2 + 2x_3^2 + 3x_2 x_3$ are, respectively,

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3/2 \\ 0 & 3/2 & 2 \end{bmatrix}.$$

The order of the matrix A is the number of variables available in the concerned problem. If H is the Hessian matrix of $\sum_{i,j=1}^{n} a_{ij} x_i x_j$, then $H = 2A$.

Convex programming problem. There are four types of convex programming problems.

(a) Minimize $f(x)$ subject to $g_i(x) \le 0$ ($i = 1, 2, \ldots, m$), where f and all g_i are convex.

(b) Maximize $f(x)$ subject to $g_i(x) \le 0$ ($i = 1, 2, \ldots, m$), where f is concave and all g_i are convex.

(c) Minimize $f(x)$ subject to $g_i(x) \ge 0$ ($i = 1, 2, \ldots, m$), where f is convex and all g_i are concave.

(d) Maximize $f(x)$ subject to $g_i(x) \ge 0$ ($i = 1, 2, \ldots, m$), where f and all g_i are concave.

Kuhn-Tucker conditions. Let f and $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, 2, \ldots, m$ be continuously differentiable functions.

Problem type	Necessary condition for critical point	Sufficient condition for optimal point	Conclusion for optimal point
minimize $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ Subject to $g_i(\mathbf{x}) \le 0$ $(1 \le i \le m)$	$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} = 0 \quad (1 \le j \le n)$ $\lambda_i g_i(\mathbf{x}) = 0 \quad (1 \le i \le m)$ $g_i(\mathbf{x}) \le 0 \quad (1 \le i \le m)$ $\lambda_i \ge 0 \quad (1 \le i \le m)$	f is convex All g_i are convex	Global minimum
maximize $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ Subject to $g_i(\mathbf{x}) \le 0$ $(1 \le i \le m)$	$\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} = 0 \quad (1 \le j \le n)$ $\lambda_i g_i(\mathbf{x}) = 0 \quad (1 \le i \le m)$ $g_i(\mathbf{x}) \le 0 \quad (1 \le i \le m)$ $\lambda_i \ge 0 \quad (1 \le i \le m)$	f is concave All g_i are convex	Global maximum
minimize $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ Subject to $g_i(\mathbf{x}) \ge 0$ $(1 \le i \le m)$	$\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} = 0 \quad (1 \le j \le n)$ $\lambda_i g_i(\mathbf{x}) = 0 \quad (1 \le i \le m)$ $g_i(\mathbf{x}) \ge 0 \quad (1 \le i \le m)$ $\lambda_i \ge 0 \quad (1 \le i \le m)$	f is convex All g_i are concave	Global minimum
maximize $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ Subject to $g_i(\mathbf{x}) \ge 0$ $(1 \le i \le m)$	$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} = 0 \quad (1 \le j \le n)$ $\lambda_i g_i(\mathbf{x}) = 0 \quad (1 \le i \le m)$ $g_i(\mathbf{x}) \ge 0 \quad (1 \le i \le m)$ $\lambda_i \ge 0 \quad (1 \le i \le m)$	f is concave All g_i are concave	Global maximum

Example. Solve the problem

$$\begin{array}{ll} \text{maximize } 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2 & \\ \text{subject to} \qquad\qquad\quad x_2 \le 8, & \\ \qquad\qquad\qquad\quad x_1 + x_2 \le 10, & \end{array}$$

using Kuhn-Tucker conditions.

Solution. Let $f(x_1, x_2) = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$, and $g_1(x_1, x_2) = x_2 - 8$ and $g_2(x_1, x_2) = x_1 + x_2 - 10$.

Notice that $\frac{\partial f}{\partial x_1} = 12 + 2x_2 - 4x_1$, $\frac{\partial f}{\partial x_2} = 2 + 2x_1 - 4x_2$, $\frac{\partial^2 f}{\partial x_1^2} = -4$, $\frac{\partial^2 f}{\partial x_2^2} = -4$, $\frac{\partial^2 f}{\partial x_1 x_2} = 2$.

So the Hessian matrix is

$$\begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix},$$

which is negative definite. Therefore f is strictly concave. Now Hessian matrices of g_1, g_2 are zero matrices, which are positive semi-definite. So g_1 and g_2 are convex functions. So the given problem is a convex programming problem for maximization, and it satisfies Kuhn-Tucker sufficient conditions.

The Kuhn-Tucker necessary conditions are

$$\begin{aligned}
&(a) \quad \frac{\partial f}{\partial x_j} - \lambda_1 \frac{\partial g_1}{\partial x_j} - \lambda_2 \frac{\partial g_2}{\partial x_j} = 0 \quad (j = 1, 2) \quad \Rightarrow \quad && 12 + 2x_2 - 4x_1 - \lambda_2 = 0, \\
& && 21 + 2x_1 - 4x_2 - \lambda_1 - \lambda_2 = 0, \\
&(b) \quad \lambda_i g_i(x_1, x_2) = 0 \quad (i = 1, 2) \quad \Rightarrow \quad && \lambda_1(x_2 - 8) = 0, \\
& && \lambda_2(x_1 + x_2 - 10) = 0, \\
&(c) \quad g_i(x_1, x_2) \leq 0 \quad (i = 1, 2) \quad \Rightarrow \quad && x_2 - 8 \leq 0, \\
& && x_1 + x_2 - 10 \leq 0, \\
&(d) \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0.
\end{aligned}$$

- **Case-1:** $\lambda_1, \lambda_2 \neq 0$**.** Then $x_1 = 2$ and $x_2 = 8$ which imply from conditions (a) that $\lambda_1 = -27$ and $\lambda_2 = 20$. It does not satisfy (d).

- **Case-2:** $\lambda_1 \neq 0, \lambda_2 = 0$**.** Then $x_2 = 8$. Now first condition of (a) implies that $x_1 = 7$. It does not satisfy $x_1 + x_2 - 10 \leq 0$.

- **Case-3:** $\lambda_1 = \lambda_2 = 0$**.** Then from conditions (a), we have $4x_1 - 2x_2 - 12 = 0$ and $2x_1 - 4x_2 + 21 = 0$, that is, $x_1 = \frac{15}{2}$ and $x_2 = 9$. It does not satisfy (d).

- **Case-4:** $\lambda_1 = 0, \lambda_2 \neq 0$**.** Then $x_1 + x_2 = 10$, and conditions (a) imply that

  $$12 + 2x_2 - 4x_1 - \lambda_2 = 0,\ 21 + 2x_1 - 4x_2 - \lambda_2 = 0 \Rightarrow x_2 - x_1 = \frac{3}{2}.$$

  So we have $x_1 = \frac{17}{4}$, $x_2 = \frac{23}{4}$ and $\lambda_2 = \frac{13}{2}$. It does not violate any Kuhn-Ticker condition. So an optimal solution is given by $x_1 = \frac{17}{4}$, $x_2 = \frac{23}{4}$, $\lambda_1 = 0$ and $\lambda_2 = \frac{13}{2}$. Therefore the maximum value of the objective function is $f\left(\frac{17}{4}, \frac{23}{4}\right) = \frac{947}{8}$.

$\square$

Note. Without convexity assumptions on f and g_i, the Kuhn-Tucker conditions are not sufficient for a point to be a local minimum or global minimum point. For example, consider the problem

$$\begin{aligned}
\text{minimize } & -x_2 \\
\text{subject to } & \quad x_1^2 + x_2^2 \leq 4, \\
& -x_1^2 + x_2 \leq 0.
\end{aligned}$$

Let $f(x_1, x_2) = -x_2$, and $g_1(x_1, x_2) = x_1^2 + x_2^2 - 4$ and $g_2(x_1, x_2) = -x_1^2 + x_2$. The Kuhn-Tucker necessary conditions are

$$(a) \qquad \frac{\partial f}{\partial x_j} + \lambda_1 \frac{\partial g_1}{\partial x_j} + \lambda_2 \frac{\partial g_2}{\partial x_j} = 0 \quad (j = 1, 2) \qquad \Rightarrow \qquad \begin{aligned} 2\lambda_1 x_1 - 2\lambda_2 x_2 &= 0, \\ -1 + 2\lambda_1 x_2 + \lambda_2 &= 0, \end{aligned}$$

$$(b) \qquad \lambda_i g_i(x_1, x_2) = 0 \quad (i = 1, 2) \qquad \Rightarrow \qquad \begin{aligned} \lambda_1(x_1^2 + x_2^2 - 4) &= 0, \\ \lambda_2(-x_1^2 + x_2) &= 0, \end{aligned}$$

$$(c) \qquad g_i(x_1, x_2) \le 0 \quad (i = 1, 2) \qquad \Rightarrow \qquad \begin{aligned} x_1^2 + x_2^2 - 4 &\le 0, \\ -x_1^2 + x_2 &\le 0, \end{aligned}$$

$$(d) \qquad \lambda_1 \ge 0, \quad \lambda_2 \ge 0.$$

The point $(0, 0)$ satisfies the Kuhn-Tucker necessary conditions with $\lambda_1 = 0$ and $\lambda_2 = 1$, but it is neither local minimum nor global minimum point. Here g_2 is concave.

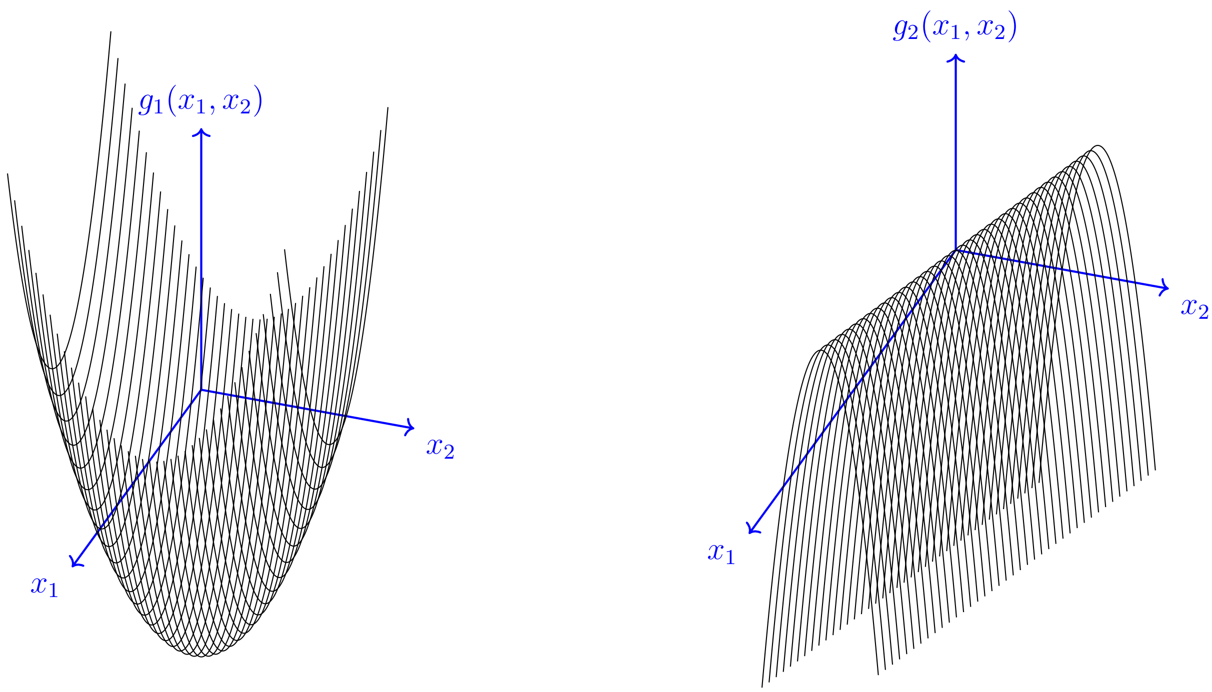

Figure 3: $g_1(x_1, x_2) = x_1^2 + x_2^2 - 4$, and $g_2(x_1, x_2) = -x_1^2 + x_2$ (right side).

Exercises. Solve the following problems using Kuhn-Tucker conditions.

1. Minimize $2x_1 + x_2$ subject to $x_1^2 + x_2^2 \le 4$ and $x_1 \le x_2$.
2. Minimize $x_1^2 + x_2^2 - 2x_1$ subject to $x_1^2 + x_2 \le 1$.
3. Minimize $(x_1 - 2)^2 + (x_2 - 1)^2$ subject to $x_1 + x_2 \le 2$ and $x_1^2 \le x_2$.
4. Minimize $(x_1 - 1)^2 + (x_2 - 5)^2$ subject to $x_2 \le 4 + x_1^2$ and $x_2 \le 3 + (x_1 - 2)^2$.