Random Variable

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Random Variable
&
Prob. Distribution

→ Random exp.
↳ Outcomes

failure Success

S

A random variable X is a funtⁿ

$X : S \longrightarrow \mathbb{R}$

s.t.

$\{x \le a\} = \{\omega \mid X(\omega) \le a\}$

is an event for all a.

Random Variable (X) :- we mean a real number X associate with the outcomes of a random experiment.

Ex: Tossing a coin 3 times ← Random exp.

{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT} ← outcomes

X : No. of Heads

Outcomes	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
X: No of Heads or Value of X	3	2	2	2	1	1	1	0

A random variable (r.v.)

$X : S \longrightarrow \mathbb{R}$

R.V → Discrete R.V, Continuous R.V

Discrete R.V: If it assume only a finite or countably infinite set of values.

Ex: 1) No of students in a class room.
2) No. of NIT's in India.
3) No. of defective balls in a bag.

Continuous R.V: A random variable is continuous if its distributive function is given by an integral $F(x) = \int_{-\infty}^{x} f(t)\,dt$ whose integrand $f(x)$, called the density of the distribution, is nonnegative and is continuous except finitely many points.

Ex: 1) weight of 50 students in a class room belongs to [52, 75] kg.

2) Speed $S = \frac{D}{T}$ (A, S, D, B)

P.D.F.

$F : \mathbb{R} \longrightarrow [0,1]$

$F(x) = P(X \le x)$, $-\infty < x < \infty$

↳ Prob. dist function

Prob. Distribution of a r.v → Discrete, Continuous

r.v. $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ ↓ $p_1, p_2, \ldots, p_n$

$p_i = P(X = x_i) \quad \forall\, i = 1, 2, \ldots, n, \ldots$

p_i is called prob. mass function (p.m.f) if following conditions are satisfied:

1) $p_i \ge 0$
2) $\sum_i p_i = 1$

Pb1 Suppose a coin is tossed two times. Construct the probability mass function and probability distribution function of random variable corresponding to number of Heads.

Tossing a coin 2 times

Outcomes	HH	HT	TH	TT
X: No of Heads	2	1	1	0

X	$p_i = P(X = x_i)$	
$x_1 = 0$	$1/4$	← {TT}
$x_2 = 1$	$1/2$	← {HT, TH}
$x_3 = 2$	$1/4$	← {HH}

$$F(x) = P(X \le x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{4}, & \text{if } 0 \le x < 1 \\ \frac{3}{4}, & \text{if } 1 \le x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}$$

↳ Prob. dist. functⁿ

Satisfied
1) $p_i \ge 0$
2) $\sum p_i = 1$
} p_i is called p.m.f and

Pb2 5 defective pieces of ABC units which are accidently mixed with 15 good pieces and looking at them. Assume that 4 pieces are picked up at random. Construct a probability mass function and probability distribution function of random variable corresponding to number of defective pieces.

Sol Let X = No. of defective pieces:

$P(X=0) = \frac{{}^{15}C_4}{{}^{20}C_4} = \alpha_1$ $\qquad P(X=2) = \frac{{}^{5}C_2 \cdot {}^{15}C_2}{{}^{20}C_4} = \alpha_3$

$P(X=1) = \frac{{}^{5}C_1 \cdot {}^{15}C_3}{{}^{20}C_4} = \alpha_2$ $\qquad P(X=3) = \frac{{}^{5}C_3 \cdot {}^{15}C_1}{{}^{20}C_4} = \alpha_4$

$P(X=4) = \frac{{}^{5}C_4}{{}^{20}C_4} = \alpha_5$

p.m.f →

X	0	1	2	3	4
p_i	α_1	α_2	α_3	α_4	α_5

1) $p_i = \alpha_i \ge 0 \quad \forall\, i$

2) $\sum p_i = \sum \alpha_i = 1$ (Varify)

$$F(x) = P(X \le x) = \begin{cases} 0, & \text{if } x < 0 \\ \alpha_1, & \text{if } 0 \le x < 1 \\ \alpha_1 + \alpha_2, & \text{if } 1 \le x < 2 \\ \alpha_1 + \alpha_2 + \alpha_3, & \text{if } 2 \le x < 3 \\ \sum_{i=1}^{4} \alpha_i, & \text{if } 3 \le x < 4 \\ \sum_{i=1}^{5} \alpha_i = 1, & \text{if } x \ge 4 \end{cases}$$

Pb3 A bag contains 6 red and 4 white balls. Three balls are drawn at random. Obtain the probability mass function and probability distribution function of the number of white balls drawn.

Sol let X = No. of white balls drawn.

$P(X=0) = \frac{{}^{6}C_3}{{}^{10}C_3} = \frac{5}{30}$ (corrected from $\frac{2}{30}$)

$P(X=1) = \frac{{}^{6}C_2 \cdot {}^{4}C_1}{{}^{10}C_3} = \frac{15}{30}$

$P(X=2) = \frac{{}^{6}C_1 \cdot {}^{4}C_2}{{}^{10}C_3} = \frac{9}{30}$

$P(X=3) = \frac{{}^{4}C_3}{{}^{10}C_3} = \frac{1}{30}$

X	0	1	2	3
p_i	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$

1) $p_i \ge 0 \quad \forall\, i$
2) $\sum p_i = 1$
} Hence p_i is p.m.f.

Q1: Two cricket players, Markus and Dean, decide to throw balls at a wicket, in alternate fashion, starting with Markus. The winner is the player who is first to hit the wicket. The probability that Markus hits the wicket is 0.2 for any of his throws. The probability that Dean hits the wicket is p for any of his throws. If Markus throws first, the probability he wins the game is 5/13.
Determine the value of p.

Sol: Let the prob. of success of Dean be p, $0<p<1$.
Let Markus throw first;

$$P(X=1) = 0.2$$
$$P(X=2) = 0.8\,(1-p) \times 0.2$$
$$P(X=3) = 0.8\,(1-p)\,0.8\,(1-p) \times 0.2$$
$$P(X=4) = 0.8\,(1-p)\,0.8\,(1-p)\,0.8\,(1-p) \times 0.2$$
$$\vdots$$

Now form an expression for a Markus win, know to be $5/13$.

$$P(X=1) + P(X=2) + P(X=3) + \cdots\cdots = \frac{5}{13}$$
$$\Rightarrow \quad 0.2 + 0.8\,(1-p)\,0.2 + [0.8(1-p)]^2\,0.2 + \cdots\cdots = \frac{5}{13}$$
$$\Rightarrow \quad 0.2\left[1 + 0.8\,(1-p) + [0.8\,(1-p)]^2 + \cdots\cdots\right] = \frac{5}{13}$$
$$\Rightarrow \quad 0.2\,\frac{1}{1-0.8(1-p)} = \frac{5}{13}$$
$$\Rightarrow \quad p = 0.04$$

Ans.

Q2: Suppose the number of hits a web site receives in any time interval is a Poisson random variable. A particular site gets on average 5 hits per second.

a) What is the probability that there will be no hits in an interval of two seconds?

Solution: X ~ Poisson(2 × 5)

$$P(X = k) = e^{-10}\,\frac{10^{-k}}{k!}$$

$$P(X = 0) = e^{-10}\,\frac{10^{0}}{0!} = e^{-10}$$

b) What is the probability that there is at least one hit in an interval of one second?

Solution: $\lambda = 5$, X ~ Poisson(5), $P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-5}\frac{5^0}{0!} = 1 - e^{-5}$

Q3: A company sells LED bulbs in packages of 20 for \$25. From past records, it known's that a bulb will be defective with probability 0.01. The company agrees to pay a full refund if a customer finds more than 1 defective bulb in a pack. If the company sells 100 packs, how much should it expect to refund ?

Sol: Let X = no. of defective bulbs in a pack.

X can be taken $0, 1, 2, 3, \ldots, 20$.

X is a binomial random variable with parameter $(20, 0.01)$

Probability that refund will be processed

$$p = P(X>1) = 1 - P(X=0) - P(X=1)$$

$$= 1 - {}^{20}C_0 (0.01)^0 (0.99)^{20} - {}^{20}C_1 (0.01)^1 (0.99)^{19} \quad (*)$$

Y = No. of packages returned.

$y_0 \quad y_1 \quad y_2 \quad \ldots \quad y_{100}$

Y can be $0, 1, 2, 3, \ldots, 100$.

Y is a binomial random variable with parameter $(100, p)$

$$P(Y=k) = {}^{100}C_k \, p^k (1-p)^{100-k}$$

$$\text{Expected refund} = \$\, 25 \left(\sum_{y_i} y_i \, P(y_i) \right)$$

$$= \$\, 25 \left(\sum_{k=1}^{100} k \, P(Y=k) \right)$$

Pb1 Find the m.g.f. of the Poisson distribution. Does $M_X(t)$ exist for all values of t?

Sol The p.m.f of the Poisson distribution with the parameter λ is

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad ; \quad x = 0, 1, 2, \ldots$$

Thus, m.g.f. of it is

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{e^t\lambda}$$

$$= e^{-\lambda(1-e^t)}$$

$\because e^x = \sum \frac{x^n}{n!}$ and is convergent for all x.

which exist for all values of t.

Pb2 Find the m. g. f. of the Poisson distribution and hence find its mean and variance. Does $M_X(t)$ exist for all values of t ?

How to calculate the Mean and Variance from m. g. f. ?

For any +ve integer r, we denote Moments: $\mu_r = E(x^r)$

1st Method :-

$$\mu_r = \frac{d^r}{dt^r}(M_X(t)) \text{ at } t = 0$$

$$\boxed{\text{Mean} = E(X) = \mu_1} \quad (\because r = 1)$$

$$\boxed{\text{Variance} = E(X^2) - (E(X))^2 = \mu_2 - \mu_1^2}$$

2nd Method :-

μ_r = Coefficient of $\frac{t^r}{r!}$ in the series expansion of $M_X(t)$

For more details:

$$M_X(t) = E(e^{tX})$$

$$= E\left[1 + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \cdots\right]$$

$$= 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \cdots$$

$$= 1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \cdots + \mu_r \frac{t^r}{r!} + \cdots$$

Pb1 Given M.G.F. $M(t) = e^{3(e^t-1)}$

Find $E(X)$, $Var(X)$?

Sol

$M(t) = e^{3(e^t-1)}$

$M'(t) = 3e^t e^{3(e^t-1)}$

$M''(t) = 3e^t e^{3(e^t-1)} + 9e^{2t} e^{3(e^t-1)}$

$E(X) = M'(0) = 3$

$E(X^2) = M''(0) = 3 + 9 = 12$

$Var(X) = E(X^2) - (E(X))^2$

$= 12 - 3^2$

$= 3$

Pb 2 Let X be uniformly distributed over (a, b). Find E(X), Var(X) and moment generating function ?