NLPP

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Recall:- LPP → Linear Programming Problem :-

$$\text{opt } (\text{Max}/\text{Min})\ Z = C^T X = f(x)$$
$$\text{s.t. } AX \le, =, \ge b$$
$$X \ge 0$$

objective $f(x)$; constraints → Linear.

$c_1x_1 + c_2x_2 + \cdots + c_nx_n$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \le b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \le b_m \end{cases}$$

Method :→ Graphical / Simplex Method & its variant.

Note: Using Fundamental Theorem of LPP.
optimum solⁿ lies at one of the extreme points/BFS ; provided convex polyhedron feasible region.

NLPP → Non Linear Programing Problem :-

Remark :- The optimal solution can be found any where, depending on problem ; it may exist on boundary of feasible region, or even at interior point ; so we don't have general technique to solve all NLPP with one method.

General NLPP :- Opt $f(x)$; s.t. $g_i(x) \le, \ge, = b$;
X – unrestricted or restricted.

- $f(x)$ or $g_i(x)$ or both are non-linear.

Examples of NLPP

1) $\text{Max } f(x) = x_1^2 + x_2^2 + 2x_2$ (without constraints)

2) $\text{Max } f(x) = x_1^2 + x_2^2$ s.t.
$x_1 + x_2 \ge 4$; $2x_1 + x_2 \ge 5$; $x_1, x_2 \ge 0$ (with constraints but with inequality)

3) $\text{Min } f(x) = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$
s.t. $x_1 + x_2 + x_3 = 11$; $x_1, x_2, x_3 \ge 0$
(with constraints ; with equality)

4) Quadratic forms
$f(x) = x_1^2 + x_2^2 + 2x_1x_2$ (without const.)

5) $f(x) = x_1^2 + x_2^2 + 5x_1$ (Not quadratic)

NLPP

Without constraint.
- Quadratic form.
- Non quadratic form

With constraint
- Inequality constraints → KKT conditions
- Equality constraints → Langrange's Multiplier

Search Techniques → Trial & optimize.

Problem.

A manufacturing company produces two products: Radios and Tv sets. Sales price relationship for these two products are given below:

Products	Quantity Demanded	Unit price
Radio	1500-5P_1	P_1
Tv	3800-10P_2	P_2

The total cost functions for these two products are given by $200x_1 + 0.1x_1^2$ and $300x_2 + 0.1x_2^2$ respectively. The production takes place on two assembly lines. Radio sets are assembled on Assembly line 1 and TV sets are assembled on Assembly line II. Because of the limitations of the assembly line capacities, the daily production is limited to nomore than 80 radio sets and 60 TV sets. The production of both types of products require electronic components. The production of each of these sets requires five units and six units of electronic equipment components respectively. The electronic components are supplied by another manufacturer, and the supplyis limited to 600 units per day. The company has 160 employees, the labor supply amounts to 160 man-days. The production of one unit of radio set requires 1 man-day of labor, whereas 2 man-days of labor are required for a TV set. How many units of radio and TV sets should the company produce in order to maximize the total profit. Formulate the problem as a non-linear Programming problem.

Sol: Assumption :- Whatever produced is sold in market.

Let x_1 and x_2 quantities of radio sets and T.V sets demanded, respectively.

$$x_1 = 1500 - 5p_1 \Rightarrow p_1 = 300 - 0.2x_1$$

$$x_2 = 3800 - 10p_2 \Rightarrow p_2 = 380 - 0.1x_2$$

Let C_1 and C_2 be the total cost of production of these units of radio sets & T.V. sets, respectively.

$$C_1 = 200x_1 + 0.1x_1^2$$

$$C_2 = 300x_2 + 0.1x_2^2$$

Total Revenue $= p_1x_1 + p_2x_2 = (300 - 0.2x_1)x_1 + (380 - 0.1x_2)x_2 = 300x_1 - 0.2x_1^2 + 380x_2 - 0.1x_2^2 = R$

Total Profit $=$ (Total Revenue) $-$ (Total cost of production) $= R - (C_1 + C_2)$

$$\left.\begin{aligned} \text{Max } Z &= 100x_1 - 0.3x_1^2 + 80x_2 - 0.2x_2^2 \\ \text{s.t.}\quad & 5x_1 + 6x_2 \le 600 \\ & x_1 + 2x_2 \le 160 \\ & 0 \le x_1 \le 80 \;;\quad 0 \le x_2 \le 60 \end{aligned}\right\} \text{NLPP formulation}$$

An optimization problem is a problem, where we maximize/minimize an objective function subject to some given conditions. If the objective function or the conditions are nonlinear, then we say the optimal problem is a nonlinear programming problem or nonlinear optimization problem. We know that for LPP, the optimal solution is achieved at some extremal points of the feasible region. It may not be true for nonlinear programming. For example, consider the problem:

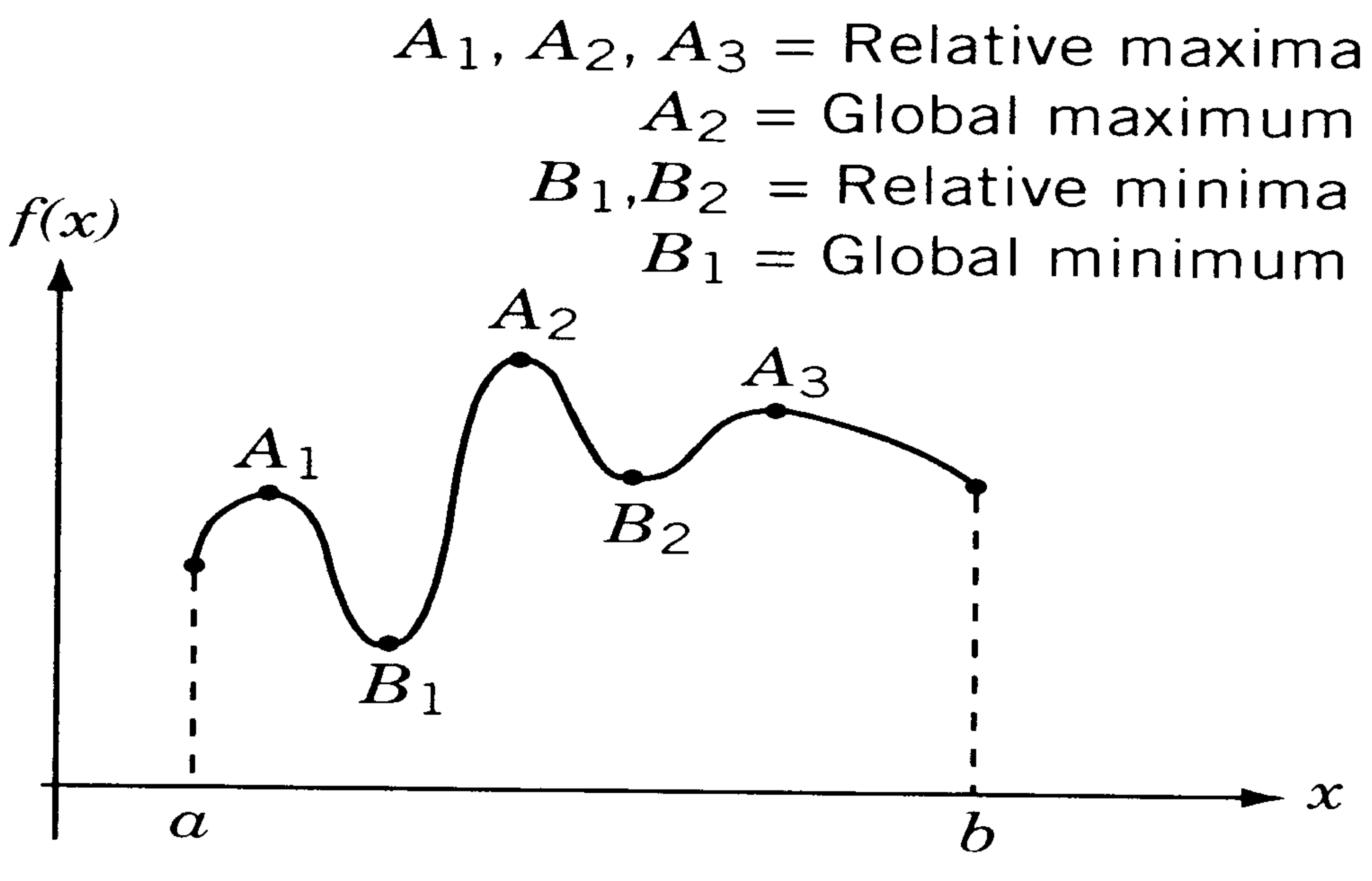
Maximize z = $x_1 x_2$
subject to $4x_1 + x_2 \leq 8$, $x_1, x_2 \geq 0$.

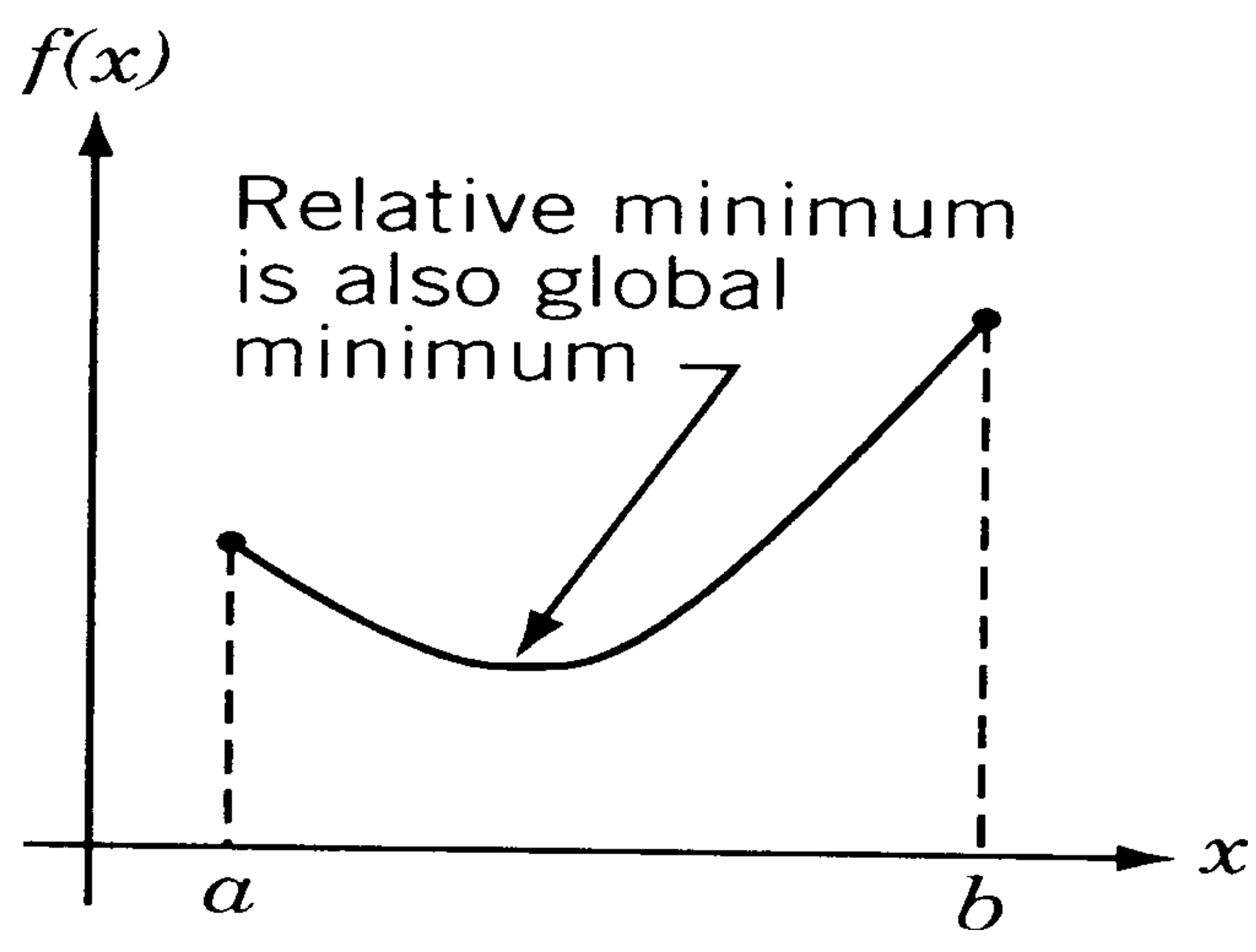
The maximum value of k for which the parabola $x_1 x_2$ = k has a common point with the feasible region is 4. The parabola $x_1 x_2$ = 4 touches the region at (1, 4), which is not an extreme point.

Let f : $R^n \rightarrow R$. The function f has **global minimum** at x^* if f(x) ≥ f(x^*) for all $x \in R^n$, and has **global maximum** at x^* if f(x) ≤ f(x^*) for all $x \in R^n$. We say f(x) has a **local minimum** at $x = x^*$ if there exists r > 0 such that f(x) ≥ f(x^*) for all $x \in R^n$ with $\|x\text{-}x^*\|$ < r. We say f(x) has a **local maximum** at $x = x^*$ if there exists r > 0 such that f(x) ≤ f(x^*) for all $x \in R^n$ with $\|x\text{-}x^*\|$ < r.

n = 1. We say f(x) has a **local minimum** at x = x∗ if there exists r > 0 such that f(x) ≥ f(x^*) for all $x^* - r < x < x^* + r$.
We say f(x) has a **local maximum** at x = x∗ if there exists r > 0 such that f(x) ≤ f(x^*) for all $x^* - r < x < x^* + r$.

n = 2. We say f(x_1, x_2) has a **local minimum** at (x_1, x_2) = (x_1^*, x_2^*) if there exists r > 0 such that f(x_1, x_2) ≥ f(x_1^*, x_2^*) for all (x_1, x_2) $\in R^2$ with $(x_1 - x_1^*)^2 + (x_2 - x_2^*)^2 < r^2$.
We say f(x) has a **local maximum** at (x_1, x_2) = (x_1^*, x_2^*) if there exists r > 0 such that f(x_1, x_2) ≤ f(x_1^*, x_2^*) for all (x_1, x_2) $\in R^2$ with $(x_1 - x_1^*)^2 + (x_2 - x_2^*)^2 < r^2$.





Pb. Find the natures of the extreme points of the function

① $f(x) = 12x^5 - 45x^4 + 40x^3 + 5$

② $f(x) = x^3 - 6x^2 + 9x + 5$

③ $f(x) = 2 + (x-1)^4$

Sol ② The necessary condition for extreme points is

$$f'(x) = 0$$

$$\Rightarrow 3x^2 - 12x + 9 = 0$$

$$\Rightarrow x^2 - 4x + 3 = 0$$

$$\Rightarrow (x-3)(x-1) = 0$$

$x = 3, 1 \leftarrow$ extreme points

Now, $f''(x) = 6x - 12$

$f''(1) = -6 < 0 \Rightarrow x = 1$ is a local maximum point.

$f''(3) = 6 > 0 \Rightarrow x = 3$ is a local minimum point.

1. Quadratic form
2. Non quadratic form

Quadratic form :-

$$f(x) = c_{11}x_1^2 + c_{22}x_2^2 + \cdots + c_{nn}x_n^2 + c_{12}x_1x_2 + c_{13}x_1x_3 + \cdots + c_{1n}x_1x_n + \cdots + c_{n-1,n}x_{n-1}x_n$$

$$= X^TAX \;;$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $A = (a_{ij})_{n\times n} \longrightarrow$ square matrix

$a_{ii} = c_{ii}$; $a_{ij} = a_{ji} = \dfrac{c_{ij}}{2}$; $i \neq j$

Ex 1 $f(X) = 2x_1^2 - x_2^2 + 4x_1x_2 + x_3^2$

Write $f(x)$ in the form of X^TAX.

Sol

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} ; \quad A = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 2 & 2 & 0 \\ x_2 & 2 & -1 & 0 \\ x_3 & 0 & 0 & 1 \end{array}$$

$$f(X) = X^TAX$$

Ex 2 $f(X) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 - 6x_1x_3 - 5x_2x_3$

Sol

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad A = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 2 & -5/2 \\ -3 & -5/2 & -7 \end{bmatrix}$$

Non-Quadratic form :-

NOTE: How can we determine whether a function f $(x_1, x_2, \ldots, x_n)$ of n variables is convex or concave on a subset of R^n? We assume that f $(x_1, x_2, \ldots, x_n)$ has continuous second-order partial derivatives.

$H(x) \longrightarrow$ Hessian Matrix

$$H(x) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Ex 1 $f(x) = 2 + 2x_1 + 3x_2 - x_1^2 - x_2^2$

find Hessian matrix ?

Sol

$\dfrac{\partial f}{\partial x_1} = 2 - 2x_1$; $\dfrac{\partial f}{\partial x_2} = 3 - 2x_2$

$\dfrac{\partial^2 f}{\partial x_1^2} = -2$; $\dfrac{\partial^2 f}{\partial x_1 \partial x_2} = 0$;

$\dfrac{\partial^2 f}{\partial x_2 \partial x_1} = 0$; $\dfrac{\partial^2 f}{\partial x_2^2} = -2$

$$H(X) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

Clairot's theorem:
If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Some more examples of quadratic and non-quadratic forms & associated Matrix.

Pb 1 $f(x) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

Sol

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; \quad A = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix} \rightarrow \text{Symmetric Matrix}$$

$$f(X) = X^TAX$$

Pb 2 $f(x) = 5x_1^2 - 7x_2$

find Hessian matrix ?

Sol

$\dfrac{\partial f}{\partial x_1} = 10x_1$; $\dfrac{\partial f}{\partial x_2} = -7$

$\dfrac{\partial^2 f}{\partial x_1^2} = 10$; $\dfrac{\partial^2 f}{\partial x_1 \partial x_2} = 0$

$\dfrac{\partial^2 f}{\partial x_2 \partial x_1} = 0$; $\dfrac{\partial^2 f}{\partial x_2^2} = 0$

$$H(X) = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$$

Remark $\boxed{H = 2A}$

Pb 3 $f(x) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

find $H = ?$

Sol

$\dfrac{\partial f}{\partial x_1} = 2x_1 - 4x_2 + 8x_3$

$\dfrac{\partial f}{\partial x_2} = 4x_2 - 4x_1$

$\dfrac{\partial f}{\partial x_3} = -14x_3 + 8x_1$

$$H(X) = \begin{bmatrix} 2 & -4 & 8 \\ -4 & 4 & 0 \\ 8 & 0 & -14 \end{bmatrix} = 2A$$

NOTE:
- In quadratic form, Hessian Matrix is always twice of symmetric matrix.
- **The Hessian matrix is a symmetric matrix** containing all the second derivatives of the multivariate function.

To find Maxima/Minima of Non-linear problem

$f(X) <$ Quadratic form. / Non-quadratic form.

Step 1:- Find the stationary points, that is; X^* s.t. $\nabla f(X^*) = 0$

or $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \cdots = \frac{\partial f}{\partial x_n} = 0$

Step 2:- Check the matrix corresponding to quadratic and Non-quadratic formed definitness. { Remark: $\boxed{H(X) = 2A}$ }

Step 3:- X^* is a point of relative minimum if matrix is positive definite.

X^* is a point of relative maximum if matrix is negative definite.

X^* is a saddle point if matrix is indefinite.

Definitness of a matrix / Quadratic form

Positive Definite :- The quadratic form is +ve definite if $f(X) > 0$ for all $X \neq 0$.

Example: $f(X) = x_1^2 + 2x_2^2 + x_3^2 > 0$ for $X \neq 0$; $X = (x_1, x_2, x_3)$ atleast one of $x_i \neq 0$.

Positive Semi-Definite :- The quadratic form is +ve semi-definite if $f(X) \geq 0$ for all $X \neq 0$ and there exist atleast one non-zero vector $X \neq 0$ for which $f(X) = 0$.

Ex: $f(X) = (x_1 - x_2)^2 + 2x_3^2$

Varification: $f(X) \geq 0 \;\; \forall X$ but for $X = (1,1,0) \neq 0$; $f(X) = 0$

Negative Definite :- If $f(X) < 0$; for all $X \neq 0$.

Negative semi-definite :- If $f(X) \leq 0$ for all $X \neq 0$; $\exists$ atleast one non zero $X \neq 0$ s.t. $f(X) = 0$.

Indefinite : If none of the above hold; it is negative definite.

Matrix Minor Test to check definitness of Matrix (A; $A = a_{ij\, n\times n}$) :-

Positive Definite :- $D_1 = a_{11} > 0$; $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$; $\cdots\cdots$ $D_n = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0$

Positive Semi-Definite :- $D_1 > 0$; $D_i \geq 0$; $i = 2, 3, \ldots, n$.

Negative-Definite :- $D_1 < 0$; $D_2 > 0$; $D_3 < 0$; $\cdots\cdots$, $(-1)^n D_n > 0$.

Negative Semi-Definite :- $D_1 < 0$; $D_2 \geq 0$; $D_3 \leq 0$; $\cdots\cdots$ $(-1)^n D_n \geq 0$.

Indefinite :- None of above.

Pb ① $f(X) = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$;

② $f(X) = 2x_1^2 - 7x_2$

③ $A = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Sol ①
$$A = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 1 & -2 & 4 \\ x_2 & -2 & 2 & 0 \\ x_3 & 4 & 0 & -7 \end{array}$$

$D_1 = 1 > 0$;

$D_2 = \begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = -2 < 0$;

$D_3 = \begin{vmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{vmatrix} = -18 < 0$

$\boxed{D_1 > 0 \; ; \; D_2 < 0 \; , \; D_3 < 0}$ Indefinite form.

②
$$A = \begin{array}{c|cc} & x_1 & x_2 \\ \hline x_1 & 2 & 0 \\ x_2 & 0 & 0 \end{array}$$
; $\boxed{D_1 = 2 > 0 \; ; \; D_2 = 0}$ +ve Semidefinite form

or $H = \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix}$ $\because H = 2A$ or $A = H/2$ (for definitness: Same definition can apply for H as in A.)

③ $A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$D_1 = 4 > 0$

$D_2 = \begin{vmatrix} 4 & 2 \\ 2 & 9 \end{vmatrix} = 32 > 0$

; $D_3 = \begin{vmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 64 > 0$;

+ve definite.

Pbl. Find stationary point and classify this as point of maxima or minima.

① $f(x) = 2 + 2x_1 + 3x_2 - x_1^2 - x_2^2$

Sol Step-1 $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$ (To find stationary point)

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} &= 2 - 2x_1 \\ \frac{\partial f}{\partial x_2} &= 3 - 2x_2 \end{aligned}\right\} \Rightarrow \frac{\partial f}{\partial x_i} = 0,\ i = 1, 2 \Rightarrow \left.\begin{aligned} 2 - 2x_1 &= 0 \\ 3 - 2x_2 &= 0 \end{aligned}\right] \Rightarrow x_1 = 1 \ \& \ x_2 = 3/2 \quad \therefore X^* = (x_1, x_2) = (1, \tfrac{3}{2}) \longrightarrow \text{Stationary point}$$

Step 2

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \quad D_1 = -1 < 0; \quad D_2 = 1 > 0;$$

or

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}. \quad D_1 = -2 < 0; \quad D_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0;$$

$\rightarrow$ Thus, Negative definite.

Step 3 $\therefore (1, 3/2)$ is a point of local maximum; $f(1, 3/2) = \frac{21}{4}$ Ans

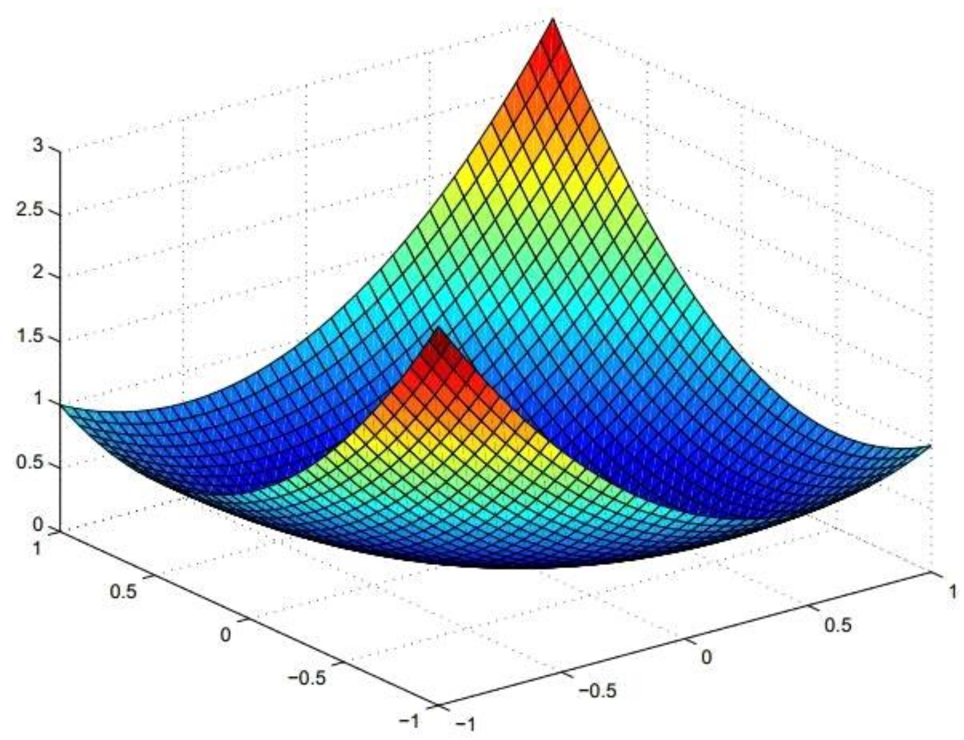
② $f(x) = 25x_1^2 - 8x_1x_2 + x_2^2$

Sol:

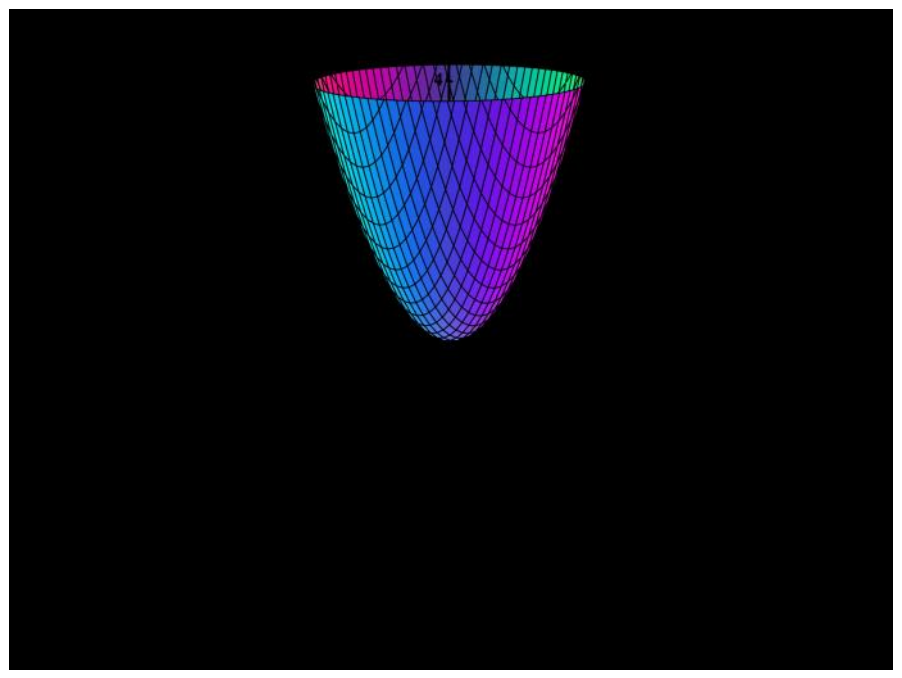
$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} &= 50x_1 - 8x_2 \\ \frac{\partial f}{\partial x_2} &= -8x_1 + 2x_2 \end{aligned}\right\} \quad \frac{\partial f}{\partial x_1} = 0 = \frac{\partial f}{\partial x_2} \Rightarrow X^* = (0, 0) \text{ stationary point.}$$

$$A = \begin{pmatrix} 25 & -4 \\ -4 & 1 \end{pmatrix}; \quad D_1 = 25; \quad D_2 = |A| = 9 > 0. \quad \text{Thus, +ve definite.}$$

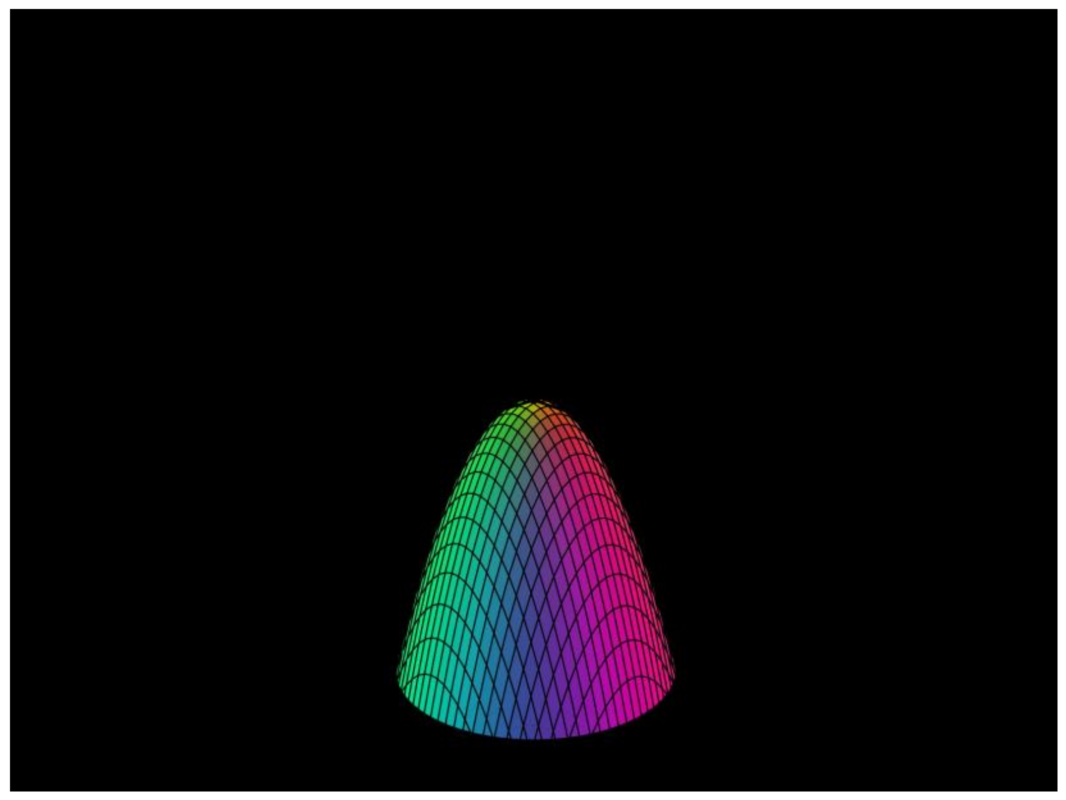
$\therefore (0, 0)$ is a point of local/relative minima; $f(0, 0) = 0$. Ans.



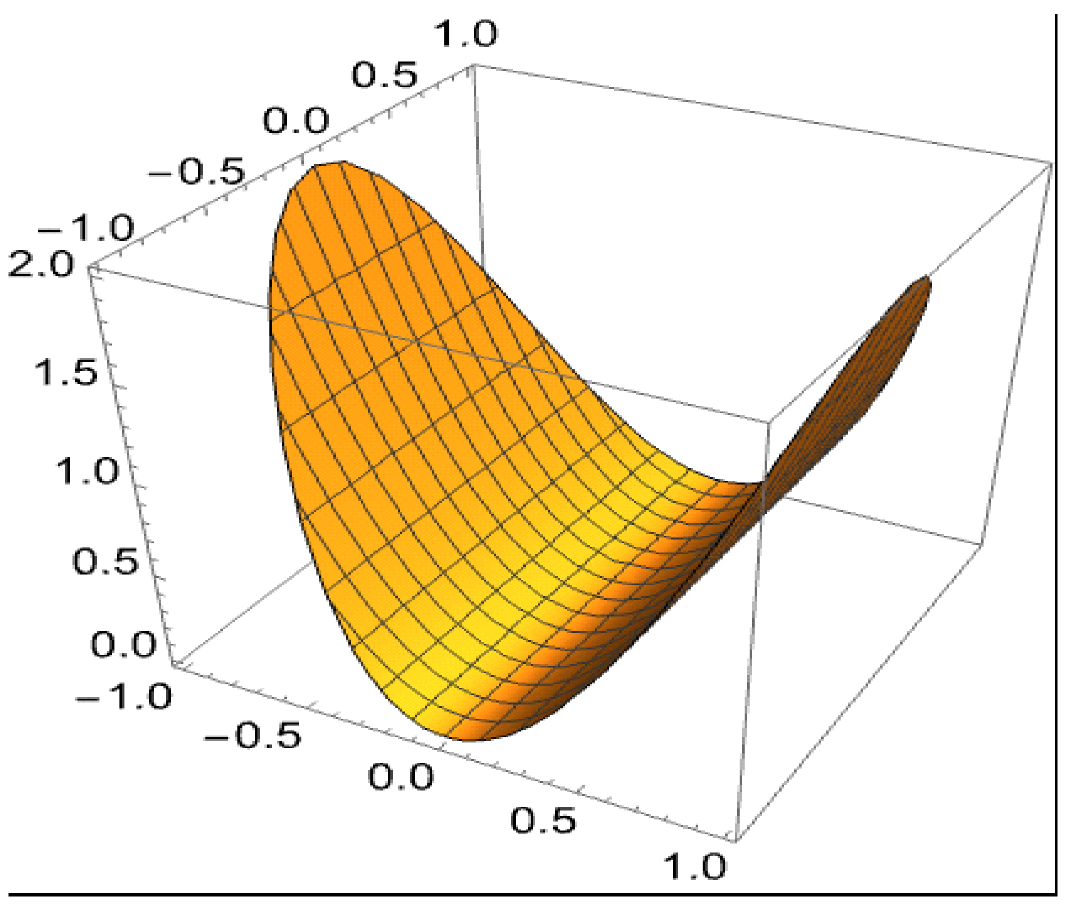
A Positive definite form



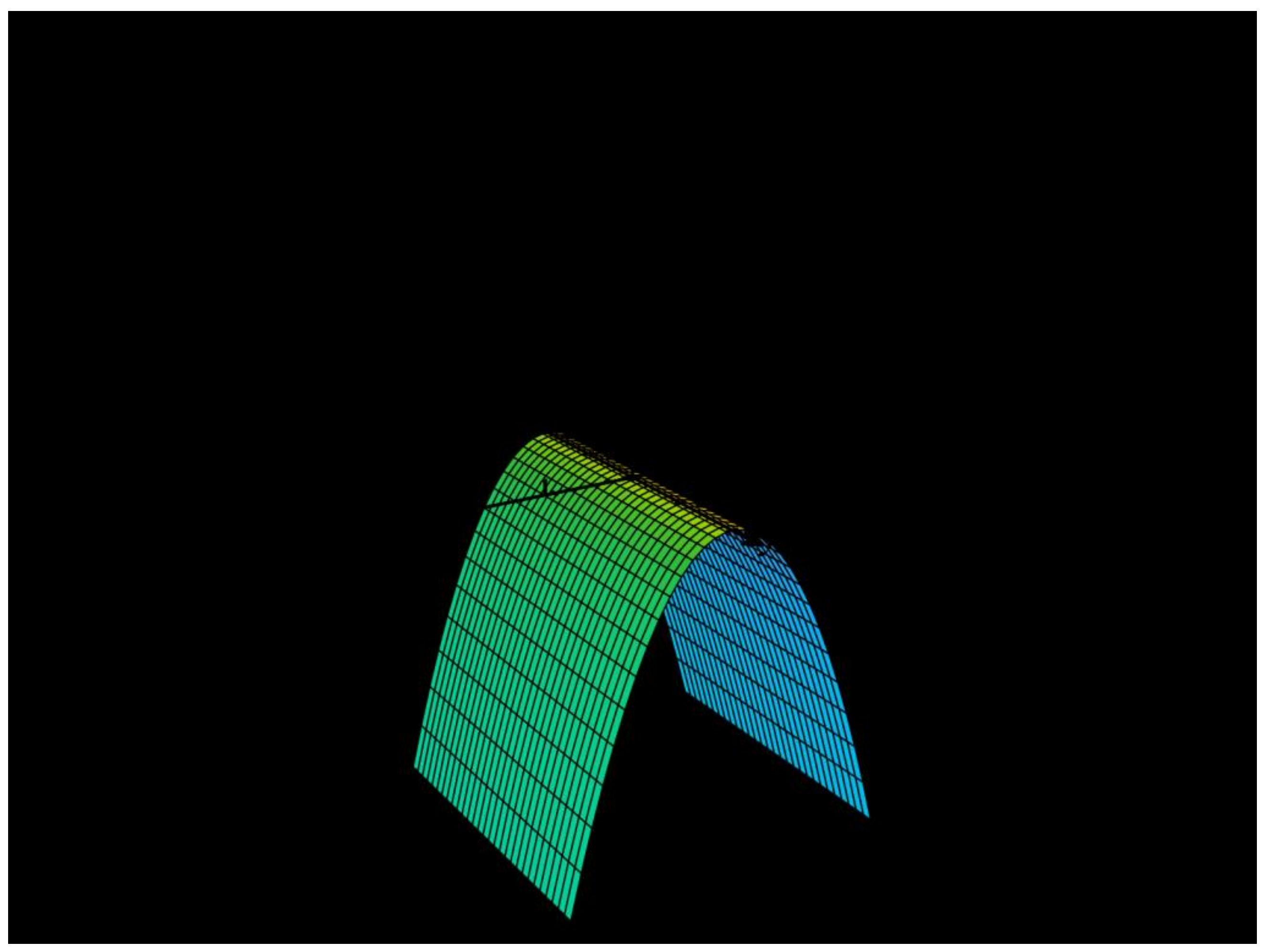
A Positive definite form



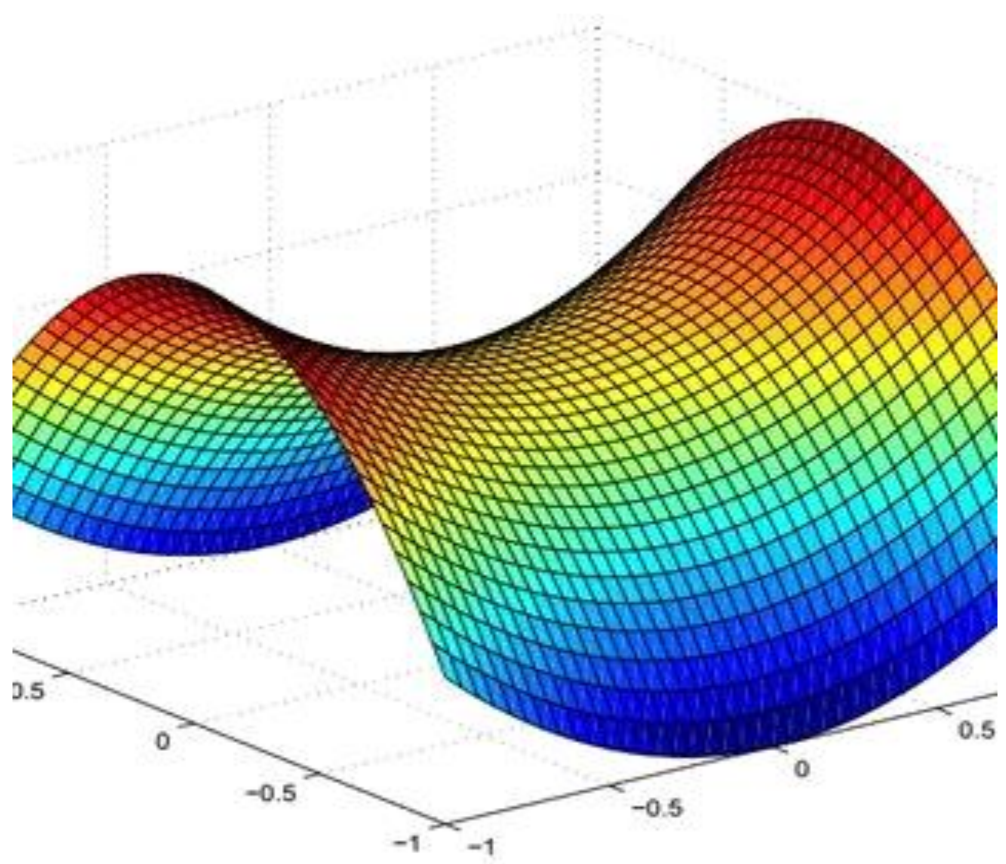
A Negative definite form



A Positive semi-definite form



A Negative semi-definite form



A Indefinite form

Unconstrained optimization problem. Let $f : \mathbb{R}^n \to \mathbb{R}$. An unconstrained optimization problem is to minimize/maximize $f(\mathbf{x})$, where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.
Necessary condition: If $\mathbf{x}^* \in \mathbb{R}^n$ is a local minimum or local maximum point of f over $\mathbb{R}^n$, then

$$(\nabla f)(\mathbf{x}^*) = 0, \quad \text{that is,} \quad \frac{\partial f}{\partial x_1}(\mathbf{x}^*) = \frac{\partial f}{\partial x_2}(\mathbf{x}^*) = \cdots = \frac{\partial f}{\partial x_n}(\mathbf{x}^*) = 0.$$

Sufficient condition: If $\mathbf{x}^* \in \mathbb{R}^n$ satisfies $(\nabla f)(\mathbf{x}^*) = 0$ and f is strictly convex (strictly concave) function in a neighbourhood of $\mathbf{x}^*$, then $\mathbf{x}^*$ is a local minimum (local maximum) point of f over $\mathbb{R}^n$.

Consider the Hessian matrix

$$(\nabla^2 f)(\mathbf{x}^*) = \left[\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(\mathbf{x}^*)\right]_{n\times n}.$$

If the Hessian matrix is positive definite (negative definite), then f is strictly convex (strictly concave).

Note: The points $\mathbf{x}^*$ satisfying $(\nabla f)(\mathbf{x}^*) = 0$ are called critical points.

$n = 1$. Let $f : \mathbb{R} \to \mathbb{R}$. An unconstrained optimization problem is to minimize/maximize $f(x)$, where $x \in \mathbb{R}$.

Necessary condition: If $x^* \in \mathbb{R}$ is a local minimum or local maximum point of f over $\mathbb{R}$, then

$$f'(x^*) = 0.$$

Sufficient condition: Let $f'(x^*) = f''(x^*) = \cdots = f^{(k-1)}(x^*) = 0$, but $f^{(k)}(x^*) \neq 0$. The point x^* is a

(a) local minimum point if $f^{(k)}(x^*) > 0$ and k is even,

(b) local maximum point if $f^{(k)}(x^*) < 0$ and k is even,

(c) neither a local maximum or local minimum if k is odd.

$n = 2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$. An unconstrained optimization problem is to minimize/maximize $f(x, y)$, where $(x, y) \in \mathbb{R}^2$.

Necessary condition: If $(x^*, y^*) \in \mathbb{R}^2$ is a local minimum or local maximum point of f over $\mathbb{R}^2$, then

$$\left(\frac{\partial f}{\partial x}\right)(x^*, y^*) = \left(\frac{\partial f}{\partial y}\right)(x^*, y^*) = 0.$$

Sufficient condition: The point $(x^*, y^*) \in \mathbb{R}^2$ satisfying the necessary condition is a local minimum (local maximum) point if

$$\begin{bmatrix} \left(\frac{\partial^2 f}{\partial x^2}\right)(x^*, y^*) & \left(\frac{\partial^2 f}{\partial x \partial y}\right)(x^*, y^*) \\ \left(\frac{\partial^2 f}{\partial y \partial x}\right)(x^*, y^*) & \left(\frac{\partial^2 f}{\partial y^2}\right)(x^*, y^*) \end{bmatrix} \qquad \text{(Hessian matrix)}$$

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is positive definite (negative definite).

If (x^*, y^*) is a critical point, and if for every $r > 0$, there exist $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{R}^2$ satisfying $(\alpha_1 - x_1^*)^2 + (\alpha_2 - x_2^*)^2 < r^2$ and $(\beta_1 - x_1^*)^2 + (\beta_2 - x_2^*)^2 < r^2$ such that $f(\alpha_1, \alpha_2) > f(x_1^*, x_2^*)$ and $f(\beta_1, \beta_2) < f(x_1^*, x_2^*)$, then (x_1^*, x_2^*) is called a saddle point. If the determinant of Hessian matrix is negative, then the point (x^*, y^*) is a saddle point.

Note: A real symmetric matrix A of order n is said to be positive definite (P.D.) if $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq 0$. A real symmetric matrix A of order n is said to be negative definite (N.D.) if $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq 0$.

A real symmetric matrix A is positive definite if and only if all eigenvalues of A are positive. A real symmetric matrix A is negative definite if and only if all eigenvalues of A are negative.

A sufficient condition for a real symmetric matrix A to be positive definite is that all leading principal minors are positive, that is,

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \quad \cdots, \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} > 0.$$

A sufficient condition for a real symmetric matrix A to be negative definite is that all leading principal minors alternate in sign starting from negative, that is,

$$a_{11} < 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} > 0, \quad \cdots.$$

Example. Find the natures of the extreme points of the function

$$f(x) = 12x^5 - 45x^4 + 40x^3 + 5.$$

Solution. The necessary condition for extreme points is

$$f'(x) = 60x^4 - 180x^3 + 120x^2 = 60x^2(x-1)(x-2) = 0.$$

The extreme points are $x = 0, 1, 2$.

$$f''(x) = 240x^3 - 540x^2 + 240x.$$

Now $f''(1) = -60 < 0$. So $x = 1$ is a local maximum point. Again $f''(2) = 240 > 0$. So $x = 2$ is a local minimum point. Since $f''(0) = 0$, we require

$$f'''(x) = 720x^2 - 1080x + 240.$$

Since $f'''(0) = 240 \neq 0$ and 3 is odd, $x = 0$ is neither a local minimum point nor a local maximum point. □

Example. Find the natures of extreme points of the function

$$f(x,y) = x^3 + y^3 + 2x^2 + 4y^2 + 6.$$

Solution. The necessary conditions for extreme points are

$$\frac{\partial f}{\partial x} = 3x^2 + 4x = 0, \quad \text{and} \quad \frac{\partial f}{\partial y} = 3y^2 + 8y = 0.$$

The extreme points are $(0,0), (0,-8/3), (-4/3,0), (-4/3,-8/3)$. The Hessian matrix is

$$H = \begin{bmatrix} 6x+4 & 0 \\ 0 & 6y+8 \end{bmatrix}.$$

The leading principal minors of H are $H_1 = 6x+4$ and $H_2 = (6x+4)(6y+8)$.

1. For (0,0), $H_1 = 4, H_2 = 32$. So H is positive definite and hence (0,0) is a local minimum point.
2. For $(0,-8/3)$, $\det H = -32 < 0$. So $(0,-8/3)$ is a saddle point.
3. For $(-4/3,0)$, $\det H = -32 < 0$. So $(-4/3,0)$ is a saddle point.
4. For $(-4/3,8/3)$, $H_1 = -4, H_2 = 32$. So H is negative definite and hence $(-4/3,-8/3)$ is a local maximum point.

□

Note: We have $f\left(-\frac{4}{3},r\right) - f\left(-\frac{4}{3},0\right) = r^3 + 4r^2 > 0$ for all $r > 0$. Again for every $0 < r < 2$,

$$\begin{aligned} f\left(-\frac{4}{3}+r,0\right) - f\left(-\frac{4}{3},0\right) &= \left(-\frac{4}{3}+r\right)^3 + 2\left(-\frac{4}{3}+r\right)^2 - \left(-\frac{4}{3}\right)^3 - 2\left(-\frac{4}{3}\right)^2 \\ &= r\left[\left(-\frac{4}{3}+r\right)^2 - \frac{4}{3}\left(-\frac{4}{3}+r\right) + \left(-\frac{4}{3}\right)^2\right] + 2r\left(-\frac{8}{3}+r\right) \\ &= r\left(\frac{16}{9} - \frac{8}{3}r + r^2 + \frac{16}{9} - \frac{4}{3}r + \frac{16}{9} - \frac{16}{3} + 2r\right) \\ &= r(r^2 - 2r) < 0. \end{aligned}$$

Therefore every neighbourhood of $\left(-\frac{4}{3},0\right)$ contain two points such that the value of f at one point is larger than $f\left(-\frac{4}{3},0\right)$, and the value of f at other point is smaller than $f\left(-\frac{4}{3},0\right)$. Therefore $\left(-\frac{4}{3},0\right)$ is a saddle point. Similar calculations can be done to show that $\left(0,-\frac{8}{3}\right)$ is a saddle point.

Exercises.

1. Find the critical points and their natures for the following functions.

   (a) $f(x) = x^2 - 6x^2 + 9x + 5$.

   (b) $f(x) = 2 + (x-1)^4$.

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. Find two positive numbers whose product is 100 and whose sum is a minimum.

4. Find the point on the parabola $y^2 = 2x$ that is closest to the point (1,4).

5. Find the area of the largest rectangle that can be inscribed in a semicircle of radius 5. (The word inscribed means that the rectangle has two vertices on the semicircle and two vertices on the diameter.)

6. Find the critical points and their natures for the following functions.

   (a) $f(x, y) = x^3 - y^3 - 2xy + 6$.

   (b) $f(x, y) = x^4 - 2x^2 + y^3 - 3y$.

   (c) $f(x, y) = x^2 + y^4 + 2xy$.

   (d) $f(x, y) = y \cos x$.