Date
11/10/23

Note:-

① λ is an eigen value of A, λ^R is an eigen value of A^R.

$$Ax = \lambda x \quad x \neq 0$$
$$A^2x = A \cdot Ax = A(\lambda x)$$
$$= \lambda \cdot (Ax)$$
$$= \lambda \cdot \lambda x$$
$$= \lambda^2 x, \quad x \neq 0.$$

λ^2 is an eigen value of A^2.

② if λ is an eigen value of A, then $\lambda + K$ is an eigen value of $(A+KI)$; $K \in \mathbb{C}$

Pf

$$(A+KI)_{n\times n} x = Ax + KIx = Ax + K \cdot x$$
$$= \lambda x + Kx$$
$$= (\lambda + K)x \quad x \neq 0$$

Eigen values

Hermitian — $A = \bar{A}^T$ — Real

Skew-Hermitian — $A = -\bar{A}^T$

unitary — $A \times \bar{A}^T = I$

(If A is skew-hermitian)

Theorem:- Let $A \in M_{n\times n}(\mathbb{C})$ then the eigen values of A is zero (or) Purely Imaginary.

Pf $A = -\bar{A}^T$

Suppose λ is an eigen value of A, $\boxed{Ax = \lambda x}$ $x \neq 0$

$$\bar{x}^T Ax = \bar{x}^T(-\bar{A}^T)x \quad — ①$$

$$\bar{A}x = \bar{\lambda}x$$
$$(-\bar{A}^T)x = (-\bar{\lambda})x \quad — ②$$

Put in ①

$\bar{x}^T(\lambda x) = \bar{x}^T(-\bar{\lambda})x$

$\lambda \bar{x}^T . x = -\bar{\lambda}(\bar{x}^T)x$

$\Rightarrow (\lambda+\bar{\lambda})(\bar{x}^T x) = 0$ $\quad \{ \bar{x}^T x = |x_1|^2 + \dots |x_n|^2 \}$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Since, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, $\bar{x}^T x \neq 0$

$\therefore \lambda + \bar{\lambda} = 0.$

Let ~~a+~~ $\lambda = a+ib$, $\bar{\lambda} = a-ib$

$\Rightarrow \lambda + \bar{\lambda} = 0$

$a+ib+a-bi = 0$

$2a = 0$

$\Rightarrow \boxed{a=0}$

$\therefore \boxed{\lambda = ib}$

* **Diagonalization** :- $A \in M_{n\times n}(\mathbb{F})$.

A is diagonilizeble if ~~$A \sim B$~~. $A \sim D$.
where D is diagonal matrix. $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$.

i.e. A is diagonalizable if $\exists$ an invertible matrix P
Such that $A = P^{-1}DP$.

i.e, $AP = DP$

↳ Importance of diagonalization

① $A = P^{-1}DP$

$A^{2023} = ?$

$\therefore A = P^{-1}AP$

$A^2 = A \cdot A = P^{-1}AP\,P^{-1}DP$

$= P^{-1}D^2P.$

$D^2 = \begin{bmatrix} a^2 & 0 & 0 & - \\ 0 & b^2 & 0 & - \\ 0 & 0 & c^2 & - \\ - & - & - & - \end{bmatrix}$

$\therefore A^{2023} = P^{-1}D^{2023}P = P^{-1}\begin{pmatrix} d_1^{2023} & 0 & 0 & - \\ 0 & d_2^{2023} & 0 & -- \\ \vdots & & & \\ 0 & 0 & -- & d_n^{2023} \end{pmatrix} P$

② $\text{Rank}(A) = \text{Rank}(P^{-1}DP) = \text{Rank}(D) \Rightarrow$ No. of non-zero diagonal entries.

* **Theorem**:

$A = P^{-1}DP$

$A \in M_{n\times n}(\mathbb{C})$ is diagonilizable $\Leftrightarrow$ A has n L.I. eigen vectors.

Proof: A is diagonilizable, $\Rightarrow AP = PA$

$P = [v_1 \cdots v_n]$; $A[v_1, v_2 \cdots v_n] = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ 0 & 0 & \cdots & d_n \end{bmatrix} [v_1 \cdots v_n]$

$v_1, v_2 \cdots v_n \Rightarrow$ column matrix

$\Rightarrow [Av_1, Av_2 \cdots Av_n] = [d_1 v_1, d_2 v_2 \cdots d_n v_n]$

$\Rightarrow Av_i = d_i v_i \quad ; \; i = 1, 2 \cdots n.$

since $P = [v_1 \cdots v_n]$ is invertible, $v_i \neq 0$, $v_i = 1, 2, \cdots n$.

$\Rightarrow \{v_1, \cdots v_n\}$ are L.I.

$Av_i = d_i v_i \quad , v_i \neq 0 , \; i = 1, 2, \cdots n$

which implies A has n L.I. eigen vectors, with eigen values $d_1, d_2 \cdots d_n$.

$\Leftarrow$ Given A has 'n' L.I. eigen vectors to prove,

$Av_i = d_i v_i$: $v_i \neq 0$, take $P = [v_1 \cdots v_n]$

$AP = A[v_1 \cdots v_n] = [Av_1, Av_2 \cdots Av_n]$

$= (d_1 v_1 \cdots d_n v_n)$

$= \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} [v_1 \cdots v_n]$

$\Rightarrow \boxed{AP = DP}$

$A \in M_{n\times n}(\mathbb{C}) \Rightarrow$ ① find eigen values $\lambda_1, \ldots, \lambda_n$ ② find eigen vectors $v_1, \ldots, v_n$

then ③ $P = [v_1 \ldots v_n]$ → set of e.vectors. ④ $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

P is invertible $\Rightarrow$ A is diagonilizable

$\boxed{A = P^{-1}DP}$

Eg. check if $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$ is diagonilizable.

→ $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 6 \\ 0 & -1-\lambda \end{vmatrix} = 0$

$(2-\lambda)(-1-\lambda) = 0$

$\lambda = 2, -1$

eigen values

$Ax = \lambda x$

$(A - \lambda I)x = 0$

$\begin{bmatrix} 0 & 6 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$6x_2 = 0 \Rightarrow x_2 = 0$

$-3x_2 = 0 \Rightarrow x_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Ax = \lambda x$ (for $\lambda = 2$)

$\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\cancel{2x_1} + 6x_2 = \cancel{2x_1}$

$6x_2 = 0$

$x_2 = 0$

$-x_2 = 2x_2$

$-3x_2 = 0$

$x_2 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → v_1

for $\lambda = -1$

$\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$2x_1 + 6x_2 = -x_1$

$3x_1 + 6x_2 = 0$

$-x_2 = -x_2$

$\begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$3x_1 + 6x_2 = 0$

$x_1 = -2x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -\frac{1}{2}x_2 \end{bmatrix}$

$= \begin{bmatrix} -2x_1 \\ x_2 \end{bmatrix}$

v_2 → $= x_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

eigen value

$\lambda = 2, -1$

eigen vector corresponding to $\lambda = 2 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

eigen " " " $\lambda = -1 \Rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore \; P = \cancel{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$|P| = (1-0) = 1 \neq 0$

so P is invertible $\Rightarrow$ A is diagonalizible.

$A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$ $\qquad PA = DP$

$\begin{vmatrix} 2-\lambda & 6 \\ 0 & -1-\lambda \end{vmatrix} = 0$

$\lambda = 2, -1$

For $\lambda = 2$

$(A - \lambda I)X = 0$

$\begin{bmatrix} 0 & 6 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$6x_2 = 0$

$-3x_2 = 0$

$\boxed{x_2 = 0}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix},$

v_1

For $\lambda = -1$

$\begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3x_1 + 6x_2 = 0$

$x_1 = -2x_2$

$\frac{x_1}{2} = \frac{x_2}{-1}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

v_2

$x_2 \neq 0$

$\therefore \; P = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$|P| = -1 \neq 0$

$\therefore$ P is invertible

$\Rightarrow$ A is diagonilizable

$A = P^{-1} D P$

$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$P^{-1} = -1 \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$P^{-1} D P = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$

$$A = \begin{bmatrix} 5 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 5-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$(5-\lambda)\left[(1-\lambda)(2-\lambda)-2\right] -1(-4+2\lambda) -1(-2) = 0$

$(3-\lambda)(2-3\lambda+\lambda^2-2) -1(4+2\lambda)+2 = 0$

$(5-\lambda)(8-3\lambda+\lambda^2)+4-2\lambda+2 = 0$

$-9\lambda+3\lambda^2+3\lambda^2-\lambda^3+6-2\lambda = 0$

$-\lambda^3+6\lambda^2-\underset{11\lambda}{\cancel{9\lambda}}+6 = 0$

$\lambda^3-6\lambda^2+11\lambda-6 = 0$

$\lambda = 1, 3, 2$

For $\lambda = 1$

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1+x_2-x_3 = 0 \Rightarrow \cancel{2x_3-x_3-x_3=0}$

$-2x_1+2x_3 = 0$

$\boxed{x_3 = x_1}$

$x_2+x_3 = 0$

$x_2 = -x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

v_1

For $\lambda = 3$

$$\begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 - x_3 = 0$

$x_2 = x_3$

$-2x_1-2x_2+2x_3 = 0$

$x_2 - x_3 = 0$

$\Rightarrow -2x_1-2x_3+2x_3 = 0$

$x_1 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

v_2

for $\lambda = 2$

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 - x_3 = 0$

$-2x_1 - x_2 + 2x_3 = 0$

$x_2 = 0$

$\Rightarrow \quad x_1 = x_3$

$$V_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad P = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$|P| = 1(1-0) + 1(-1-1)$

$= 1 - 2 = -1 \neq 0$

$\therefore$ P is invertible

$\Rightarrow$ A is diagonalizable.

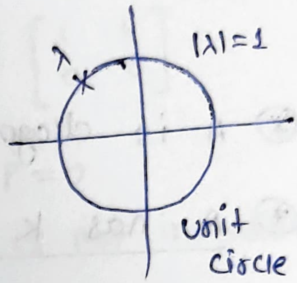
Theorem

$$U = \{\lambda \in \mathbb{C} \,/\, |\lambda| = 1\}$$

$A \in M_{n\times n}(\mathbb{C})$

If A is unitary then eigen values of A are unit module U.

$$A\bar{A}^T = I \Rightarrow \bar{A}^T = A^{-1}$$



Let λ be an eigen value of A

$\boxed{Ax = \lambda x}$, $x \neq 0$,

$$(\overline{Ax})^T = (\overline{\lambda x})^T,$$

$$\bar{x}^T \cdot \bar{A}^T = \bar{\lambda}\bar{x}^T$$

$$\bar{x}^T \cdot A^{-1} = \bar{\lambda}\bar{x}^T \quad (\because \bar{A}^T = A^{-1})$$

$$(\bar{x}^T \cdot A^{-1}) Ax = \bar{\lambda}\bar{x}^T Ax$$

$$\bar{x}^T I x = \bar{\lambda}\bar{x}^T \lambda x$$

$$\bar{x}^T x = \bar{\lambda}\lambda \bar{x}^T x$$

$$\bar{x}^T x = \bar{\lambda}\lambda \bar{x}^T x$$

$$\bar{x}^T x(1 - |\lambda|^2) = 0 \quad (\because \bar{x}^T x \neq 0)$$

$$\Rightarrow 1 - |\lambda|^2 = 0$$

$$\Rightarrow |\lambda|^2 = 1 \Rightarrow |\lambda| = 1$$

↳ If A is unitary, then $\det(A) = ?$

$$A\bar{A}^T = I$$

$$\det \bar{C} = (\overline{\det C})$$

$$\det C = \det C^T$$

$$\det(A\bar{A}^T) = \det(I)$$

$$\det(A) \cdot \det(\bar{A}^T) = 1$$

$$\det(A) \cdot (\overline{\det A}) = 1$$

$$|\det(A)|^2 = 1$$

$$\Rightarrow |\det(A)| = 1$$

Note: ① If A is unitary $\Rightarrow$ A is invertible ($\because |\det A| = 1$).

② A is unitary $\Rightarrow$ $\bar{A}, A^T, \bar{A}^T, A^{-1}$ are unitary.

Given, $A\bar{A}^T = I$, Let $B = A^T$, to prove B is unitary

i.e. $\underline{B\bar{B}^T = I}$.

⊛ A is diagonalizable $\Leftrightarrow$ A has n L.I. eigen vectors.

⊛ $\underline{A \text{ has } k \text{ distinct eigen } \cancel{\text{vectors}} \text{ values}}$ (or) $\underline{\text{eigen vectors}}$ $\underline{v_1 \cdots v_k}$ $\Rightarrow \{v_1 \cdots v_k\}$ is L.I.

⊛ $A_{n\times n}$ has n distinct eigen values $\Rightarrow$ A is diagonizable.

↳ $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$

$\because A_{2\times 2}$ has two distinct eigen values then A is diagoolizable ($\because$ by theorem ③).

* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = P^{-1}DP$

$I = I^{-1}II$

$I = I$

$\Rightarrow A = I$ is diagonilizable.

(or) $(1-\lambda)(1-\lambda) = 0$

$\lambda = 1, 1$

$(A - \lambda I)x = 0$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0.x_1 + 0.x_2 = 0$

$\Rightarrow$ e.vectors $= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{matrix} x_1, x_2 \in \mathbb{R} \\ x_1 = x_2 \neq 0 \end{matrix} \right\}$

$= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\because$ P is invertible $\Rightarrow$ A is diagonilizable.

$\ast \hookrightarrow$ $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

eigen value $\lambda^2 = 0 \Rightarrow \lambda = 0$

$\hookrightarrow 0, 0$

$(A - 0I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0x_1 + 1 \cdot x_2 = 0$

$\Rightarrow \boxed{x_2 = 0}$

Eigen vector $= \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\middle|\; x_1 \neq 0 \right\}$

$$\therefore P = \begin{array}{c} \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix} \end{array}$$

$\det P = 0$

$\therefore P$ is not invertible

$\Rightarrow P$ doesnot have two L.I.

$\Rightarrow$ A is not diagonalizable

Note:- $A_{n \times n}$ is nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$.

$\ast$ A is nilpotent then A is not diagonizable.

Proof Let A is diagonilizable ; $A \neq 0$

$A = P^{-1}DP$ — ①

$A^k = P^{-1}D^kP$

$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ 0 & 0 & \cdots & d_n \end{bmatrix}$

$\because A^k = 0 \Rightarrow D = 0$

$\Rightarrow$ $0 = P^{-1}D^kP$

$\Rightarrow D^k = 0 \Rightarrow d_i = 0$

from ① $A = P^{-1}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}P$

$\Rightarrow A = 0 \Leftrightarrow$ this is contradiction to $A \neq 0$

$\Rightarrow$ A is not diagonizable.

Module : 03

* Numerical Method :

origin	approximate
π	$\frac{22}{7} = 3.14$

$$\boxed{\text{error} = (\text{original} - \text{approximation})}$$

* $f(x) = 0$,

finding roots, solving diff. equation.
solving diff. integration.

* **Intermediate value theorem** $f : \mathbb{R} \to \mathbb{R}$

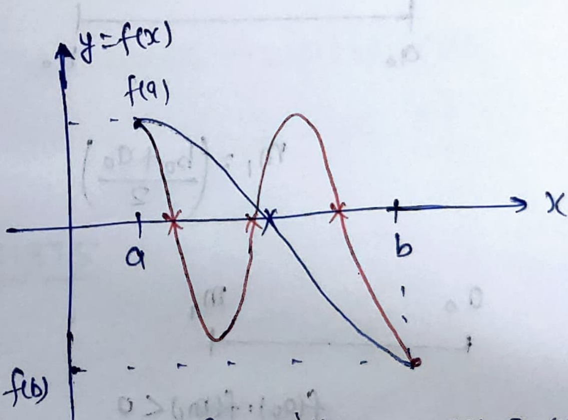
if f is continuous on $[a, b]$ and $f(a) \cdot f(b) < 0$ then f has 1 (or) odd number of roots.

let

$f(a) > 0$

$f(b) < 0$ $\Rightarrow$ $f(a) \cdot f(b) < 0$

$f(\xi) = 0$, ξ is a root of $f(x)$.



So either 1 or 3 or 5 or odd no. of Roots are there.

· **Iteration method**

$$a_0x^2 + a_1x + a_2 = 0$$

I.) x_0 is given

$$a_1x = -a_2 - a_0x^2$$

$$x = -\frac{a_2}{a_1} - \frac{a_0x^2}{a_1}$$

$$\boxed{x_{k+1} = -\frac{a_2}{a_1} - \frac{a_0}{a_1}x_k^2}$$

II.) $a_0x \cdot x = -a_1x - a_2$

$$\boxed{x_{k+1} = \frac{-a_1x_k - a_2}{a_0x_k}}$$

III.) $a_0x^2 = -a_1x - a_1$

$$\boxed{x_{k+1} = \sqrt{\frac{-a_1x_k - a_2}{a_0}}}$$

Eg. $f(x) = \cos x - xe^{x}$

$x \Rightarrow$	0	0.5	1
$f(x) \Rightarrow$	1	0.05	-2.17

sign change

$\because f(0.5) * f(1) < 0$

$\therefore f(x)$ has a root in $(0.5, 1)$.

Bisection method :

$f(x) = 0$

f has a root in (a_0, b_0)

$f(a_0) \cdot f(b_0) < 0$

a_0 —— b_0

$$m_1 = \left(\frac{b_0 + a_0}{2}\right)$$

a_0 —— m_1

$f(a_0) \cdot f(m_1) < 0$

$$\boxed{m_2 = \left(\frac{a_0 + m_1}{2}\right)}$$

Sly m_3 -- m_4 --

m_1 —— b_0

$f(m_1) \cdot f(b_0) < 0$

$$\boxed{m_2 = \frac{m_1 + b_0}{2}}$$

Sly m_3 -- m_4 --.

Eg Perform 5 iteration of the bisection method to obtain the root in $(0,1)$, when $f(x) = x^3 - 5x + 1 = 0$

$f(0) = 1$

$f(1) = -3$

$\rightarrow f(0) \cdot f(1) < 0$

$\therefore m_1 = \cancel{\frac{1+(-3)}{2}} \; \frac{0+1}{2} = \left(\frac{1}{2}\right)$

$f\left(\frac{1}{2}\right) = -1.375$

$f(0) \cdot f(1/2) < 0$

$$m_2 = \frac{0+0.5}{2} = 0.25$$

$$f(m_2) = -0.2343$$

$\Rightarrow$ $f(0) \cdot f(m_2) < 0$

$$m_3 = \frac{0+0.25}{2} = 0.125$$

$$\boxed{f(0.125) = 0.3769.}$$

Now; $f(m_2) \cdot f(m_3) < 0$

$$m_4 = \frac{0.25+0.125}{2}$$

$$m_4 = 0.1875$$

$$\boxed{f(m_4) = 0.06909}$$

$$f(m_4) \cdot f(m_2) < 0$$

$$m_5 = \frac{m_2 + m_4}{2} = 0.21875$$

$$f(m_5) = -0.08328$$

↳ Hence the root lies in (0.21875 and 0.25).

~~* Numerical~~

Newton Raphson method :-

$$\boxed{x_{k+1} = \phi(x_k)}$$

$$f(x) = a_0 x + a_1 = 0 \quad (2), \quad a_0 \neq 0$$

$$\Rightarrow a_1 = -a_0 x$$

Notation

$$f_k = f(x_k) = a_0 x_k + a_1 \quad — (1) \quad \Rightarrow \boxed{a_1 = -a_0 \cdot x_{k+1}} \quad — (3)$$

$$f_k' = a_0$$

$$(1) \Rightarrow f_k = f_k' x_k + a_1$$

$$f_k = f_k' x_k - a_0 x_{k+1} \quad \text{(from 3)}.$$

$$f_k = f_k' \cdot x_k - f_k' x_{k+1}$$

$$f_k = f_k'(x_k - x_{k+1})$$

$$x_k - x_{k+1} = \frac{f_k}{f_k'}$$

$$x_{k+1} = x_k - \frac{f_k}{f_k'}$$

$$\Rightarrow \boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}} \qquad k = 0, 1, 2 \ldots$$

Eg. 1 perform 4 iteration of N.R. method to find the approximate value of $(17)^{1/3}$ with initial approximation is $x_0 = 2$.

Sol^n

$$x = (17)^{1/3}$$

$$x^3 = 17$$

$$f(x) = x^3 - 17 = 0$$

$$f'(x) = 3x^2$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_1 = 2 - \frac{(-9)}{12} = 2.75$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.75 - \left(\frac{3.796875}{22.6875}\right)$$

$$x_2 = 2.582645$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.582645 - \frac{0.226384}{20.010165}$$

$$x_3 = 2.571331$$

$$x_4 = 2.571331 - \frac{0.000990}{(\quad)}$$

$$x_4 = 2.571281$$