Linear combination of vectors:

Let V be any vector space and let $v_1, v_2, v_3, \ldots, v_n$ be n vectors of V and $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be any n scalars then the sum $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ is called as linear combination of vectors of $v_1, v_2, \ldots, v_n$.

Eg: let $v_1, v_2, v_3 \in \mathbb{R}^3 = V$

$1, 3, 4 \in \mathbb{R} = \mathbb{F}$

then

$$\left.\begin{array}{l} v_1 + 3v_2 + 4v_3 \\ 3v_1 + v_2 + 4v_3 \\ 4v_1 + 3v_2 + v_3 \\ \vdots \\ v_1 + v_2 + v_3 \end{array}\right\} \text{These are all linear combinations of } v_1, v_2, v_3.$$

Span :- Let S be any non-empty subset of a vector space V, then the set of ~~vectors~~ all finite linear combination of vectors of S is called span of S. The span of S is denoted by $[S]$ or $\langle S \rangle$ or $L[S]$.

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span of S is also called as linear span of S.

$$[S] = \left\{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \;\middle|\; \begin{array}{l} v_1, v_2, \dots, v_n \in S, \\ \alpha_1, \alpha_2, \dots, \alpha_n \in F \end{array} \right\}$$

Eg:1. Let $S \subset V_3 = \mathbb{R}^3$ such that

$$S = \{ (1,2,1), (0,-1,3), (2,4,5) \}$$

Find span of S?

Sol:

$$[S] = \{ \alpha(1,2,1) + \beta(0,-1,3) + \gamma(2,4,5) \mid \alpha, \beta, \gamma \in F \}$$

$$= \left\{ (\alpha + 2\gamma,\ 2\alpha - \beta + 4\gamma,\ \alpha + 3\beta + 5\gamma) \;\middle|\; \alpha, \beta, \gamma \in F \right\}$$

Note: The order of $[S]$ is always infinite, where S is any subset of vector space.

Eg2. If $S = \{ (0,1,0), (1,0,0), (0,0,1) \} \subset V_3$ then find $[S]$?

$$[S] = \{ \alpha(0,1,0) + \beta(1,0,0) + \gamma(0,0,1) \mid \alpha, \beta, \gamma \in F \}$$

$$= \{ (\beta, \alpha, \gamma) \mid \alpha, \beta, \gamma \in F \}$$

$$= \mathbb{R}^3 \quad \text{if } F = \mathbb{R}$$

Here, $\mathbb{R}^3 = \{ (x, y, z) \mid x, y, z \in \mathbb{R} \}$

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Thm 2. let $S \neq \phi$ and $S \subset V$ (vector space) then span of S i.e., $[S]$ is always a subspace of vector space V.

Pf: let S be any non-empty subset of a vector space V, then

$$[S] = \left\{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \;/\; \begin{array}{l} v_1, v_2, \dots, v_n \in S, \\ \alpha_1, \alpha_2, \dots, \alpha_n \in F \end{array} \right\}$$

Now we have to show that $[S]$ is a subspace of V.

let $u, v \in [S]$

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \quad , \quad \alpha_i \in F$$

$$v = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n \quad , \quad \beta_i \in F$$

$\forall \; i = 1, 2, \dots n$

Now, $u + v = (\alpha_1 v_1 + \dots + \alpha_n v_n) + (\beta_1 v_1 + \dots + \beta_n v_n)$

$$= (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + \dots + (\alpha_n + \beta_n) v_n$$

$$= \gamma_1 v_1 + \gamma_2 v_2 + \dots + \gamma_n v_n$$

$\in [S]$ where $\gamma_i \in F$, $\forall \; i = 1, 2 \dots n$

$\Rightarrow \quad u + v \in [S]$

Next $\alpha \cdot u = \alpha \cdot (\alpha_1 v_1 + \dots + \alpha_n v_n)$

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$= \alpha . \alpha_1 V_1 + \alpha \alpha_2 V_2 + \dots + \alpha \alpha_n V_n$

$= \eta_1 V_1 + \eta_2 V_2 + \dots + \eta_n V_n$,

$\in [S]$

$\Rightarrow \alpha . u \in [S]$

Hence by Theorem 1, $[S]$ is a subspace of V.

Q Let $S = \{(1,2,1), (1,1,-1), (4,5,-2)\}$

Determine which of the following vectors are in $[S]$?

a) $(0,0,0)$ (b) $(1,1,0)$ (c) $(2,-1,-8)$

Solⁿ:

$[S] = \{ \alpha(1,2,1) + \beta(1,1,-1) + \gamma(4,5,-2) \,/\, \alpha, \beta, \gamma \in F \}$

$= \{ (\alpha + \beta + 4\gamma, 2\alpha + \beta + 5\gamma, \alpha - \beta - 2\gamma) \,/\, \alpha, \beta, \gamma \in F \}$

a) $(0,0,0) \in [S]$ for $\alpha = \beta = \gamma = 0$

b) Let $(1,1,-1) \in [S] \Leftrightarrow (1,1,0) = (\alpha+\beta+4\gamma, 2\alpha+\beta+5\gamma, \alpha-\beta-2\gamma)$

$\Leftrightarrow \alpha + \beta + 4\gamma = 1$ —— ①

$2\alpha + \beta + 5\gamma = 1$ —— ②

$\alpha - \beta - 2\gamma = 0$ —— ③

Adding ① & ③, $2\alpha + 2\gamma = 1 \Rightarrow \boxed{\alpha + \gamma = \tfrac{1}{2}}$

& subtracting ① from ②, $\boxed{\alpha + \gamma = 0}$ No solⁿ.

Hence $(1,1,-1) \notin [S]$

(c) Let $(2,-1,-8) \in [S]$

$\Leftrightarrow (2,-1,-8) = (\alpha+\beta+4\gamma,\ 2\alpha+\beta+5\gamma,\ \alpha-\beta-2\gamma)$

$$\Leftrightarrow \quad \alpha+\beta+4\gamma = 2 \quad \text{--- } (1)$$
$$2\alpha+\beta+5\gamma = -1 \quad \text{--- } (2)$$
$$\alpha-\beta-2\gamma = -8 \quad \text{--- } (3)$$

Using (1)+(3),

$2\alpha+2\gamma = -6$

$\boxed{\alpha+\gamma = -3}$

Using (2)−(1),

$\boxed{\alpha+\gamma = -3}$

Let $\gamma = k$ (scalar) then

$\alpha = -3-k$

from (1), $\beta = 2-(-3-k)-4k$

$= 5-3k$. ~~Hence~~

$(\alpha,\beta,\gamma) \longrightarrow (-3-k,\ 5-3k,\ k)$

Thus, $(2,-1,-8) \in [S]$. you have infinite choices

Ex:- for $k=0 \Rightarrow \alpha=-3,\ \beta=5,\ \gamma=0$

$(2,-1,-8) = -3(1,2,1)+5(1,1,-1)+0(4,5,-2)$

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Ex1. Let V be the Vector space of all 2x2 real matrices. Show that the sets

i) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

ii) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

Span V.

Ex2: Let V be the Vector space of all polynomials of degree ≤ 3. Determine whether or not the set

i) $S = \{ t^3, t^2, t, 1 \}$

ii) $S = \{ t^3, t^2+t, t^3+t+1 \}$

Span V. ?

Sol^n. ii) let $p(t) = \alpha t^3 + \beta t^2 + \gamma t + \delta$ be any polynomial (element) in V. We ~~need~~ need to find (if exist) scalars $\alpha_1, \alpha_2, \alpha_3$ s.t.

$$\alpha t^3 + \beta t^2 + \gamma t + \delta = \alpha_1 t^3 + \alpha_2 (t^2+t) + \alpha_3 (t^3+t+1)$$
$$= (\alpha_1 + \alpha_3) t^3 + \alpha_2 t^2 + (\alpha_2 + \alpha_3) t + \alpha_3 .$$

$\Rightarrow$ on comparing coefficient of various powers of t, we get $\alpha_1 + \alpha_3 = \alpha$, $\alpha_2 = \beta$, $\alpha_2 + \alpha_3 = \gamma$, $\alpha_3 = \delta$

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From the first three eq^n, we get

$$\left.\begin{aligned} \alpha_1 &= \alpha+\beta-\gamma \\ \alpha_2 &= \beta \\ \alpha_3 &= \gamma-\beta \end{aligned}\right\}$$

substituting all of these in the last eq^n, we obtain

$$\gamma-\beta = \delta$$

which may not be true for all elements of V.

If we take $t^3+2t^2+t+3 \in V$

here $\gamma-\beta \neq \delta$

so, this polynomial can not be written in the linear combination of elements of S.

$\therefore$ S does not span the vector space V.

<u>Thm</u> 3. Let S be any non-empty subset of a vector space V, then $[S]$ is the smallest subspace of V containing S.

<u>Pf</u>: Let $S = \{v_1, v_2, \ldots, v_n\} \subset V$

then $v_i = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 1 \cdot v_i + 0 \cdot v_{i+1} + \cdots + 0 \cdot v_n$

$\Rightarrow v_i \in [S] \quad \forall \ i = 1, 2, \ldots, n$

$\Rightarrow S \subset [S]$

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Now, we have to prove that $[S]$ is the <u>smallest</u> <u>subspace</u> of V containing S.

By Thm 2., $[S]$ is always a subspace of S.

Now, let T be any other subspace of V containing S i.e. $S \subset T$.

So, we have to show that $[S] \subset T$

Let $u \in [S]$

$\Rightarrow \quad u = \alpha_1 v_1 + \cdots + \alpha_n v_n \quad , \quad \alpha_i \in F$ and $v_1, v_2, \ldots, v_n \in S$

$\because \quad S \subset T$

$\Rightarrow \quad v_1, v_2, \ldots, v_n \in T$ and also T is subspa

$$\Rightarrow \quad \left.\begin{array}{l} \alpha_1 v_1 \in T \\ \alpha_2 v_2 \in T \\ \vdots \\ \alpha_n v_n \in T \end{array}\right\}$$

$\Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in T$

$\Rightarrow \quad u \in T$

$\Rightarrow \quad [S] \subset T$.

Hence $[S]$ is the smallest subspace containing S.

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Important results:-

1. Let U and W are subspaces of a vector space V, then

   a) $U \cap W$ is a subspace of V.

   b) $U \cup W$ is a subspace of $V \iff U \subset W$ or $W \subset U$.

2. Let $U_1, U_2, \ldots, U_n$ be n-subspaces of a vector space V. Then

$$U_1 \cap U_2 \cap \ldots \cap U_n = \bigcap_{i=1}^{n} U_i$$ is also a subspace of V.

3. If S is non-empty subset of V, then $[S]$ is the intersection of all subspaces of V containing S.

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Linear dependence & independence of vectors

Let V be a vector space. A finite set $\{v_1, v_2, \dots, v_n\}$ of the elements of V is said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0.$$

If the above equation is satisfied only for $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$, then the set of vectors is said to be linearly independent.

Pb: Check whether the following set of vectors are L.D or L.I over the field $\mathbb{R}$.

1) $\{(1,0), (0,1)\}$
2) $\{(3,2), (2,3)\}$
3) $\{(1,-1,0), (0,1,-1), (0,0,1)\}$
4) $\{(1,-1,0), (0,1,-1), (0,2,1), (1,0,3)\}$
5) $\{(1,0,1), (1,1,0), (1,-1,1), (1,2,-3)\}$
6) $\{(1,0,0), (0,1,0), (0,0,1)\}$

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Res 4. Let V be any vector space then

① $v \in V$ is L.D $\Leftrightarrow$ $v = 0$

② $v_1, v_2 \in V$ are L.D $\Leftrightarrow$ v_1 and v_2 are colinear i.e., one is scalar multiple of other.

③ $v_1, v_2, v_3 \in V$ are L.D $\Leftrightarrow$ they are coplanar i.e., one of them is the linear combination of other two.

④ If S is any subset of vector space V such that $0 \in S$, then S is L.D.

Res 5 Let V be a vector space and S be any non-empty subset of V. Then

1) If S is L.I, then every subset of S is L.I.

2) If S is L.D, then every ~~sub~~ superset of S is L.D.

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Basis & Dimensions

<u>Basis</u> :- let V be any vector space, then $B \subset V$ is said to form Basis of V, if the following two conditions are satisfied.

1) B is L.I

2) B spans V or B generates V or $[B] = V$.

Ex 1. let $B_1 = \{(1,0),(0,1)\} \subset \mathbb{R}^2$. Is B_1 is a basis of $\mathbb{R}^2$.

1) <u>B_1 is L.I</u> : for $\alpha, \beta \in F = \mathbb{R}$

let $\alpha(1,0) + \beta(0,1) = (0,0)$

$\Rightarrow (\alpha, \beta) = (0,0)$

$\Rightarrow \alpha = 0 = \beta$. Hence B_1 is L.I

2) $[B_1] = \{\alpha(1,0) + \beta(0,1) \mid \alpha, \beta \in R\}$

$= \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}$

$= \mathbb{R}^2$

$\therefore$ B_1 is a basis of $\mathbb{R}^2$. Also, B_1 is called standard Basis.

Ex 2. Is $B_2 = \{(-1,0),(2,1)\} \subset \mathbb{R}^2$ is a basis of $\mathbb{R}^2$.

1) <u>B_2 is L.I</u> : Let $\alpha, \beta \in \mathbb{R}$ s.t.

$\alpha(-1,0) + \beta(2,1) = (0,0)$

$\Rightarrow (-\alpha + 2\beta, \beta) = (0,0)$

$\beta = 0$, $\alpha = 0$. B_2 is L.I

2) $[B_2] = \{\alpha(-1,0) + \beta(2,1) \mid \alpha, \beta \in \mathbb{R}\}$

$= \{(-\alpha + 2\beta, \beta) \mid \alpha, \beta \in \mathbb{R}\} = \mathbb{R}^2$.

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Here $[B_2] = R^2$, Since every element of $\mathbb{R}^2$ can be written in the linear combination of $(-1,0)$ and $(2,1)$. Thus B_2 is a basis of $\mathbb{R}^2$.

Here, we observe that B_1 and B_2 are two basis of $\mathbb{R}^2$.

<u>Note</u>: Basis of a vector space is not unique.

<u>Standard Basis of $V_n = \mathbb{R}^n$.</u>

Let $e_1 = (1,0,0,\dots,0)$
$e_2 = (0,1,0,\dots,0)$
$\vdots$
$e_n = (0,0,0,\dots,0,1)$

Now, take a set $S = \{e_1, e_2, \dots, e_n\}$

As $S \subset V_n$

1) It can be easily verified that S is L.I.

2) $[S] = V_n$

Thus, S is called as standard basis of V_n.

Ex: 1) Standard basis of V_3 is $\{(1,0,0),(0,1,0),(0,0,1)\}$

2) " " " V_4 is $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$

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Standard basis of $M_{m\times n}(\mathbb{R})$

$$E_{11} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}, \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}, \ \cdots\cdots$$

$$E_{ij} = \begin{cases} 1 & , \text{ at } i^{th}\text{-row and } j^{th}\text{- column} \\ 0 & , \text{ else.} \end{cases}$$

Coordinate Vector:- let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of vector space V, then $v \in V$ is a vector that can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

then $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is called as the coordinate vector of V w.r.t. basis B.

Pb Find coordinate vectors of $(1, 4, 3)$ w.r.t.

1) standard basis of V_3.
2) $B = \{(-1,2,0), (0,1,0), (1,-1,1)\}$ basis of V_3.

Sol(2) $(1,4,3) = \alpha(-1,2,0) + \beta(0,1,0) + \gamma(1,-1,1)$

$(1,4,3) = (-\alpha+\gamma,\ 2\alpha+\beta-\gamma,\ \gamma)$

$\Rightarrow \boxed{\gamma = 3}$, $-\alpha+\gamma = 1$ and $2\alpha+\beta-\gamma = 4$

$\Rightarrow \boxed{\alpha = 2}$ $\qquad \boxed{\beta = 3}$

Thus, the $(2, 3, 3)$ is the coordinate vector.

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Dimension :- Let V be a vector space, then the number of elements in the basis of V is known as dimension of V.

Note : Dimension of a vector space is unique.

Remark : A vector space may have infinite dim., finite dim. or zero dimension.

- A vector space with infinite dimension is called as infinite dimensional vector space.

  Eg: $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \ldots \ldots \infty$

- A vector sp. with finite dim. is called as finite dimensional vector space.

  Eg: $V_1, V_2, V_3, \ldots, V_n$

  V_1 → one dim.

  V_2 → two dim.

  V_n → n-dim.

Example: 1) Dim. of $\mathbb{R}^n$ over $\mathbb{R} = n$

2) Dim of $\mathbb{C}^n$ over $\mathbb{C} = n$

3) Dim. of $\mathbb{C}^n$ over $\mathbb{R} = 2n$

4) Dim. of $M_{n\times n}(\mathbb{R})$ over $\mathbb{R} = n$

- $\{0\}$ is an example of zero dim. vector space of $\mathbb{R}$.

  Here, Basis $= \{\ \} = \phi$.

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<u>Thm 4.</u> Let V be a vector space which is spanned by a finite set of vectors $\{v_1, v_2, \dots, v_n\}$. ~~Then~~ If $\{w_1, w_2, \dots, w_m\}$ is L.I set of vectors of V, then $m \leq n$.

<u>OR</u> [The largest L.I subset of a vector space V is its basis itself.]

<u>Cor 1.</u> If V is a finite-dimensional vector space, then any two bases of V have the same (finite) number of elements.

<u>Cor 2.</u> Let V be a finite-dimensional vector space and let $n = \dim V$. Then

a) any subset of V which contains more than n vectors is linearly dependent.

b) no subset of V which contains less than n vectors can span V.

OR [Basis is the largest L.I. subset of V and the smallest set which can span V.]

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~~Addition of sets~~. Additions of sets in a V. sp.

let A and B be two subset of V sp. V. Then sum of A and B written as A+B, is the set of all vectors of the form u+v, $u \in A$, $v \in B$ i.e;

$$A+B = \{u+v \mid u \in A,\ v \in B\}$$

Ex:1. let $A = \{(1,2),\ (0,1)\}$ and
$B = \{(1,1), (-1,2), (2,5)\}$. Then $A+B = ?$ and $A \cup B = ?$

Sol^n: $A+B = \{(1,2)+(1,1),\ (1,2)+(-1,2),\ (1,2)+(2,5),\ (0,1)+(1,1),$
$(0,1)+(-1,2),\ (0,1)+(2,5)\}$

$= \{(1,3), (0,4), (3,7), (1,2), (-1,3), (2,6)\}$

$A \cup B = \{(1,2), (0,1), (1,1), (-1,2), (2,5)\}$

Difference bet^n $A+B$ and $A \cup B$

let $A = \{(x,0) \mid x \in \mathbb{R}\}$ and $B = \{(0,y) \mid y \in \mathbb{R}\}$

$A+B = \{(x,0)+(0,y) \mid x,y \in \mathbb{R}\} = \{(x,y) \mid x,y \in \mathbb{R}\} = \mathbb{R}^2$.

$A \cup B = \{(x,0) \cup (0,y) \mid (x,0) \in A, (0,y) \in B\} = \{(x,0), (0,y) \mid x,y \in \mathbb{R}\}$

i.e; if A is x-axis and B is y-axis then A+B is generating whole plane while $A \cup B$ is just x and y-axis.

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Sum / Direct sum of two subspaces of a V-Sp.

Thm 2 If A and B are two subspaces of vector space V, then $A+B$ is subspace of V and also

$$A+B = [A \cup B] \text{ i.e. } span(A \cup B)$$

Pf: To prove $A+B$ subspace of V

$$\boxed{A+B = \{a+b \mid a \in A, b \in B\}}$$

$\because$ A, B subspace of V

$\Rightarrow$ $0 \in A$ and $0 \in B$

$\Rightarrow$ $0 = 0 + 0 \in A+B$

$\Rightarrow$ $A+B \neq \phi$

let $a+b \in A+B$ and $c+d \in A+B$

also $\alpha, \beta \in F$.

then $\alpha(a+b) + \beta(c+d) = \underbrace{(\alpha a + \beta c)}_{\in A} + \underbrace{(\alpha b + \beta d)}_{\in B}$

$\in A+B$

$\Rightarrow$ $A+B$ is subspace of V.

To prove $A+B = [A \cup B]$

Since $A+B \subseteq [A \cup B]$ as every element of $A+B$ will be written as finite linear combination of elements of $A \cup B$.

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Now, we will prove that $[A \cup B] \subseteq A+B$.

Let $v \in [A \cup B] = \left\{ \sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j \;\middle|\; a_i \in A,\ b_j \in B,\ \alpha_i, \beta_j \in F \right\}$

So,

$$v = \underbrace{\sum_{i=1}^{n} \alpha_i a_i}_{\in A} + \underbrace{\sum_{j=1}^{m} \beta_j b_j}_{\in B}$$

$$\in A+B$$

$$\Rightarrow [A \cup B] \subseteq A+B.$$

Hence, $A+B = [A \cup B]$ //

$[A \cup B] = \left\{ \sum_{i=1}^{n} \alpha_i v_i \;\middle|\; v_i \in A \cup B \right\}$

If $v \in [A \cup B]$

$v = \sum_{i=1}^{n} \alpha_i v_i$, $v_i \in A \cup B$

$= \underbrace{\sum \beta_i a_i}_{\in A} + \underbrace{\sum \gamma_i b_i}_{\in B}$,

$\in$ ~~A+B~~ $A+B$

<u>Direct sum</u>:-

We have seen that if A and B are subspaces of vector space V then the sum $A+B$ is also a subspace of V.

If in addition $A \cap B = \{0\}$, then the sum $A+B$ is called direct sum and written as $A \oplus B$.

- The advantage of direct sum is that any element z of $A \oplus B$ will be uniquely represented as $\underbrace{z = a+b}$, $a \in A$, $b \in B$

This representation is unique for $z \in A \oplus B$.

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Ex1 let vector space $V = \mathbb{R}^3$
let U be xy-plane and W be yz plane.
then $U+W$ is ?

$U = \{(a,b,0) \mid a,b \in \mathbb{R}\}$

$W = \{(0,b,c) \mid b,c \in \mathbb{R}\}$

$U+W = \{(a,b,c) \mid a,b,c \in \mathbb{R}\}$

$= \mathbb{R}^3$

But $\mathbb{R}^3$ will not be direct sum of U & W.

Since $U \cap W = \{(0,b,0) \mid b \in \mathbb{R}\}$

$\neq \{(0,0,0)\}$

Ex2 If $U = \{(a,b,0) \mid a,b \in \mathbb{R}\} = xy$-plane
and $W = \{(0,0,c) \mid c \in \mathbb{R}\} = z$-axis.

Then $\mathbb{R}^3 = U \oplus W$ as $U \cap W = \{(0,0,0)\}$

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Thm 6 In an n-dimensional vector space, any set of n-linearly independent vectors is a basis.

Pf. Let V be an n-dim. vector sp.

Then every basis of V contains n elements.

Let $B = \{v_1, v_2, \ldots, v_n\}$ be an L.I subset of V.

Let $v \in V$, then the set

$$B' = \{v_1, v_2, \ldots, v_n, v\} \text{ is L.D}$$

$\Rightarrow$ One of vectors of B' can be written as the linear combination of other vectors.

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \quad , \alpha_i \in F$$

$\Rightarrow \quad v \in [B]$

$\Rightarrow \quad V \subseteq [B] \quad (\because v \text{ is arbitrary element of } V)$

But $[B] \subseteq V$ (always)

$\Rightarrow \quad [B] = V$

Hence, B is a basis.

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Ex Prove that set $\{(1,1,1), (1,-1,1), (0,1,1)\}$ is a basis of V_3.

Pb1. Which of the following subsets form a basis for V_3? In case, if S is not a basis for V_3, determine a basis for $[S]$?

1) $S_1 = \{(1,2,3), (3,1,0), (-2,1,3)\}$

2) $S_2 = \{(1,1,1), (1,2,3), (-1,0,1)\}$

Sol 1) Let $\alpha, \beta, \gamma \in \mathbb{R}$ s.t.

$$\alpha(1,2,3) + \beta(3,1,0) + \gamma(-2,1,3) = (0,0,0)$$

$$\left.\begin{aligned} \alpha + 3\beta - 2\gamma &= 0 \\ 2\alpha + \beta + \gamma &= 0 \\ 3\alpha + 3\gamma &= 0 \end{aligned}\right\} \qquad \begin{aligned} \beta &= \gamma \\ \alpha &= -\gamma \end{aligned}$$

Thus, S_1 is L.D. Hence, S_1 is not a basis of V_3.

Then the basis of $[S_1]$ is

$B = \{(1,2,3), (3,1,0)\}$.

Since $(-2,1,3) = \alpha(1,2,3) + \beta(3,1,0)$ for $\alpha = 1$, $\beta = -1$

ii) let $\alpha, \beta, \gamma \in \mathbb{R}$ s.t.

$$\alpha(1,1,1) + \beta(1,2,3) + \gamma(-1,0,1) = (0,0,0)$$

$$\left.\begin{aligned} \alpha + \beta - \gamma &= 0 \\ \alpha + 2\beta &= 0 \\ \alpha + 3\beta + \gamma &= 0 \end{aligned}\right\} \qquad \begin{aligned} \alpha &= 2\gamma \\ \beta &= -\gamma \end{aligned}$$

Thus, S_2 is L.D. Hence S_2 is not a basis of V_3.

Then the basis of $[S_2]$ is

$$B = \{(1,1,1), (1,2,3)\}$$

Since, $(-1,0,1) = -2(1,1,1) + 1(1,2,3)$.

Pb 2. Extend the set $\{(3,-1,2)\}$ to two different basis of V_3.

Sol: Given $S = \{(3,-1,2)\}$

$$\therefore [S] = \{\alpha(3,-1,2) \mid \alpha \in \mathbb{R}\}$$

$$= \{(3\alpha, -\alpha, 2\alpha) \mid \alpha \in \mathbb{R}\}$$

$$\because (0,0,1) \notin [S]$$

Hence $S_1 = \{(3,-1,2), (0,0,1)\}$ is L.I.

Now, $[S_1] = \{\alpha(3,-1,2) + \beta(0,0,1) \mid \alpha, \beta \in \mathbb{R}\}$

$$= \{(3\alpha, -\alpha, 2\alpha + \beta) \mid \alpha, \beta \in \mathbb{R}\}$$

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$(0,1,0) \notin [S_1]$

$(-1,0,0) \notin [S_1]$

Then

$S_2 = \{(3,-1,2),(0,0,1),(0,1,0)\}$ is L-I and hence forms a basis for V_3.

Smly, $S_3 = \{(3,-1,2),(0,0,1),(-1,0,0)\}$ is L-I and hence forms another basis for V_3.

(Extension-Theorem)

Thm 7: Let the set $\{v_1, v_2, \ldots, v_k\}$ be a L-I subset of an n-dimensional vector space V, then we can find vectors $v_{k+1}, \ldots, v_n$ in V such that the set $\{v_1, v_2, \ldots, v_k, \ldots, v_n\}$ is a basis for V.

Pf: Given $S = \{v_1, v_2, \ldots, v_k\}$ is L-I subset of V_n.

If $k = n$, then S is a basis of V_n (by Thm 6.)

If $k > n$, not possible ($\because S$ becomes L.D).

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For $k < n$, $S = \{v_1, v_2, \ldots, v_k\}$ is L.I

but $[S] \neq V$.

let $v_{k+1} \in V$ such that $v_{k+1} \notin [S]$

then the set $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$ is again L.I.

Take another $v_{k+2} \in V$ such that

$v_{k+2} \notin [v_1, v_2, \ldots, v_k, v_{k+1}]$

then set $\{v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}\}$ is L.I

proceeding Inductively, we get

set $\{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ as L.I

subset of V_n. Hence forms a basis for V_n.

(Dimension Theorem)

Theorem 3.8: If U and W are the subspaces of finite dimensional vector space V, then

$$\boxed{\dim(U+W) = \dim U + \dim W - \dim(U \cap W)}$$

Pf: As we know if U and W are subspace of V then $U \cap W$ is subspace of both U and W.

let $\dim(U) = m$, $\dim W = n$ and $\dim(U \cap W) = r$.

Suppose $S = \{v_1, v_2, \ldots, v_r\}$ be the basis of $U \cap W$.

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By extension theorem, S may be extended to the basis of U, W. ~~and~~

Let $B_U = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_{m-r}\}$

and $B_W = \{v_1, v_2, \ldots, v_r, \omega_1, \omega_2, \ldots, \omega_{n-r}\}$

~~and $B = \{v_1, v_2,$~~

Let $B = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_{m-r}, \omega_1, \omega_2, \ldots, \omega_{n-r}\}$

Here B has exactly $(m+n-r)$ element. And hence theorem is proved if we show B is a basis of $U+W$.

To prove B is basis of $U+W$, we have to show

a) B is L·I in $U+W$

b) B span $U+W$ i.e. $[B] = U+W$.

<u>B - L·I :-</u>

Let $a_1v_1 + \cdots + a_rv_r + b_1u_1 + \cdots + b_{m-r}u_{m-r} + c_1\omega_1 + \cdots + c_{n-r}\omega_{n-r} = 0$

where a_i, b_i, c_i are scalars.

$$\Rightarrow \sum_{i=1}^{r} a_i v_i + \sum_{j=1}^{m-r} b_j u_j + \sum_{k=1}^{n-r} c_k \omega_k = 0$$

$$\Rightarrow \underbrace{\sum a_i v_i + \sum b_j u_j}_{\in U} = \overbrace{-\sum c_k \omega_k}^{\in W} \quad \text{—— (1)}$$

$$= v \text{ (say)}$$

Since LHS of ① is in $U \Rightarrow v \in U$
$\therefore$ RHS of ① is in $W \Rightarrow v \in W$

Combinedly $v \in U \cap W$

$\therefore \quad v = d_1 v_1 + d_2 v_2 + \dots + d_r v_k$

$\Rightarrow \quad -\sum c_k w_k = \sum d_i v_i$

$\Rightarrow \quad \sum d_i v_i + \sum c_k w_k = 0$

As $\{v_i, w_k\}$ "B_W" are basis of W.

Therefore all scalars $d_i = c_k = 0$, $i = 1, \dots r$, $k = 1, \dots n-r$

Thus eq^n ① becomes

$$\sum a_i v_i + \sum b_j u_j = 0$$

As again $\{v_i, u_j\}$ "B_U" are L.I as basis of U.

$\Rightarrow \quad a_i = 0, \ b_j = 0 \quad \forall \ i = 1, \dots, r$, $j = 1, 2, \dots, m-r$.

Thus all $a_i, b_j, c_k = 0 \Rightarrow$ Elements of B are L.I.

<u>B-span (U+W)</u> let $z \in U+W$ then

$z = u + w, \quad u \in U, \ w \in W$

$= \left(\sum \alpha_i v_i + \sum \beta_j u_j\right) + \left(\sum \alpha_i' v_i + \sum \gamma_k w_k\right)$

$= \sum (\alpha_i + \alpha_i') v_i + \sum \beta_j u_j + \sum \gamma_k w_k$

$\Rightarrow \quad U + W \subseteq [B]$ and $[B] \subseteq U + W$ (always)

$\Rightarrow \quad [B] = U + W$.

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Pb1. What are the dim. of the following subspaces of $\mathbb{R}^3(\mathbb{R})$?

1) $A_1 = \{(x, y, z) \mid x - y = 0,\ y + z = 0\}$

2) $A_2 = \{(x, y, x) \mid x, y \in \mathbb{R}\}$

3) $A_3 = \{(x, y, z) \mid 7x + 9y + 4z = 0\}$

4) $A_4 = \{(0, y, z) \mid y, z \in \mathbb{R}\}$

Sol 1) $x - y = 0 \Rightarrow x = y$

$y + z = 0 \Rightarrow z = -y$

so, $A_1 = \{(y, y, -y) \mid y \in \mathbb{R}\}$

Basis $= \{(1, 1, -1)\}$

dim $= 1$

2) $A_2 = \{(x, y, x) \mid x, y \in \mathbb{R}\}$

Basis $= \{(1, 0, 1), (0, 1, 0)\}$

dim $= 2$

3) $A_3 = \{(x, y, z) \mid 7x + 9y + 4z = 0\}$

$= \left\{\left(x, y, -\left(\frac{7x + 9y}{4}\right)\right) \mid x, y \in \mathbb{R}\right\}$

Basis $= \{(1, 0, -\frac{7}{4}), (0, 1, -9/4)\}$

dim $= 2$.

4) Basis $= \{(0, 1, 0)\ (0, 0, 1)\}$

dim $= 2$

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P62. Consider vector space $M_n(\mathbb{R})$. What are the dimension of its following subspaces?

1) $A_1 = \{ A \in M_n(\mathbb{R}) \mid A' = A \}$

2) $A_2 = \{ A \in M_n(\mathbb{R}) \mid A' = -A \}$

3) $A_3 = \{ A \in M_n(\mathbb{R}) \mid trace(A) = 0 \}$

where, $trace(A) = \sum_{i=1}^{n} a_{ii}$

4) $A_4 = \{ A \in M_n(\mathbb{R}) \mid \sum_{j=1}^{n} a_{ij} = 0 \quad \forall i \}$

5) $A_5 = \{ A \in M_n(\mathbb{R}) \mid \sum_{i=1}^{n} a_{ij} = 0 \quad \forall j \}$

6) $A_6 = \{ A \in M_n(\mathbb{R}) \mid \sum_{i=1}^{n} a_{ij} = 0 \ \forall j \text{ and } \sum_{j=1}^{n} a_{ij} = 0 \ \forall i \}$

7) $A_7 = \{ A \in M_n(\mathbb{R}) \mid A' = A \text{ and } tr(A) = 0 \}$

AS62. Ans

$\dim M_n(\mathbb{R}) = n^2$

1) $\dim(A_1) = \frac{n(n+1)}{2}$

2) $\dim A_2 = \frac{n(n-1)}{2}$

3) $\dim A_3 = n^2 - 1$

4) $\dim A_4 = n^2 - n$

5) $\dim A_5 = n^2 - n$

6) $\dim A_6 = (n-1)^2$

7) $\dim A_7 = \frac{n(n+1)}{2} - 1$

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Pb3. Consider Vector space $M_n(\mathbb{R})$ and it's subspaces

$$W_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a+b=0 \,,\; c+d=0 \right\}$$

$$W_2 = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

then find

1) $\dim W_1$ 2) $\dim W_2$

3) $\dim (W_1 \cap W_2)$ 4) $\dim (W_1 + W_2)$

5) Give a basis of $(W_1 + W_2)$.

Solⁿ: $\dim(M_2(\mathbb{R})) = 4$

1) $\dim W_1 = 4 - 2 = 2$

2) $\dim W_2 = 1$

3) $$W_1 \cap W_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{array}{l} a+b=0,\; c+d=0 \\ b=0,\; c=0 \text{ and } a=d \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$\dim (W_1 \cap W_2) = 0$.

(Here Basis of $(W_1 \cap W_2) = \phi$)

4) $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$= 2 + 1 - 0 = 3$

5) $$B_{W_1} = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

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$B_{W_2} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

So, $B_{W_1+W_2} = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Pb 4 Consider a vector space $P_{10}(\mathbb{R})$ and it's subspaces.

$$P_{10}(\mathbb{R}) = \left\{ \sum_{i=0}^{10} a_i x^i = p(x) \,/\, a_i \in \mathbb{R} \right\}$$

↳ (polynomial in x with degree upto 10.)

Find dim of the following

1) $W_1 = \{ P(x) \mid P(0) = 0 \}$

2) $W_2 = \{ P(x) \in P_{10}(\mathbb{R}) \mid P(1) = 0 \}$

3) $W_3 = \{ P(x) \in P_{10}(\mathbb{R}) \mid P(1) = 0,\ P(2) = 0,\ P(3) = 0 \}$

4) $W_4 = \{ P(x) \in P_{10}(\mathbb{R}) \,/\, \frac{d}{dx}(P(x)) = 0 \}$

Sol^n: $\dim P_{10}(\mathbb{R}) = 10 + 1 = 11$

Basis $= \{ 1, x, x^2, \ldots, x^{10} \}$

1) If $P(0) = 0 \Rightarrow a_0 = 0$

So, $P(x) = a_1 x + a_2 x^2 + \ldots + a_{10} x^{10}$

Basic$(W_1) = \{ x, x^2, \ldots, x^{10} \}$

$\dim(W_1) = 10$

2) $\dim W_2 = 11 - 1 = 10$ ← No. of L.I restriction.

3) $\dim W_3 = 11 - 3 = 8$

4) If $\frac{d}{dx}(P(x)) = 0 \Rightarrow P(x) =$ constant poly. So, $\dim W_4 = 1$

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