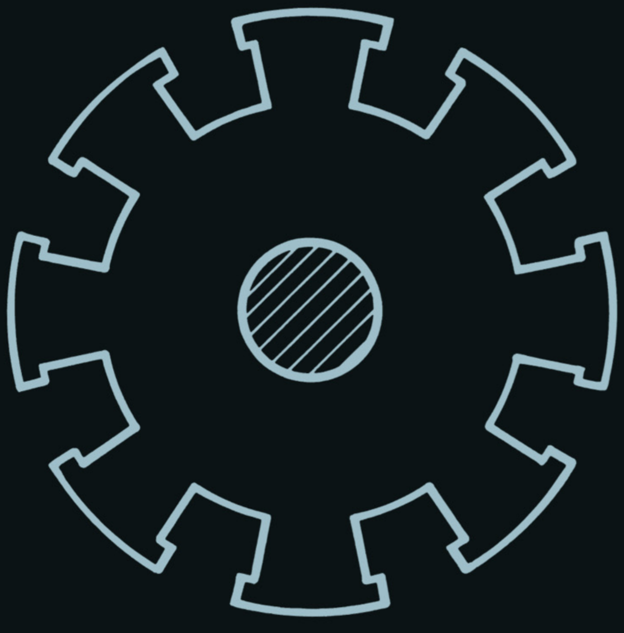
Electrical Machine Dynamics



D. P. Sen Gupta
and J.W. Lynn

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D. P. SEN GUPTA

Department of Electrical Engineering
The Indian Institute of Science
Bangalore
India

and

J. W. LYNN

Formerly
Department of Electrical Engineering and Electronics
The University of Liverpool
Liverpool
England

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Contents

Preface

The book is designed to give a straightforward, logical presentation of the methods available for predicting the dynamic behaviour and response of electrical machines. Three particular aspects of machine theory have been utilised to achieve this.

1. The early chapters contain a simple presentation of electromagnetic theory underlying the concepts of self and mutual inductance in saturable machines containing several windings.
2. The generalised form of electrical machine theory is developed in simple matrix terms. The d.c. machines, induction motor and alternator are treated in detail. Extension to other types of machines not treated in the book is straightforward.
3. A torque-balance analysis of machines is presented, from which the dynamic response and the synchronising and damping forces can be investigated and computed. The effect of a generator voltage regulator is introduced and its influence on power system dynamic response is discussed.

The material is developed to the level of final year honours undergraduate and first year post-graduate students in electrical engineering. The authors also had in mind power system engineers concerned with the design, performance and control of turbo-alternators and large salient-pole generators for hydro-electric schemes, together with ancillary electrical drives, and mechanical engineers concerned with industrial electrical machine systems.

In presenting the dynamical analysis of electrical machine systems, the authors are very conscious that a vast amount of important machine theory has been taken for granted, or only very briefly discussed in the text. The book is clearly based on the work of three or four generations of outstanding electrical machine engineers who provided industry with a wide variety of convenient, easily controlled and highly efficient drive systems. Names such as Miles

Walker, Say, Gibbs, Adkins, Concordia, Park, Alger and Kron, to mention only a few well known to students, span both post-war and pre-war technologies. The authors hope that the present book may make a contribution to the dynamical aspects of machine and power system engineering.

They wish to express their thanks to their former colleagues, Drs C. V. Jones and B. W. Hogg for continuing interest and discussion. They are very much indebted to past research students who carried out the experimental work and computation for virtually all of the results presented here. Dr M. Y. M. Yau developed the earlier work on regulator effects in Chapters 7 and 8, in the laboratories at the University of Liverpool, and the more recent work was done by Mr N. G. Narahari at the Indian Institute of Science, Bangalore. The electromechanical studies using the alternator and induction motor in the laboratory were carried out by Dr J. S. Gorley and the Ward–Leonard system dynamics by Dr M. W. Main, both at that time at the University of Liverpool. The latter experiments were suggested by Mr H. J. T. Whitehead and Mr J. Warnock of B. N. F. L. Windscale in discussions on power system dynamics and the authors wish to acknowledge the valuable insight into practical power system dynamic problems which emerged from these talks.

Mrs Joan Johnston carefully and patiently typed the manuscript and her assistance throughout is gratefully acknowledged. The authors wish to express their thanks to Mr Malcolm Stewart and Mr Richard Powell of The Macmillan Press Ltd for their guidance at all stages in the preparation of the manuscript.

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J.W.L.

1

Introduction

1.1 ELECTRICAL MACHINES

Following the discovery of the phenomenon of electromagnetic induction in the mid-nineteenth century, the design and construction of electrical machines developed very rapidly. By the beginning of the twentieth century, industry was using machines which were efficient, mechanically strong, and beautifully made—indeed many of these machines are still in excellent working condition. For about twenty years from 1900, much research and development went into optimising cost and reducing loss. Design techniques were largely tailored to this end. Here one must refer to the early voluminous work of Miles Walker, *The Specification and Design of Dynamo-electric Machinery* (Longman Green, London, 1915).

In the early part of this century, control gear was simple and robust, to meet conditions of starting, running, and load and speed control in steady state operation. The Tirrill voltage regulator and the centrifugal governor, in improved design, were almost the only pieces of equipment that could be classed as automatic. Control equipment for prime movers for generation in power stations was again fairly simple but reliable, in service with large triple-expansion reciprocating engines and the first generation of steam turbines.

In the first three or four decades, the problems of operation in the steady state had largely been satisfactorily solved and efficiency was the main concern of designers and operators. There was, on the whole, adequate protection against the effects of transient disturbances and faults, although on some of the large systems dynamic instability did on occasion cause serious trouble. Thus for half a century, electrical machine engineering met the demands of industry very creditably indeed and research, teaching, and design in industrial applications were of a high standard. With the more recent introduction of fast-acting static and solid-state devices in regulators, excitation systems, and control equipment, the dynamic response of the machines has become of increasing importance.

Research in these areas has been greatly assisted by the concurrent development of the large fast digital computers.

The design of an electrical machine is a highly specialised task, with a number of iterative processes involved in the correlation of all the variables. The main two features are, of course, the current-carrying copper coils in slots on the rotor and stator and the iron circuits carrying the necessary magnetic field flux, in which both the stator and rotor coils are immersed. In a motor one set of coils, say those on the stator, produces a magnetic field and current flowing in the other set interacts with this to produce a rotational couple on the rotor. The roles of the stator and rotor coils may be reversed in some machines. In a generator, torque is applied to the shaft, with the consequent generation of voltage in the rotating coils. Again the roles of the rotor and stator coils may sometimes be reversed.

The basic quantities which are usually specified in the design of a machine are the operating voltage, speed, rotor torque, power, and efficiency. The machine is designed to meet the required values of these under all normal operating conditions, within given tolerance limits. It should also be of sufficient mechanical strength and should not suffer any serious damage under forseeable fault conditions external to the machine. The copper power-loss due to coil resistance, the iron power-loss due to flux variations, and mechanical loss due to windage and friction all tend to reduce the efficiency of the machine (which is usually about 85–95 per cent), and in a good design these losses are kept to a minimum.

Electrical power is proportional to the product of voltage and current, and an electrical machine which is designed for a given power output and speed would have a wide range of possible dimensions. It could be an 'iron-dominant' machine, with a relatively large amount of iron, high flux values, high generated voltage and smaller gauge coils. On the other hand it could be a 'copper-dominant' machine with smaller iron circuit, lower magnetic flux values, lower voltage, and correspondingly greater current and larger copper coils. Very often the operating voltage and speed are the main constraints upon the design. If a generator is to be operated in isolation from others then the voltage is not so important, provided that the load equipment is obtainable at the specified voltage. However, most machines will be designed for standard values of impressed or generated voltage, to allow for interchangeability of equipment.

Aircraft alternators operate at high speeds (about 8000–10000 r.p.m.) driven from the engines and they tend to be of the lower flux, lower voltage (about 100 V) heavy current type. There is a saving in weight with a high-speed low voltage 'copper machine' as compared with one which is designed with higher flux values and larger iron paths.

The shape of an electrical machine is largely determined by the relative values of the voltage, current, and speed. The aircraft high-speed machine usually has the shape shown in Figure 1.1(a), with a rotor diameter which is small compared to its length and with comparatively low moment of inertia per kV A. Most

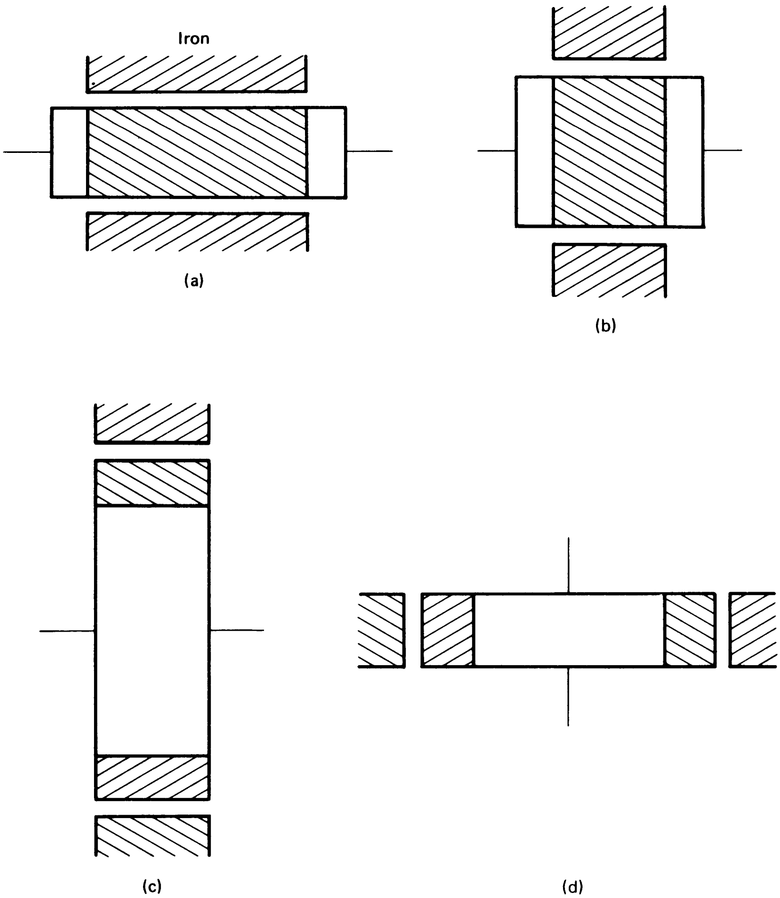

Figure 1.1 Machine shape in relation to speed. (a) high speed, (b) medium speed, and (c) and (d) low speed

industrial medium-speed machines, say 750–3000 r.p.m. have the shape shown in Figure 1.1(b) in which the rotor length is about the same as its diameter, with operating voltages about 200–11000 V. Large low-speed generators operating at say 150–400 r.p.m. which are used in hydro schemes, generating at voltages up to about 11 kV (11000 V) have rotor diameter very large compared to the axial length, in order to produce a satisfactory peripheral speed, as shown in Figure 1.1(c). They are often constructed for vertical mounting, as in Figure 1.1(d). These machines have relatively high polar moment of inertia per kV A. Turbo-alternators, which generate at 6.6, 11 or 33 kV and operate at either 1500 or 3000 r.p.m. (1800 or 3600 in USA) are invariably of the shape shown in Figure 1.1(a). This design matches the characteristics and inertia of the turbine much better than those of Figure 1.1(b) or 1.1(c).

It will be remembered that

$$\text{torque} \propto \text{magnetic flux} \times \text{current}$$

$$\text{power} \propto \text{torque} \times \text{angular velocity}$$

$$\text{voltage} \propto \text{flux} \times \text{angular velocity}$$

Also, the electrical and mechanical components of power loss are proportional to current, voltage, and speed. We have, therefore an interesting optimisation problem in meeting a given specification.

One important factor which increases the complexity of electrical machine engineering is non-linearity in production of magnetic flux in iron by current in the coils. All machines are nonlinear to some extent. In some the non-linearity is not important over most of the working range. Others, for example self-excited d.c. shunt generators and motors, either in isolation or in parallel, will work satisfactorily in a stable state only because of the non-linear magnetising characteristics of the iron. This may present some problems in control of the machine output and speed by automatic regulators. On the other hand, the system shown in Figure 1.2, a separately excited d.c. motor driven by a series connected d.c. generator, will not operate in a stable state at all but will break into non-linear mechanical oscillation of the rotor. This system has been described and analysed by Govinda Rao.[1]

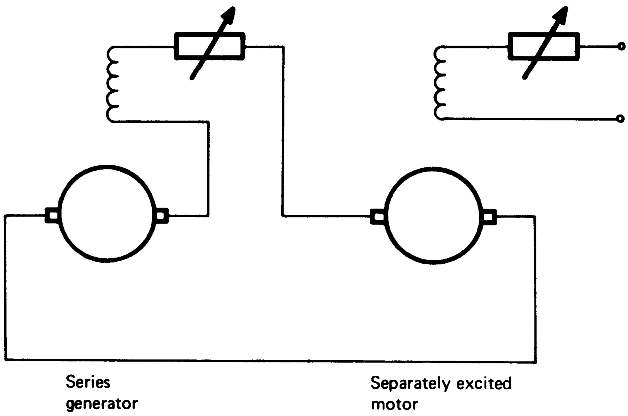

Figure 1.2 An unstable machine system

In the present book, the steady state and dynamic performance of machines is treated in an elementary manner. The aim is to provide a simple but sound mathematical basis for the subject and at the same time to develop the physical concepts and to investigate, in particular cases, the interactions which determine the machine behaviour.

1.2 DYNAMIC STABILITY

Even in the first decades of this century, machines increased very rapidly in number and in size. Problems of parallel operation began to arise, sometimes accentuated by the action of automatic voltage regulators. With the use of large synchronous a.c. generators and motors, it was found at an early stage that there were well defined limits to the steady power load which the machines could deliver in synchronism, quite apart from the thermal limitations. The machines also had to be designed and operated in such a manner that they would remain in synchronism following sudden changes in load and the effects of system faults. It was soon found that synchronous machines have definite natural mechanical rotational modes of continuous small oscillation while running in synchronism, the phenomenon known as 'hunting'. These modes are closely related to the electrical design parameters and the machine inertia constant. Hunting conditions are examined in Chapters 7 and 8. The importance of transient dynamic response times is exemplified by the loading conditions imposed on the electrical system of a large jet aircraft during a landing sequence, or in the automatic control of machines in a steel rolling-mill, or in the acceleration of an electrical locomotive. Precise control and response are required in many industrial processes. The machine is now very often part of an integrated, closely coupled feedback system and its satisfactory electrodynamic performance is an important aspect of the overall system specification.

In investigating the dynamic behaviour of a machine we have a torque-balance problem. The inertia of the rotor absorbs or contributes transient stored mechanical energy and therefore torque, which balances the difference between the instantaneous value of the electrically generated torque and that impressed upon the shaft by the load and by mechanical friction and windage loss. Thus a change in load or electrical operating conditions will cause a change in speed. It is important that speed changes following alterations in operating conditions should be easily controllable, within predictable bounds and such that the performance of the machine is satisfactory. Transient speed fluctuations following sudden changes should be small. In other words the machine should be dynamically stable.

It is quite possible, as we shall see, that the machine will behave satisfactorily over part of its normal operating range and then, beyond a quite sharply defined state become increasingly unstable in response to small changes in load. Since the torque-balance depends on magnetic flux and current and these have a non-linear relationship to one another, it is clear that the dynamic problem in general is a non-linear one. However, when speed and torque perturbations are small, the response about a given operating condition can be closely approximated by linearised small-oscillation equations. Under these conditions mechanical damping and synchronising torque coefficients can be defined, from which the stability boundaries can be calculated with very satisfactory accuracy.

In the dynamic studies in the following chapters, we examine some of the effects of automatic voltage regulators. We have not, however, included any special solid-state control equipment for supply sources, since it is in the dynamic response of the machines that our interest is centred. Having developed techniques for examining the basic types of machine, we feel that these can be used in industrial studies in conjunction with solid-state variable-voltage and variable-frequency sources.

The characteristics of regulators on the other hand form an essential part of machine stability theory and a regulator or governor should clearly be treated as an integral part of the machine systems. It is important, in designing machine systems and their regulators and control equipment, to understand how the latter two, together with the machine windings contribute to the dynamic response, how hunting forces arise, and how one may predict the dynamic performance under given conditions. For this reason the nature of the forces and the action of the regulator are treated at some length.

1.3 GENERALISED MACHINE THEORY

In the following chapters we shall not consider the design of machines in detail. We shall assume that the designer has met specified values of power, torque, speed, voltage, and frequency, and has produced a machine which has satisfactory capability and efficiency. We shall then use the resulting parameters, namely the values of resistance and inductance of the coils, to examine the steady-state and dynamic response characteristics of various machines.

We shall include the three main industrial machines, namely the d.c. generator and motor, the three-phase induction motor, and the three phase alternator. We hope to demonstrate by use of the generalised machine theory that the application of the analysis to other types of machine (such as the a.c. commutator motors and the reluctance motor) is straightforward. Cross-field machines are not often used now and these have been omitted.

The book is written in terms of the generalised theory. The advantage here is that both a.c. and d.c. machines can be described from the same standpoint and the torque concepts, etc., are identical in form. The analysis is then applied to different types of machine in an interconnected system.

The machines discussed are, to some extent, idealised, with respect to waveform, iron loss, and magnetic saturation. This does not introduce as much error as one might perhaps expect. When single machines are considered the idealised treatment can be modified quite easily to include realistic non-linearities, which are sometimes smoothed out or neglected in the generalised theory. We give examples of the computer treatment of saturation. When we deal with interconnected machines, the addition of the external system often reduces local machine-variations to second order effects. In short, the generalised approach has been found in practice to give such a satisfactory degree of accuracy that it is only in special cases that modifications are necessary. These are treated in the text in numerical examples.

During the past decade, as techniques of optimal control, system identification, and stochastic estimation theory have developed, these have been increasingly applied to power system analysis and design and to studies of the dynamics of machine systems. In Chapter 4, in application to machines, we introduce the concepts of state variables, on which modern control and general system engineering are based.

1.4 REFERENCE

1. Govinda Rao, C. V., 'A Solution of the Non-linear Differential Equation of the Rotating Machine Oscillator', *J. Franklin Institute*, 29 (Jan. 1958).

2

A review of basic machine theory

2.1 INTRODUCTION

There are many types of rotating electromechanical energy converter. When these transform mechanical energy to electrical energy, they are called generators. When they convert electrical to mechanical energy, they are operating as motors. Most energy converters can operate either as generators or motors. We shall, however, refer to these rotating electromechanical energy converters simply as electrical machines.

Electrical machines range from the very small, say a few watts output, to the very large machines with output of hundreds of millions of watts. When they are used as power station generators, it is economical to have them very large and it is not uncommon to have 500 MW generated from a single unit. The trend is towards still higher ratings.

Before induction motors were discovered by Nicola Tesla in 1880, there was not a very good case for generation of alternating current (a.c.) power supply, and the major breakthrough with alternating current did not come until 1889 when Dobrowolsky of AEG Co. invented the squirrel cage rotor for the 3-phase induction motor. Direct current (d.c.) was generated, transmitted, and used in almost all electrical drives. Today, nearly 80 per cent of industrial motors are induction motors. Almost all of the electrical power that is generated is from a.c. machines for the convenience of generation, transmission, and use. However, high voltage d.c. transmission is becoming increasingly necessary these days, for economical despatch of power from power stations situated in remote areas.

Although induction motors are widely used, they have an inherent limitation, namely, that they are virtually constant speed machines. In spite of world-wide research in speed-control of induction motors, a cheap and efficient method has yet to be found. Hence, d.c. motors are still widely used.

Electrical machines, despite the differences in their construction and characteristics, have a lot in common.[5] In earlier days, different types of machine were analysed in quite different ways, until in 1934 Gabriel Kron[2] showed that all machines are basically the same, and can be analysed with the same set of generalised tensor and matrix equations. Kron's researches started a trend towards generalised machine concepts in the analysis of all types of machines and this method of analysis, from the general to the particular, will be used in the latter part of this book.

Kron has shown, by applying the theory of tensor transformation, that the difference between various types of electrical machine is essentially due to the 'observer's position'. It is not intended to go into the details of tensor analysis in this book. Instead a physical explanation of the implications of the transformations will be given.

2.2 INDUCTIVE COUPLING

In this book we shall be primarily concerned with magnetically coupled coils in relative motion and with their basic parameters, resistance and inductance. Since inductive coupling of two or more coils will form the basis of what is to follow, the first few sections of this chapter will be devoted to a recapitulation of fundamental concepts.

2.3 SELF-INDUCTANCE

Self-inductance of a coil is due to voltage induced by the current which flows through it. This voltage is proportional to the rate of change of the current. The effect is manifested as a kind of 'inertia' which causes a delay in current response following a change in the impressed voltage. This statement, which draws an analogy between electrical and mechanical systems, will be clarified as we proceed. When a steady direct current flows through the coil, the inductance does not make itself felt but it is present and will affect the response to any change in the current or voltage.

When the coil has no 'core' and the magnetic flux is wholly in air, the amount of flux set up by a given current will be considerably less than is the case if the coil has an iron core. In the latter case the coil will have a much higher self-inductance. The coefficient of self-inductance can be defined by Faraday's law. This states that the e.m.f. induced in a coil is equal to the negative of the time rate of change of the magnetic flux linking the coil. Flux linkage is clearly the product of the total flux through the coil and the number of turns.

Faraday's law can be expressed in the form

$$\text{induced voltage } e = -\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}(N\Phi)$$

or

$$e = -N\frac{d\Phi}{dt} \tag{2.1}$$

In the simplified circuit shown in Figure 2.1 lines numbered 4 and 5 link three turns and lines numbered 1, 2 and 3 link one turn each. Therefore for the whole coil the flux linkage is $2 \times 3 + 3 \times 1 = 9$.

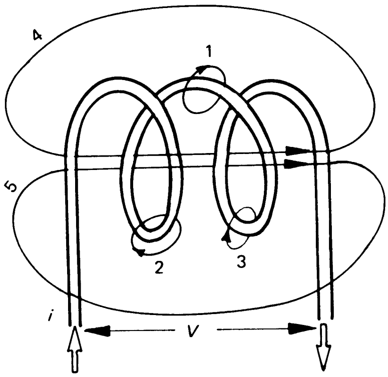

Figure 2.1 Concept of flux linkage

The flux density B inside the coil (and therefore the number of flux lines) depends upon the current flowing in the coil and the number of turns. This product is the well-known magnetomotive force (m.m.f.), whose units are ampere turn. The relationship between the magnetic flux density B (Wb/m^2) and the magnetic intensity H (ampere turn/m) is shown in Figure 2.2. It is clear

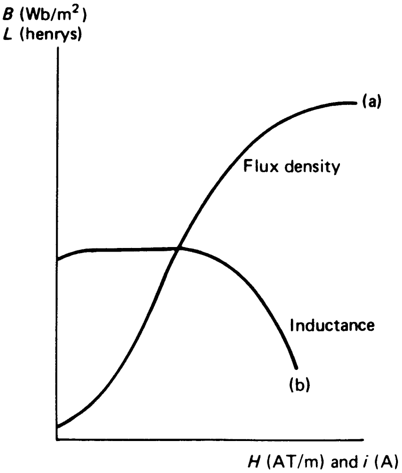

Figure 2.2 Magnetic saturation and its effect on inductance

that this relationship is not a linear one and that there is a pronounced saturation effect.

The number of flux lines Φ is related to the magnetomotive force Ni as follows

$$\Phi = Ni\mathscr{P} \quad \text{(Wb)} \tag{2.2}$$

where $\mathscr{P}$ is the permeance of the magnetic path and is given by

$$\mathscr{P} = \mu A/l \tag{2.3}$$

μ being the magnetic permeability of the medium, l the length of the magnetic path (m), and A its area (m^2).

The relationship is obviously very complicated if A and/or μ should vary along the magnetic path. There is an analogy here with an electrical circuit, in which conductance is given by $\sigma A/l$, σ being the conductivity of the medium.

Considering again the concept of flux linkage, we now find that if a coil having N turns produces Φ flux lines then the total flux linkage will be given by

$$\psi = N\Phi = N^2 i\mathscr{P} \tag{2.4}$$

This clearly is an idealisation because all flux lines produced by the coil will not link all the turns. However, with an iron core the amount of leakage is usually small.

From the Faraday–Lenz law

$$e = -\frac{d\psi}{dt} = -\frac{d}{dt}(N^2\mathscr{P}i) \tag{2.5}$$

If the number of turns and the magnetic permeance do not change with time then

$$e = -N^2\mathscr{P}\frac{di}{dt} = -L\frac{di}{dt} \tag{2.6}$$

where e is the induced 'back' voltages opposing the applied voltage and L is the coefficient of self-inductance. If the coil has no resistance, then the current which will flow in response to an impressed voltage V will be such that the following relationship is satisfied

$$V = -e = (N^2\mathscr{P})di/dt \tag{2.7}$$

Clearly the term $N^2\mathscr{P}$ is a property of the coil; by definition it is the coefficient of self-inductance. Being dependent upon the permeability μ, the self-inductance will be a nonlinear function of the current in a ferromagnetic circuit. However, in many cases the performance of electrical machines may be quite closely determined with the assumption of constant inductances.

The self-inductance of the coil is analogous to the moment of inertia of a rotating mass in a mechanical system, in which

$$\text{torque } T = J\,d\omega/dt \tag{2.8}$$

where ω is the angular velocity of the mass (rad/s) and J is the moment of inertia (kg m^2).

2.3.1 Calculation of self-inductance

The simple analogy between electrical resistance and magnetic reluctance is illustrated in Figure 2.3. Here a coil of 100 turns is wound on an iron core of cross-section one centimetre square, in which there is a gap of 0.01 cm. The magnetic permeability of the iron is 10^{-2} Wb/ampere-turn m.

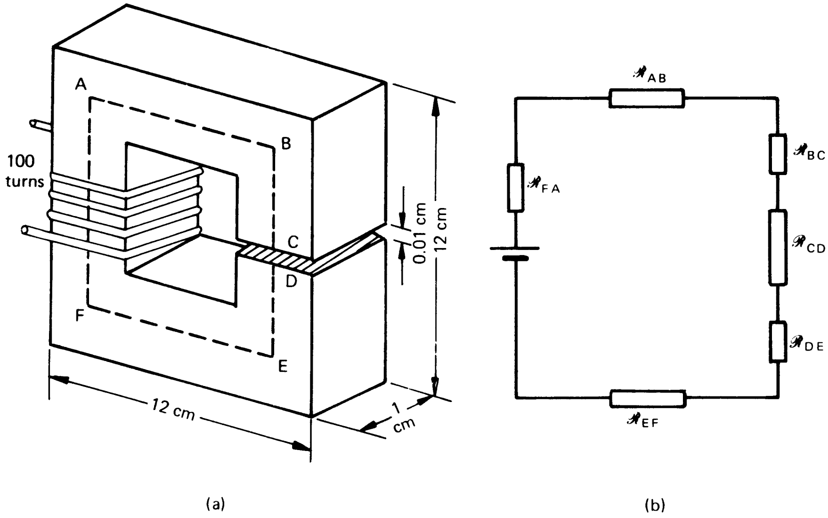

Figure 2.3 Calculation of self inductance of a coil. (a) Magnetic structure and (b) electric circuit analogue

The total magnetic reluctance is the sum of the separate values of reluctance in each element of the path length

$$\mathscr{R} = \frac{l_1}{a_1\mu_1} + \frac{l_2}{a_2\mu_2} + \cdots$$

This equation is similar in form to that for electrical resistance of a series circuit, when the magnetic permeability μ is replaced by the electrical resistivity σ.

In the example shown, the mean length of the magnetic path in iron is 0.439 9 m and the cross-sectional area is 0.000 1 m^2. The path length in air is 0.000 1 m and the cross-sectional area is 0.000 1 m^2, as before. The total reluctance is therefore 1.24×10^6 ampere-turn/Wb. The magnetic permeance is $\mathscr{P} = 1/\mathscr{R} = 0.81 \times 10^{-6}$ Wb/ampere-turn. The inductance is

$$L = N^2\mathscr{P} = 0.008\ 1 \text{ H} = 8.1 \text{ mH}$$

At 50 Hz the reactance is

$$X = \omega L = 2.54\ \Omega$$

Without this very small air gap, the figures are

$$\mathscr{R} = 44 \times 10^4 \text{ ampere turn/Wb}$$

$$\mathscr{P} = 2.273 \times 10^{-6} \text{ Wb/ampere turn}$$

$$L = 0.022\,73 \text{ H} = 22.73 \text{ mH}$$

At 50 Hz, $X = 7.14\ \Omega$

2.4 ELECTROMECHANICAL ANALOGY

At this point a more complete analogy between the properties of an electrical and a mechanical system may be presented. Two types of mechanical system will be considered, one rotational and the other translational.

Table 2.1 A comparison between electrical and mechanical systems

Mechanical system		*Electrical system*
translational	*rotational*	
displacement x, m	angular displacement θ, rad	charge q, C
velocity $v = \dot{x}$, m/s	angular velocity $\omega = \dot{\theta}$, rad/s	current $= i = \dot{q}$, A
force F, N	torque T, N m	voltage V, V
momentum Mv, kg m/s	angular momentum, $J\dot{\omega}$	flux linkage ψ, Wb-turn
frictional resistance R_f, N s/m	frictional resistance R_f, N s/m	electrical resistance R, Ω
mass M, kg	moment of inertia J, kg m^2	inductance L, henry
spring constant K_t, N m	spring constant K_r, N m	elastance $1/C$, F^{-1}

The analogy given in Table 2.1 is not unique. Some authors prefer a dual analogy, shown in Table 2.2.

Table 2.2 A second form of analogy between electrical and mechanical systems

Mechanical quantities	*Electrical quantities*	*Dual*
F or T	V	I
v or ω	i	V
M or J	LC	C
K_t or K_r	$1/C$	$1/L$

It is also interesting to study the mechanical analogues of Kirchhoff's two fundamental laws of electrical circuits. The mechanical laws due to d'Alembert state that at the kth mechanical node of a system

$$\sum_{i=1}^{r}\left[\frac{\mathrm{d}}{\mathrm{d}t}(m_{ki}\dot{x}_{ki}) - F_{ki}\right] = 0 \qquad i = 1 \cdots 5\ k = 1 \cdots n \tag{2.9}$$

and around a mechanical loop (kth loop)

$$\sum_{i=1}^{r} \dot{x}_{ki} = 0 \tag{2.10}$$

The ith inertial force on the kth node is

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_{ki}\dot{x}_{ki})$$

and the ith applied mechanical force including constraints is F_{ki}. The above analogy is not complete without a reference to the well-known principle of conservation of momentum. The analogous concept in an electromagnetic system is the theorem of constant flux-linkage. This theorem states that:

> The flux linkage of any closed circuit of finite resistance and e.m.f. cannot change instantly.

In the special case in which the resistance and voltage are zero, the theorem may be restated

> The flux linkage of any closed circuit having no resistance and no e.m.f. remains constant.

This theorem can be proved as follows. Consider a coil with resistance, flux linkage, and voltage source (but no series capacitors). The voltage equation can be written

$$Ri + \frac{\mathrm{d}\psi}{\mathrm{d}t} = V \tag{2.11}$$

Integrating the above equation, we have

$$\left[R\int_0^{\Delta t} i\,\mathrm{d}t\right] + \Delta\psi = \int_0^{\Delta t} V\,\mathrm{d}t \tag{2.12}$$

If R and V are finite, then as $\Delta t \to 0$ the definite integrals approach zero and therefore $\Delta\psi$ tends to zero. This shows that flux linkage cannot change instantaneously.

If V and R are zero then

$$\mathrm{d}\psi/\mathrm{d}t = 0 \quad \text{and} \quad \psi = \text{constant} \tag{2.13}$$

In electrical machines if zero resistance can be achieved using superconductivity, then ideally no voltage will be required to maintain a constant field flux.

The mechanical principle of conservation of momentum is similar, and states that in a non-dissipative system, mv is constant in the absence of external impressed forces (following from Newton's first law of motion).

The concept of constant flux-linkage is found to be very helpful in understanding the transient behaviour of electrical circuits and machines.

It should be further noted that in contrast to the mechanical inertia of a rotating mass, the inductance of a coil, especially with an iron core, may be current-dependent and nonlinear.

2.5 MAGNETIC SATURATION

Since we are dealing with the electromagnetic interaction among various coils, the effects of magnetic saturation are of importance in our study. What we are primarily concerned with is the effect of saturation on the inductance of a coil. We have seen earlier that

$$L = N^2 \mathscr{P} = N^2 \mu A / l$$

where μ, the permeability of the magnetic material is defined as the ratio B/H. B is the flux density ($\mathrm{Wb/m^2}$) and H is the intensity of magnetisation (in ampere-turn/m). As the magnetic material becomes saturated, the proportionality is lost, many more ampere turns must be used and therefore much more current is necessary to produce additional flux lines. This is called the hard magnetisation zone as opposed to soft magnetisation where B and H have a linear relationship. Obviously with saturation the B/H ratio (μ) diminishes, resulting in an effective reduction in the self-inductance of the coil.

From the point of view of analysis this phenomenon creates a great deal of complication. If we consider a very simple situation where $L\mathrm{d}i/\mathrm{d}t = V$ (i.e. the resistance of the coil is neglected), the equation is linear as long as the coefficient of the derivative of i is a constant. With L constant, the solution of this equation is straightforward. However, when saturation takes place the inductance L is no longer a constant and becomes a function of the current. The solution of this very simple equation now becomes quite complicated. Figure 2.2 shows the variation of inductance with current.

A more detailed discussion of the effect of saturation and different methods of taking this into account are discussed in reference 1.

2.6 STORED ENERGY

In this section, we shall show how the energy stored in a magnetic field is to be evaluated, in order to establish more deeply the mechanical analogue with which we started.

If a voltage V is applied to a coil with no resistance then

$$V = \mathrm{d}\psi/\mathrm{d}t \quad (\mathrm{V}) \tag{2.14}$$

and the power input is

$$Vi = i\mathrm{d}\psi/\mathrm{d}t \tag{2.15a}$$

Assuming that saturation does not exist and that there is no residual magnetism, the energy required to establish a flux linkage ψ_1 in time t is

$$E = \int_0^t Vi\mathrm{d}t = \int_0^{\psi_1} i\mathrm{d}\psi \tag{2.15b}$$

which is the area 0ab in Figure 2.4(a). This energy is stored in the

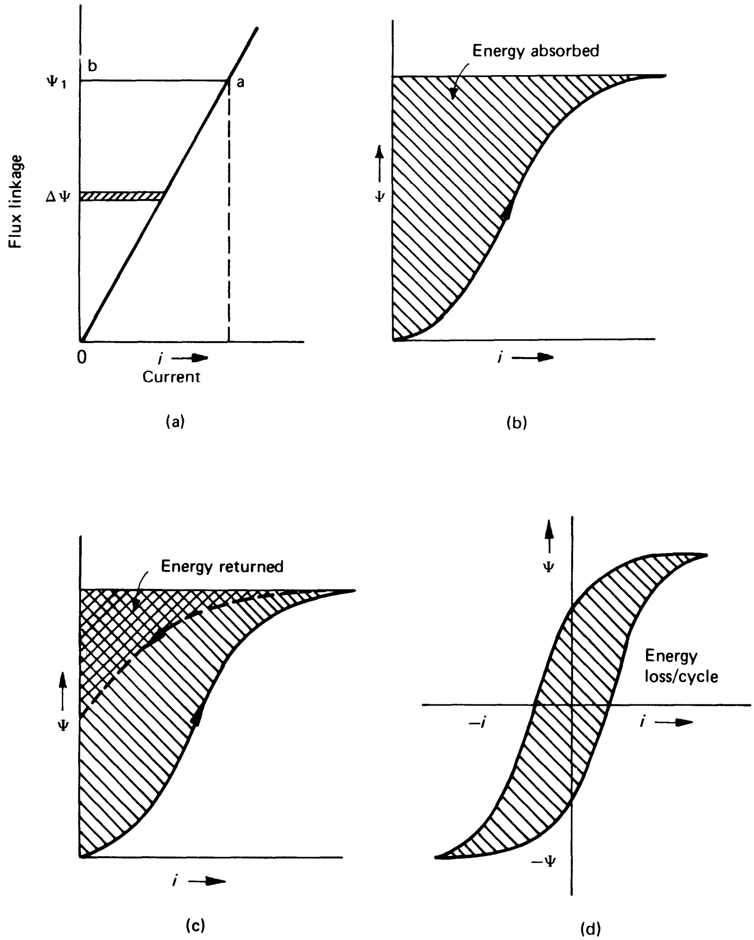

Figure 2.4 Stored magnetic energy and hysteresis loss. (a) In an air core, (b) in a ferromagnetic core, (c) magnetic energy and (d) hysteresis loss

electromagnetic space. Since saturation has been neglected a proportionate increase in energy will be required to establish increased flux. Equation (2.15*b*) also makes the assumption that no mechanical work has been done by the coil, as in a relay—since naturally the energy required for mechanical work would have to come from the electrical source. This aspect will be considered later. For the time being we shall assume that there is no change in the electromagnetic configuration of the system.

When a ferromagnetic core is present, which is normally the case, the flux-linkage build-up is limited by saturation. If there were no resistance present in the circuit the current would tend to infinity with a d.c. source, since the opposing e.m.f. $\mathrm{d}\psi/\mathrm{d}t$ would disappear. In reality the resistance of the coil and the internal resistance of the source will limit the current.

When the source is a.c. the current will not tend to infinity since the opposing induced e.m.f. will always be present. The energy and power delivered to the coil will be fed back to the source and again into the coil, at the rate of twice the source frequency. This power, which causes no drain on the source is not 'real power' but the well-known 'watt-less' reactive power. Let us elucidate this point a little further.

If, in a pure inductor, a sinusoidal impressed voltage V causes current i to flow, then since in this case the induced voltage will be equal to the impressed voltage

$$V = -e = L\frac{\mathrm{d}i}{\mathrm{d}t} = \omega L \hat{I} \cos \omega t = \hat{V} \cos \omega t \tag{2.16}$$

The voltage and current will be as shown in Figure 2.5(a). The power delivered by the source at any particular instant is the product of the voltage and current at that instant

$$Q = \hat{V}\hat{I} \cos \omega t \sin \omega t = \frac{\hat{V}\hat{I}}{2} \sin 2\omega t \tag{2.17}$$

The quantity Q changes direction every quarter cycle of the voltage or the current. Thus, integrating over a voltage cycle, the total Q is zero. If, however, resistance is present in the circuit, power loss takes place irrespective of the direction of flow of the alternating current. This loss must be supplied by the source and constitutes real power. In this case the voltage and current will be as shown in Figure 2.5(b) where it is seen that the current i will have two components, i_1 in phase with V giving a uni-directional power product and i_2 in quadrature giving the reactive product Q which is not uni-directional. Ideally therefore an inductive coil would not absorb any real power, but of course there is always resistance present which causes real power dissipation. In a large power system the transmission resistance loss, together with iron loss in the transformers, costs many thousands of pounds annually in fuel consumption.

In any ferromagnetic coil, magnetic hysteresis causes power loss in the form of heat because not all of the energy absorbed from the source to set up the

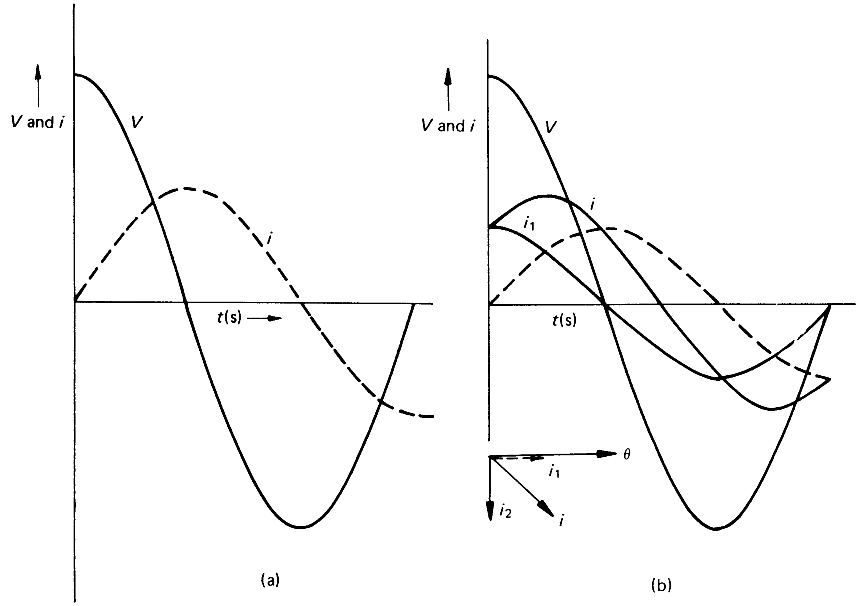

Figure 2.5 Voltage and current relationship, (a) In a pure inductor and (b) in an inductor with resistance

magnetic field over one half cycle is returned to the source during the reverse half cycle. The hatched area in Figure 2.4(d) gives a measure of the energy dissipated during every cycle, due to hysteresis.

An important question in machine analysis is how one may include the effects of hysteresis and eddy current in the iron. It is very difficult to do this accurately by analytical methods but good approximate methods are available, which will be discussed later.

The expression for the stored magnetic energy in a coil is given in equation (2.15*b*). In order to express this in a more tangible form we can write the energy expression

$$\mathscr{E} = \int_0^{\psi} i\mathrm{d}(N\Phi) = \int_0^{i} i\mathrm{d}(N^2 i\mathscr{P}) = N^2 \int_0^{i} i\mathrm{d}(\mathscr{P}i) \tag{2.18}$$

If we assume for the moment that the magnetic structure is fixed, that is, that no part of it moves relatively, during the interval under consideration, then the magnetic permeance

$$\mathscr{P} = \mu A/l = \text{constant} \tag{2.19}$$

and

$$\mathscr{E} = N^2\mathscr{P} \int_0^{i} i\mathrm{d}i = \tfrac{1}{2}Li^2 \quad \text{(J)} \tag{2.20}$$

Referring to table 2.1 we can see that this expression for magnetic energy has the same form as those for linear or rotating mechanical systems in which

kinetic energy $= \frac{1}{2}mv^2$	for mass m, linear velocity v
$= \frac{1}{2}J\omega^2$	for a rotating mass, moment of inertia J, angular velocity ω

Although electrostatic devices rarely feature in electromagnetic machines (except for the use of capacitors in single phase induction motors) a brief review of a capacitor as the dual of an inductor will not be out of place. This will also help to introduce the concept of potential energy in an electrical system, based on the analogues, current–velocity and voltage–force.

We know that in a capacitor

$$V = q/C \tag{2.21}$$

or

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{i}{C} \tag{2.22}$$

Thus

$$Vi\,\mathrm{d}t = CV\,\mathrm{d}V$$

and

$$\int_0^t Vi\,\mathrm{d}t = C\int_0^{V_t} V\,\mathrm{d}V \tag{2.23}$$

The energy input stored in a capacitor is, therefore,

$$\mathscr{E}_\mathrm{c} = \tfrac{1}{2}CV^2 = \tfrac{1}{2}q^2/C \tag{2.24}$$

Mechanically, when a spring with spring constant K is stretched through a distance x the stored potential energy is

$$\int_0^x F\,\mathrm{d}x = \int_0^x Kx\,\mathrm{d}x = \tfrac{1}{2}Kx^2 \tag{2.25}$$

We can therefore conclude, following the force–voltage analogy, that an inductor accounts for the kinetic energy of an electrical system while a capacitor accounts for the potential energy.

Following this, we can write two sets of equations, one for a static electrical network and another for a mechanical system as in Figure 2.6

$$L\ddot{q} + R\dot{q} + q/C = V, \quad \text{electrical network} \tag{2.26}$$

$$m\ddot{x} + D\dot{x} + Kx = F, \quad \text{mechanical system} \tag{2.27}$$

The implications of this analogy will be emphasised several times during the rest of the book. In Chapter 3, the analogy will be presented in its most general form as Lagrange's equation.

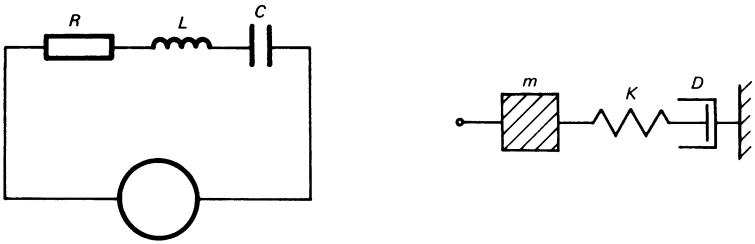

Figure 2.6 RLC circuit and its mechanical analogue

In the sections that follow, two further highly relevant characteristics of an electromagnetic system will be dealt with in some detail. One is that of coupled circuits and the other is the mechanical force or torque produced by an electromagnetic system.

2.7 COUPLED CIRCUITS

A transformer has a set of stationary coils which are electromagnetically coupled. The purpose of recapitulating coupled circuits is to understand terms like leakage inductance, mutual inductance, and magnetising inductance, since these concepts are of fundamental importance in the study of interacting coils.

We know that in a transformer

$$V_1/V_2 = N_1/N_2$$

and the nearest mechanical analogue would be Brahma's hydraulic press where

$$F_1/F_2 = A_1/A_2$$

In Figure 2.7(a) a voltage V_1 may be applied. As long as coil 2 is open its presence makes no difference to the characteristics of coil 1. However, a voltage V_2 will appear across coil 2. This voltage will be caused by magnetic flux Φ_{1m} produced by coil 1 and coupling the two coils. In a transformer, Φ_{1m} is a very major part of the total flux Φ_{11} produced by coil 1. The leakage flux is given by

$$\Phi_{1l} = \Phi_{11} - \Phi_{1m}$$

The latter is flux produced and linked by coil 1, none of which links coil 2. If coil 1 carries current i_1 then the total available m.m.f. is $N_1 i_1$. This m.m.f. will drive the small amount of leakage flux Φ_{1l} through the high reluctance path in air, but will also drive most of the flux through the low reluctance path in the iron.

The total flux set up by coil 1 is

$$\begin{aligned}\Phi_{11} &= \Phi_{1m} + \Phi_{1l} \\ &= N_1 i_1[\mathscr{P}_{\text{iron}} + \mathscr{P}_{\text{air}}] \qquad (2.28)\end{aligned}$$

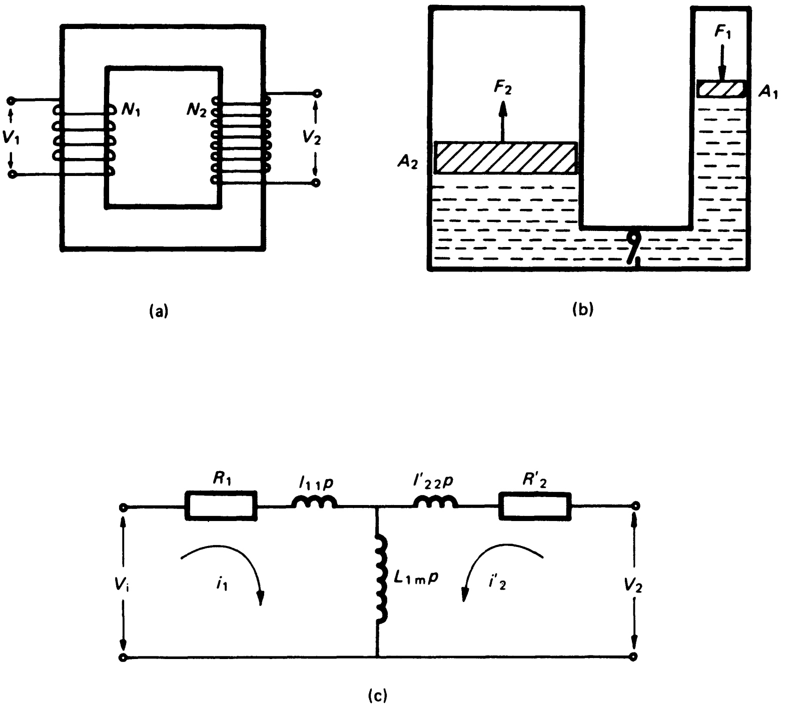

Figure 2.7 Coupled circuits. (a) Electromagnetic coupling, (b) mechanical analogue and (c) electrical equivalent circuit

The flux linkage of coil 1 is ψ_{11}, where

$$\psi_{11} = N_1 \Phi_{11}$$

Thus

$$\psi_{11} = (N_1)^2[\mathscr{P}_i + \mathscr{P}_a]i_1 \tag{2.29}$$

Neglecting resistance, we have

$$\begin{aligned} V_1 \doteq \frac{d\psi_{11}}{dt} = (N_1)^2\left[\mathscr{P}_i + \mathscr{P}_a\right]\frac{di}{dt} &= (N_1)^2\mathscr{P}_i\frac{di_1}{dt} + (N_1)^2\mathscr{P}_a\frac{di_1}{dt} \\ &= L_{1m}\frac{di_1}{dt} + l_{11}\frac{di_1}{dt} \\ &= L_{11}\frac{di_1}{dt} \end{aligned} \tag{2.30}$$

The term $(N_1)^2\mathscr{P}_i$ comprises what is known as the magnetising inductance L_{1m} of the coil and $(N_1)^2\mathscr{P}_a$ is the leakage inductance l_{11}. Together they give the total self inductance of coil 1. Thus self inductance = magnetising inductance + leakage inductance. In a similar manner coil 2 has self inductance L_{22}, magnetising inductance L_{2m} and leakage inductance l_{22}.

Let us now consider the case in which the coils carry current i_1 and i_2 respectively by applying a voltage V_1 to coil 1 and closing the coil 2 through a load impedance (or vice versa).

The flux through coils 1 and 2 respectively will be

$$\Phi_1 = \Phi_{11} + \Phi_{21} \tag{2.31a}$$

$$\Phi_2 = \Phi_{22} + \Phi_{12} \tag{2.31b}$$

Flux linkages of these coils are

$$\psi_1 = N_1\Phi_1 = N_1\Phi_{11} + N_1\Phi_{21} \tag{2.32a}$$

and

$$\psi_2 = N_2\Phi_2 = N_2\Phi_{22} + N_2\Phi_{12} \tag{2.32b}$$

These equations can now be further expanded as follows:

$$\psi_1 = [(N_1)^2\mathscr{P}_i + (N_1)^2\mathscr{P}_{a1}]\, i_1 + [N_1N_2\mathscr{P}_i]\, i_2 \tag{2.33a}$$

$$\psi_2 = [(N_2)^2\mathscr{P}_i + (N_2)^2\mathscr{P}_{a2}]\, i_2 + [N_2N_1\mathscr{P}_i]\, i_1 \tag{2.33b}$$

Clearly the permeance $\mathscr{P}_i$ of the single iron core is the same in both cases but the permeances $\mathscr{P}_{a1}$ and $\mathscr{P}_{a2}$ of the leakage paths in air may be quite different for coils 1 and 2.

As we have seen earlier, $((N_1)^2\,\mathscr{P}_i + (N_1)^2\mathscr{P}_{a1})$ gives the self-inductance of coil 1. The term $N_1N_2\,\mathscr{P}_i$ represents the mutual inductance M_{12} between coils 1 and 2 and the flux linkages can be expressed

$$\psi_1 = L_{11}i_1 + M_{12}i_2 \tag{2.34a}$$

$$\psi_2 = L_{22}i_2 + M_{21}i_1 \tag{2.34b}$$

and

$$M_{12} = M_{21} = M$$

Some simple but rather interesting relations can now be derived.
Since

$$L_{1m} = (N_1)^2\mathscr{P}_i$$

and

$$M_{12} = N_1N_2\mathscr{P}_i$$

we can see that

$$L_{1m}/M_{12} = N_1/N_2$$

or

$$\frac{\text{magnetising inductance}}{\text{mutual inductance}} = \frac{N_1}{N_2} \tag{2.35}$$

It is also evident that

$$M^2 = L_{1m} L_{2m}$$

Further, if we assume that $L_{1m} = kL_{11}$ and $L_{2m} = kL_{22}$ then

$$M^2 = k^2 L_{11} L_{22}$$

or

$$M = k\sqrt{L_{11} L_{22}} \tag{2.36}$$

and k is the coupling coefficient.

The voltage equations for the two coils, including resistance, may now be written

$$V_1 = (R_1 + L_{11} p) i_1 + M_{12} p i_2 \tag{2.37a}$$

$$V_2 = (R_2 + L_{22} p) i_2 + M_{21} p i_1 \tag{2.37b}$$

(where $p = \mathrm{d}/\mathrm{d}t$) with voltage and current measured at the respective terminals.

To calculate the total stored energy of this magnetically coupled system we proceed from equations (2.37) as follows,

$$V_1 i_1 = R_1 i_1^2 + L_{11} i_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + M_{12} i_1 \frac{\mathrm{d}i_2}{\mathrm{d}t} \tag{2.38a}$$

$$V_2 i_2 = R_2 i_2^2 + L_{22} i_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} + M_{21} i_2 \frac{\mathrm{d}i_1}{\mathrm{d}t} \tag{2.38b}$$

Adding these equations we have the total power input

$$(V_1 i_1 + V_2 i_2) = (R_1 i_1^2 + R_2 i_2^2) + L_{11} i_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_{22} i_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} + M_{12} \frac{\mathrm{d}}{\mathrm{d}t}(i_1 i_2) \tag{2.39}$$

and integrating this, to calculate the energy input over time t

$$\mathscr{E} = \underbrace{(R_1 i_1^2 + R_2 i_2^2)t}_{\text{ohmic loss}} + \underbrace{\tfrac{1}{2} L_{11} i_1^2 + \tfrac{1}{2} L_{22} i_2^2 + M_{12} i_1 i_2}_{\text{stored magnetic energy}} \tag{2.40}$$

We shall make use of this expression in a more general form in later analysis.

In the case of coupled circuits, particularly with transformers, it is customary to represent the secondary quantities as viewed from the primary side (or perhaps the other way round). In order to express the secondary (coil 2) parameters in terms of the primary, we multiply equation (2.37*b*) by $a = N_1/N_2$.

Equation (2.37*b*) now becomes

$$aV_2 = a^2(R_2 + L_{22}p)\frac{i_2}{a} + Mai_1 \tag{2.41}$$

In an ideal transformer the secondary voltage and current when referred to the primary would be identical to the primary values, since in that case

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} \quad \text{and} \quad \frac{i_1}{i_2} = \frac{N_2}{N_1}$$

or

$$V_1 = aV_2 \quad \text{and} \quad i_1 = i_2/a$$

The secondary impedance Z_{22} has the value $a^2 Z_{22}$ when referred to the primary side and this same factor applies to any load impedance connected to the transformer. If, however, we wish to represent a transformer by an electrical network, a minor adjustment of equation (2.37) is made, viz,

$$V_1 = (R_1 + l_{11}p)i_1 + L_{1m}pi_1 + aMp\frac{i_2}{a}$$

$$aV_2 = a^2(R_2 + l_{22}p)\frac{i_2}{a} + a^2L_{2m}\frac{i_2}{a} + aMpi_1$$

writing

$$V'_2 = aV_2 \text{ and } i'_2 = i_2/a$$

and substituting

$$L_{1m} = aM_{12} = aM_{21} = aM = a^2L_{2m}$$

we have

$$V_1 = (R_1 + l_{11}p)i_1 + L_{1m}p(i_1 + i_2) \tag{2.42a}$$

$$V'_2 = (R'_2 + l'_{22}p)i'_2 + L_{1m}p(i_1 + i'_2) \tag{2.42b}$$

where L_{1m} is the common magnetising inductance referred to the primary side. These equations may be represented by the electrical network shown in Figure 2.7(c). In matrix form they become

$$\begin{array}{|c|}\hline V_1 \\ \hline V'_2 \\ \hline \end{array} = \begin{array}{|c|c|}\hline R_1 + L_{11}p & L_{1m}p \\ \hline L_{1m}p & R'_2 + L'_{22}p \\ \hline \end{array} \cdot \begin{array}{|c|}\hline i_1 \\ \hline i'_2 \\ \hline \end{array} \tag{2.43}$$

where $L_{11} = (l_{11} + L_{1m})$
$L'_{22} = (l'_{22} + L_{1m})$

When the secondary quantities are not referred to the primary side the matrix equations have the form

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} R_1 + L_{11}p & M_{12}p \\ M_{21}p & R_2 + L_{22}p \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{2.44}$$

With the transformer secondary side connected to a load impedance $Z_L = R_L + L_L p$ the equations referred to the primary side will become

$$\begin{bmatrix} V_1 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + L_{11}p & L_{1m}p \\ L_{1m}p & (R'_2 + R'_L) + (L'_{22} + L'_L)p \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i'_2 \end{bmatrix} \tag{2.45}$$

where, as usual, we represent the referred value of the mutual inductance $M'_{12} = M'_{21} = aM$ by the symbol L_{1m}.

In application to practical cases, the leakage reactances of the primary and secondary windings (together with the resistances) have a dominant influence on the behaviour of the transformer when on load and in terms of parallel operation, voltage drop, and short-circuit current. These reactances are very small in comparison with the full magnetising reactance (usually one or two per cent) and they cannot be determined with sufficient accuracy by taking the difference between L_{11} and L_{1m} as measured by the open-circuit test. However, the combined leakage reactance of both windings can be easily determined by a short-circuit test and this value can be used with the measured primary reactance to give the other inductance parameters. This important subject is treated in considerable detail in reference 1. We shall return to it in Chapter 7 in relation to synchronous machines, in which the same problem arises in a more complicated form.

2.8 GENERATED VOLTAGE

So far we have considered coupled coils which are stationary with respect to each other. Let us now consider a coil rotating in a magnetic field as in Figure 2.8. Faraday's law states that the voltage generated in this coil will be

$$e = -\mathrm{d}\psi/\mathrm{d}t, \tag{2.5}$$

where $\psi = N\Phi$ is the flux-linkage of the coil, N being the number of turns and Φ the flux enclosed by the coil.

If B is the flux density of the magnetic field, then the flux-linkage will be NBa, where a is the area of the coil 'exposed' to the lines of flux at a particular instant,

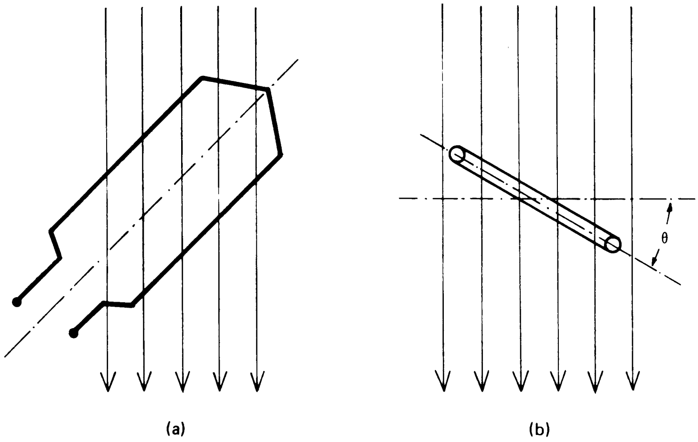

Figure 2.8 Coil rotating in a magnetic field

namely the projected area of the coil on a plane perpendicular to the lines of flux.

As the coil rotates, the area a changes from zero to A, where A is the area of the coil in its own plane.

At any instant, if θ is the angle between the plane of the coil and the plane normal to the lines of flux,

$$a = A\cos\theta$$

Since $\theta = \omega t$, where ω is the angular velocity of the coil in radians per second,

$$a = A\cos\omega t$$

Therefore, ψ at any instant is given by

$$\psi = NBA\cos\omega t = N\Phi\cos\omega t \qquad (2.46)$$

where Φ is the maximum flux linking the coil (when $\theta = 0$).
Consequently,

$$e = -\mathrm{d}\psi/\mathrm{d}t = N\Phi\omega\sin\omega t \qquad (2.47)$$

or

$$e = e_{\mathrm{m}}\sin\omega t, \text{ where } e_{\mathrm{m}} = N\Phi\omega \qquad (2.48)$$

This indicates that the voltage generated in a coil, as it rotates in a constant magnetic field, is always alternating. If B is constant throughout the area of the coil the voltage wave form is sinusoidal.

Alternating current machines produce this alternating voltage, which is generated from a set of coils.

In direct current machines also, there is a set of coils which is rotated in a magnetic field and the *voltage generated in the coils is also alternating*. However, the output from a d.c. generator is unidirectional.

We shall presently see that it is the 'observer's measurement position' that introduces the basic difference between these two types of machine.

Figure 2.9 illustrates a basic alternator, in which the two ends of the rotating coil are connected to two brass slip-rings mounted on the shaft which carries the coil. There are two brushes which touch the slip-rings continuously. If we connect these brushes to a voltmeter or an oscilloscope we shall observe the voltage that is being generated in the coil at any particular instant. Although the measuring equipment is stationary in space, the observer, as it were, rotates with the coil and records at every instant the voltage that is being generated in the coil. The reference axis of the observer in this case is known as the slip-ring axis (to be elaborated later). Clearly the observer is 'rotating' by being continuously connected to the rotating coil through the slip-ring.

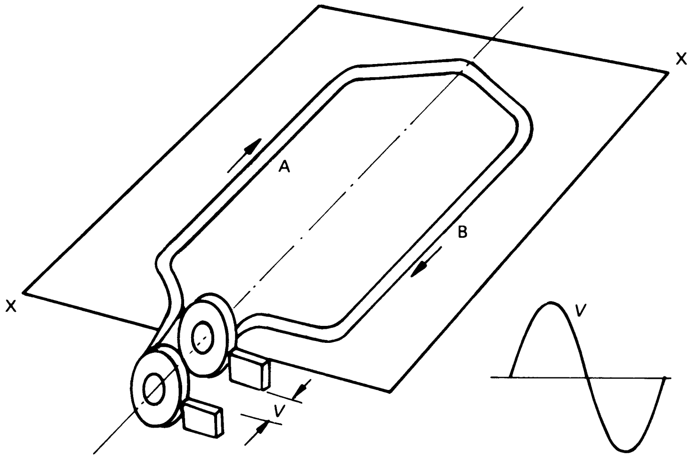

Figure 2.9 Coil and slip rings

When a coil-side passes above the plane through XX and the coil is rotating in the clockwise direction, the current in coil-side A flows away from the slip-ring A to which it is connected. When the same coil-side passes below the plane, the current flows towards the same slip-ring A. Obviously the voltage polarity at the two brushes A and B will alternate and will assume alternately positive and negative values. The result is an alternator in which the observer rotates with the coil.

2.9 THE STATIONARY OBSERVER

In a d.c. machine, instead of slip rings we have a commutator. In its most elementary form a commutator could be described as two copper half-rings insulated from each other (Figure 2.10).

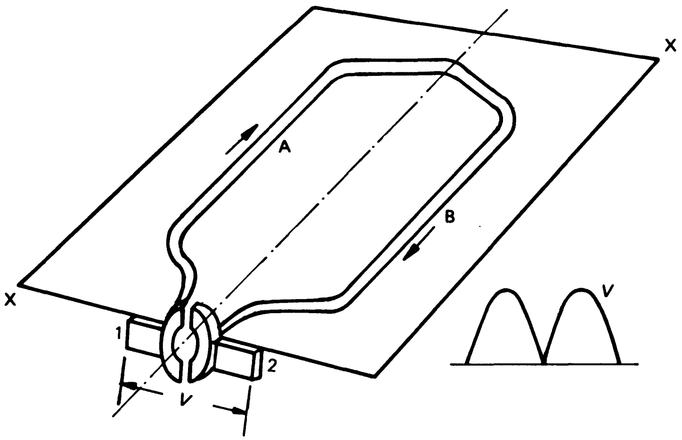

Figure 2.10 Coil and a commutator

It is clear from Figure 2.10 that brush 1 will be in contact only with that coil side which passes the upper side of the plane XX, and brush 2 will be connected to the coil-side that passes through the lower side. In this arrangement current will always flow from brush 1 to brush 2 through the coil. This means that the current through the brushes is always uni-directional. The voltage waveform will be as shown in Figure 2.10.

Here the observer does not rotate with the coil all the way. In the particular case in which there are two commutator segments, he goes only half way. But in an actual machine, there are a very large number of commutator segments and the observer is almost stationary in space. He is in contact, through the commutator, with a set of coils, each generating voltage of the same polarity but of different magnitude connected in series. The observer 'on the brushes' records the sum, as in Figure 2.11.

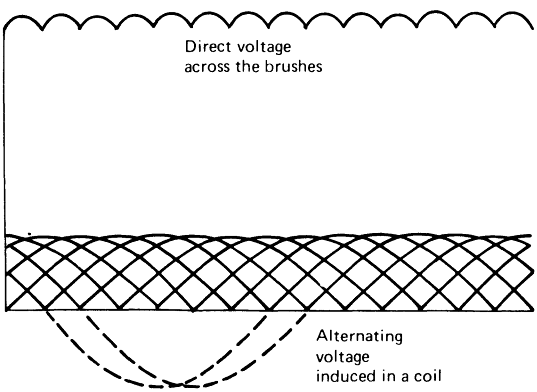

Figure 2.11 Alternating voltage in individual coils
and direct voltage across brushes

2.10 ACTUAL MACHINES[3]

In actual machines there is more than one coil. In alternators, these coils are usually divided into phases spaced out according to the number of poles for which the machine is designed. The coils forming a phase are connected in series. The phases are then connected in star or in delta, as required. In a 3-phase machine there are, in fact, three alternating voltages, one from each phase, and the phase coils are distributed in space in such a manner that these three voltages have a time phase difference of 120 electrical degrees. If the magnetic field is produced by two poles only, then electrical and mechanical degrees are the same. Otherwise, electrical degrees = number of mechanical degrees multiplied by the number of pole pairs.

In a d.c. machine also, there are several coils, which are connected to form a winding either in a lap or wave arrangement (Figure 2.12). Each coil connection or joint is connected to a commutator segment, as in Figure 2.13. However, there

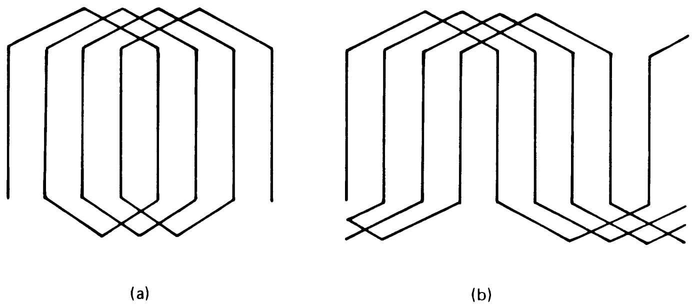

Figure 2.12 Winding arrangements. (a) Lap winding and (b) wave winding

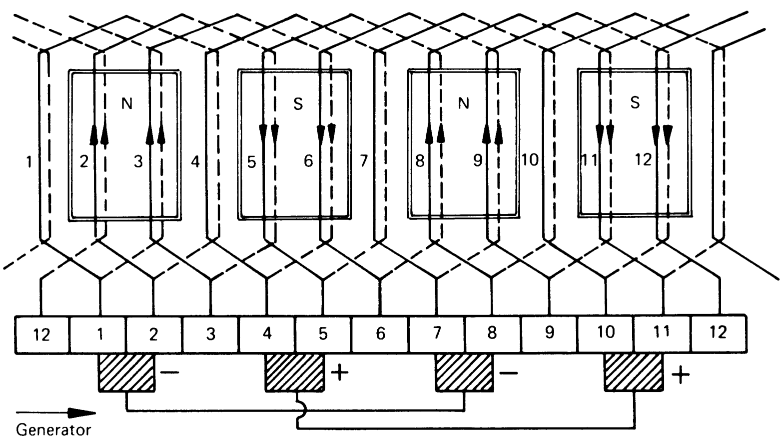

Figure 2.13 Developed view of 4-pole, simple lap winding

is a continuous short-circuiting effect when a brush is in contact with two commutator segments at once, as the segments pass under the brush. For this reason the brushes are so placed that the short-circuited turns of the armature winding will be in the neutral axis between the poles, generating minimum voltage in these turns. It must be remembered that the armature coils in groups generate their voltages in series. The summation of generated voltage in a complete set of coils is shown by the top curve in Figure 2.11. In principle, with commutator machines one may regard the observer as stationary in space, whilst for slip ring machines, the observer is rotating with the rotor. Of course, there are many differences in the design and manufacturing details of these machines but so far as the basic principle is concerned there is only the difference between the positions of the observer relative to the moving coils.

It will be shown in the next chapter that the performance equations for various kinds of electrical machines can be deduced from those of one generalised machine—Kron's 'primitive machine'.

2.11 ELECTRIC MOTOR OPERATION

Basic motor operation has been described by some authors as 'the tendency of the two magnets to align themselves'. It could also be described as the attempt to align a magnetic material in a magnetic field, or move it into a stronger magnetic field.

The two magnets in a motor are those which we call the stator and the rotor. These can be energised individually, as in d.c. motors and synchronous motors. Alternatively, one magnet can induce magnetism in the other, as in the induction motor. Motor action is also achieved from what is known as reluctance torque. This arises as the ferromagnetic material tries to reach a position in which there will be minimum reluctance to the flux path.

Basic similarity exists in all types of motor, but the *observer* may either be at standstill, rotating, or in fact oscillating.[4]

When a current-carrying conductor is placed in a magnetic field, it experiences a force as a result of the interaction between this field and the field set up by the current in the conductor. In a rotating system, this force manifests itself as torque. The force or torque experienced by the conductor is due to the distortion or asymmetry of the resultant field, and the conductor tends to move in a direction that would help to restore the symmetry. In a d.c. motor, when the field coil has been energised and no current flows through the armature, the flux pattern is as shown in Figure 2.14(b). Since the flux pattern is symmetrical, there is no mechanical torque and the rotor does not rotate. When the brushes are connected to a d.c. supply, the armature conductors carry a current. In principle, the armature conductors could be replaced by a single equivalent coil, lying in a plane normal to the brush axis, which produces a magnetic field along that axis. The flux lines produced by the equivalent coil distort the main field flux as shown in Figure 2.14(c) and the rotor tries to move into a position that will ease

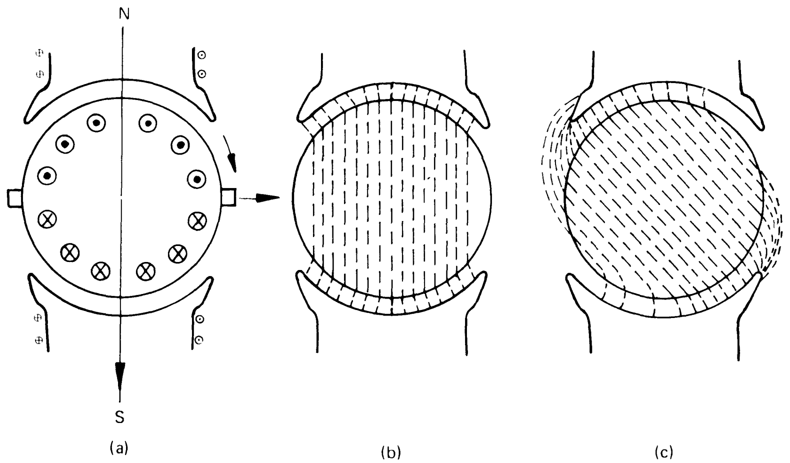

Figure 2.14 Electric motor operation. (a) Direction of currents and fluxes, (b) flux pattern with no armature current and (c) flux pattern with armature current

the 'strain' due to this distortion. In other words, the rotor magnet tries to align itself with the stator magnet, and thus the rotor rotates in the direction indicated. The coil which has been in contact with the brushes moves away and the next coil takes its place. The spatial distribution of the currents, however, does not change. As a result, the equivalent armature coil behaves as though there is no change in its position or in its current. In other words, the distortion of the magnetic field continues to exist although the conductors and therefore the rotor, rotate. A dynamic equilibrium is thus reached and the rotor continues to rotate. With increased load, the armature current increases and the field pattern becomes further distorted as it supplies the necessary torque.

In a.c. motor operation, a similar phenomenon occurs but it is more easily visualised by means of 'rotating field theory'.

2.12 ROTATING FIELD THEORY

The armature (usually the stator) of a synchronous or induction motor is normally wound for three phases and connected to a three-phase supply. A typical winding diagram is given in Figure 2.15. A balanced three-phase voltage

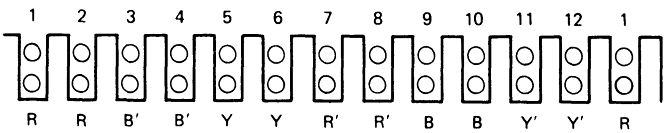

Figure 2.15 Three-phase double-layer winding

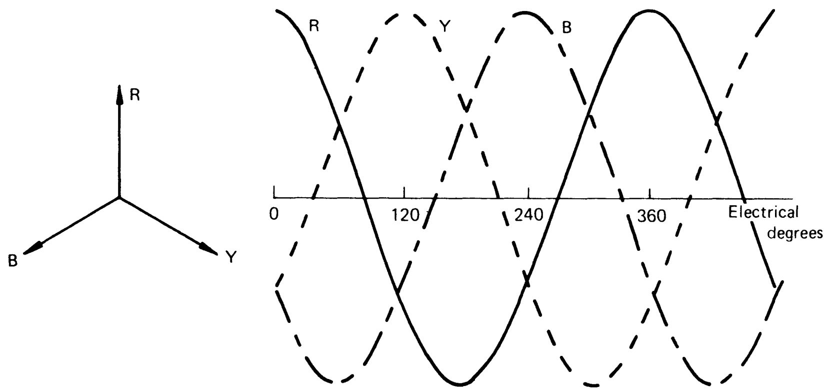

Figure 2.16 Balanced three-phase voltage/current waveform

waveform can be represented as in Figure 2.16. In Figure 2.17 let each of the three phases be represented by a coil. If we start counting time from the instant at which the current in phase R is maximum, we can see that the currents in phases Y and B are negative and each equal to half of their maximum value. In balanced conditions the maximum values in the three phases are the same. Let the current flowing in the side r of the coil rr′ in phase R be considered positive, and let the m.m.f. produced by this coil for the maximum current be H. The m.m.f. values produced by coils bb′ and yy′ for negative half-currents are $H/2$ along the directions shown in Figure 2.17(a).

The magnitude of the resultant m.m.f. vector is $3/2\,H$ as shown in Figure 2.17(a).

Let us consider the situation after a lapse of time corresponding to 30 electrical degrees. For a 50 Hz supply in a two-pole machine, 30 electrical degrees correspond to $30/(360 \times 50)$ or $1/600$ of a second.

Now the current in phase Y is zero, whilst the currents in phases R and B are each $\sqrt{3}I_m/2$, but with opposite signs. The resultant m.m.f. vector is still $3H/2$ but it has moved in space through 30 mechanical degrees (Figure 2.17b).

After $1/200$ of a second (i.e. 90 electrical degrees) the current in phase R is zero and the currents in phase B and Y are each $\sqrt{3}I_m/2$, but with opposite signs. The resultant m.m.f. vector is as shown in Figure 2.17(c). Its magnitude is still $3H/2$ but it has rotated in space through 90 mechanical degrees.

That this is a perfectly general and continuous rotation effect may be shown by simple analysis. Consider balanced three-phase currents flowing in the three-phase stator windings of the machine in Figure 2.17. The phase currents will be separated by $2\pi/3$ electrical radians *in time* and the windings are displaced by $2\pi/3$ mechanical radians *in space*. In a 2-pole machine electrical and mechanical radians are equal.

Let the flux density produced by each phase be proportional to the m.m.f.

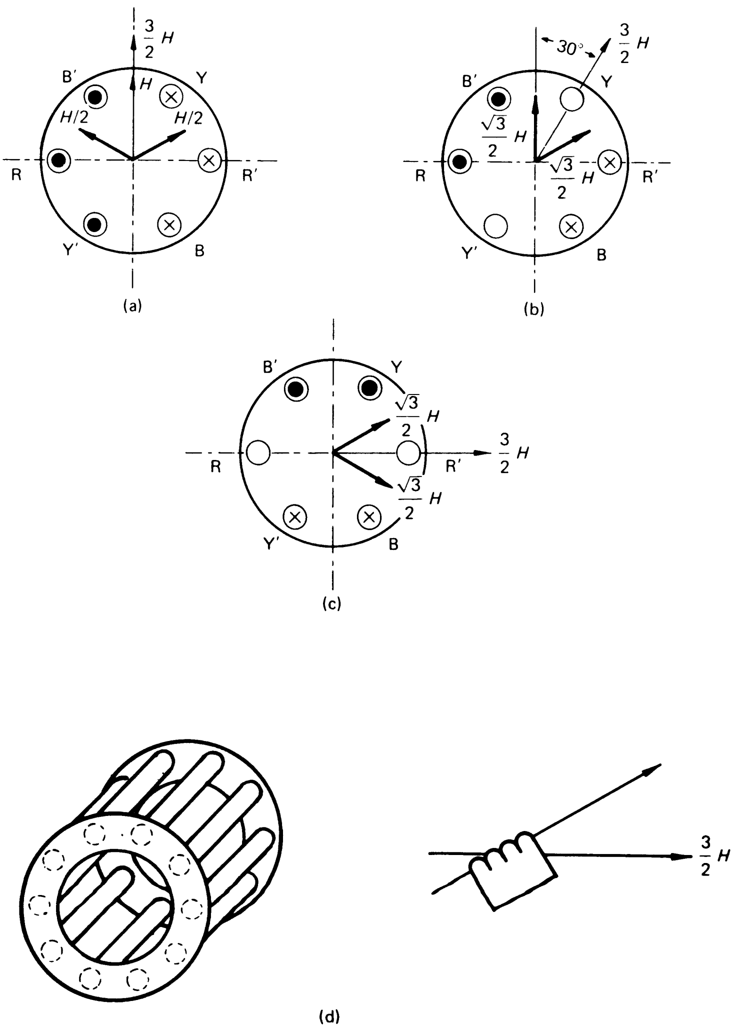

Figure 2.17 Production of rotating magnetic field. (a) $\theta = 0$, (b) $\theta = 30°$, (c) $\theta = 90°$ and (d) rotor circuit

Then

$$B_{\text{phase A}} = K\hat{\imath}\cos\omega t \tag{2.49a}$$

$$B_{\text{phase B}} = K\hat{\imath}\cos(\omega t - 2\pi/3) \tag{2.49b}$$

$$B_{\text{phase C}} = K\hat{\imath}\cos(\omega t + 2\pi/3) \tag{2.49c}$$

The resultant flux density is therefore found by adding these components vectorially, allowing for the spatial displacement, giving

$$B = Ki\left[\cos\omega t + \cos(\omega t - 2\pi/3)e^{-j2\pi/3} + \cos(\omega t + 2\pi/3)e^{j2\pi/3}\right] \quad (2.50)$$

Substituting $e^{\pm j2\pi/3} = -\frac{1}{2} \pm j\sqrt{3}/2$ and expanding

$$B = \tfrac{3}{2}K\hat{i}e^{-j\omega t} = \tfrac{3}{2}\hat{B}_{ph}e^{-j\omega t} \quad (2.51)$$

where $\hat{B}_{ph}$ is the maximum value of the flux density wave set up by one phase. This flux wave rotates at the synchronous anguar velocity $\omega_s = 2\pi f$ in the 2-pole case.

Therefore, although the coils are stationary, three-phase 50 Hz currents flowing through three-phase windings produce a resultant m.m.f. of constant magnitude which is not stationary in space but rotates at the rate of 360 electrical degrees in 1/50 of a second. In a 2-pole machine this rate corresponds to one revolution in 1/50 of a second or 3000 r.p.m. In a 4-pole machine, the resultant m.m.f. will still rotate at the same speed in terms of electrical degrees but in this case 360 electrical degrees will now correspond to 180 mechanical degrees (1/2 revolution) in 1/50 of a second or 1500 r.p.m. Writing the relationship between electrical and mechanical angular velocities, electrical rad/s $= p \times$ (mechanical rad/s) where p is the number of pole pairs.

The angular velocity of the resultant m.m.f. wave is given by

$$\omega_s = 2\pi f/p \ \text{ mechanical rad/s} \quad (2.52)$$

where f is the supply frequency and ω_s is known as the synchronous speed.

2.12.1 Operation of an induction motor

If a three-phase supply is connected to the stator circuit of an induction motor it produces an m.m.f. of constant amplitude, which rotates at synchronous speed. This could be considered as an electromagnet energised by direct current and rotating at synchronous speed. This m.m.f. sweeps past the rotor conductors and induces a voltage in them. If the rotor conductors are connected to form closed paths (Figure 2.17d) current will flow through each path. The rotor is therefore magnetised by the field set up by its own current. This magnet then 'chases' the stator magnet. The rotor cannot attain synchronous speed. If it did, then with no relative velocity between the stator m.m.f. and the rotor conductors, no voltage would be induced in the latter. As a result, the rotor would cease to develop its own field, and since torque is produced by the interaction of stator and rotor fields, there would be no torque. The rotor, therefore, settles at a sub-synchronous speed at which just enough torque is produced to overcome the torque imposed by the load, that is at a speed at which dynamic equilibrium is maintained. Increase of the load torque reduces the speed, and hence increases the relative velocity between the stator m.m.f. and the rotor conductors. This increases the rotor induced voltage and consequently increases the motor torque to meet the increase in the load torque. Thus, with increasing load the motor torque increases until a certain 'pull-out' speed is reached, after which the generated motor torque falls with drop in speed. Further increase in load torque will cause stalling of the rotor. The maximum torque developed by the motor is the pull-out torque. Normally, an induction motor is designed with the pull-out torque well in excess of the rated or full load torque, to allow ample momentary overload capacity.

2.12.2 Operation of a synchronous motor

In the case of a synchronous motor the rotor field winding is energised by direct current from an external source known as the exciter. Hence its magnetism does not have to depend on the relative velocity between the stator (armature) m.m.f. and the rotor. On the contrary, the rotor magnet aligns with the stator rotating magnetic field and rotates synchronously with the stator m.m.f. Hence the terms synchronous motor and synchronous speed. If the motor is loaded then the rotor magnet falls back by a certain angle, still maintaining synchronous speed. This is known as the load angle. Its vital role in stable operation and control of synchronous machines will be examined in later chapters.

2.13 STEADY-STATE EQUATIONS OF D.C. MACHINES

The voltage that is generated across the brushes in a d.c. machine can be derived as follows (Figure 2.18):

$$\text{generated voltage } E \propto \mathrm{d}\Phi/\mathrm{d}t$$

If a conductor of length l(m) cuts a magnetic field of flux density B(Wb/m^2) at velocity v(m/s),

$$E = Blv \quad \text{(V)}$$

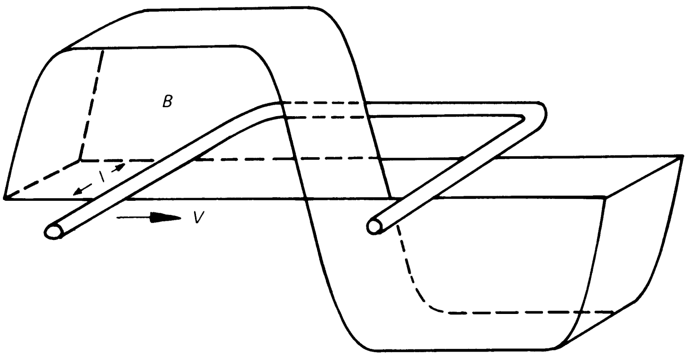

Figure 2.18 Voltage generation in a conductor

If there are Z_s conductors in series

$$E = BlvZ_s$$

If these are around the surface of an armature of diameter D(m) rotating at n(rev/s)

$$E = Bl\pi DnZ_s$$

and the total flux crossing the air gap is

$$Bl\pi D = \Phi$$

Thus

$$E = \Phi Z_s n$$

The equation is sometimes written in terms of the total number of armature conductors Z, the number of parallel paths A on the armature and the magnetic flux per pole Φ_p.
In this case we have

$$E = \Phi_p Zn(2p/A)$$

where $2p$ is the number of poles and $A = 2$ for a wave winding and $A = 2p$ for a lap winding.

When the d.c. generator is connected to a load, the terminal voltage drops as shown in Figure 2.19(c). This is due to the effects of armature resistance and some suppression of the main field by the additional field now set up by the armature current—an effect known as armature reaction. Neglecting for the moment the effect of armature reaction, and considering the separately excited machine shown in Figure 2.19(a), the terminal voltage is given by

$$V_t = E - R_a I_a \tag{2.53}$$

where E is the internally generated voltage
or

$$V_t = K\Phi\omega - R_a I_a \tag{2.54}$$

the last term being the voltage drop in the armature due to the load current. When the machine operates as a motor, voltage is applied across the brushes and as the conductors rotate in a magnetic field, the voltage E is also generated in the coils. The generated voltage in motor action is termed the back e.m.f. Since the voltage applied to terminals of a motor must be greater than the back e.m.f. in order to drive current through the armature, the applied voltage equation is now

$$V_t = E + R_a I_a \tag{2.55}$$

where E is the back e.m.f. and the last term is again the voltage drop in the armature. The voltages V_t and E are normally very nearly equal since the voltage drop in the armature is relatively small.

A starting resistance is always used in the armature circuit of a d.c. motor,

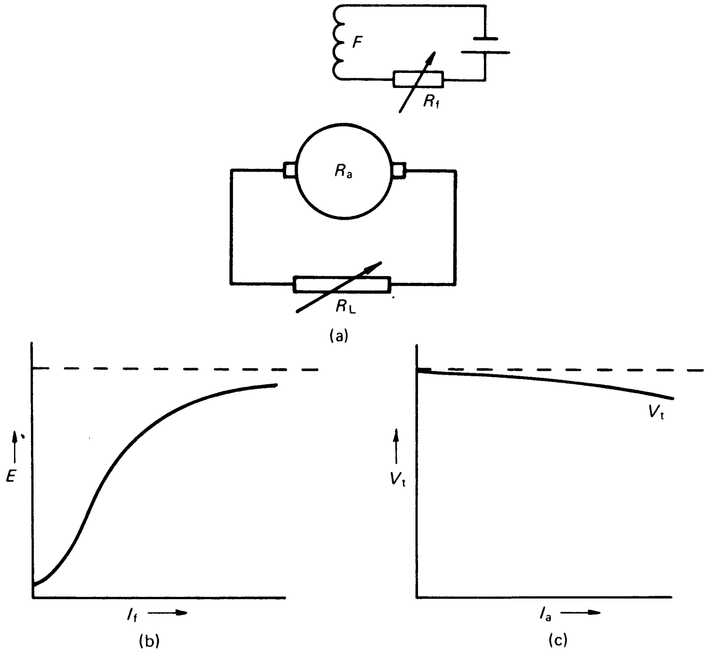

Figure 2.19 Separately excited generator and its characteristics. (a) Separately excited generator, (b) open circuit characteristic and (c) load characteristic

otherwise the input current would be excessive before the motor had gained enough speed to generate sufficient back e.m.f.

The motor and generator equations are usually combined thus

$$V_t = E \pm R_a I_a \tag{2.56}$$

The torque equation for motor operation is obtained as follows. The voltage equation is $V_t = E + R_a I_a$. Multiplying both sides by I_a, we have

$$V_t I_a = E I_a + R_a I_a^2 \tag{2.57}$$

where $V_t I_a$ is the input power, $R_a I_a^2$ is the copper loss in the armature, and EI_a is the electrical power output.

The electrical output power EI_a provides the mechanical power at the shaft. If T is the mechanical torque (N m), then

$$\omega T = E I_a \quad \text{(W)} \tag{2.58}$$

where ω is the rotor angular velocity (mechanical rad/s).

Since the generated voltage is proportional to the field current I_f, in the absence of saturation, we may write

$$T \propto I_f I_a \tag{2.59}$$

Also, in motor operation, since

$$V_t \approx E \quad \text{and} \quad E = K\Phi\omega \tag{2.60}$$

we may write

$$V_t \approx K\Phi\omega \tag{2.61}$$

Thus,

$$\omega \propto 1/\Phi \tag{2.62}$$

if the voltage is constant.
Neglecting saturation, we may write

$$\omega \propto 1/I_f \tag{2.63}$$

This shows that if the flux is weakened the speed will increase, and vice versa.

When the field winding is energised, a magnetomotive force is set up which drives flux through the magnetic circuit. The overall magnetising B–H characteristic of d.c. machines can be obtained by plotting the field current against the open-circuit voltage, the speed being maintained constant. The magnetic intensity H is given by the field ampere turns per unit length *of the magnetic circuit*. Since the number of turns of the field winding and the length of the magnetic circuit are constant, H is directly proportional to the field current. Similarly, the generated voltage is proportional to the flux-density as long as the speed of the generator is kept constant. Hence, if we plot the open-circuit voltage against field current for a constant speed, we can obtain the magnetising characteristic of the machine. This is also called the open-circuit characteristic (Figure 2.19b).

2.13.1 The load characteristic of d.c. generators

The load characteristic represents the relationship between the terminal voltage and the load current. Figures 2.19 to 2.22 illustrate the load characteristics of d.c. generators for different modes of excitation.

2.13.1(i) *Separately-excited d.c. generator (Figure 2.19)*

The field current is independent of the terminal voltage V_t. The latter falls with the load because (*a*) the voltage drop in the armature increases with increasing load current and (*b*) there is nett loss of flux due to armature reaction (the distortion of the main magnetic field by the field produced by the armature).

2.13.1(ii) *Shunt generator (Figure 2.20)*

The field current in the shunt generator is established and maintained by the terminal voltage. This means, of course, that in order that the generator can excite itself, there must be some residual magnetism in the poles. The open-circuit voltage characteristic is shown in Figure 2.20(b). It can be seen from this curve that there is an upper limit to the total resistance in the field circuit, above which the generator will not excite. The critical value R_c is the tangent to the open-circuit voltage curve. The resistance R_1 would be too high.

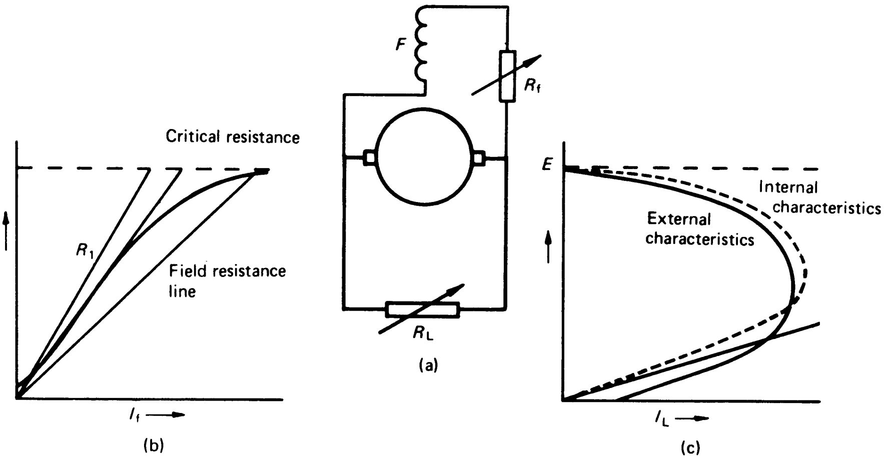

Figure 2.20 Shunt generator and its characteristics. (a) Shunt generator, (b) open-circuit characteristic and (c) load characteristic

As the electrical load on the machine is increased there is increasing voltage drop in the armature. As the terminal voltage falls so does the current in the field circuit, causing still further drop in terminal voltage. This will be self stabilising up to a certain load current as shown in Figure 2.20(b). It will be seen, however, that a point on the load characteristic is reached when the voltage will become rapidly suppressed by this effect which is accentuated by armature reaction. There is a lower limit to the value of load resistance, indicated by the tangent line to the internal generated voltage curve. If a load resistance lower than this critical value is connected, the generator will not build up voltage.

2.13.1(iii) *Series generator (Figure 2.21)*

If we ignore the current through the bypass control resistor across the field winding, the load current is the same as the field current. The voltage characteristic on load is therefore as shown in Figure 2.21(b). The bypass resistor provides a means of controlling the current through the field winding and thus the terminal voltage.

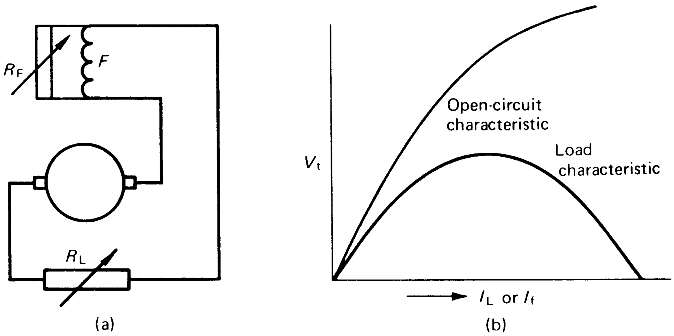

Figure 2.21 Series generator and its load characteristic. (a) Series generator and (b) load characteristic

2.13.1(iv) *Compound generator (Figure 2.22)*

An extra field winding F_1 is provided on the main poles, which is connected to carry the total armature current including that taken by the field (long shunt),

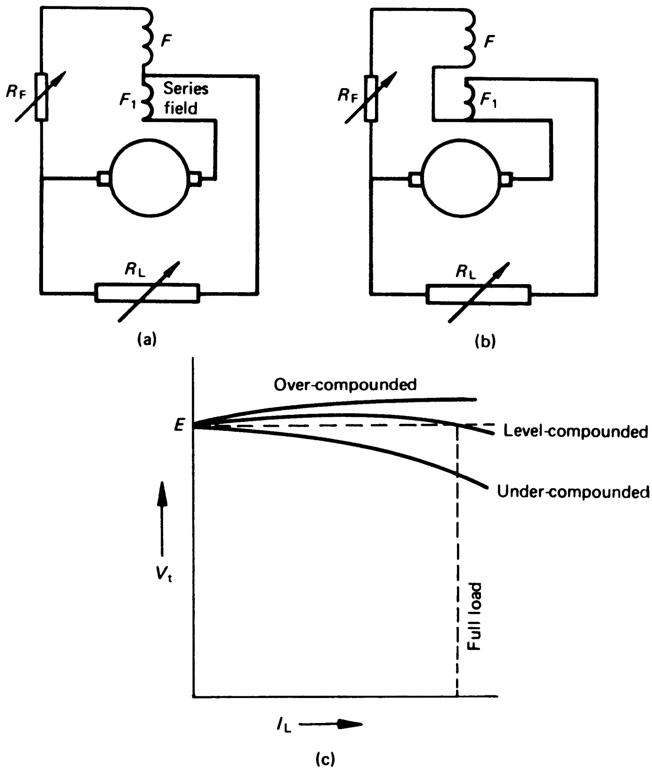

Figure 2.22 Compound generators. (a) Long shunt, (b) short shunt and (c) load characteristics

either to assist (cumulative compound) or oppose (differentially compounded) the main field F. By suitably adjusting the current through the extra field winding the load characteristics may be varied as shown in Figure 2.22(c).

2.14 MOTOR CHARACTERISTICS

2.14.1 Shunt motor (*Figure 2.23*)

The terminal voltage is applied to the armature and field circuits in parallel, the field current being controlled quite separately from the armature current by means of an added rheostat. On starting the machine a resistor rated to carry the full load current is inserted in series with the armature and is cut out as the motor speed rises. If the load on the shaft increases when the motor is running at speed, the increased input current taken to meet the additional torque will cause voltage drop at the armature, due to the winding and brush resistance and armature reaction. The generated back e.m.f.

$$E = K\Phi\omega = V - IR$$

therefore decreases. Since the main flux remains approximately constant the speed ω will drop almost linearly with increasing load when the applied voltage is constant. The normal drop in speed from no-load to full-load is about ten per cent with constant field current.

The speed can be restored to its normal value (or indeed increased in value) by reducing the field current, hence reducing the main flux. The motor will then meet the additional shaft torque (plus loss) at, say, the original speed, by taking additional armature current. When the motor is running at speed taking armature current i the generated back voltage E is such that the power EI provides the input torque to meet the shaft load.

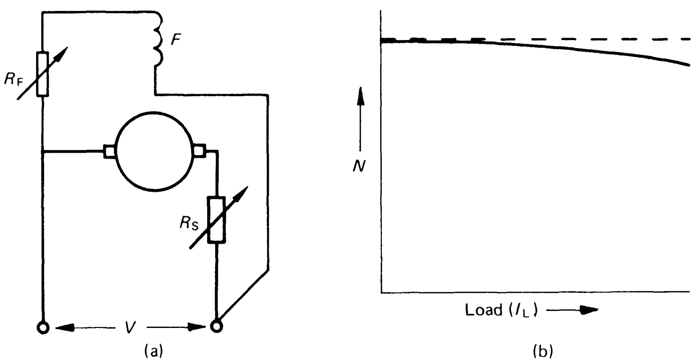

Figure 2.23 Shunt motor and its load characteristic. (a) Shunt motor and (b) load characteristic

Since

$$\omega = \frac{V - IR}{K\Phi} \tag{2.64}$$

a reduction in Φ will lead to an increase in ω. This will mean a higher value of input power ωT and input current. The converse occurs if the field current is increased, namely, speed and input current will fall. The speed range (maximum/minimum) available with a normal d.c. shunt motor is about 5:1.

It will be noticed that external resistance in the armature circuit will cause voltage drop at the motor terminals. This will reduce the value of E and lead to a drop in speed. The following two points may be mentioned regarding such resistance.

(*a*) If the resistance of the cabling to the motor is too high troublesome speed fluctuations with load variation may occur.
(*b*) The speed of a d.c. shunt motor can be controlled by introducing variable resistance in the armature circuit. This is obviously a costly method, since such resistors must dissipate large amounts of power as heat. A better method is to apply a variable input voltage from a separate d.c. generator, as in the Ward–Leonard system. The advantage of the latter over simple field control is that a much greater speed range (about 20:1) can be obtained very efficiently. The modern trend, however, is to use thyristors to provide variable direct voltage by suitably controlling the firing of the thyristors.

2.14.2 Compound motor (*Figure 2.24*)

A compensating field winding is provided on the main poles in series with the armature (or input) current. The magnetic field set up by these coils either

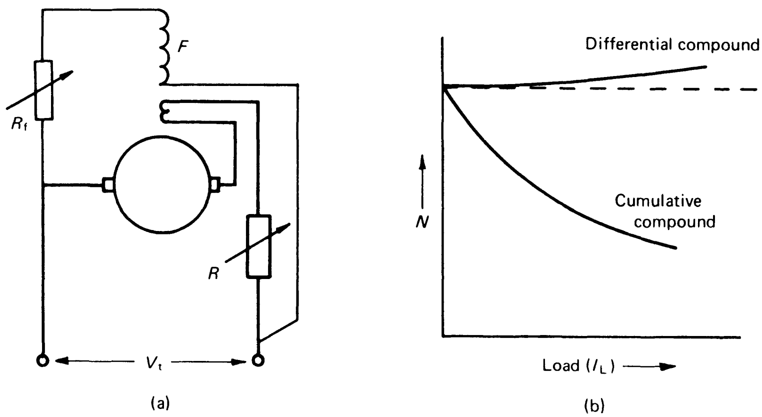

Figure 2.24 Compound motor and its load characteristics.
(a) Compound motor and (b) load characteristic

cumulatively assists or differentially opposes the main field (the latter connection is rarely used). Different load/speed characteristics obtainable are shown in Figure 2.24(b).

2.14.3 Series motor (*Figure 2.25*)

In the d.c. series motor the field winding is rated to carry the full load current in series with the armature. The motor will build up speed against the load torque until, at equilibrium, the generated back e.m.f. E, due to the rotation of the armature conductors in the magnetic field set up by the series coils, is such that $E = (V - IR)$ provides the shaft power and torque.

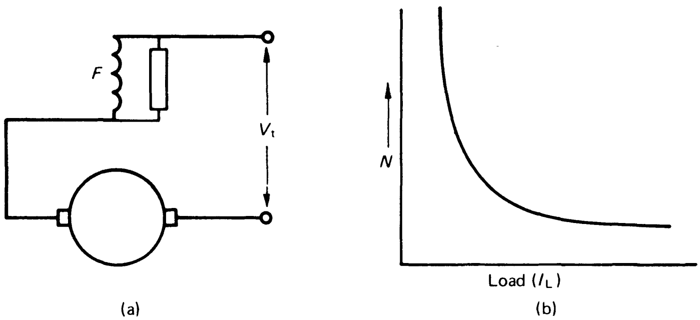

Figure 2.25 Series motor and its load characteristic. (a) Series motor and (b) load characteristic

With this type of excitation, the flux Φ is approximately proportional to the load current and consequently, since V and E are nearly constant, the speed $\omega \propto 1/\Phi \propto 1/I_{load}$, giving the 'inverse law' speed–torque curve shown in Figure 2.25(b). The series motor is started through a heavy duty resistor in the armature circuit. As in the case of the shunt motor, this is cut out in stages as the speed builds up. This type of motor will produce a high torque at the lower end of the speed range. It must always be started on load and is therefore used for traction, cranes, and other heavy duty industrial drives which are permanently connected to the shaft. The inverse speed–torque law shows that the series motor will tend to accelerate to excessive and dangerous speed if the load is removed. The motor speed can be controlled to a limited extent by use of a variable by-pass resistor in parallel with the field windings.

2.15 SYNCHRONOUS MACHINES

Synchronous machines may be broadly divided into two classes depending on whether the prime mover is a steam turbine or a hydraulic (water) turbine. Since steam turbines are inherently high-speed machines, generators driven by steam turbines usually have two (or four) poles and are known as turbogenerators. The field winding is set into slots on the rotor to form non-salient poles, keeping the surface of the rotor a comparatively smooth cylinder, as in Figure 2.26(a). When driven at 3000 r.p.m. (or 1500 r.p.m.) these generators produce 3-phase alternating voltage at 50 Hz (60 Hz at 3600 r.p.m. in the USA). This follows from the relationship between speed and frequency established in Section 2.12.

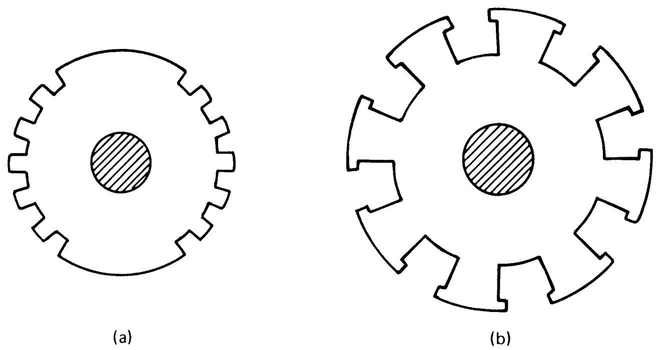

Figure 2.26 Rotor configurations of alternators. (a) Cylindrical rotor and (b) salient pole rotor

Water turbines on the other hand are low-speed machines, in the range 100–400 r.p.m. Hence generators driven by such prime movers have a large number of poles to produce the required frequency. These salient poles are located on the circumference of a large-diameter rotor, in order to produce an adequate peripheral speed of the rotating field flux set up by the poles.

We have seen that the inertia constant of a turbo-alternator differs greatly from that of a large hydrogenerator. In a mixed hydro and steam interconnected system, this difference has a pronounced effect on the dynamic stability and control of the machines in synchronous operation.

The air gap in a salient pole machine is obviously not uniform around the periphery. In a cylindrical-rotor machine in which the air gap is nearly uniform (except of course for the slots) the field winding is distributed over several slots to produce a m.m.f. with stepped waveform. One of the effects of distributing a winding in this manner is to improve the sinusoidal m.m.f. waveform by reducing the magnitude of the higher-order harmonics in the m.m.f. wave. In a salient-pole machine this may be achieved only by suitably contouring the pole face.

2.15.1 The m.m.f. and flux density waves

Figure 2.27 is a developed diagram of a cylindrical rotor with 2/3 of its periphery wound. Coil 1,1′ would produce a rectangular magnetomotive force (m.m.f.);

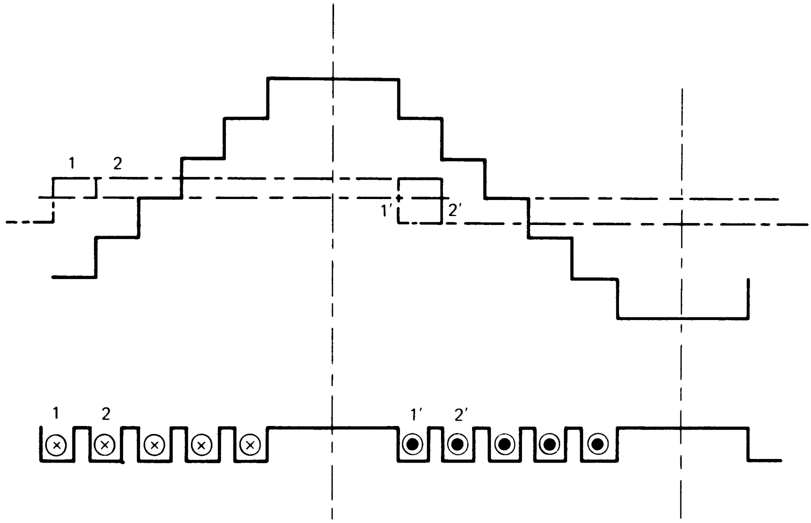

Figure 2.27 MMF wave of a cylindrical rotor machine

coil 2,2′ would produce another rectangular m.m.f. displaced from the previous one by one rotor slot pitch (the angle subtended at the centre of the rotor by two consecutive slots). In this manner all of the coils produce individual rectangular m.m.f. waveforms which, when combined, give a stepped trapezoidal waveform as illustrated. In this manner, a distributed winding modifies a rectangular wave into a stepped trapezoidal wave which is a closer approximation to a sine wave.

The air gap being uniform, the permeance of the magnetic path to flux waves remains fairly constant. Hence the flux or flux-density wave is similar to the m.m.f. wave in shape.

In the case of a salient pole machine, the m.m.f. wave is as shown in Figure 2.28. However, by suitably grading the air gap from the centre of the pole to the pole tips, the permeance is made to decrease in a manner that produces an approximately sinusoidal flux density wave, as shown.

2.15.2 The effects of non-uniform air gap

The non-uniform air gap in a salient-pole machine gives rise to non-uniform permeance of the magnetic path. Assuming the magnetic core to be infinitely permeable, the main reluctance to the flux is offered by the air gap and hence this gap length is of vital importance in all rotating electrical machines. The effect of this non-uniformity on the inductance of the coils may be understood by a simple illustration.

A coil is wound around a magnetic structure as shown in Figure 2.29. It is required to measure the self-inductance of the coil. Let us assume that the iron

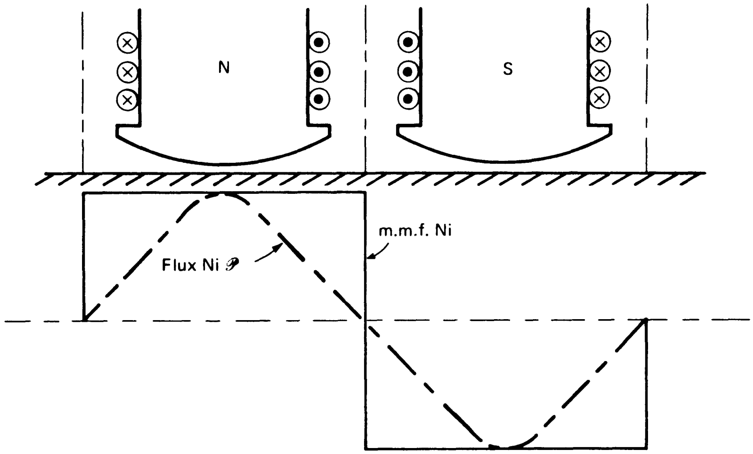

Figure 2.28 MMF and flux wave of a salient pole rotor

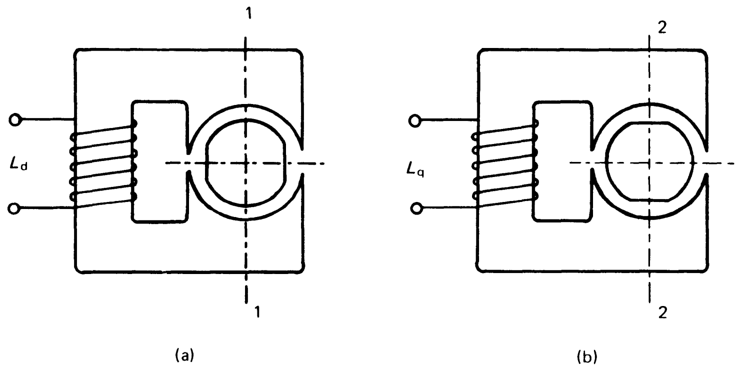

Figure 2.29 Variation of inductance due to saliency. (a) Small airgap in direct axis and (b) large airgap in quadrature axis

path offers no reluctance to the flux; reluctance is then offered only by the air gap (which is very nearly the case in practice). Reluctance is the inverse of permeance, and the self-inductance of the coil is $N^2\mathscr{P}$. If the air gap is constant the inductance of the coil is independent of the rotor position.

If, however, the rotor is shaped as shown in Figure 2.29 then for position 1 of the rotor as shown in Figure 2.29, the air gap is small, and hence the permeance large. As a result the inductance of the coil is large. In position 2, the air gap is larger and the permeance smaller, and therefore the inductance is smaller.

If L_d is the inductance of the coil for rotor position 1 and L_q that for rotor position 2, then $L_d > L_q$. In a machine the same thing happens.

If A, B, C represent the 3-phase armature coils in Figure 2.30(a), their self-

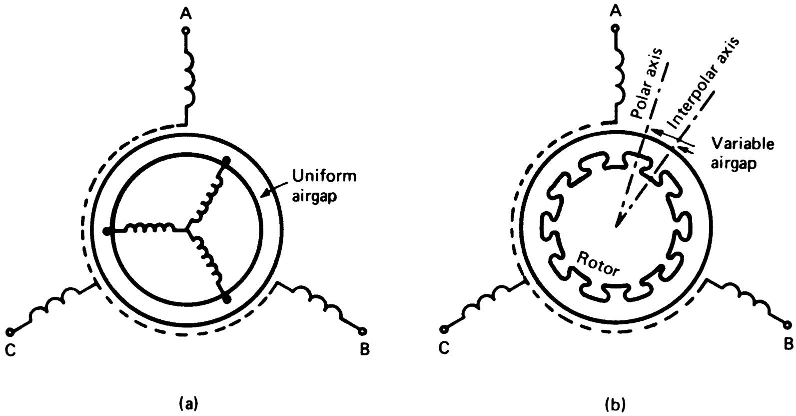

Figure 2.30 Inductance and airgap. (a) Uniform airgap and (b) variable airgap

inductances remain constant if the air gap is constant. For a cylindrical-rotor machine, such an assumption is valid but in salient pole machines the inductance will vary with the variation of the air gap in the flux path, as shown in Figure 2.30(b). Provided the pole shoes are suitably graded, we may assume a simple relationship between L_d and L_q in the form of a truncated Fourier series

$$L_d = \frac{(L_d + L_q)}{2} + \frac{(L_d - L_q)}{2} \cos 2\theta \tag{2.65}$$

The inductance or the reactance of the armature (usually the *stator*) is of vital importance in the performance of synchronous machines, as will be seen later.

Synchronous motors are commonly of salient pole construction. The non-uniform air gap with salient pole construction gives rise to what is known as reluctance torque, which tends to hold the field poles in synchronism with the rotating field set up by the three-phase armature current. This torque can be used for running synchronous motors without any applied field excitation and they are then known as 'reluctance motors'. Until recently such motors were available only at fractional-kilowatt rating but improved methods of producing increased saliency effects have resulted in reluctance motors with several kilowatts rating.

2.15.3 Amortisseur windings

Cylindrical rotors for synchronous machines are made of solid forged steel. While an alternator is running in a balanced 3-phase state, at constant speed, the flux in the air gap will be of constant magnitude and rotating in synchronism with the field structure. There is then no alternation in the flux carried by the rotor. If, however, the machine rotor should depart from synchronous speed, even for a short time, or undergo sustained oscillation, eddy currents will be set up in the solid steel poles, which can cause severe local heating. There is therefore, a special winding—a cage of bars—set into the pole faces, which will produce heavy damping current. These are called amortisseur windings. During steady balanced operation they have no effect. They must, however, be designed with an optimum resistance. If this is too high, insufficient damping current will flow; on the other hand if it is too low, heavy amortisseur current will be produced during disturbances but its phase relationship is then such that this has little damping effect.

2.15.4 Methods of field excitation

The power required to provide the field excitation of a synchronous machine is relatively small, usually less than two per cent of the rating of the generator even on small machines. The excitation power required for present large turbogenerators is of the order of 0.4 per cent of the generator rated power. Moreover, the exciter voltage for even large generators is quite low. The armature or output voltage, however, can be very much higher, 11 kV and 33 kV being common for the larger size machines. Consequently, it is normal engineering practice for all medium and large size generators to have the armature coils on the stator and the field windings on the rotor. On smaller machines, however, the field windings may be on a salient-pole stator and the armature on the rotor. From the standpoint of mathematical analysis, it makes no difference whether the armature is on the stator or on the rotor, since it is only the relative speed of the stator and rotor coils that matters in the analysis.

Conventionally, the field excitation of an alternator is obtained from a d.c. generator mounted on the same shaft as the rotor. For rotating-field generators the excitation current is fed through brushes to a pair of slip rings on the rotor. The field of the exciter is energised by pilot exciter, which is subject to automatic or manual control.

Static excitation schemes have also become available with the development of solid-state devices which rectify an input a.c. voltage, often that of the generator itself, and feed the rectified output to the generator field winding through slip rings. A fairly recent development is that of the brushless excitation system in which the excitation power unit is a directly driven rotating-armature exciter supplying a rotating rectifier mounted on the same shaft. The d.c. terminals of the rectifier are connected directly to the generator rotor winding by leads secured along the coupled shafts. At present, rotating silicon diodes are being used for rectification but there is continuing development work on brushless thyristor excitation.

2.15.5 Mode of operation

When a 3-phase alternator is run with its field winding energised, a 3-phase armature voltage is generated, the frequency depending on the speed at which the field is rotated and the number of poles.

Alternators are usually connected to a power system or grid to ensure reliable supply of electric power. These alternators run in synchronism and supply electrical power at a standard frequency. When an alternator is to be connected to the grid certain procedures have to be followed. The process of synchronising is perhaps best explained by a simple mechanical analogy. When a gear has to be meshed with another gear which is already rotating, its peripheral speed and direction of rotation have to be in perfect consonance with those of the gear with which it is going to be meshed.

In a similar manner, when an alternator is to be connected to a power grid, its frequency, voltage, phase sequence, and phase position must be perfectly matched with those of the mains supply. When it is synchronised, the machine could in theory operate either as a motor or as a generator, depending on whether it draws power from the grid, or supplies power to it. In practice, however, large power system alternators never operate in the motoring state, except inadvertently under transient conditions. Synchronous motors are specially designed for industrial drives.

The rotor in either case rotates in the same direction for given direction of phase rotation R–Y–B and always at synchronous speed. Physically, synchronous operation may be visualised as the rotor magnet having been locked on to the stator magnetic poles (produced by the 3-phase armature current), both of which are revolving at synchronous speed.

If the machine is run as a generator, the rotor magnet rotates at an angular displacement, the load angle, *synchronously* ahead of the rotating field structure. As a motor, the rotor trails the stator rotating field by the load angle δ.

2.15.6 Vector diagram

When a phase current I_p flows through the armature, there is an impedance drop in each phase and hence the terminal phase/neutral voltage is

$$V_p = E_p - Z_p I_p \tag{2.66}$$

(If the armature coils are connected in star, then $I_{phase} = I_{line}$ but $E_{line} = \sqrt{3}\, E_{phase}$. With delta connection of the coils, these relations are reversed.) The impedance drop ZI is similar in its effect to armature resistance drop considered earlier in the case of d.c. machines.

Let us consider an alternator which delivers a current I_p at a lagging power factor angle ϕ. If we neglect armature resistance for the time being, we can draw the vector diagram as shown in Figure 2.31.

The field flux Φ_F produces the e.m.f. E_p, the open circuit voltage, which lags behind Φ_F by 90°. When current flows in the armature windings, the field m.m.f. and the armature m.m.f. combine giving the resultant gap flux Φ_G which produces the terminal voltage V_p. So we can either look at this problem from the point of view of generated voltage, impedance drop, and terminal voltage, or of the field flux—armature reaction and the resultant gap flux. They mean the

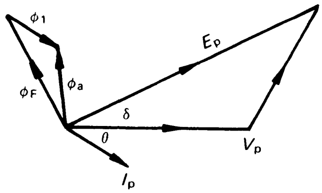

Figure 2.31 Vector diagram of a synchronous generator

same. It is hardly necessary to point out that Φ_I (the flux due to armature reaction) is directly proportional to IX, the reactance voltage drop. The total reactance drop IX which includes the effects of armature leakage reactance and armature reaction, is called 'synchronous reactance'.

2.15.7 Generator and motor operation

A synchronous machine connected to the grid can be considered to be connected to an infinite busbar (Figure 2.32). The infinite busbar is assumed to be at a constant voltage and frequency, and has zero impedance and infinite inertia. Physically, it means the connection of a very large number of synchronous machines in parallel, whose moments of inertia combine to give theoretically infinite inertia and whose impedances combine to give zero. In this case the oscillations of a particular machine connected to the infinite busbar would not transmit the oscillations to the grid. If the impedance is taken as zero, then the current that this machine might deliver to or draw from the infinite bus will not affect the voltage of the latter.

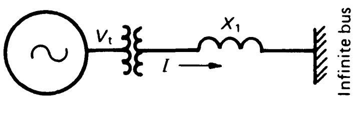

Figure 2.32 Synchronous generator on infinite busbar

The concept of an infinite busbar is a useful one. It renders the study of the behaviour of a synchronous machine relatively simple and the results obtained are reasonably satisfactory. Obviously, this concept is an over simplification.

Another approximation made in this section is to neglect resistance of the armature and the tie-line connecting the machine to the infinite busbar. In later sections these resistances will be duly taken into account.

2.15.8 Active and reactive power

Consider a synchronous machine connected to an infinite busbar through a line reactance X_1. Let the machine terminal voltage be V' and the voltage of the infinite busbar be V_p per phase. The power delivered to the busbar is P. The active power delivered depends on the mechanical power input to the turbine. The output current and power-factor are therefore determined jointly by the mechanical input and the excitation current in the field winding. Their product $I \cos \phi$ is a constant as long as the active power P (controlled by the turbine) and the busbar voltage V are maintained constant, since the active power $P = 3V_pI_p\cos\phi$. The vector diagram, Figure 2.33(a), will help to clarify this.

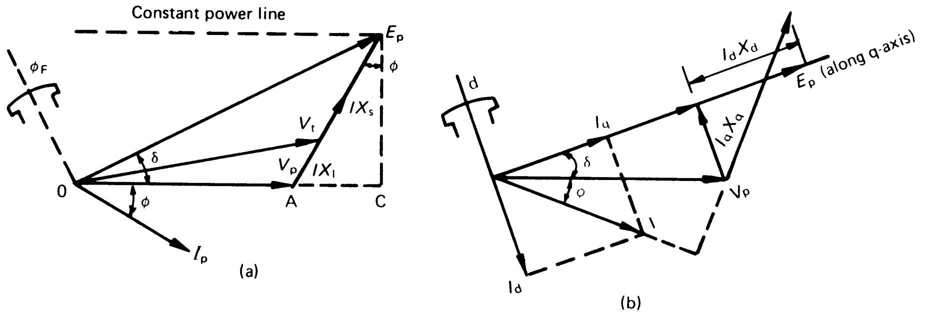

Figure 2.33 (a) Vector diagram of a synchronous generator on infinite busbar through a reactance and (b) vector diagram of a salient-pole synchronous generator on infinite busbar

In Figure 2.33(a) 0A represents the busbar voltage. AE is the total voltage drop due to synchronous reactance X_s of the machine, and the line reactance X_l and 0E represents the induced phase voltage E due to the field current I_f. Let $I(X_s + X_l)$ be represented as IX. Then the power per phase

$$P_p = V_pI_p\cos\phi = \frac{V_pI_p\cos\phi}{X} = \frac{V_pE_p\sin\delta}{X} \tag{2.67}$$

As long as P, V, and X are kept constant, $E_p \sin\delta$ will be constant. Hence the locus of the vector E_p must be a straight line parallel to the vector V_p, called the constant power line. (When armature and line resistances are included, this locus is no longer a straight line, but becomes an arc of a circle.) If E_p is altered by changing I_f, AE changes along the constant power line but EC remains constant. Since AE ($= I_pX_p$) will change, I_p also changes and since $I_p\cos\phi$ = constant, $\cos\phi$ must change also. The armature current I is obviously minimum when $\cos\phi$ is unity. With further reduction in I_f, the current I_p increases again and now leads the voltage. The relationship between the field and armature current at constant power is brought out clearly by the well-known V-curves shown in Figure 2.34. Total power is $3(V_pE_p\sin\delta)/X$. For star connection $V = \sqrt{3}V_p$.

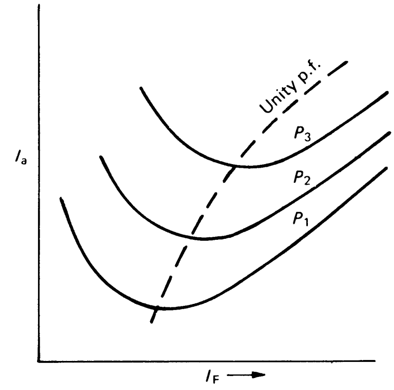

Figure 2.34 V-curves for a synchronous generator

Hence the total power is $(VE\sin\delta)/X$. For delta connection $V = V_p$, but synchronous reactance per phase is $3X$. Therefore, we have the total power once again to be $(VE\sin\delta)/X$.

2.15.9 Expression for reactive power

Reactive power per phase, usually designated by Q, is given by

$$Q = 3V_pI_p\sin\phi \tag{2.68}$$

where, again, V_p and I_p are voltage and current per phase. We can obtain an expression for the reactive power in terms of the load angle, as in the case of active power. From Figure 2.33(a), $I\sin\phi$ is given by $AC/X = (0E\cos\phi - V_p)/X$.

Hence the reactive power per phase is

$$Q = \frac{3(E_pV_p\cos\delta - V_p^2)}{X} = \frac{EV\cos\delta - V^2}{X} \tag{2.69}$$

2.15.10 Power-angle characteristic

We have seen that the power equation is

$$P = \frac{V_pE_p\sin\delta}{X}$$

This equation gives the relationship between the power delivered and the load angle δ.

When the machine is not delivering any load the angle δ is zero and E and V are coincident. Physically, this means that the axis of the pole (the field magnet) is perpendicular to the terminal voltage vector V. In other words the revolving armature field and the magnetic field of the rotor are coincident. As soon as the machine is loaded, these magnets are 'pulled' apart slowly and synchronously through the angle δ. The power P and the load angle δ in the case shown are obviously related by a sine curve. With various levels of excitation we have a family of such curves, as in Figure 2.35. These show maximum power, at constant excitation and without a voltage regulator, when the load-angle $\delta = \pi/2$ electrical radians, beyond which the machine loses synchronism. This may be explained by the energy-balance concept (see the equal-area criterion, Chapter 4).

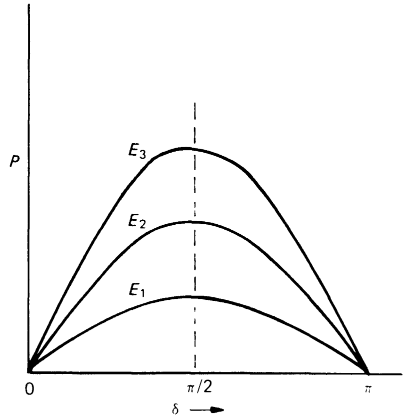

Figure 2.35 Power-angle curves for a synchronous generator

The power-angle characteristic is sinusoidal as long as the air gap between the stator and the rotor is uniform.

If there is any saliency, the power-angle equation becomes

$$P_{\text{phase}} = \frac{E_p V_p \sin\delta}{X_d} + \frac{V_p^2}{2}\left\{\frac{1}{X_q} - \frac{1}{X_d}\right\}\sin 2\delta \tag{2.70}$$

where X_d is the synchronous reactance along the pole-axis or direct axis and X_q is the synchronous reactance along the interpolar axis or the quadrature axis.

The power-angle curve now has a second-harmonic term, illustrated in Figure 2.36.

The term involving $\sin 2\delta$ is independent of the excitation voltage E. This indicates that a synchronous machine can be made to develop a torque which is independent of the excitation, if saliency is present. This is the principle used in reluctance motors.

The relation stated in equation (2.70) can be obtained as follows.

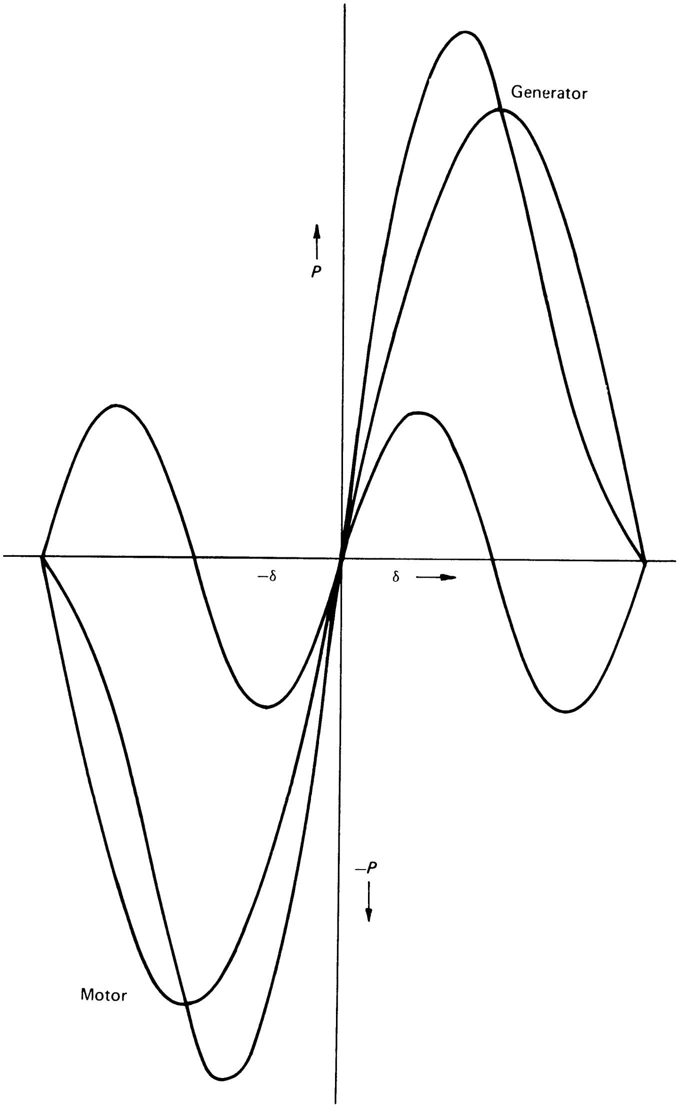

Figure 2.36 Power-angle characteristics of synchronous machine with saliency

Let us consider the vector diagram in Figure 2.33(b). By considering the components of the voltage V_p along the direct and quadrature axes, the following relations obtain.

$$V_p \cos\delta = E_p - X_d I_d \tag{2.71}$$

$$V_p \sin\delta = \dot{X}_q I_q \tag{2.72}$$

I_d and I_q are the direct and quadrature axis components of the armature current I_p. The power per phase is obtained by taking the sum of the products of the in-phase components of V_p and I_p. Hence, the total power is

$$P = 3(V_d I_d + V_q I_q) \tag{2.73}$$

If we write

$$I_d = \frac{E_p - V_p \cos\delta}{X_d}; \quad I_q = \frac{V_p \sin\delta}{X_q}$$

and

$$V_d = V_p \sin\delta; \qquad V_q = V_p \cos\delta$$

and substitute these values in equation (2.73), equation (2.70) follows directly after rearranging the various terms.

2.16 EFFECT OF A VOLTAGE REGULATOR

A continuously acting voltage regulator can improve the stability margin and a synchronous machine can then operate beyond the 90° limit.

The main purpose of an automatic voltage regulator is to maintain the terminal voltage as far as possible constant. This is achieved by sensing the terminal voltage, comparing it with a reference and feeding back the error to control the field excitation. It is a negative feedback. When the terminal voltage drops, the field excitation is increased and as a result the generated voltage E is increased. This can maintain the terminal voltage at a constant value when the connected load or voltage of the system changes.

If we consider an alternator connected directly to an infinite busbar, then obviously, by definition the voltage cannot change. However, the voltage regulator can be so arranged that as the machine is loaded (by increasing the power input) the regulator will alter the excitation in such a manner that the output power will always be delivered at, say, constant power factor. In these cases the regulator will require signals from current transformers in addition to the voltage signals. An example shows, however, that care is required in the use of regulators for control of reactive-MVA. If the power-factor is to be maintained at unity, then

$$E_p \cos\delta = V_p$$

and since

$$P = \frac{E_p V_p \sin\delta}{X}$$

if E_p, V_p, and X are constant

$$P = K \sin\delta$$

which gives an upper limit of P at $\delta = 90°$ electrical. Here, however, by substitution

$$P = \frac{V_p^2}{X}\tan\delta$$

and P becomes infinity when $\delta = 90°$.

On a power system the action of the regulator changes E but the electrodynamics of the machine will change the load angle δ also, which keeps P at a realistic value. Kimbark has shown that for $X = 1.0$ per unit, a typical alternator gives maximum power transfer at 117° with an increase of 15 per cent in P_{max}.

It is clear therefore that we must examine closely the effects of voltage regulators on a power system. In practice they are very useful devices which can reduce voltage fluctuations and maintain controlled flow of reactive power. They also play an important part in transient and hunting stability of synchronous machines, since they have a significant effect on both damping and synchronising power produced during disturbances.

2.17 INDUCTION MOTOR

The principle of operation of an induction motor from the point of view of rotating field theory has been described in Section 2.12. The induction motor rotor may have either a 3-phase winding or a cage-type construction on the iron cylinder (Figure 2.17d). The latter type is much more common. There is usually a 3-phase winding on the stator, which may be connected in either star or delta. To analyse the performance of induction machines it is convenient to use an equivalent circuit.

2.17.1 Equivalent circuit

An induction motor is often likened to a transformer whose secondary is free to rotate. The other difference is that there is an air gap separating the stator and rotor windings, which the flux produced by the stator must penetrate. In a transformer, of course, the main flux passes through the high permeance iron core. Nevertheless, the analogy is quite close and the equivalent circuits representing the two are very similar. Although we are considering a 3-phase induction motor, it is sufficient to deal with the equations for one phase only during balanced operation, since the machine structure is completely symmetrical about the axis of the shaft.

There is one very important point we have got to bear in mind in deriving the equivalent circuit of the induction motor. The voltages and currents in the secondary circuit, the rotor winding, or cage, are not at supply frequency but at slip frequency, because of the relative motion between the stator and the rotor.

Thus in addition to modifying the secondary parameters due to the turns-ratio as in a transformer, we have to take care of the difference in frequencies in these two circuits. We have the problem of having to join two circuits carrying current at different frequencies. Since the rotor circuit carries current at a frequency sf we have to multiply the referred leakage reactance x'_2 (expressed in terms of the supply frequency) by a factor s, the slip, defined as $(\omega_s - \omega)/\omega_s$ where ω_s is the synchronous angular velocity and ω the speed of the rotor in mechanical radian/second. The voltage in the rotor circuit referred to the stator is sE'_2, where E_2 is the induced voltage when the rotor is at standstill. This is because the magnitude of the induced voltage depends on the relative velocities of the revolving field (ω_s) and the rotor conductors. Let us divide all quantities in the secondary circuit by the slip s. If we do so, then the secondary circuit takes the form shown in Figure 2.37(a). Now we can connect the secondary circuit to the primary because the former has been converted to its equivalent at the supply frequency. The secondary resistance R_2 as it appears from the primary side is R'_2/s, where $R'_2 = k^2R_2$ and k is the stator/rotor turns ratio per phase (see Section 2.7). This can be split into $R'_2 + R'_2(1-s)/s$ and the circuit can be redrawn as shown in Figure 2.37(b). The resistance $R'_2(1-s)/s$ is equivalent to a load resistance connected to the secondary circuit of a transformer. If the machine is loaded, then it slows down and the slip increases, so $R'_2(1-s)/s$ decreases and the induction motor draws more current. In the limit when the rotor is at standstill (the starting condition), $s = 1$ and therefore the secondary of the equivalent circuit is short-circuited. When the machine runs on no-load, the slip is very nearly zero since the machine runs at almost synchronous speed.

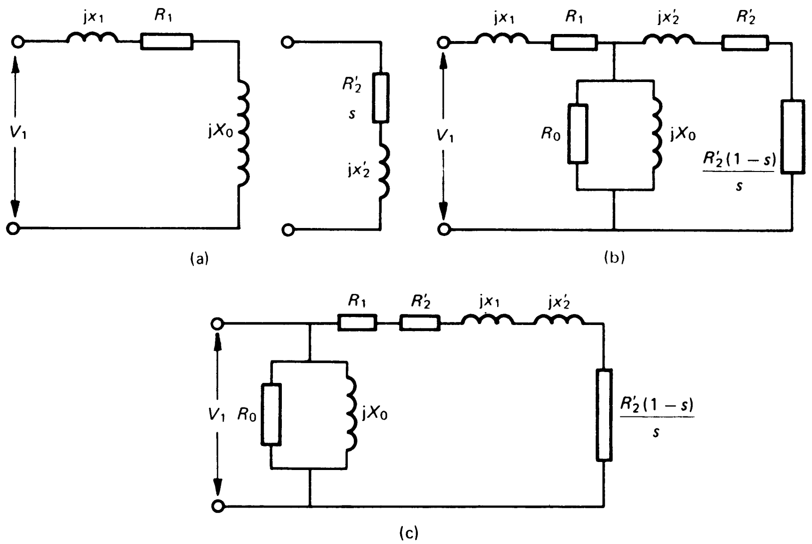

Figure 2.37 Equivalent circuit of an induction motor. (a) Secondary circuit, (b) exact equivalent circuit and (c) approximate equivalent circuit

This condition may be represented by open-circuiting the secondary winding in the equivalent circuit. We can account for the core loss by inserting a fictitious resistance across the magnetising reactance just as in a transformer, and the equivalent circuit now accurately represents an induction motor.

2.17.2 Calculation of torque from equivalent circuit

Let R_1 and x_1 represent the stator resistance and leakage reactance of an induction motor per phase, and R'_2 and x'_2 represent the corresponding terms of the rotor circuit referred to the stator. The magnetising reactance common to the stator and the rotor is X_0.

A fictitious resistance R_0 represents the iron loss. This takes place mainly in the stator core since the slip frequency of the rotor flux is very low.

The quantity $R'_2(1-s)/s$ is a variable fictitious resistance which represents the load.

An approximate equivalent circuit is as shown in Figure 2.37(c), in which the magnetising current is independent of the stator leakage reactance drop.

In this simplified circuit the current in the secondary of the rotor circuit is

$$I'_2 = \frac{V_p}{\sqrt{(R_1 + R'_2/s)^2 + (x_1 + x'_2)^2}} \tag{2.74}$$

The total 3-phase power output is

$$P = \frac{3(I'_2)^2 R'_2(1-s)}{s}$$

or

$$\omega T = \frac{3V_p^2 R'_2(1-s)}{s[(R_1 + R'_2/s)^2 + (x_1 + x'_2)^2]} \tag{2.75}$$

where

$$\omega = \omega_s(1-s).$$

Again ω is the rotational speed of the rotor and ω_s the synchronous speed (mechanical rad/s).
Thus

$$T = \frac{3V_p^2 R'_2}{s[(R_1 + R'_2/s)^2 + (x_1 + x'_2)^2]\omega_s} \tag{2.76}$$

At starting, $s = 1$

$$T = \frac{3V_p^2 R'_2}{\omega_s Z^2} \tag{2.77}$$

where $Z^2 = (R_1 + R'_2/s)^2 + (x_1 + x'_2)^2$.

When $s \approx 0$, (R_2'/s) is much greater than R_1 or $(x_1 + x_2')^2$ and we can write

$$T = \frac{3V_p^2 R_2'}{\omega_s \dfrac{(R_2')^2}{s}} = \frac{3V_p^2 s}{\omega_s R_2'} \tag{2.78}$$

or

$$T \propto s$$

(Note that when the slip is zero there is no torque.) If, however, s is not very small we can neglect the effect of $(R_1 + R_2'/s)$ in the denominator in equation (2.76) which reduces to

$$T = \frac{3V_p^2 R_2'}{s(x_1 + x_2')^2 \omega_s}$$

or

$$T \propto 1/s$$

A typical torque–speed characteristic of an induction motor is given in Figure 2.38. One can see that for low values of slip, T and s are related by a straight line and for larger values of s, they are approximately related by a rectangular hyperbola. If, however, the rotor resistance is large this relation does not obtain.

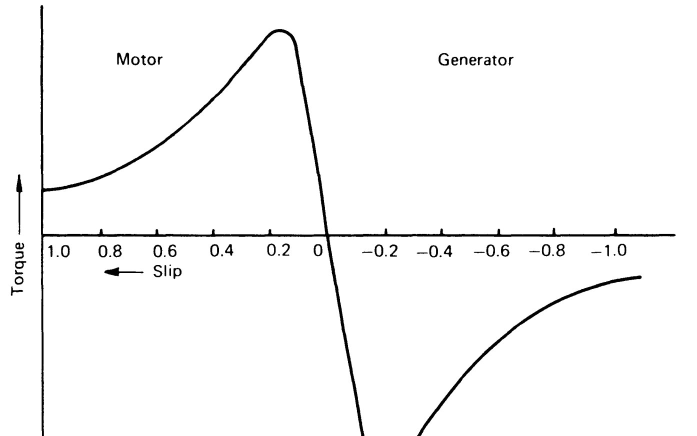

Figure 2.38 Speed–torque characteristics of an induction machine

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3

Electrodynamical equations and their solution

In Chapter 2, we dealt with mutually coupled stationary coils (Section 2.7). This was followed by a review of the performance characteristics of a.c. and d.c. generators and motors.

The present chapter deals with some basic principles of electrodynamics.[1] The fundamental equation is once again the Faraday–Lenz law. For stationary coils, $e = -\mathrm{d}\psi/\mathrm{d}t$ or, $e = -N^2\mathscr{P}\mathrm{d}i/\mathrm{d}t$ since the inductance $N^2\mathscr{P}$ does not change (saturation is neglected). If the coils are in motion, this relation is no longer true and we have to write

$$\begin{aligned} e &= -\frac{\mathrm{d}(Li)}{\mathrm{d}t} \\ &= -L\frac{\mathrm{d}i}{\mathrm{d}t} - i\frac{\mathrm{d}L}{\mathrm{d}t} \\ &= -L\frac{\mathrm{d}i}{\mathrm{d}t} - i\frac{\partial L}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} \end{aligned} \tag{3.1}$$

if L changes with x.

3.1 A SPRING AND PLUNGER SYSTEM

Let us consider the case of a plunger moving inside a coil. As the plunger moves in, the inductance of the coil increases with the increase of permeance of the magnetic path. Although this problem has been analysed in detail in Section 3.6, a qualitative analysis is presented in this section.

As the switch is closed, applying to the coil a voltage V, we have

$$V = Ri + \frac{d}{dt}(Li)$$

$$= Ri + L\frac{di}{dt} + i\frac{\partial L}{\partial x} \cdot \frac{dx}{dt} \tag{3.2}$$

The power input to the coil is

$$Vi = Ri^2 + iL\frac{di}{dt} + i^2\frac{\partial L}{\partial x}\frac{dx}{dt}$$

$$= Ri^2 + \frac{d}{dt}(\tfrac{1}{2}Li^2) + \tfrac{1}{2}i^2\frac{\partial L}{\partial x}\frac{dx}{dt} \tag{3.3}$$

The term Ri^2 is the power loss due to the resistance of the coil and $d/dt\,(\frac{1}{2}Li^2)$ represents the rate of change of the stored magnetic energy (i.e. reactive power).

The term $\frac{1}{2}i^2(\partial L/\partial x)dx/dt$ obviously represents the power necessary to accelerate the plunger and overcome the tension of the spring. This term represents the mechanical power. Since it has electrical origin it may be called electromechanical power. We now have

$$P_e = \left(\tfrac{1}{2}i^2\frac{\partial L}{\partial x}\right)\left(\frac{dx}{dt}\right) \tag{3.4}$$

Therefore electromechanical force $f_e = \frac{1}{2}i^2\dfrac{\partial L}{\partial x}$ (3.5)

Obviously this force changes as i and $\partial L/\partial x$ change. If m is the mass of the plunger its equation of motion is

$$m\frac{d^2x}{dt^2} = (F - Kx) = \tfrac{1}{2}i^2\frac{\partial L}{\partial x} - Kx \tag{3.6}$$

or

$$m\frac{d^2x}{dt^2} + Kx - \tfrac{1}{2}i^2\frac{\partial L}{\partial x} = 0 \tag{3.7}$$

This is the electrodynamic equation of the plunger, where K is the stiffness of the spring.

Equations (3.2) and (3.7) taken together give the complete electrodynamic equations of the system. We shall discuss fuller implications of these in a later section. However, it is worthwhile at this stage to examine equation (3.5) a little more closely.

Since equation (3.5) involves the square of the current, this implies that irrespective of the direction of the currents through the coil the force on the plunger acts only in one direction. Reversing the polarity of the d.c. source, or replacing it by an a.c. source will not alter the direction of force.

The next question to be considered is 'in which direction does the force act?' Let us examine equation (3.6) to obtain the answer to this. Let x be taken as positive in the right hand direction (Figure 3.1). Clearly f_e is positive if $\partial L/\partial x$ is positive which implies that the plunger will move in such a direction that L increases with x or that the permeance increases with x, i.e., it will tend to move into the coil. This movement will be opposed by the spring and the acceleration will depend on the resultant of these opposing forces.

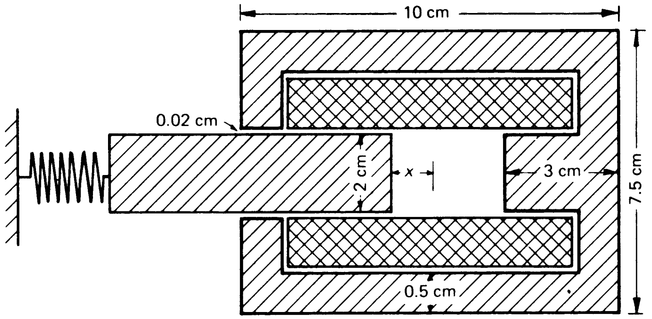


Figure 3.1 Spring and plunger system

If the plunger overshoots through the coil, the force f_e will reverse its direction and combine with the restoring force from the spring to pull the plunger back into the coil.

It may be generally said that the electromechanical force will act in such a direction as to make the reluctance of the magnetic path a minimum.

3.2 ROTATIONAL MOTION

In considering the mechanical torque between coils capable of rotation, we shall start with the simplest configuration, in which there is only one fixed coil and a rotating armature as in Figure 3.2. If the rotor has a structure as shown, obviously, the flux experiences least reluctance when axes a and b coincide and $\theta = 0$. Hence the tendency of the flux will be to rotate the rotor in such a way that θ becomes zero or, the flux can take the path having the least reluctance. It is clear from the example in Chapter 2, that the major opposition, or reluctance, to the flux path comes from the air gap. If we assume that the total permeance of the flux path is $\mathscr{P}_d$ when $\theta = 0$ and $\mathscr{P}_q$ when $\theta = \pi/2$, at any intermediate position the permeance $\mathscr{P}_\theta$ follows the curve given by

$$\mathscr{P}_\theta = \frac{\mathscr{P}_d + \mathscr{P}_q}{2} + \frac{\mathscr{P}_d - \mathscr{P}_q}{2} \cos 2\theta$$

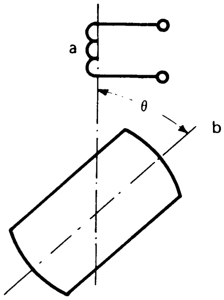


Figure 3.2 Singly excited system

If the coil 1 has N turns, then we can see that the self-inductance of coil 1 varies with rotor position and is $N^2\mathscr{P}_\theta$

If a voltage V is applied to the coil, then (ignoring resistance)

$$V = \frac{\mathrm{d}}{\mathrm{d}t}(Li) = L\frac{\mathrm{d}i}{\mathrm{d}t} + i\frac{\mathrm{d}L}{\mathrm{d}t}$$

$$= L\frac{\mathrm{d}i}{\mathrm{d}t} + i\frac{\partial L}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{3.8}$$

Following the same argument as in equation (3.3) we have

$$Vi = \frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}Li^2\right] + \tfrac{1}{2}i^2\frac{\partial L}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{3.9}$$

where the first term gives the rate of change of the stored magnetic energy and the second term gives the mechanical power. Since $P_{\text{mech}} = T\frac{\mathrm{d}\theta}{\mathrm{d}t} = T\omega$ the torque on the armature (from equation 3.9) is

$$T = \tfrac{1}{2}i^2\frac{\partial L}{\partial \theta} = \tfrac{1}{2}i^2 N^2 \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\mathscr{P}_\mathrm{d} + \mathscr{P}_\mathrm{q}}{2} + \frac{\mathscr{P}_\mathrm{d} - \mathscr{P}_\mathrm{q}}{2}\cos 2\theta\right]$$

$$= -\tfrac{1}{2}i^2 N^2(\mathscr{P}_\mathrm{d} - \mathscr{P}_\mathrm{q})\sin 2\theta \tag{3.10}$$

The negative sign indicates that this is a restoring torque. The dynamic equation for the rotor whose inertia is assumed to be J is

$$J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \tfrac{1}{2}i^2 N^2(\mathscr{P}_\mathrm{d} - \mathscr{P}_\mathrm{q})\sin 2\theta = 0 \tag{3.11}$$

If we assume that the current is constant, this equation can be written

$$J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + k\sin 2\theta = 0 \tag{3.12}$$

This equation is clearly non-linear when θ is large and is analogous to that of a pendulum, whose dynamics may be represented by the equation

$$m\frac{d^2\theta}{dt^2} + mgh\sin\theta = 0$$

if we neglect damping forces.

For very small angular displacements both of these equations may be linearised and they reduce to the simple harmonic form

$$J\frac{d^2\Delta\theta}{dt^2} + K\Delta\theta = 0 \tag{3.13}$$

Linearisation of non-linear equations is discussed in fuller detail in Section 3.7.

It is obvious that in the case of a single coil shown, the torque arises by virtue of the fact that the reluctance, or the permeance of the magnetic path, is a function of θ. If we had a cylindrical rotor the air gap would be independent of θ. In other words $\mathscr{P}_d - \mathscr{P}_q = 0$ and no torque would be experienced by the rotor. The torque that is generated in this manner is called 'reluctance torque'. This will be discussed in greater detail later.

In electrical machines the rotor is also provided with coils, the current in which sets up additional forces. The study of these forces and their effects upon the operating characteristics of the machines forms the subject matter of the following chapters.

3.3 MUTUALLY COUPLED COILS

We have considered earlier the case of mutually coupled coils which were stationary. We shall now consider the case in which one of the coils rotates with respect to the other as in Figure 3.3.

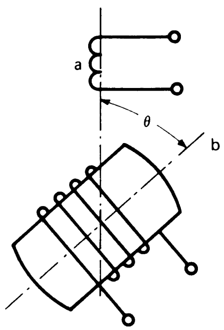


Figure 3.3 Doubly excited system

Neglecting the effects of resistance as in earlier cases (not that its inclusion makes the equations any more complicated) we have the voltage equations

$$V_1 = \frac{\mathrm{d}}{\mathrm{d}t}(L_{11}i_1 + M_{12}i_2) \tag{3.14}$$

$$V_2 = \frac{\mathrm{d}}{\mathrm{d}t}(L_{22}i_2 + M_{12}i_1) \tag{3.15}$$

Expanding these equations we have

$$V_1 = L_{11}\frac{\mathrm{d}i_1}{\mathrm{d}t} + i_1\frac{\partial L_{11}}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + M_{12}\frac{\mathrm{d}i_2}{\mathrm{d}t} + i_2\frac{\partial M_{12}}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{3.16}$$

$$V_2 = L_{22}\frac{\mathrm{d}i_2}{\mathrm{d}t} + i_2\frac{\partial L_{22}}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + M_{12}\frac{\mathrm{d}i_1}{\mathrm{d}t} + i_1\frac{\partial M_{12}}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{3.17}$$

Proceeding as before, the total power input is

$$\begin{aligned}(V_1 i_1 + V_2 i_2) = {} & i_1 L_{11}\frac{\mathrm{d}i_1}{\mathrm{d}t} + i_1 M_{12}\frac{\mathrm{d}i_2}{\mathrm{d}t} + i_1^2\frac{\partial L_{11}}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + i_1 i_2\frac{\partial M_{12}}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ & + i_2 L_{22}\frac{\mathrm{d}i_2}{\mathrm{d}t} + i_2 M_{12}\frac{\mathrm{d}i_1}{\mathrm{d}t} + i_2^2\frac{\partial L_{22}}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} + i_1 i_2\frac{\partial M_{12}}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\end{aligned} \tag{3.18}$$

From equation (2.40) we have seen that the stored magnetic energy in the coupled circuit system is

$$\tfrac{1}{2}L_{11}i_1^2 + \tfrac{1}{2}L_{22}i_2^2 + i_1 M_{12} i_2$$

and the rate of change of this stored energy is given by its time derivative

$$\begin{aligned}(V_1 i_1 + V_2 i_2) = {} & \frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}L_{11}i_1^2 + \tfrac{1}{2}L_{22}i_2^2 + i_1 M_{12} i_2\right] \\ & + \left[\tfrac{1}{2}i_1^2\frac{\partial L_{11}}{\partial\theta} + \tfrac{1}{2}i_2^2\frac{\partial L_{22}}{\partial\theta} + i_1 i_2\frac{\partial M_{12}}{\partial\theta}\right]\frac{\mathrm{d}\theta}{\mathrm{d}t}\end{aligned} \tag{3.19}$$

Clearly the coefficient of $\mathrm{d}\theta/\mathrm{d}t$ is the mechanical torque available. Hence

$$T = \tfrac{1}{2}i_1^2\frac{\partial L_{11}}{\partial\theta} + \tfrac{1}{2}i_2^2\frac{\partial L_{22}}{\partial\theta} + i_1 i_2\frac{\partial M_{12}}{\partial\theta} \tag{3.20}$$

3.4 LAGRANGE'S EQUATION

In an electrical machine there will be two (or more) sets of coils to be considered, located respectively on the stator and rotor. There will be magnetic coupling within each set and also between the windings of different sets. The electromechanical equations of voltage and torque for such systems can be derived from Lagrange's dynamical equations of motion.

Basically, Lagrange's equation is another way of presenting Newton's second law of motion. Let us consider a conservative system without dissipation. A simple mechanical example is a mass + spring system without a dashpot.

If the mass is m, the spring constant is K and x represents the displacement, the kinetic energy of the mass is

$$\mathscr{T} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\dot{x})^2$$

When the spring is stretched through a distance x from the initial position, the potential energy stored in the spring is

$$\mathscr{V} = \int_0^x F\,\mathrm{d}x = \int_0^x Kx\mathrm{d}x = Kx^2/2$$

By Newton's second law of motion, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = -F$$

Where m is the mass (kg), v is the velocity (m/s) and F is the force (N).

The negative sign appears in this case because $F = Kx$ is a restoring force and acts in a direction opposite to $\mathrm{d}(mv)/\mathrm{d}t$, the rate of change of momentum of the mass m.

If we differentiate $\mathscr{T}$, with respect to $\dot{x}$, we have

$$\frac{\partial \mathscr{T}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left[\tfrac{1}{2}m(\dot{x})^2\right] = m\dot{x} = mv \tag{3.21}$$

Differentiating the potential energy $\mathscr{V}$, with respect to x we have

$$\frac{\partial \mathscr{V}}{\partial x} = \frac{\partial}{\partial x}(\tfrac{1}{2}Kx^2) = Kx = F \tag{3.22}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = -F,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{T}}{\partial \dot{x}}\right) + \frac{\partial \mathscr{V}}{\partial x} = 0 \tag{3.23}$$

This is Lagrange's equation of motion for a conservative system. That this equation may be applied to an electrical system is shown below. From the analogy given in Table 1.1, the counterpart of a mass + spring system is an LC circuit. Since we are not considering any dissipation as yet, resistance R is not included.

The stored magnetic energy $\frac{1}{2}Li^2$ is the analogue of kinetic energy and the stored energy in the capacitor $\frac{1}{2}q^2/C$, is the analogue of potential energy.

Also

$$\frac{\partial \mathcal{T}}{\partial \dot{x}} = \frac{\partial}{\partial i}(\tfrac{1}{2}Li^2) = Li \tag{3.24}$$

and

$$\frac{\partial \mathcal{V}}{\partial q} = \frac{\partial}{\partial q}\left(\tfrac{1}{2}\frac{q^2}{C}\right) = \frac{q}{C} \tag{3.25}$$

Substituting these values into Lagrange's equation, we now have

$$\frac{\mathrm{d}}{\mathrm{d}t}(Li) + \frac{q}{C} = 0 \tag{3.26}$$

or

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{q}{C} = 0 \tag{3.27}$$

This is the familiar equation for an $L + C$ circuit when there is no dissipation (i.e. resistance) and no forcing function (i.e. applied voltage).

This illustration, although a trivial exercise, has deep implications in the study of electrodynamical systems.

In the original example of a mass and a spring there are several constraints. The elements have only one degree of freedom, x, the kinetic energy of the system is not a function of the coordinate x explicitly but of the velocity $\dot{x}$ only. On the other hand the potential energy is a function of x and not of $\dot{x}$. In a more general case one can define a Lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{V}$ and rewrite the elementary equation in a more general form for a system with n degrees of freedom

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_n}\right) - \frac{\partial \mathcal{L}}{\partial x_n} = 0 \tag{3.28}$$

In the particular case considered earlier, where $n = 1$, this reduces immediately to equation (3.23).

Equation (3.28) represents Lagrange's equation for a conservative system with no applied force (or voltage) and completely loss free.

In order to include the effects of loss let us define a velocity-dependent (or current-dependent) function $\mathcal{F}$, the Rayleigh dissipation function,

$$\mathcal{F} = \sum_{n=1}^{k} \tfrac{1}{2}R_n(\dot{x}_n)^2 \tag{3.29}$$

$$\frac{\partial \mathcal{F}}{\partial \dot{x}_n} = R_n\dot{x}_n \text{ represents force (or voltage)} \tag{3.30}$$

If we now consider applied forces (voltage) V then we can rewrite Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}_n}\right)-\frac{\partial \mathscr{L}}{\partial x_n}+\frac{\partial \mathscr{F}}{\partial \dot{x}_n}=V_n \tag{3.31}$$

This equation, after suitable substitution, will directly yield the familiar voltage/force equations

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}+R\frac{\mathrm{d}q}{\mathrm{d}t}+\frac{q}{C}=V \tag{3.32a}$$

or

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}+R_{\mathrm{F}}\frac{\mathrm{d}x}{\mathrm{d}t}+Kx=F \tag{3.32b}$$

We do not really need Lagrange's equation to derive trivial equations such as these but it is very useful indeed to set up the equations for much more complicated systems.

It is interesting to note that if we are dealing with a purely mechanical system the coordinates x_n represent only length (or angular displacement θ). In the case of a stationary electrical network x_n represents charge q. This implies that an electrical 'space' can be considered with charge as its coordinates. It is, however, difficult to visualise such an abstract space.

3.4.1 Application of Lagrange's equation to electromechanical systems

3.4.1(i) *Case 1: Spring plunger and dashpot*

Let us consider the earlier example in Section 3.1. The total kinetic energy of the system as a whole is

$$\mathscr{T}=\underset{\text{elect}}{\tfrac{1}{2}Li^2}+\underset{\text{mech}}{\tfrac{1}{2}mv^2} \tag{3.33}$$

The potential energy is

$$\mathscr{V}=\underset{\text{mechanical}}{\tfrac{1}{2}Kx^2} \tag{3.34a}$$

The dissipation function is

$$\mathscr{F}=\underset{\text{elect}}{\tfrac{1}{2}Ri^2}+\underset{\text{mech}}{\tfrac{1}{2}Dv^2} \tag{3.34b}$$

If we designate electrical charge q passing through the coil by the coordinate

x_1 and the mechanical displacement x by the coordinate x_2, the Lagrangian $\mathscr{L}$ is

$$\mathscr{L} = \mathscr{T} - \mathscr{V} = \tfrac{1}{2}L\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 - \tfrac{1}{2}Kx_2^2 \tag{3.35}$$

$$\mathscr{F} = \tfrac{1}{2}R\dot{x}_1^2 + \tfrac{1}{2}D\dot{x}_2^2 \tag{3.36}$$

Let us apply equation (3.31) to derive the voltage/force equations of the plunger. If $n = 1$, in equation (3.31), we shall obtain the voltage equation

$$\frac{\partial \mathscr{L}}{\partial \dot{x}_1} = L\dot{x}_1 \tag{3.37}$$

(since m and K are not affected by the current and saturation is neglected)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}_1}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(L\dot{x}_1) = L\ddot{x}_1 + \dot{x}_1\frac{\mathrm{d}L}{\mathrm{d}t} \tag{3.38}$$

$$\frac{\partial \mathscr{L}}{\partial x_1} = 0 \tag{3.39}$$

(m, L, K etc. are not affected by the charge q)

$$\frac{\partial \mathscr{F}}{\partial \dot{x}_1} = R_1\dot{x}_1 \tag{3.40}$$

$$V_n = V_1 = \text{voltage applied to the coil}$$

Combining all the terms in (3.37) we have the voltage equation

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri + i\frac{\mathrm{d}L}{\mathrm{d}t} = V_1 \tag{3.41}$$

If we now put $n = 2$, we have the following relations

$$\frac{\partial \mathscr{L}}{\partial \dot{x}_2} = m\dot{x}_2 \tag{3.42}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}_2) = m\ddot{x}_2$$

$$\frac{\partial \mathscr{L}}{\partial x_2} = -Kx_2 + \tfrac{1}{2}\dot{x}_1^2\frac{\partial L}{\partial x_2} \tag{3.43}$$

$$\frac{\partial \mathscr{F}}{\partial \dot{x}_2} = D\dot{x}_2$$

$$V_2 = \text{applied force} = F = 0$$

giving the dynamic equation of the plunger

$$m\ddot{x}_2 + D\dot{x}_2 + Kx_2 - \tfrac{1}{2}i^2\frac{\partial L}{\partial x_2} = 0 \tag{3.44}$$

Equations (3.41) and (3.44) are exactly the same as those obtained in Section 3.1.

Clearly this is a complicated method to obtain the dynamic equation for a spring and plunger system. However, in more complex electromechanical devices, Lagrange's equations provide a systematic approach to the problem. Let us go back again to the problem considered in the Section 3.3 (Figure 3.3).

3.4.1 (ii) *Case 2: Rotating mass with doubly-excited coils*

Let x_1 represent the charge q_1 through coil 1
x_2 represent the charge q_2 through coil 2
x_3 represent the displacement θ of the rotor

Total kinetic energy of the system is

$$\begin{aligned}\mathscr{T} &= \mathscr{T}_e + \mathscr{T}_m = \tfrac{1}{2}L_{11}i_1^2 + \tfrac{1}{2}L_{22}i_2^2 + i_1 i_2 M_{12} + \tfrac{1}{2}J\omega^2 \\ &= \tfrac{1}{2}L_{11}\dot{x}_1^2 + \tfrac{1}{2}L_{22}\dot{x}_2^2 + \dot{x}_1\dot{x}_2 M_{12} + \tfrac{1}{2}J\dot{x}_3^2 \end{aligned} \tag{3.45}$$

$$\mathscr{V} = 0 \text{ in this case}$$

$$\begin{aligned}\mathscr{F} &= \tfrac{1}{2}R_1 i_1^2 + \tfrac{1}{2}R_2 i_2^2 + \tfrac{1}{2}R_F\omega^2 \\ &= \tfrac{1}{2}R_1\dot{x}_1^2 + \tfrac{1}{2}R_2\dot{x}_2^2 + \tfrac{1}{2}R_F\dot{x}_3^2\end{aligned} \tag{3.46}$$

Proceeding as before, we have

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial\dot{x}_1} &= \frac{\partial\mathscr{L}}{\partial i_1} = \frac{\partial\mathscr{T}}{\partial i_1} = L_{11}i_1 + M_{12}i_2 \\ \frac{\partial\mathscr{L}}{\partial\dot{x}_2} &= \frac{\partial\mathscr{L}}{\partial i_2} = \frac{\partial\mathscr{T}}{\partial i_2} = L_{22}i_2 + M_{12}i_1 \\ \frac{\partial\mathscr{L}}{\partial\dot{x}_3} &= \frac{\partial\mathscr{L}}{\partial\omega} = \frac{\partial\mathscr{T}}{\partial\omega} = J\omega\end{aligned} \tag{3.47}$$

In equation (3.45), the mechanical inertia J is independent of any electrical charge or current but the electrical inductances are functions of the mechanical variable θ. As a result we have,

$$\frac{\partial\mathscr{L}}{\partial x_3} = \frac{\partial\mathscr{L}}{\partial\theta} = \tfrac{1}{2}i_1^2\frac{\partial L_{11}}{\partial\theta} + \tfrac{1}{2}i_2^2\frac{\partial L_{22}}{\partial\theta} + i_1 i_2\frac{\partial M_{12}}{\partial\theta}$$

We also have

$$\begin{aligned}\frac{\partial\mathscr{F}}{\partial\dot{x}_1} &= \frac{\partial\mathscr{F}}{\partial i_1} = R_1 i_1 \\ \frac{\partial\mathscr{F}}{\partial\dot{x}_2} &= \frac{\partial\mathscr{F}}{\partial i_2} = R_2 i_2 \\ \frac{\partial\mathscr{F}}{\partial\dot{x}_3} &= \frac{\partial\mathscr{F}}{\partial\omega} = R_F\omega\end{aligned} \tag{3.48}$$

Collecting the equations corresponding to $n = 1, 2$, and 3 successively, we have

$$V_1 = R_1 i_1 + p[L_{11} i_1 + M_{12} i_2]$$

$$V_2 = R_2 i_2 + p[L_{22} i_2 + M_{12} i_1]$$

$$0 = R_F \omega + Jp\omega - \left[\tfrac{1}{2} i_1^2 \frac{\partial L_{11}}{\partial \theta} + \tfrac{1}{2} i_2^2 \frac{\partial L_{22}}{\partial \theta} + i_1 i_2 \frac{\partial M_{12}}{\partial \theta}\right] \quad (3.49)$$

In the third equation $V_n = V_3 = 0$ if no torque is applied to the rotor. These equations give the complete electrodynamic description of a machine with two coils energised.

3.4.1(iii) *Case 3: An electromechanical transducer*

Figure 3.4 is an electromechanical transducer. It may be considered to be a representation of a pick-up. The system is capable of vertical motion. The mass m_1 is a permanent magnet. The relative motion between m_1 and the coil m_2 induces voltage in the coil. The spring and the damping actions are represented by lumped parameters. It is required to derive the electrodynamic equations of the transducer (we shall neglect hysteresis).

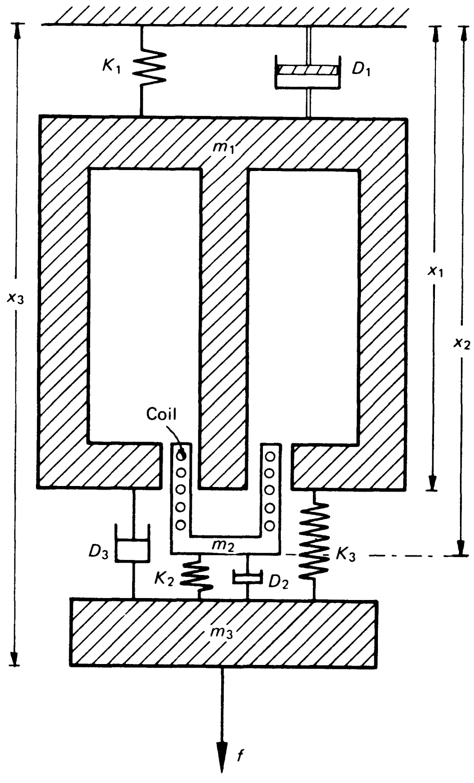


Figure 3.4 Electromechanical transducer

Once again we shall resort to Lagrange's equations. The electrodynamic equations can certainly be deduced by observation. There is, however, every possibility of overlooking something! This is unlikely to happen when one follows a fairly routine procedure.

Using the same nomenclature as before, we proceed as below.

Here x_1, x_2, x_3 are the displacements and $\dot{x}_4$ is the current

$$
\begin{aligned}
\mathscr{T} &= \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 + \tfrac{1}{2}m_3\dot{x}_3^2 + \tfrac{1}{2}L\dot{x}_4^2 \\
\mathscr{V} &= \tfrac{1}{2}K_1x_1^2 + \tfrac{1}{2}K_2(x_3-x_2)^2 + \tfrac{1}{2}K_3(x_3-x_1)^2 \qquad (3.50)\\
\mathscr{F} &= \tfrac{1}{2}R\dot{x}_4^2 + \tfrac{1}{2}D_1\dot{x}_1^2 + \tfrac{1}{2}D_3(\dot{x}_3-\dot{x}_4)^2 + \tfrac{1}{2}D_2(\dot{x}_3-\dot{x}_2)^2
\end{aligned}
$$

The Lagrangian $\mathscr{L} = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 + \tfrac{1}{2}m_3\dot{x}_3^2 + \tfrac{1}{2}L\dot{x}_4^2$

$$- \tfrac{1}{2}K_1x_1^2 - \tfrac{1}{2}K_2(x_3-x_2)^2 - \tfrac{1}{2}K_3(x_3-x_1)^2 \qquad (3.51)$$

$$\left(\frac{\partial \mathscr{L}}{\partial \dot{x}_1}\right) = m_1\dot{x}_1$$

$$
\begin{aligned}
\left(\frac{\partial \mathscr{L}}{\partial x_1}\right) &= -K_1x_1 + K_3(x_3-x_1) + \tfrac{1}{2}\dot{x}_4^2\frac{\partial L}{\partial x_1} \\
&= -(K_1+K_3)x_1 + K_3x_3 + \tfrac{1}{2}\dot{x}_4^2\frac{\partial L}{\partial x_1}
\end{aligned}
$$

$$\frac{\partial \mathscr{F}}{\partial \dot{x}_1} = D_1\dot{x}_1 - D_3(\dot{x}_3-\dot{x}_1) = (D_1+D_3)\dot{x}_1 - D_3\dot{x}_3 \qquad (3.52)$$

Therefore the equation for mass m_1 is

$$
\begin{aligned}
m_1\ddot{x}_1 + (K_1+K_3)x_1 - K_3x_3 + (D_1+D_3)\dot{x}_1 - D_3\dot{x}_3 \\
= \tfrac{1}{2}\left(\frac{\partial L}{\partial x_1}\right)\dot{x}_4^2
\end{aligned}
$$

In a similar manner equations for masses m_2 and m_3 are

$$m_2\ddot{x}_2 + K_2(x_3-x_2) + D_2(\dot{x}_3-\dot{x}_2) = \tfrac{1}{2}\left(\frac{\partial L}{\partial x_2}\right)\dot{x}_4^2 \qquad (3.53)$$

and

$$m_3\ddot{x}_3 + K_3(x_3-x_1) + D_3(\dot{x}_3-\dot{x}_1) = F$$

The equation for the electrical response is

$$L\ddot{x}_4 + R\dot{x}_4 + \dot{x}_4\frac{\mathrm{d}L}{\mathrm{d}t} = -e(t) \qquad (3.54)$$

Therefore

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri + i\frac{\partial L}{\partial(x_2-x_1)}\cdot\frac{\mathrm{d}(x_2-x_1)}{\mathrm{d}t} = -e(t) \qquad (3.55)$$

or

$$L\frac{\mathrm{d}i}{\mathrm{d}t}+Ri+i\frac{\partial L}{\partial \xi}\cdot\frac{\mathrm{d}\xi}{\mathrm{d}t} = -e(t) \tag{3.56}$$

where $e(t)$ is the generated voltage in the coil. If the load impedance is $(R_L + L_L p)$, the equation is further modified to

$$(L+L_L)\frac{\mathrm{d}i}{\mathrm{d}t}+(R+R_L)i+i\frac{\partial L}{\partial \xi}\cdot\frac{\mathrm{d}\xi}{\mathrm{d}t} = 0 \tag{3.57}$$

where

$$\xi = (x_2 - x_1)$$

Once again the equations are nonlinear. We shall discuss the method of solution of these equations in the next chapter.

3.5 SOLUTION OF THE ELECTRODYNAMICAL EQUATIONS

Formulation of the electrodynamic equations is fairly straightforward as long as one proceeds systematically. The solution of the equations, however, is not always simple. Most of these equations are inherently nonlinear as will be seen in examples considered later. The electrodynamic equations are usually nonlinear because of the terms such as $i(\partial L/\partial x)(\mathrm{d}x/\mathrm{d}t)$, in equation (3.2) and $\frac{1}{2}i^2(\partial L/\partial x)$ in equation (3.7). In certain cases $\partial L/\partial x$ may be a constant, but that alone may not render the equations linear. The product of the dependent variables, current and velocity in the voltage equations and the square of the current in the dynamical equation, make it difficult to derive a closed form solution. By considering small perturbations about the operating point these equations can be linearised (as in equation 3.13) and a solution obtained. This has been done in Chapter 7, and is standard practice in considering small oscillations in electrical machines. Linearisation of the equations where fairly large displacements are involved leads to considerable errors and in such cases the standard practice is to obtain numerical solutions.

There are a large number of numerical methods of solving ordinary differential equations. We shall discuss two methods that are commonly used.

3.5.1 Euler's method

Let us consider the following differential equation

$$y' = f(x, y) \tag{3.58}$$

where $f(x, y)$ is real, continuous, and single valued in some region R of the xy plane.

By Taylor's series we have,

$$y_{n+1} = y_n + h y'_n + \frac{h^2}{2!} y''_n + \text{higher powers of } h \tag{3.59}$$

where h is a small interval defined by $x_{n+1} - x_n$. If h is sufficiently small we can write

$$y_{n+1} = y_n + h y'_n \tag{3.60}$$

where $n = 0, 1, 2, \ldots$ Here we have truncated the higher powers of h. Starting from the initial value of the function (i.e. $n = 0$) we find the derivative of the function (or its slope) at that point and obtain y_1. Then we calculate the slope at y_1 and calculate y_2 and proceed in this manner.

A simple and more accurate formula is

$$y_{n+1} = y_{n-1} + 2h y'_n \tag{3.61}$$

This again follows from Taylor's series expansion

$$y_{n+1} = y_{n-1} + 2h y'_n + \frac{h^3 y'''_n}{3!} + \text{higher powers of } h$$

Equation (3.61) is more accurate than equation (3.60), since the neglected terms in the second formula are smaller than in the first.

One of the major disadvantages of Euler's method is that the interval or the step length h must be very small in order to provide correct results. Selecting large step sizes often leads to divergence or numerical instability.

3.5.2 Runge–Kutta method

The Runge–Kutta method is very widely used for numerical solution of differential equations because of its high accuracy and the large step lengths that one may select.

Runge–Kutta method of order 2:

y_0 at x_0 is given

$$y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2) \tag{3.62}$$

where

$$k_1 = hf(x_n, y_n)$$

$$k_2 = f(x_n + h, y_n + k_1)$$

Equation (3.62) is an improvement over Euler's equation.

Runge–Kutta method of order 4:

y_0 at x_0 is given

$$y_{n+1} = y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{3.63}$$

where

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + h/2,\ y_n + k_1)$$
$$k_3 = hf(x_n + h/2,\ y_n + k_2)$$
$$k_4 = hf(x_n + h,\ y_n + k_3)$$

Let us consider a simple example whose solution we already know and use Runge–Kutta method 4 to obtain a solution to the problem.
Let

$$\frac{dy}{dx} = y, \text{ and } y(0) = 1$$

Let us select $h = 0.02$

1st step

$$y_0 = 1$$
$$k_1 = hf(x_0, y_0) = 0.02$$
$$k_2 = hf(x_0 + h/2,\ y_0 + k_1) = (0.02)(1.02) = 0.020\,4$$
$$k_3 = hf(x_0 + h/2,\ y_0 + k_2) = 0.02(1.020\,4) = 0.020\,408$$
$$k_4 = hf(x_0 + h,\ y_0 + k_3) = 0.02(1.020\,408) = 0.020\,409\,16$$
$$\tfrac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 0.020\,404$$

Hence

$$y_1 \text{ at } x = 0.02 \text{ is}$$
$$y_0 + \tfrac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1.020\,404$$

2nd step

$$y_1 \doteqdot 1.020\,404$$
$$k_1 = hf(x_1, y_1) = 0.02(1.020\,404) = 0.020\,408$$
$$k_2 = hf(x_1 + h/2,\ y_1 + k_1) = 0.02(1.020\,408) = 0.020\,409\,16$$
$$k_3 = hf(x_1 + h/2,\ y_1 + k_2) = 0.02(1.020\,409\,16) = 0.020\,409\,183\,2$$
$$k_4 = hf(x_1 + h,\ y_1 + k_3) = 0.02(1.020\,409\,183\,2) = 0.020\,409\,183\,664$$
$$\tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.020\,408\,94$$

Therefore

y_2 at $x = 0.04$ is

$y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.040\,812$

$$\text{Now } \frac{\mathrm{d}y}{\mathrm{d}x} = y \text{ has a solution } y = \mathrm{e}^x$$

and for $x = 0.04$, $y = 1.040\,810\,774$.

Using Euler's method (equation 3.60) we have $y = 1.040\,4$. If in the latter a step length of 0.005 is used, we have $y = 1.040\,8$ which is a considerable improvement. This example clearly shows that the step length has to be very small if an accurate result is to be obtained using Euler's method. The step length can be considerably larger if Runge–Kutta (order 4) is used.

3.6 SOLVING A SPRING AND PLUNGER PROBLEM

Equations for this system have been derived in Section 3.1, and later in 3.4.1. Here we consider a specific case in an attempt to understand the effects of different parameters on the motion of the plunger and its stability.

Figure 3.1 represents such a plunger. The coil is 16 AWG, with 1000 turns and resistance 1.6 Ω. A d.c. supply of 10 V is suddenly applied to the coil. The mass of the plunger is 0.193 kg.

The spring constant K varies from 1000 to 2000 N/m. Damping due to friction varies from 0 to 5 N/m s.

First, we have to derive an expression for the self-inductance of the coil, which is obviously a function of x. Following the method outlined in Section 2.3.1 we have the reluctances of various magnetic paths from the dimensions of the plunger system. The maximum length of the air gap is 3 cm.

$$\mathscr{R}_\mathrm{a} = (0.03 - x) \times 2533 \times 10^6 \qquad \text{(variable air gap)}$$

$$\mathscr{R}_\mathrm{g} = 1.013 \times 10^6 \qquad \text{(tolerance gap)}$$

$$\mathscr{R}_\mathrm{i} = 3.456 \times 10^6 \qquad \text{(iron path)}$$

$$\mathscr{R}_\mathrm{ip} = (0.065 + x) + 0.636\,6 \times 10^6 \qquad \text{(plunger path)}$$

The reluctance of the two ends of the cylinder is neglected. Total reluctance is

$$\mathscr{R}_\mathrm{T} = (\mathscr{R}_\mathrm{a} + \mathscr{R}_\mathrm{ip}) + (\mathscr{R}_\mathrm{g} + \mathscr{R}_\mathrm{i})/2$$
$$= 78.311 - 2532.36x \times 10^6$$

Therefore

$$L = N^2/\mathscr{R}_\mathrm{T} = \frac{0.012\,76}{1 - 32.33x} = \frac{A}{1 - Bx}$$

subject to the constraint that $x \not> 0.03$ m, because of the stopper placed at $x = 0.03$ m and

$$\frac{\partial L}{\partial x} = \frac{AB}{(1-Bx)^2} = \frac{0.412\,8}{(1-32.33x)^2}$$

The electrodynamic equations are

$$V = Ri + Lpi + i\frac{\partial L}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} \tag{3.2}$$

$$0 = D\dot{x} + m\ddot{x} + Kx - \tfrac{1}{2}i^2\frac{\partial L}{\partial x} \tag{3.44}$$

These non-linear equations were solved by Euler's method with a step length of 0.002 s.

For a step-by-step method of solution it is convenient to represent the equations in the following form

$$\left.\begin{aligned} pi &= -\frac{R}{L}i - \frac{1}{L}i\frac{\partial L}{\partial x}v + \frac{V}{L} && (a)\\ pv &= -\frac{D}{m}v - \frac{Kx}{m} + \frac{1}{2m}i^2\frac{\partial L}{\partial x} && (b)\\ px &= v && (c)\end{aligned}\right\} \tag{3.64}$$

For initial conditions we have

$$x = 0,\ i = 0, \quad \text{and} \quad v = 0 \quad \text{and} \quad V = 10\ \text{V}$$

Using equation (3.6) and having chosen $h = 0.002$, we solve the equations for different combinations of K and D and the results are plotted in Figures 3.5 and 3.6.

If $K = 1000$ N/m, the plunger will be drawn into the solenoid at high speed and hit the stopper, although $D = 3$ N/m s. The inductance equations do not allow for the impact with the stopper. The inductance L becomes very high with no air gap and the programme will fail.

Of course, it is easy to instruct the programme to stop when $x \geqslant 0.03$ m.

On the other hand if $K = 2000$ N/m the plunger reaches a maximum distance of 0.86 cm and is pulled back by the spring. After a few oscillations it settles down at $x = 0.64$ cm.

The current rises sharply at first, almost exponentially, before the plunger has begun to move. It then overshoots its steady-state value of $V/R = 6.25$ A and oscillates—as the plunger oscillates, before reaching its steady-state value.

The velocity obviously changes direction as the plunger oscillates, like a pendulum, finally becoming zero.

The position of equilibrium is of particular interest to us. We can in fact predict this point without solving the nonlinear equations. We proceed as below.

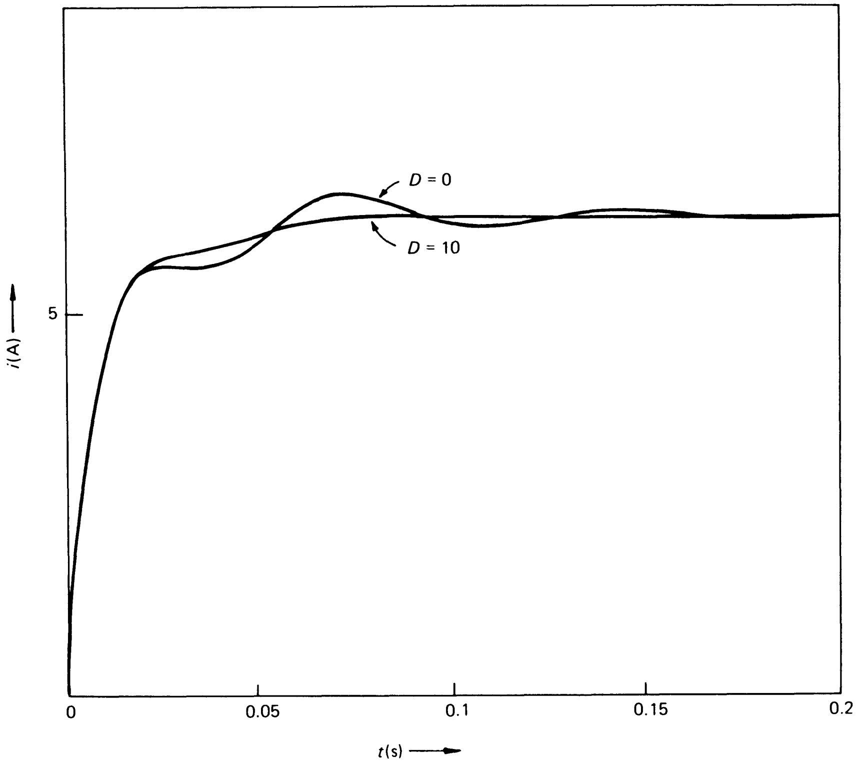


Figure 3.5 Variations of current for variable damping in a spring/plunger system

We set all terms on the left hand side of equations (3.64) equal to zero, since at steady-state, i, v, and x do not change with time. (If the coil is energised by an alternating voltage then $pi \neq 0$. It is equal to $j\omega i$ where ω is the angular frequency of the excitation voltage.)
First from equation (3.64c)

$$px = v = 0$$

Substitute this result in equation (3.64a) which now becomes $V = Ri$, giving the steady state current $i_0 = V/R$. Substituting this relation in (3.64b) we have

$$0 = -Kx \quad + \quad \tfrac{1}{2} i_0^2 \frac{\partial L}{\partial x} \tag{3.65}$$

restoring force — electromagnetic force

In the present problem $i_0 = 6.25$ A and $\dfrac{\partial L}{\partial x} = \dfrac{0.412\,8}{(1 - 32.33x)^2}$. We now plot two

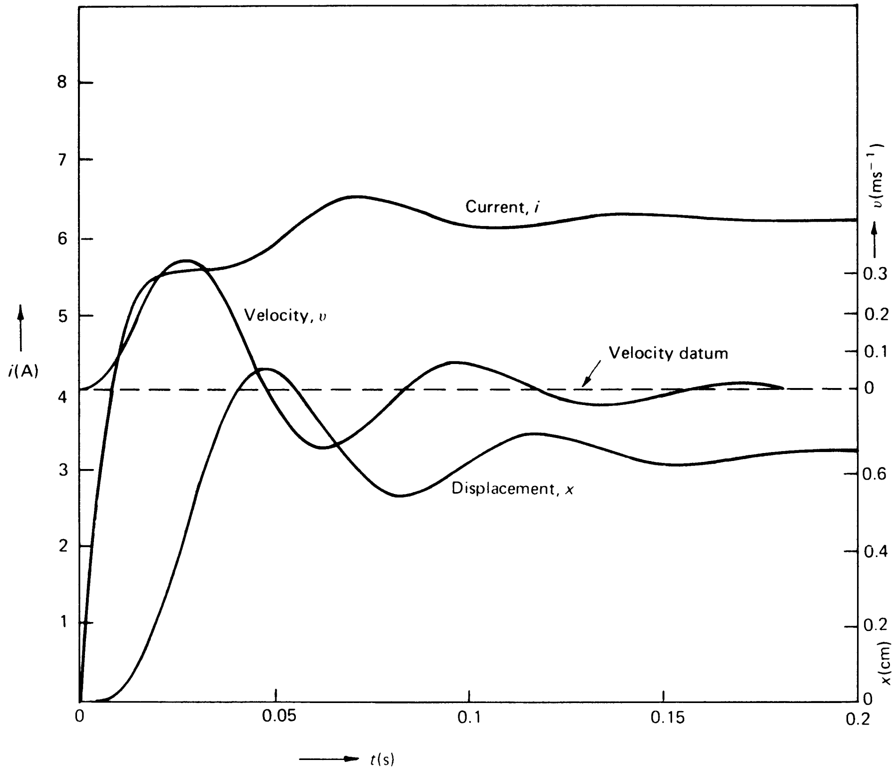


Figure 3.6 Current, velocity and displacement in a spring/plunger system

sets of graphs,

$$y = Kx, \quad K = 1000, 1500, 2000$$

and

$$y = \frac{i_0^2}{2} \cdot \frac{0.4128}{(1 - 32.33x)^2}$$

The point or points of intersection of these graphs obviously satisfy equation (3.65).

If $K = 1000$, it is clear that the electromagnetic force will always be greater than the restoring force due to the spring and the plunger will continue to accelerate until it hits the stopper. The same will happen when $K = 1500$, the acceleration being lower in this case.

It is to be remembered that the force of acceleration or deceleration at any position of the plunger is given by the difference between the heights of the ordinates as shown in Figure 3.7.

It may also be observed in the same figure that the straight line for K

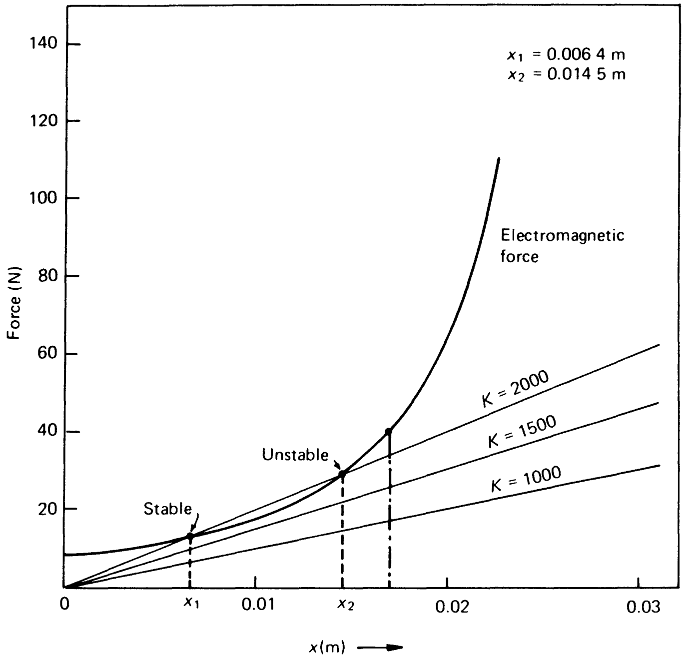


Figure 3.7 Force/displacement of the spring/plunger system

$= 2000\,\mathrm{N/m}$, intersects the graph representing the electromagnetic force, at two points; one at $x_1 = 0.006\,4\,\mathrm{m}$ (0.64 cm) and the other at $x_2 = 0.014\,5$ cm.

Solution of equations (3.64) and Figure 3.7 gives $x_1 = 0.64$ cm as has been stated earlier. Theoretically, the plunger may settle at either of these two positions but in reality it will never settle at x_2 which is an unstable position.

Physically, we can understand this in the following manner.

When the plunger comes to x_1 and overshoots due to its inertia it is acted upon by a decelerating force (Figure 3.7) which brings it back to x_1, then it overshoots in the opposite direction only to be accelerated by the electromagnetic force which is greater than the restoring force for $x < x_1$. At $x = x_2$, however, the situation is totally different. If the plunger approaches from the direction $x < x_2$, it overshoots x_2 due to its inertia and is acted upon by an accelerating force (the same as in the case where $K = 1000$) until it hits the stopper. On the other hand if the plunger reaches x_2 from $x > x_2$ and overshoots, which it will, it is acted upon by the decelerating force which tends to reduce x. It is therefore obvious that x_2 is an unstable equilibrium position. We shall come across a similar phenomenon in the case of acceleration of an induction motor.

We shall now study this question of stability/instability from a mathematical point of view. For this purpose, we shall linearise equations (3.2) and (3.44) about their positions of equilibrium at $x = x_1$ and $x = x_2$.

3.7 LINEARISATION OF THE DYNAMIC EQUATIONS

Let us now linearise equations (3.2) and (3.44) about their points of equilibrium x_1 and x_2. What we shall do in fact is to displace the plunger by a small amount from its equilibrium position. If the equilibrium is stable, the plunger will return to the position it occupied before. If the equilibrium is unstable, the plunger will move away from its equilibrium point, irrespective of the direction of displacement.

Let us take small increments of the terms in equations (3.2) and (3.44)

$$\Delta V = \Delta(Ri) + \Delta(Lpi) + \Delta\left(i\frac{\partial L}{\partial x}v\right) \tag{3.66}$$

and

$$0 = \Delta(Dv) + \Delta(mpv) + \Delta(Kx) - \Delta\left(\tfrac{1}{2}i^2\frac{\partial L}{\partial x}\right) \tag{3.67}$$

Equation (3.66) when expanded gives

$$\Delta V = R\Delta i + Lp\Delta i + pi\Delta L + \Delta i\left(\frac{\partial L}{\partial x_0}\right)v_0$$

$$+ i_0\left(\frac{\partial L}{\partial x_0}\right)\Delta v + i_0 v_0 \Delta\left(\frac{\partial L}{\partial x}\right) \tag{3.68}$$

where i_0, v_0, and $(\partial L/\partial x_0)$ designate the steady state values at the point of equilibrium.

Obviously $v_0 = 0$, since the plunger is at rest. We may put $V = 0$ if we consider that the d.c. source has negligible internal resistance. Also, the term $pi\Delta L$ is negligible, since the coil carries direct current. Equation (3.66) now reduces to

$$0 = R\Delta i + Lp\Delta i + i_0\left(\frac{\partial L}{\partial x_0}\right)\Delta v \tag{3.69}$$

In equation (3.67) all terms except $\Delta\left(\tfrac{1}{2}i^2\frac{\partial L}{\partial x}\right)$ are fairly simple.

We know also that

$$\tfrac{1}{2}i^2\frac{\partial L}{\partial x} = \tfrac{1}{2}\frac{i^2 AB}{(1-Bx)^2} = f(x, i) \tag{3.70}$$

By Taylor's series expansion we have

$$f(x,i) = f(x_0, i_0) + \left.\frac{\partial f(x,i)}{\partial x}\right|_{x_0, i_0} (x - x_0) + \left.\frac{\partial f(x,i)}{\partial i}\right|_{x_0, i_0} (i - i_0) + \frac{1}{2!}\left.\frac{\partial^2 f(x,i)}{\partial x^2}\right|_{x_0, i_0} (x - x_0)^2 + \frac{1}{2!}\left.\frac{\partial^2 f(x,i)}{\partial i^2}\right|_{x_0, i_0} (i - i_0)^2 + \frac{1}{2!}\left.\frac{\partial^2 f(x,i)}{\partial i \partial x}\right|_{x_0, i_0} (i - i_0)(x - x_0) \tag{3.71}$$

When we consider small increments we have

$$f(x_0 + \Delta x, i_0 + \Delta i) - f(x_0, i_0)$$

neglecting higher powers of Δx and Δi, and using the first order terms,

$$\left.\frac{\partial f(x,i)}{\partial x}\right|_{x_0, i_0} \Delta x + \left.\frac{\partial f(x,i)}{\partial i}\right|_{x_0, i_0} \Delta i \tag{3.72}$$

In this particular case,

$$\Delta\left(\tfrac{1}{2} i^2 \frac{\partial L}{\partial x}\right) = i_0 \frac{AB}{(1 - Bx_0)^2} \Delta i + i_0^2 \frac{AB^2}{(1 - Bx_0)^3} \Delta x \tag{3.73}$$

Here x_0 is the position of equilibrium, which can be either x_1 or x_2.

Re-arranging equations (3.66) and (3.67) we have

$$\begin{bmatrix} p\Delta i \\ p\Delta v \\ p\Delta x \end{bmatrix} = \begin{bmatrix} -\dfrac{R}{L_0} & -\dfrac{i_0}{L_0} \cdot \dfrac{AB}{(1 - Bx_0)^2} & \\ \dfrac{i_0}{m} \cdot \dfrac{AB}{(1 - Bx_0)^2} & -\dfrac{D}{m} & -\dfrac{k}{m} - \dfrac{AB^2 i_0^2}{(1 - Bx_0)^3 m} \\ & 1 & \end{bmatrix} \begin{bmatrix} \Delta i \\ \Delta v \\ \Delta x \end{bmatrix} \tag{3.74}$$

This method of arranging the equations is particularly convenient and in a condensed form can be written

$$p\Delta\mathbf{x} = \mathbf{A}\Delta\mathbf{x} \tag{3.75}$$

This is the state-variable representation of the system (see Section 5.10).

Whether the equilibrium is stable or unstable depends on the A-matrix. Let us investigate whether the plunger is stable or not at $x_0 = x_1$ and $x_0 = x_2$. We shall make use of the Routh–Hurwitz criterion and also calculate eigenvalues to determine stability.

3.7.1 Routh–Hurwitz criterion

If the characteristic equation of a system is given by a polynomial

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots a_0 = 0$$

then, in order to test the stability of the system using Routh–Hurwitz criterion, we formulate an array known as the Routh array, as follows:

$$\begin{array}{ccccccc}
a_n & a_{n-2} & a_{n-4} & a_{n-6} & - & - & - \\
a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & - & - & - \\
b_1 & b_2 & b_3 & - & - & - & - \\
c_1 & c_2 & c_2 & - & - & - & - \\
d_1 & d_2 & d_3 & - & - & - & - \\
- & - & - & - & - & - & - \\
- & - & - & - & - & - & -
\end{array}$$

where

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$$

and

$$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

and so on. Having formed the Routh array we study the first column. If the sign of the terms in that column changes m times, this means that the polynomial has m roots with positive real part, which indicates instability. Even if the sign changes only once, it will mean that the system is unstable.

Let us now apply this simple test to our problem.

Case 1:

$$x = x_1 = 0.006\,4$$

$$L_0 \text{ at } x_1 = 0.016\,09$$

$$\left(\frac{\partial L}{\partial x}\right) \text{at } x_1 = 0.656\,3$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial x}\right) \text{at } x_1 = 26.753\,5$$

$$i_0 = 6.25 \text{ A}$$

The A-matrix at $x = x_1$ is

$$\mathbf{A} = \begin{bmatrix} -99.4 & -254.9 & 0 \\ 21.25 & -2.59 & -4948 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation is obtained from $|s1 - A|$ where 1 is a unit matrix and s the Laplace transform operator. The characteristic equation of the system at $x = 0.006\,4$ is

$$s^3 + 102s^2 + 10\,622s + 491\,831 = 0$$

The Routh array for this equation is

1	10 622
102	491 831
5 800	0
491 831	—
—	

There is no change in sign in the first column which shows that equation has no root with positive real part and therefore the system is stable at $x = 0.006\,4$. The computed results have already confirmed this conclusion.

Case 2:

$$x = x_2 = 0.014\,5$$

$$L_0 \text{ at } x_2 = 0.024\,02$$

$$\left(\frac{\partial L}{\partial x}\right) \text{ at } x_2 = 1.462\,8$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial x}\right) \text{ at } x_2 = 89$$

$$i_0 = 6.25 \text{ as before}$$

The A-matrix at $x = x_2$ is

$$\mathbf{A} = \begin{bmatrix} -66.6 & -380.6 & 0 \\ 47.4 & -2.59 & 7\,650 \\ 0 & 1 & 0 \end{bmatrix}$$

and the characteristic equation is

$$s^3 + 69.2s^2 + 10\,552s - 509\,490 = 0$$

It is obvious from the negative sign of one of the terms in this equation that the system is unstable at this point of equilibrium. The Routh array is formed nevertheless and it is,

$$\begin{array}{ll} 1 & 10\,552 \\ 69.2 & -509\,490 \\ 17\,915 & \\ -50\,940 & \end{array}$$

There is one change in sign in the first column which indicates that there is one root which has a positive real part. Point x_2 thus gives unstable equilibrium.

We had arrived at this conclusion from physical considerations. Of course we could do so because the present system is very simple. For larger and more complicated systems, we can often determine by mathematical analysis the stable and unstable states.

3.8 REFERENCE

1. Fitzgerald, A. E. and Kingsley, C. (Jr)., *Electric Machinery*, McGraw-Hill , New York (1952).

4

Electrical machine dynamics

4.1 INTRODUCTION

In this chapter, we shall look at the dynamic performance of some typical machines using approximate methods of solution. The analytical technique developed in the previous chapter leads to more exact solutions and will be used later, in the form of generalised machine theory. There is, however, the danger that by formulating equations in a 'mechanical manner' and solving them by using a digital computer, one may lose sight of the physical processes that are involved. The approximate methods often help us to obtain a physical insight into the behaviour of the systems.

The dynamical performance of electrical machines is determined by the interaction of all the mechanical forces acting on the rotating mass. Some of these forces may be impressed from sources outside the machine, some will be generated by the action of electrical current flowing in the various coils and the magnetic field in which the rotor is immersed. Clearly the polar moment of inertia of the total rotating structure will play an important part in the dynamical response of any machine. In steady state operation mechanical torque impressed on the rotating shaft will balance the torque of electromagnetic origin and in this equilibrium condition, in which the rotor is rotating at a constant velocity, the moment of inertia will play no part. As soon as the equilibrium condition is upset, however, for any reason, by a transient disturbance or superimposed small oscillation (hunting), there is a torque difference which will be balanced by forces arising from the kinetic energy stored in the rotor. This results in the appearance of the torque component $J\dot{\omega}$ acting upon the rotor, which will cause positive or negative acceleration, tending towards the restoration of equilibrium either at constant angular velocity or sustained 'hunting'.

In the following sections we shall consider the acceleration of a few types of machines when voltages are suddenly applied to them.

4.2 DYNAMICS OF D.C. MACHINES

4.2.1 Separately excited d.c. motor

Let us consider the starting of a separately excited d.c. motor. As has been explained earlier, d.c. motors are always started with reduced armature voltage and the applied voltage is increased as the back e.m.f. builds up with the speed. This particular motor may be started with or without load.

The voltage and torque equations of a separately excited d.c. motor are:

$$V = R_a i_a + L\frac{di_a}{dt} + K\Phi\omega \tag{4.1}$$

$$J\frac{d\omega}{dt} = K\Phi i_a - T_{load} \tag{4.2}$$

These are transient equations whereas those in Section 2.1.3 are for steady state.

It has been shown in equation (2.48) that the generated e.m.f. in a d.c. machine is

$$E = N_a\Phi\omega \quad (V)$$

In a motor, the generated e.m.f. is also called the back e.m.f. Here Φ is the flux produced by the field winding. It penetrates the air gap and links N_a armature turns.

Following from Section 2.3, we can write

$$\Phi = N_f i_f \mathscr{P}_i$$

We now have

$$E = (N_a N_f \mathscr{P}_i) i_f \omega = M i_f \omega$$

and clearly $(N_a N_f \mathscr{P}_i)$ is a rotational mutual inductance term as shown in Section 5.6. It must be pointed out here that this mutual inductance is not the same as that in coupled stationary circuits (Section 2.7). This term is responsible for generated voltage. We shall discuss this in greater detail in Chapter 5.

What we wish to emphasise here is that a stationary observer will measure a generated voltage E which is proportional to the field excitation i_f and the angular velocity ω of the rotor. The constant of proportionality has the dimension of inductance. The observer will consider that the armature coil is fixed in space, because of the action of the commutator. Therefore, its inductance does not vary. In writing the transient equation for the armature we consider three terms.

$R_a i_a$ = armature voltage drop

$L_a\dfrac{di_a}{dt}$ = inductive voltage which disappears as soon as steady state current is reached

$Mi_f\omega$ = back e.m.f. or generated e.m.f.

In a motor, the sum of all these terms is balanced by the applied voltage. In terms of the field current, we rewrite equations (4.1) and (4.2)

$$V = R_a i_a + L_a \frac{di_a}{dt} + M i_f \omega \tag{4.3}$$

$$J\frac{d\omega}{dt} = M i_f i_a - T_{load} \tag{4.4}$$

These equations could easily have been derived using Lagrange's equations. This system, however, is too simple to invoke Lagrange's equations.

Two major assumptions have been made.

(i) It has been assumed that the brushes have not been shifted from the neutral axis position.
(ii) Saturation has been neglected.

As soon as we write $\phi = N_f i_f \mathscr{P}_i$ and treat $\mathscr{P}_i$ as a constant term, a linear relation between flux and current is implied. Saturation plays an important role in d.c. machines. In fact there is no need to neglect the effect of saturation when we solve the dynamic equations step by step. The method of including saturation is discussed in Section 2.7. However, our main objective here is to understand the dynamic behaviour of the machines and it is better if non-linear effects such as saturation, commutation, etc., are excluded from our analysis.

In equations (4.3) and (4.4) field current i_f is constant and is not a function of either the armature current or the rotor speed, the two dependent variables in the equations. The differential equations are linear and can be solved easily. In this elementary analysis we shall look at the acceleration of the separately excited motor when it is operating on no load. We thus simplify the equations somewhat by the omission of the load torque term T_L. In some cases the load torque is a function of the motor speed, for example when the motor drives a fan. Before we attempt to solve these equations let us obtain an approximate solution. We shall assume that the armature inductance L is low and hence that the rotor takes much longer to build up its speed than the time taken for the current to reach a steady value. In other words neglecting the electrical transient we rewrite equations (4.3) and (4.4).

$$V = Ri + M i_f \omega \tag{4.5}$$

$$J\frac{d\omega}{dt} = M i_f i \tag{4.6}$$

Solving these we have

$$i = \frac{V}{R}\varepsilon^{-t/\tau_m} \tag{4.7}$$

$$\omega = \frac{V}{M i_f}(1 - \varepsilon^{-t/\tau_m}) \tag{4.8}$$

where

$$\tau_m = \frac{JR}{(Mi_f)^2}$$

This may be considered to be the electromechanical time constant; as distinct from τ_e the electrical time constant of the armature, which is equal to L/R. [It is an interesting exercise to confirm that $JR/(Mi_f)^2$ has the dimension of time.] The armature current and the angular velocity of the separately excited d.c. motor are as shown in Figure 4.1. The equations clearly show that an increase in field current, in other words a stronger magnetic field, reduces τ_m and increases the acceleration of the rotor to its rated speed. Large values of rotor inertia or armature resistance obviously delay the building up of the speed. The starting resistance has to be large, however, to limit the initial value of the armature current. We shall now obtain solutions to equations (4.3) and (4.4) which include the armature inductance.

4.2.2 Laplace transform method

The voltage V applied to the d.c. motor is a step input. Taking the Laplace transform of equations (4.3) and (4.4), we have,

$$\frac{\overline{V}(s)}{s} = RI(s) + L[SI(s) - i(0)] + Mi_f\Omega(s) \tag{4.9}$$

$$J[s\Omega(s) - \omega(0)] = Mi_f I(s) \tag{4.10}$$

(having assumed that the machine starts on no load). The initial values $\omega(0)$ and $i(0)$ are obviously zero since the motor starts from rest and the armature current is initially zero. Substituting for $\Omega(s)$ from equation (4.10) in (4.9) we have,

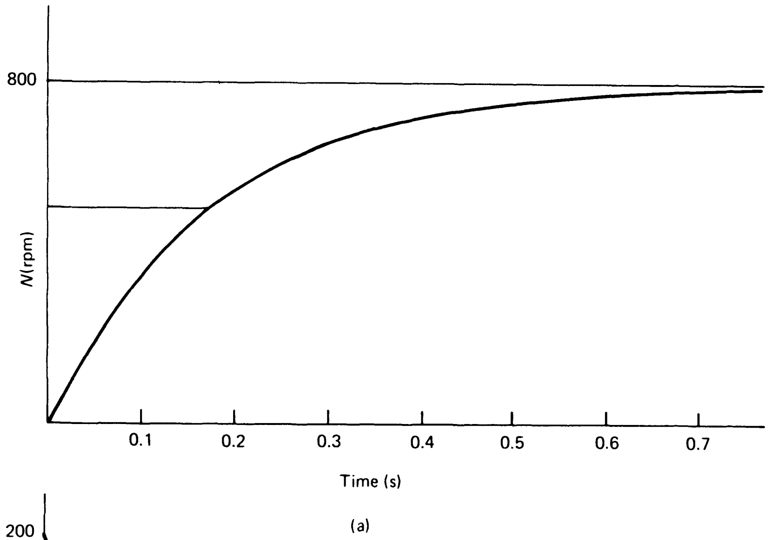
$$I(s) = \frac{\overline{V}(s)}{L\left[s^2 + \dfrac{1}{\tau_e}s + \dfrac{1}{\tau_e\tau_m}\right]}$$

where τ_e and τ_m are the electrical and mechanical time constants. In a similar manner the angular velocity in the frequency domain (s-domain) is

$$\Omega(s) = \frac{\overline{V}(s)}{Mi_f}\frac{1}{\tau_m\tau_e}\frac{1}{s\left(s^2 + \dfrac{1}{\tau_e} + \dfrac{1}{\tau_e\tau_m}\right)} \tag{4.11}$$

The current i and angular velocity ω expressed in time domain are

$$i(t) = \frac{V}{R}C(\varepsilon^{-\alpha_1 t} - \varepsilon^{-\alpha_2 t}) \approx \frac{V}{R}C\varepsilon^{-\alpha_1 t} \tag{4.12}$$



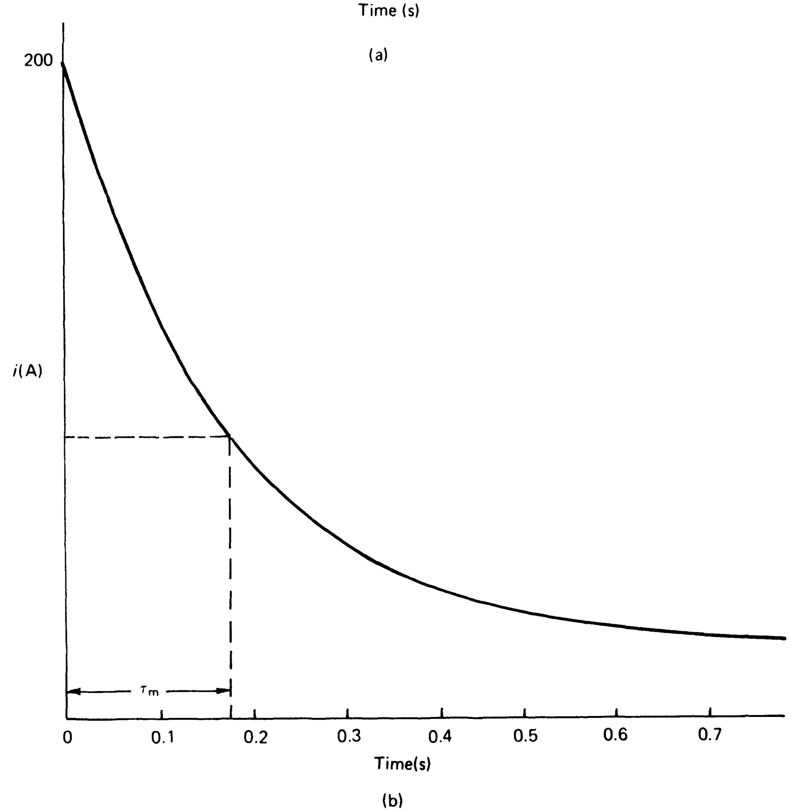


Figure 4.1 Transient speed and current in a separately excited d.c. motor (a) speed–time and (b) current–time

and

$$\omega(t) = \frac{V}{Mi_f}\left[1 - \frac{(1+C)}{2C}\varepsilon^{-\alpha_1 t} - \frac{(1-C)}{2C}\varepsilon^{-\alpha_2 t}\right] \tag{4.13}$$

where

$$\alpha_1 = +\frac{1}{2\tau_e} - \sqrt{\frac{1}{4\tau_e^2} - \frac{1}{\tau_e \tau_m}}$$

$$\alpha_2 = +\frac{1}{2\tau_e} + \sqrt{\frac{1}{4\tau_e^2} - \frac{1}{\tau_e \tau_m}}$$

$$C = \frac{1}{\left(1 - \frac{4\tau_e}{\tau_m}\right)}$$

Equations (4.12) and (4.13) give the solutions for the current and the angular velocity of the machine when the armature voltage is switched on.

4.2.3 State variables

Equations (4.3) and (4.4) can be conveniently presented in the following form to obtain an easy solution by using either an analogue or a digital computer. So far as the present problem is concerned it is unnecessary to resort to either of these methods, but computers are sometimes the only means to solve nonlinear differential equations. For this purpose, we write the equations

$$pi = -\frac{R}{L}i - \frac{Mi_f}{L}\omega + \frac{V}{L} \tag{4.14}$$

$$p\omega = \frac{Mi_f}{J}i - \frac{T_L}{J} \tag{4.15}$$

or

$$\begin{bmatrix} pi \\ p\omega \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{Mi_f}{L} \\ \frac{Mi_f}{J} & 0 \end{bmatrix} \cdot \begin{bmatrix} i \\ \omega \end{bmatrix} + \begin{bmatrix} \frac{V}{L} \\ -\frac{T_L}{J} \end{bmatrix} \tag{4.16}$$

Equation (4.16) is the state variable representation of the differential equations. This will be dealt with in further detail in later chapters. In the present case equations (4.14) and (4.15) are used to build an analogue model which will represent the electrodynamic performance of the motor. Figure 4.2 represents such an analogue model. It consists of two summing and two integrating amplifiers. It will be a useful exercise to add all the terms that are fed into a summer (with their signs duly taken into account) and equate them to zero (Kirchhoff's law). One will obtain equations (4.14) and (4.15) and the structure of the circuit model becomes obvious.

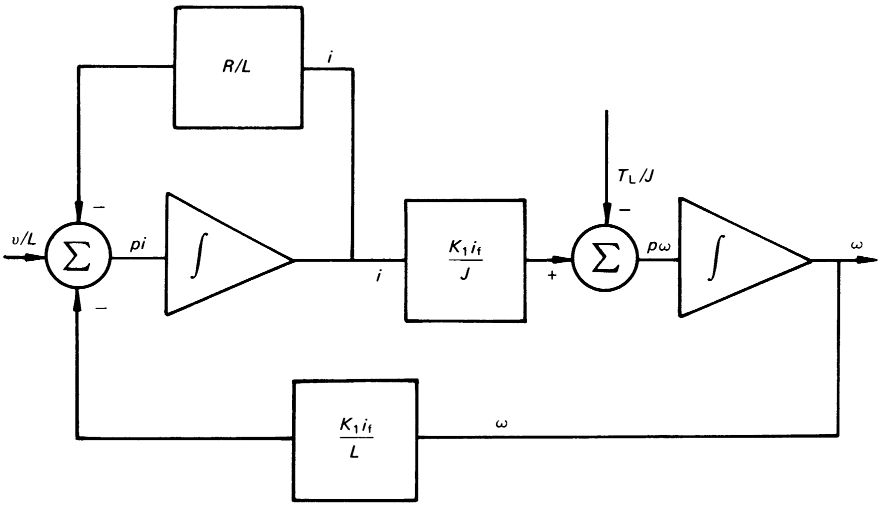


Figure 4.2 Analogue model of a separately excited d.c. motor

4.2.4 Transient solution

In this section we shall consider a particular case and compare the solutions provided by two methods.

Problem: A separately excited d.c. motor with no impressed shaft torque has its field energised. A supply of 100 V is suddenly applied to the armature terminals. Calculate the transient current and speed as the armature accelerates. The parameters are as follows:

$$\text{armature resistance } R = 0.5\,\Omega$$

$$\text{field current } i_f = 5\text{ A}$$

$$\text{moment of inertia } J = 0.5\text{ kg m}^2$$

$$\text{Armature self inductance } L = 0.000\,5\text{ H}$$

Inductance M responsible for the generated, or back, e.m.f. $= 0.238\,8$ H

(1) Approximate solution:

$$i = \frac{V}{R}\varepsilon^{-t/\tau_m} \tag{4.7}$$

$$\omega = \frac{V}{M i_f}(1 - \varepsilon^{-t/\tau_m}) \tag{4.8}$$

$$\tau_m = 0.175\,4 \text{ s} \qquad \tau_e = 0.001 \text{ s}$$

It is justifiable to ignore τ_e, since $\tau_m \gg \tau_e$. The resulting graphs are shown in Figure 4.1.

(2) Solution including armature inductance:

$$i = \frac{V}{R}C(\varepsilon^{-\alpha_1 t} - \varepsilon^{-\alpha_2 t}) \tag{4.12}$$

$$\omega = \frac{V}{M i_f}\left[1 - \frac{1+C}{2C}\varepsilon^{-\alpha_1 t} - \frac{1-C}{2C}\varepsilon^{-\alpha_2 t}\right] \tag{4.13}$$

$$\alpha_1 = 5.735\,4 \qquad C = 1.011\,6$$

$$\alpha_2 = 994.264 \qquad 1/\alpha_1 = 0.174\,36 \text{ s}, \quad 1/\alpha_2 = 0.001 \text{ s}$$

The following table provides the relative values of the two exponentials for different values of time

Table 4.1 The exponentials in equations (4.12) and (4.13)

t	$\varepsilon^{-\alpha_1 t}$	$\varepsilon^{-\alpha_2 t}$
0	1	1
0.001	0.994 3	0.369 9
0.002	0.988 6	0.136 9
0.003	0.982 9	0.050 6
0.004	0.977 3	0.018 7
0.005	0.971 7	0.006 9
0.01	0.944 3	0.000 05

From this table it is clear that there is a good deal of justification in neglecting $\varepsilon^{-\alpha_2 t}$. Since $C \approx 1$, and $1/\alpha_1 \approx \tau_m$, one can see that in practical cases equations (4.12) and (4.13) reduce to (4.7) and (4.8) without significant error. Physically, this means that *electrical inertia* of the machine being small, the current reaches its maximum value V/R almost instantaneously and then as the rotor starts to accelerate, back e.m.f. builds up, and opposes the applied voltage. The armature current now begins to decay depending on the electromechanical time constant.

Figure 4.1 shows i and ω calculated using Euler's method. The step length used was $\Delta t = 0.000\,1$ s initially, increased to 0.001 s, later. It may be clearly seen that the current very nearly reaches $V/R = 200$ A and the rate of its decay is governed by the electromechanical time constant which, from the graph, is 0.179. This is very close to τ_m or $1/\alpha_1$.

The rotor accelerates, as shown in Figure 4.1 reaching its maximum speed of 716 r.p.m. in less than a second. By reducing the armature resistance (which for a machine of this size is about 0.05 Ω) to one tenth of its value, the speed will not increase very substantially. It will go up to about 791 r.p.m. On the other hand if the field resistance is doubled (the field current becomes 2.5 A) the steady state value of the armature current will increase, with constant load torque, and the speed will increase to 1263 r.p.m. Quite often the load torque is not constant but changes with the speed. If $T_L = k\omega$, we substitute this relation in equation (4.4) and the equations are still linear. If, however, T_L varies as the square or the cube of the speed, the dynamical equations are no longer linear and may have to be solved numerically.

4.2.5 Series-excited d.c. machine

In this machine the field winding consists of a coil of few turns of heavy conductor rated to carry the full armature current. The coils are in series with the armature. The field current in the series motor is therefore no longer constant. If we substitute the armature current i for the field current i_f in equations (4.14) and (4.15) we have the equations for a series motor,

$$V = Ri + L\frac{di}{dt} + Mi\omega \tag{4.17}$$

$$J\frac{d\omega}{dt} = Mi^2 - T_L \tag{4.18}$$

Clearly these are nonlinear and it is now difficult to obtain a closed form solution. The equations can again be written

$$pi = \frac{V}{L} - \frac{R}{L}i - \frac{M}{L}i\omega \tag{4.19}$$

$$p\omega = \frac{M}{J}i^2 - \frac{T_L}{J} \tag{4.20}$$

Problem: A 10 kW 1500 r.p.m. series motor has a supply voltage V suddenly applied to the terminals. The motor is made to accelerate against a constant load torque T_L. Calculate the transient current and the speed.

The parameters are as follows:

$$V = 230\text{ V},\quad R = 1\,\Omega,\quad L = 0.05\text{ H}$$

$$J = 0.5\text{ kg m}^2,\quad M = 0.027\text{ H},\quad T_L = 55\text{ N m}.$$

This problem was solved using Euler's method with a step length of 0.000 1 s increasing later to 0.001 s. The results are plotted in Figure 4.3.

The electrical time constant is 0.05 s, which is considerably longer than we had in the case of the separately excited motor. Following the application of the voltage, the current builds up very quickly, exponentially, for about 0.025 s when the rotor begins to move. The current response is then no longer a simple exponential but is determined by the armature time-constant and the rate of build up of the generated back e.m.f. as the armature accelerates, as shown.

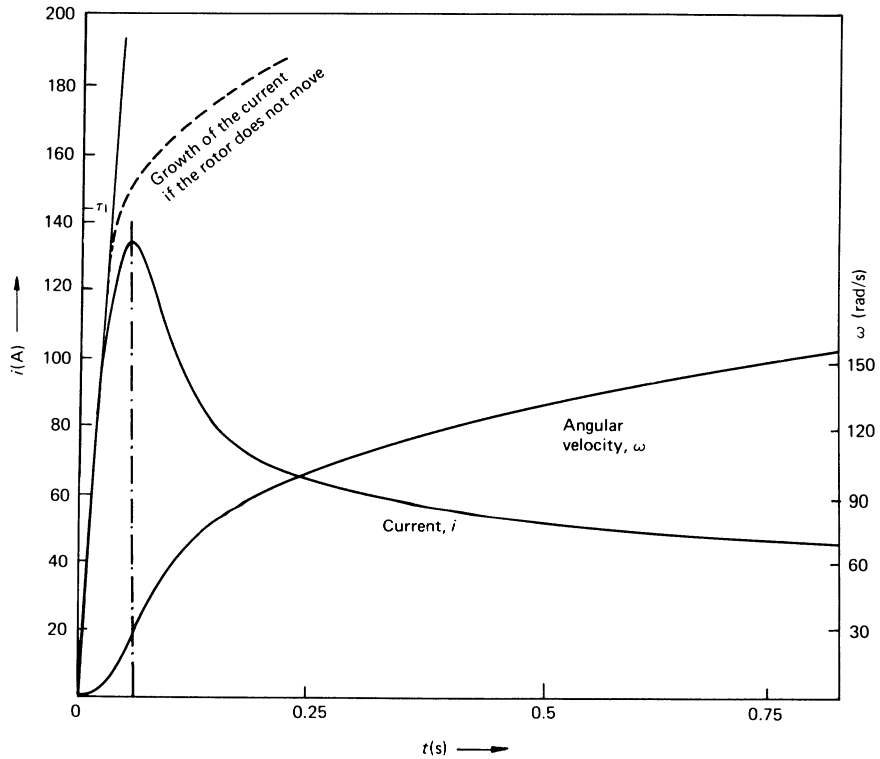


Figure 4.3 Transient speed and current in a series motor

The analogue model is given in Figure 4.4. It has a squarer to obtain i^2 and a quarter-square multiplier to obtain $i\omega$. It is well-known that a series motor reaches very high speeds limited only by friction and windage if there is no impressed load upon the shaft.

This may be seen by solving the equations approximately. If we consider $L\mathrm{d}i/\mathrm{d}t = 0$, in equation (4.17) and solve equation (4.18) by putting $T_L = 0$, the resulting solution is

$$t = \frac{J}{MV^2}\left(\omega R^2 + \omega^2 RM + \frac{\omega^3}{3}M^2\right) \tag{4.21}$$

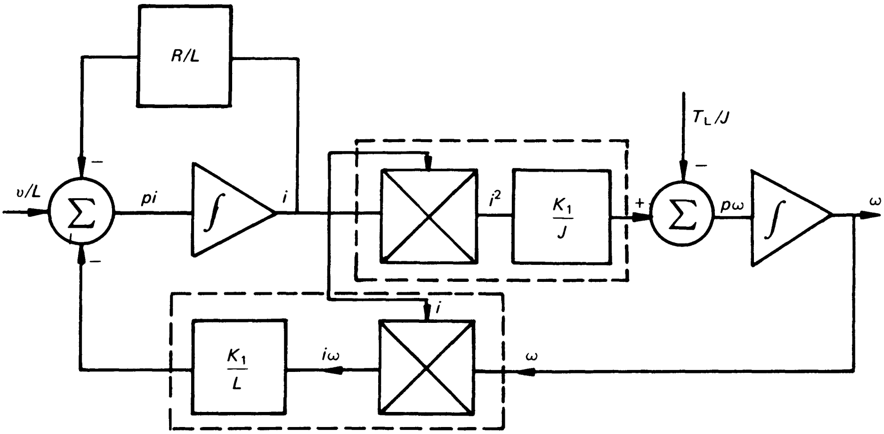


Figure 4.4 Analogue model of a series motor

which shows that $\omega \to \infty$ as $t \to \infty$. The increase in speed of the rotor with no load is shown in Figure 4.5.

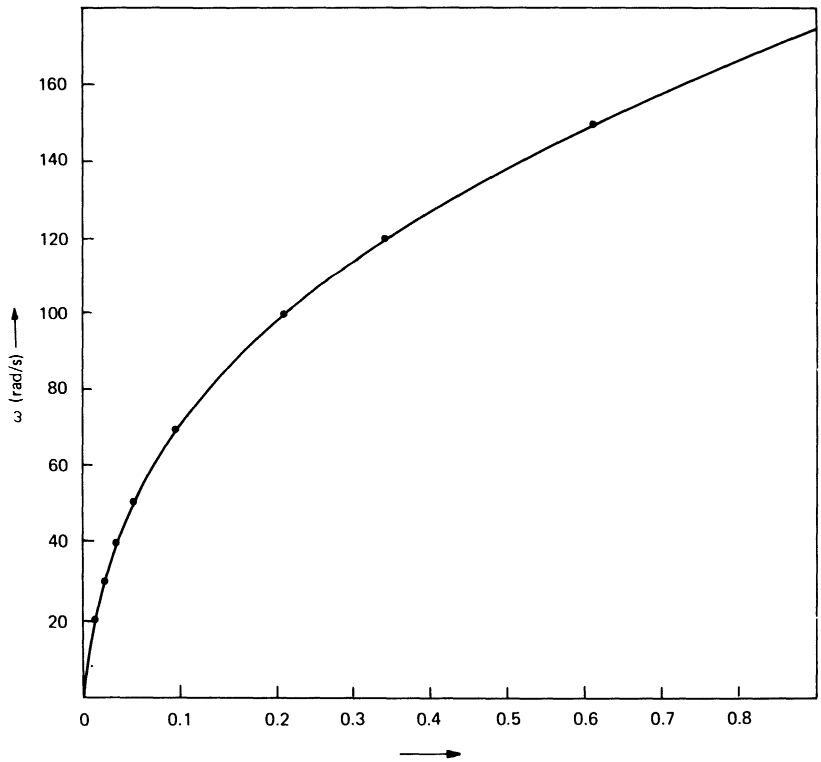


Figure 4.5 Runaway speed characteristic of a series motor

4.3 INDUCTION MOTOR

We shall now determine the acceleration of an induction motor when a 3-phase voltage is applied to the stator terminals. If we refer to the equivalent circuit (Figure 2.37) we can see that at standstill, (slip $s = 1$) the induction motor is like a short circuited transformer. The stator current therefore has a very high value on starting. The stator flux, however, also induces a voltage in the rotor and rotor current is set up. The stator and the rotor fluxes now interact and the rotor starts to rotate. When the rotor picks up speed, the stator current begins to decrease.

The direct on-line start of an induction motor may result in voltage dip on the power system if the induction motor rating is large. However, the power system can usually absorb the shock of starting a large induction motor on line.

Two approximate methods are described here to calculate the transient speed of an induction motor when a voltage is applied to the stator. Both methods are essentially the same.

4.3.1 Determination of acceleration: method I

The first method is based on the slip-torque characteristic of the motor. It provides the transient speed or slip but does not give the current as a function of time.

It has been shown in Chapter 2 that at very low values of slip, i.e. near to the synchronous speed, the electromechanical torque of an induction motor is linear, or $T \propto s$. On the other hand at larger values of slip $T \propto 1/s$.

From Figure 4.6 we clearly obtain the following relationships

$$\frac{s}{s_m} = \frac{T}{T_m} \quad \text{for low values of slip} \tag{4.22}$$

$$\frac{s_m}{s} = \frac{T}{T_m} \quad \text{for larger values of slip} \tag{4.23}$$

where T_m is the maximum or the pull-out torque and s_m is the slip at which it occurs. The maximum torque T_m is given by

$$T_m = \frac{1}{\omega_s}\left[\frac{1.5\,V^2}{R_1 + \sqrt{R_1^2 + x_l^2}}\right] \tag{4.24}$$

and

$$s_m = \frac{R'_2}{\sqrt{R_1^2 + x_l^2}} \tag{4.25}$$

Equations (4.22) and (4.23) give the following relation

$$\frac{T_m}{T} = \tfrac{1}{2}\left[\frac{s_m}{s} + \frac{s}{s_m}\right] \tag{4.26}$$

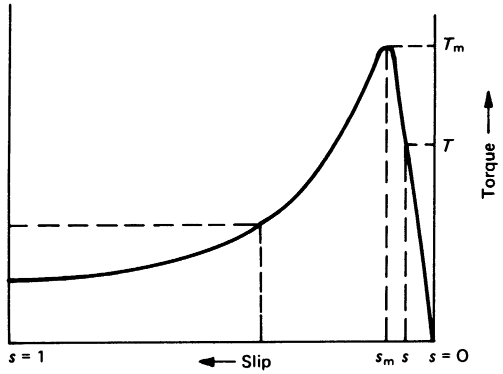


Figure 4.6 Speed–torque characteristic of an induction motor

or

$$T = \frac{2T_m s s_m}{s_m^2 + s^2} \tag{4.27}$$

If the inertia of the rotor is J and there is no initial load, we have

$$J\frac{d\omega}{dt} = 2T_m\frac{s s_m}{s_m^2 + s^2} \tag{4.28}$$

or

$$t = K\left[\frac{1 - s^2}{4s_m} - \frac{s_m}{2}\ln(s)\right] \tag{4.29}$$

where

$$K = \frac{J\omega_s}{T_m}$$

Equation (4.29) will be solved presently and the results will be compared with those obtained by method II.

4.3.2 Determination of acceleration: method II

This is based on an approximate equivalent circuit for the induction motor. Let us consider one of the phases. The voltage equation is

$$V_R = V_m \sin \omega t = \left(R_e + \frac{R'_2\omega}{\omega_s - \omega}\right)i_R + lpi_R \tag{4.30}$$

where l is the leakage inductance, $V_m \sin\omega t$ is the voltage on the red phase, i_R is the instantaneous phase current, $R_e = R_1 + R'_2$ = the equivalent resistance, and $R'_2\omega/(\omega_s - \omega)$ is the effective load resistance.

The equations for the yellow and blue phases are identical except that

$$
\begin{aligned}
V_Y &= V_m \sin(\omega t - 120^\circ) \\
V_B &= V_m \sin(\omega t + 120^\circ)
\end{aligned}
\tag{4.31}
$$

The values of torque produced by these phases add linearly. Torque produced by one of the phases, at a particular instant, is

$$(i_{\text{phase}})^2 \cdot \frac{R_2'}{\omega_s - \omega}$$

It can be seen that the total torque due to i_R, i_Y, and i_B is approximately

$$T = \frac{3}{2} I_m^2 \frac{R_2'}{\omega_s - \omega}$$

I_m is the peak value of the phase current. In the derivation of this relation one makes the assumption that the phase currents are sinusoidal in nature. Oscillograms of phase currents of an induction motor during starting[3] show that the current reaches a fairly large value and maintains nearly constant peak value (varying sinusoidally at supply frequency) while the motor accelerates. Then the current drops sharply to its steady state value.

As a first approximation, we are justified in replacing p by $j\omega_s$ so that lp becomes jx_l and the r.m.s. value of the phase current at any instant is

$$i = \frac{V_{\text{rms}}}{\sqrt{\left(R_1 + \dfrac{R_2'\omega_s}{\omega_s - \omega}\right)^2 + x_l^2}} \tag{4.32}$$

We can now write the dynamic equation of the motor,

$$J\frac{d\omega}{dt} = 3i^2 \frac{R_2'}{\omega_s - \omega} \tag{4.33}$$

this is again a nonlinear equation and is best solved step by step.

Let us apply methods I and II to a particular case.

Problem: An induction motor is started on a reduced voltage of 160 V (r.m.s.) The parameters of the motor are as follows

$$R_1 = 0.1\ \Omega,\ R_2' = 0.1\ \Omega,\ x_l = 0.5\ \Omega \text{ and } J = 1.0 \text{ kg m}^2$$

Plot the envelope of the current and the angular velocity.

Methods I and II are used and the results, plotted in Figure 4.7, indicate that these methods give very nearly the same answers.

It should be pointed out that these methods are not valid (especially method I) when we consider induction motors with double cage or deep-bar rotors. In deep-bar rotors, the rotor resistance is high at starting. This is because at starting, the frequency of the rotor current is the same as the supply frequency. This gives rise to 'skin effect' and reduces the effective area of the conductors.

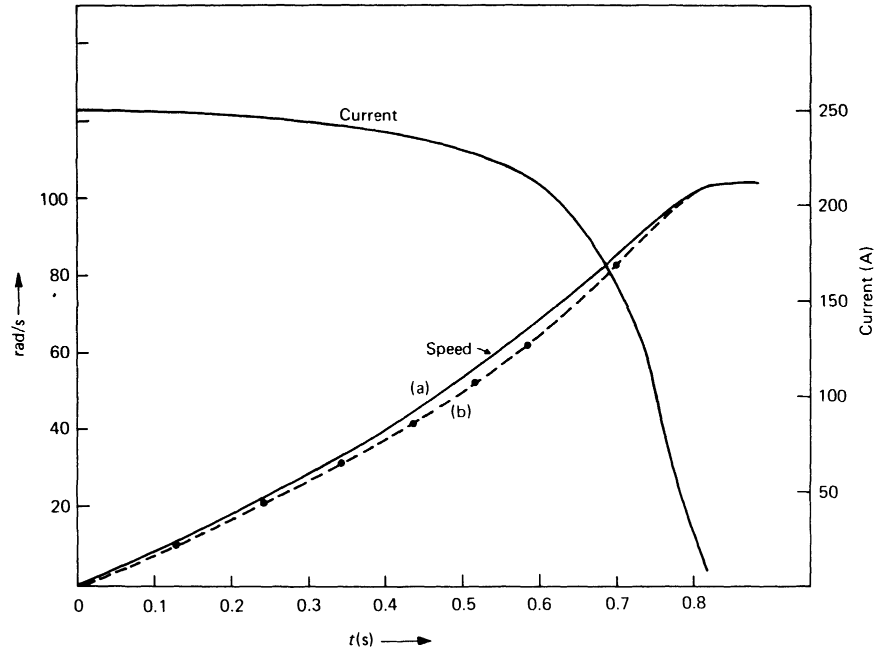


Figure 4.7 Current and speed characteristic of an induction motor at starting

The resistance is therefore increased. As the rotor speeds up the frequency of the rotor current drops and the current density in the rotor bars becomes fairly uniform. The rotor resistance is therefore, reduced. In using method II, the value of R_2' is changed at every step as it changes with rotor speed. In the case of method I, the formula used cannot take into account variable rotor resistance.

Although equation (4.33) is nonlinear, we can obtain a solution to it in a straightforward manner. Substituting equation (4.32) in equation (4.33),

$$J\frac{\mathrm{d}\omega}{\mathrm{d}t}=\frac{3V^2}{\left(R_1+\dfrac{R_2'\omega_s}{\omega_s-\omega}\right)^2+x_l^2}\quad\frac{R_2'}{\omega_s-\omega} \tag{4.34}$$

Using the relation $\omega=\omega_s(1-s)$, we solve the above equation and the following relation is obtained.

$$\frac{1-s^2}{4s_m}+\frac{s_m}{2}\ln\left(\frac{1}{s}\right)+2s_m\left(\frac{R_1}{R_2'}\right)=\frac{1}{\omega_s J}\left(\frac{1.5V^2}{\omega_s\sqrt{(R_1^2+x_l^2)}}\right)t \tag{4.35}$$

This equation may be compared with equation (4.29). They are very similar except that equation (4.29) does not contain the term $2s_m R_1/R_2'$ and on the right hand side of equation (4.35) we have additional terms (see equation 4.24).

The speeds calculated using methods I and II, however, agree very closely. Equation (4.32) is used in order to calculate the current i. Equations (4.32) and

(4.33) may also be solved step by step. Figure 4.7 gives the envelope of the current waveform. It will be seen that the current maintains a fairly large value for about 0.5 s in this particular case before it begins to drop. After 0.6 s it drops sharply to its steady state value. In the present case, we did not include any load or friction and therefore the no-load current will be zero. The magnetising current has to be added vectorially to obtain the total current.

The fact that the current stays constant at a high value for a certain length of time is responsible for the hazards that may follow the on-line start of an induction motor. The current remains fairly constant because in the initial part of the accelerating process, when slip is large, the effective resistance R_e is small compared to x_l, the total leakage reactance. Besides x_l and R_e are added as $\sqrt{x_l^2 + R_e^2}$ and therefore the change in R_e as a result of the change in R'_2/s is not evident until a very small value of slip is reached. Turning our attention to acceleration, we can see that the term $(1 - s^2)/4s_m$ dominates during the earlier part. When the motor reaches the straight line part of the slip torque characteristic (Figure 4.6), the term $\ln(1/s)$ becomes dominant.

In order to understand the acceleration of the rotor against a load, let us consider the following situation.

The induction motor considered in the earlier example is now connected to a load and we assume that the load speed–torque characteristic is $T_L = 2\omega$ N m.

The slip–torque characteristic of the motor is shown in Figure 4.8 along with the torque characteristic of the load which is a straight line. At any slip (or speed) of the motor, the available torque is given by the difference between the driving torque and the load torque as shown by ab. It is this torque difference which causes the rotor to accelerate. We can therefore write the following relation

$$J\frac{d\omega}{dt} = \Delta T \tag{4.36}$$

where ΔT is the torque available at any instant of time.
Then

$$t = J\int\frac{d\omega}{\Delta T} \tag{4.37}$$

If we plot $J/\Delta T$ at different values of ω, then the area under the curve stretching from $\omega = 0$ to any value of ω gives the time required for the motor to reach that value of ω. Curve A is $J/\Delta T$ drawn for no load and curve B has been drawn for full load represented by $T_L = 2\omega$. The area under curve A is 0.81 which is actually the time required to reach the maximum speed. The area under curve B is 1.66 which is the time (in seconds) necessary for the rotor to reach the speed of 94.05 radian/s (905 r.p.m.), its steady state value. The steady state speed obviously is at the intersection of the torque and the load characteristics (point 0 in Figure 4.8).

Since the load torque and the driving torque are equal at this point, the motor will not accelerate any further and dynamic equilibrium is established.

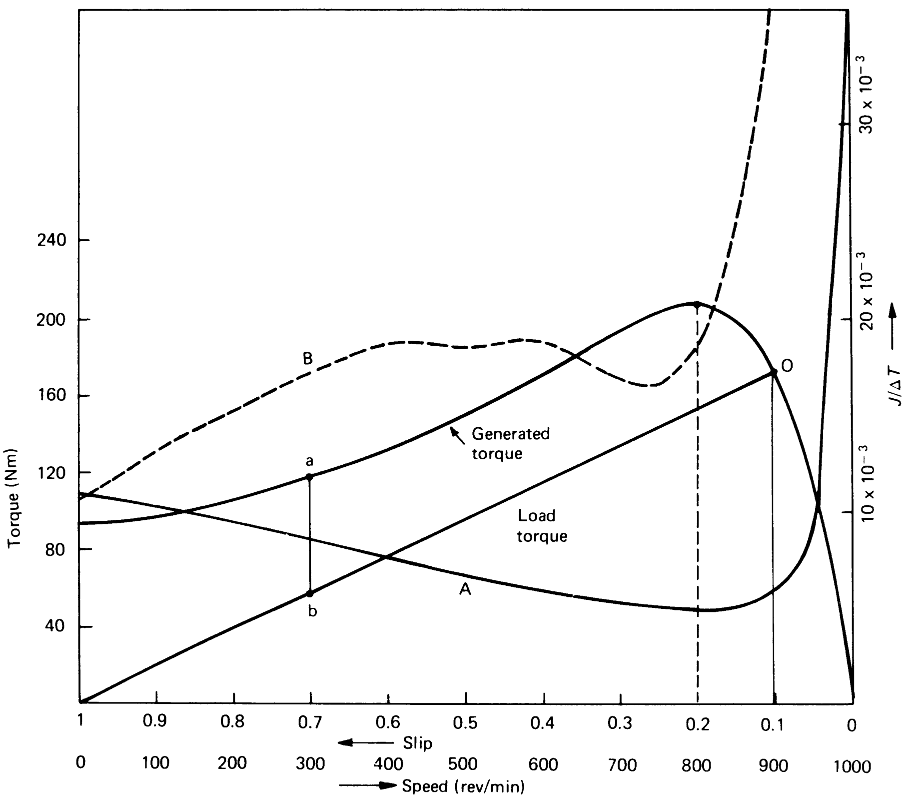


Figure 4.8 Graphical method of computation of acceleration of an induction motor. (A) $J/\Delta T$ for no load and (B) for load

If the speed increases beyond this point, the load torque exceeds the driving torque and decelerates the motor. On the other hand, if the speed momentarily decreases, the surplus driving torque will accelerate the motor till the point of equilibrium at 0 is reached. In other words stable operation is ensured when the load characteristic intersects the torque characteristic of the motor at a point where $\Delta T/\Delta\omega$ is negative. We encountered a similar situation in the plunger and spring problem. If the magnetic field of an induction motor has not been designed properly harmonic torques may be produced. Figure 4.9 represents a typical torque–speed characteristic of a motor in which a dominant seventh harmonic torque is present. If this induction motor is switched on and accelerated against a load it cannot reach its normal speed. It will, instead, settle down at almost one seventh of its speed. This phenomenon is known as crawling. As may be observed, the load characteristic intersects the torque characteristic at three points, namely a, b, and c. For reasons given earlier point b is unstable ($\Delta T/\Delta\omega$ is positive). Point c is stable but before the rotor can reach c it meets point a, which satisfies the conditions necessary for stable operation and the rotor crawls at this speed. If however, the rotor is accelerated out of this 'dip', it accelerates normally until it reaches point c.

Modern induction motors do not normally suffer from these problems of harmonics.

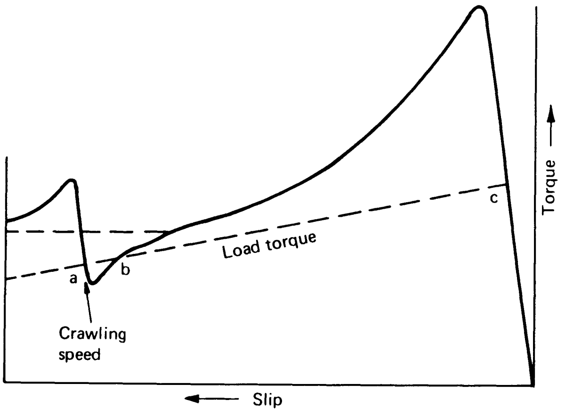


Figure 4.9 Crawling in an induction motor

4.4 THE ALTERNATOR

In the earlier sections of this chapter, we have studied the acceleration of direct current and induction motors. In studying alternators and synchronous motors, the dynamic or transient behaviour following some disturbance is of vital importance. In this section we shall describe approximate methods of studying this behaviour, once again with the object of gaining a physical understanding of the problem.

Usually there are two types of rotor disturbance that are studied and analysed in synchronous generators. One of these is a transient disturbance which generally results in large oscillations and sometimes leads to loss of stability. The other is of small magnitude and may be self-excited or induced by a relatively minor factor, such as a change in the operating condition.

The essential difference between these two types is that in the case of transient disturbances, the large rotor oscillations are described by nonlinear differential equations. With small excursions of the rotor the machine equations can be linearised.

We have seen that in the case of separately excited d.c. motors, the electrical and mechanical transients during starting can be separated without much loss in accuracy. This would not be possible if the disturbance were prolonged and the rotor oscillations sustained.

4.4.1 Electrical transients

When an alternator is short circuited, we can assume that the rotor speed remains unchanged for a short time and only the currents and fluxes change. The behaviour of the rotor following the short circuit may then be studied separately, as has been done in later sections in this chapter. Sudden short circuit of an alternator results in very large initial currents in the 3-phases (Figure 4.10)

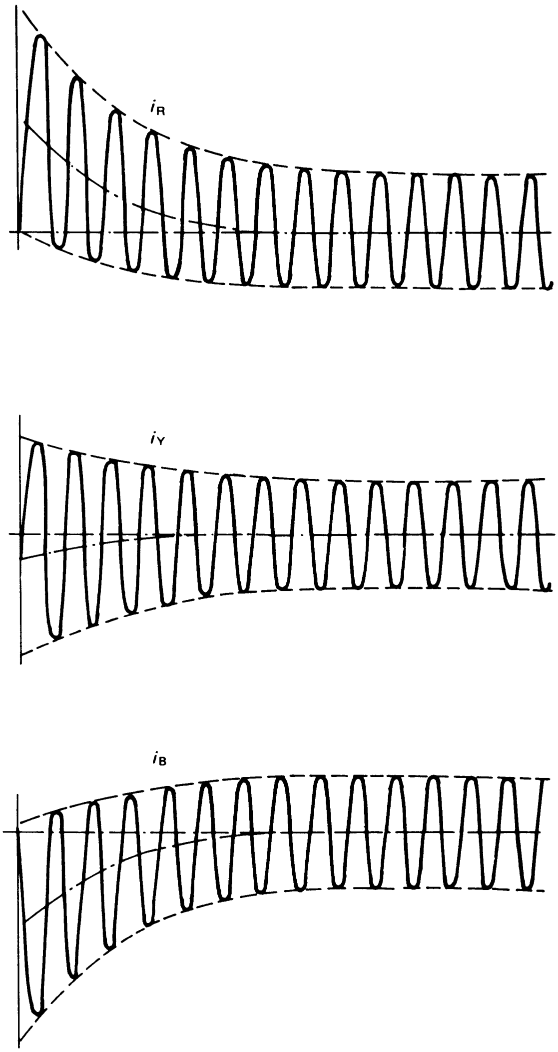


Figure 4.10 Balanced short-circuit current in three phases of an alternator

of the armature. These currents settle down to a steady state value, the sustained short-circuit current of the alternator. The field current increases rapidly and then decays to its original d.c. value. Amortisseur or damper windings which do not carry any current at steady state, have transient current induced. In order to understand the occurrence of these transient currents we may either make use of the theorem of constant flux linkage (Section 2.4) or mathematically model the machine and observe that these are inherent in the system. Application of a short circuit is comparable to the sudden application of voltage to a system with closed circuits. While studying the starting of a d.c. motor we observed how the current rises to a high value immediately after the application of the armature voltage.

Let us first consider the simple case of a transformer, to comprehend the constant flux linkage theorem. Let the secondary coil be short circuited and a direct voltage applied to the primary side. Initially, the flux linkage ψ_2 of coil 2 is zero and if its resistance is neglected then ψ_2 will continue to be zero. Hence

$$\psi_2 = L_2 i_2 + M i_1 = 0$$

$$i_2 = -\frac{M}{L_2} i_1$$

Here M is the mutual inductance between the primary and the secondary coils and L_2 is the self inductance of the secondary. So far as the primary circuit is concerned

$$\psi_1 = L_1 i_1 + M i_2$$

$$= \left(L_1 - \frac{M^2}{L_2}\right) i_1 = L_{s\text{-}c} i_1$$

The inductance of the primary with short circuited secondary is not L_1 but $(L_1 - M^2/L_2) \ll L_1$ and it is this reactance which controls the short circuit current. This relation will immediately follow if we consider the equivalent circuit of a transformer, neglect its resistance and short circuit the secondary side. The leakage inductances dominate because, by virtue of constant flux linkage, the flux cannot flow through the short circuited coils and is constrained to flow in the low permeance leakage paths.

In the case of an alternator on short circuit, the sudden increase of current in the armature induces currents in the amortisseur and field windings to maintain their flux linkages unaltered. However, since these windings have resistance, the currents decay, leaving the armature to carry the steady state short circuit current.

Study of the armature current of an alternator on short circuit shows the presence of two exponentials and a sinusoid. The first exponential is due to the combination of the armature, the field, and the amortisseur windings. This gives the low value of reactance of an armature immediately after a short circuit, x'' known as the subtransient reactance. This is given by the parallel combination of the leakage reactance of the amortisseur and field winding combined with the armature reactance (Appendix A3). The initial short circuit current is therefore high. The current through the amortisseur winding decays fairly rapidly but the transient current in the field circuit decays more slowly. The reactance of the armature circuit is now x' known as the transient reactance of the alternator. The reactance x' is always greater than x'' because the amortisseur winding is no longer involved. Finally, because of the presence of field resistance, the transient current in the field decays and the armature reactance is x its synchronous reactance—as though the amortisseur and the field windings had been open circuited.

The subtransient, transient, and synchronous reactances play a very important role in the stability of generators. The effects of the field circuit and the amortisseur windings are automatically included when we write the machine equations in the generalised form.

In the section that follows we shall study the dynamics of the rotor using a graphical approach which is helpful in illustrating the physical aspects of the problem.

4.4.2 The dynamics of the system

The dynamical performance of the alternator depends on its mode of operation—of which there are three, which will be treated in some detail in the form of examples given below. In the first case, in which a single alternator supplies its own load, the terminal voltage depends on the speed and excitation of the machine. The frequency depends on the speed. The machine operating power factor at the terminals depends on the nature of the load. Here the dynamical response of the machine may be determined in association with its prime mover.

In the second mode of operation the alternator operates synchronously in parallel with a very large system, in which the voltage and frequency are constant and are independent of any load supplied to the system or extracted from it by the alternator. This implies that the busbar has zero impedance behind it, or an infinite number of machines feeding it in parallel—hence it is called an infinite busbar. In this case the frequency and terminal voltage of the alternator must at all times be precisely those of the infinite system. The machine runs synchronously and if it accelerates or decelerates slightly, large synchronising currents flow to or from the busbar, in such a direction and phase that they tend to restore the synchronous condition. Under these conditions the load supplied by the alternator is a function of the driving power and torque impressed by the prime mover on the shaft. If this is increased by a small amount, the machine rotor will accelerate slightly away from its original synchronous position. Current and power will flow into the busbar, restoring the synchronous condition but at a synchronous angular position slightly in advance of the previous position. This is the load angle δ which we have already met in Chapter

2. The effect of a disturbance in the operating condition is to produce changes in the synchronous angle. Depending on the nature of the disturbances, the angle δ may take up a new value. It may oscillate at a frequency governed, among other things, by the moment of inertia of the rotating mass, or it may simply increase indefinitely, in other words the alternator may lose synchronism altogether. The current, voltage, and frequency will all undergo changes as the angle δ swings. Large excursions of δ will be subject to nonlinear conditions in the machine, but small oscillations about a constant mean operating state will be governed by small changes in the machine quantities, and the analysis of small oscillations or 'hunting' is therefore linear.

In the third mode of operation, a group of alternators operates in synchronism to supply a load. This is not a large enough system to present infinite busbar conditions for any one machine yet the machines are tied together by synchronising forces. In this case one type of disturbance may affect one machine as though it were on an infinite busbar, with little effect on the others, while another type of disturbance may affect the dynamical performance of the whole group. In the latter case the speed and frequency of the group may be affected but synchronism be maintained. Here the infinite bus concept cannot be used and a multivariable nonlinear analysis results, which is very complicated indeed. Some large power systems (i.e. about 40 GW) which have previously been considered to present infinite bus conditions to individual machines or stations, must now be analysed in this category, with the advent of very high voltage interconnections, transferring several thousand megawatts per circuit. The fact that an entire national grid system no longer presents an infinite busbar to a single power station has increased enormously the difficulties in designing the system for overall dynamic, transient, and hunting stability. The situation is further complicated by the use of fast-acting voltage regulators and turbine-governor equipment. This full analysis of multi-machine power systems is beyond the scope of the present book, but the techniques given below, in problems of alternator and synchronous machine stability, form the established basis for all studies of power system dynamics.

The elementary treatment of transient stability of an alternator synchronised with an infinite busbar, will be found in virtually all works on power system analysis.[1, 2] The basic facts may be summarised as follows.

The alternator in its simplest form may be represented by a generated voltage and an internal impedance. For more detailed analysis the field circuit parameters and the machine time constants must be considered.

If the elementary synchronous system is as shown in Figure 2.32 the dynamical behaviour is given by the equation

$$J\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} = \text{torque input} - \text{torque output} \quad (\mathrm{N\,m}) \tag{4.38}$$

where δ is the synchronous load angle in mechanical radians and J is the moment of inertia of the rotating mass in $\mathrm{kg\,m^2}$.

In terms of power, at synchronous mechanical angular velocity ω rad/s

$$M\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} = \text{power input} - \text{power output} \quad (\mathrm{W}) \tag{4.39}$$

where $M = J\omega$.

In terms of the stored energy per megavolt ampere, the constant H is used (see Appendix A1) where

$$H = \tfrac{1}{2}J\omega^2/G = \tfrac{1}{2}M\omega/G \quad (\text{J/V A-rating}) \tag{4.40}$$

thus

$$M = H/\pi f \tag{4.41}$$

per unit of the machine rating. The dynamical equation may now be written

$$\text{accelerating power} = (\text{power input} - \text{power output}) = \frac{H}{\pi f}\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} \tag{4.42}$$

During steady state operation $P_{\text{in}} = P_{\text{out}}$, the angular velocity ω is constant, and the load angle δ is constant for constant power output.

The acceleration clearly depends on the inertia constant H and the accelerating power differential $(P_{\text{in}} - P_{\text{out}})$. If we consider that the power input is constant during the disturbance (ignoring for the moment the effect of the governor and voltage regulator), then the accelerating power differential depends on the instantaneous power output, which is a nonlinear function of the load angle. The steady-state vector diagram for the alternator is drawn in Figures 2.31 and 2.33. From these, if for the present we ignore the line and machine resistance, the power output is seen to be

$$P_0 = \frac{EV\sin\delta}{x_{\mathrm{d}}} + \frac{V^2}{2}\left(\frac{1}{x_{\mathrm{q}}} - \frac{1}{x_{\mathrm{d}}}\right)\sin 2\delta \tag{4.43a}$$

where V is the terminal voltage. With a round-rotor turbo-alternator (and even with salient pole machines during transient operation) the second term is small and the power expression reduces to

$$P_0 = \frac{EV}{X}\sin\delta \tag{4.43b}$$

where now the reactance X includes the internal and external reactance between the internal generated voltage E and the voltage V at the point of connection with the infinite busbar. The dynamical equation, which now becomes

$$\frac{H}{\pi f}\frac{\mathrm{d}^2\delta}{\mathrm{d}t^2} = P_{\text{in}} - \frac{EV}{X}\sin\delta \tag{4.44}$$

can be solved by step-by-step techniques to give the swing curves of δ as a function of time. Before this can be done, however, several conditions must be known, namely (i) the initial operating conditions giving P_{in}, E, V and δ_0 and

(ii) the different values of the reactance during the whole time period of the disturbance. The importance of the latter is shown below. The system impedance may undergo sudden changes due to faults, automatic circuit-breaker operation, reclosure, etc., and the sudden changes taking place at given time intervals will have a critical effect on the whole of the dynamics of the system. Obviously in the machine there will be mutual coupling between the armature circuits on the one hand and the field and amortisseur circuits on the other. This coupling introduces the concepts of 'transient' and 'subtransient' reactances, as mentioned in the earlier section. Stability analyses including these effects have been extensively treated in the literature.

Figure 4.11 shows the power angle curve for an alternator connected to an infinite busbar through two transmission lines. Now consider the following sequence of events: (a) the machine is operating in the steady state, on load at load angle δ_0 and the input power remains constant, (b) a three-phase fault occurs on one line, (c) the circuit breakers on this line trip and remove the fault, (d) the alternator continues to operate through one line. In condition (d) there will be a considerable increase in the reactance X compared to condition (a) with corresponding reduction in power output for any given load angle. For the duration of the faulted condition, the fault current will be reactive, with a reduced 'real-power' component. The power demand on the prime mover will thus fall and the machine will accelerate. When the line trips we shall assume that the load angle has reached the position δ_m. The power demand will then suddenly rise to the power-angle curve corresponding to operation through the one line. There are thus three power angle curves to be considered, namely those representing the system intact, faulted, and restored, as shown in Figure 4.11.

We shall now show that the shaded areas A and B in Figure 4.11 give us information about the stability of the system. If area A is greater than area B, then there will be so much kinetic energy stored in the rotor of the turbo-alternator that it will continue to accelerate in the restored state, even though it is still connected to the infinite system through one line, and it will lose synchronism. If, however, the fault is cleared when the machine has reached an

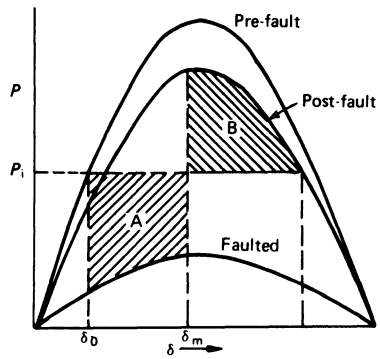


Figure 4.11 Power-angle curve for an alternator connected to an infinite busbar through two transmission lines

angle δ_m such that area A is less than area B, then the machine will remain in synchronism and settle at a new steady state load and load-angle. The problem of transient stability is therefore in two parts (1) to determine the critical clearing angle δ_c when area A = area B and (2) to arrange to trip a faulted item of plant before this critical angle is reached; in other words to carry out a 'swing-curve' (δ-time) analysis by solving the dynamical equation. Needless to say this computation becomes very lengthy when the system is large and the effects of regulators and governors have to be included. The 'equal-area criterion' of transient stability can be expressed mathematically as follows.

Consider the 2-machine system shown in Figure 4.12. The dynamical equations are

$$M_1 \frac{d^2\delta_1}{dt^2} = \Delta P_1 \tag{4.45}$$

$$M_2 \frac{d^2\delta_2}{dt^2} = \Delta P_2 \tag{4.46}$$

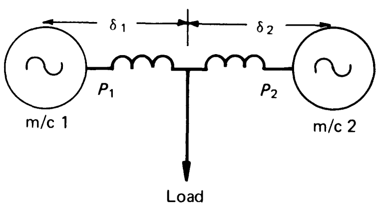


Figure 4.12 Two-machine system

Angle of separation of machine rotors,

$$\delta = \delta_1 - \delta_2 \tag{4.47}$$

and

$$\frac{d^2\delta}{dt^2} = \frac{d^2\delta_1}{dt^2} - \frac{d^2\delta_2}{dt^2} = \frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2}$$

or

$$\frac{d}{dt}\left(\frac{d\delta}{dt}\right)^2 = 2\left[\frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2}\right]\frac{d\delta}{dt}$$

$$\frac{d\delta}{dt} = \sqrt{2\int_{\delta_0}^{\delta_{max}} \left(\frac{P_1}{M_1} - \frac{P_2}{M_2}\right) d\delta} \tag{4.48}$$

The machines will swing apart until $d\delta/dt = 0$, that is, until

$$\int_{\delta_0}^{\delta_{max}} \left(\frac{P_1}{M_1} - \frac{P_2}{M_2}\right) d\delta = 0 \tag{4.49}$$

A 2-machine system can be expressed as follows, in terms of an equivalent machine synchronised with an infinite busbar as shown in Figure 4.13

$$\frac{d^2\delta}{dt^2} = \frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2} = \frac{P_{i1} - P_{01}}{M_1} - \frac{P_{i2} - P_{02}}{M_2}$$

or

$$M_0 \frac{d^2\delta}{dt^2} = P_i' - P_0' - \Delta P' \tag{4.50}$$

where

$$M_0 = \frac{M_1 M_2}{M_1 + M_2} \tag{4.51}$$

$$P_i' = \frac{M_2 P_{i1} - M_1 P_{i2}}{M_1 + M_2} \tag{4.52}$$

$$P_0' = \frac{M_2 P_{01} - M_1 P_{02}}{M_1 + M_2} \tag{4.53}$$

In this case

$$\frac{d\delta}{dt} = \sqrt{2 \int_{\delta_0}^{\delta_{max}} \left(\frac{\Delta P'}{M_0}\right) d\delta} \tag{4.54}$$

and the machines will swing apart until

$$\int_{\delta_0}^{\delta_{max}} \left(\frac{\Delta P'}{M_0}\right) d\delta = 0 \tag{4.55}$$

that is, until

$$\int_{\delta_0}^{\delta_{max}} \Delta P' \, d\delta = 0 \tag{4.56}$$

In the case shown in Figure 4.11 in which the machine remains in synchronism following the loss of one of the parallel circuits, the rotor load angle undergoes an excursion from the initial angle δ_0, to δ_{max} the angle reached before the fault is cleared. Following the tripping of the faulted circuit, the rotor returns to a

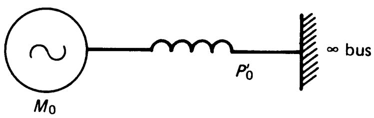


Figure 4.13 Equivalent of the two-machine system

new steady-state angle δ'_0 under the new system impedance conditions. With the input power maintained at its original value and higher systems impedance between the machine and the grid busbar, the alternator will run at the increased load angle. The value of the clearing angle δ at which stability would be lost can be calculated as shown in the following example.

A 3-phase alternator operates at 3000 r.p.m. 50 Hz at 0.6 per unit power load in synchronism with the grid. The machine excitation and the system impedances are such that the operating power-angle curves before, during, and after clearing a three phase fault, have maximum values 1.8, 0.4, and 1.5 per unit respectively. Calculate the critical clearing angle for transient stability (Figure 4.14).

System intact, $1.8 \sin \delta_0 = 0.6$, $\delta_0 = 19° \, 24'$,

System restored, $1.5 \sin \delta'_0 = 0.6$, $\delta'_0 = 23° \, 36'$

By symmetry the angle $\delta_a = 180 - 23° \, 36' = 156° \, 24'$

These values are electrical degrees. In electrical radians

$$\delta_0 = 0.337, \quad \delta_a = 2.73 \quad \delta'_0 = 0.41$$

Since this is a 2-pole machine, mechanical displacements of the rotor expressed in mechanical radians will be equal to the electrical values.

When area A = area B, the critical angle is δ_c

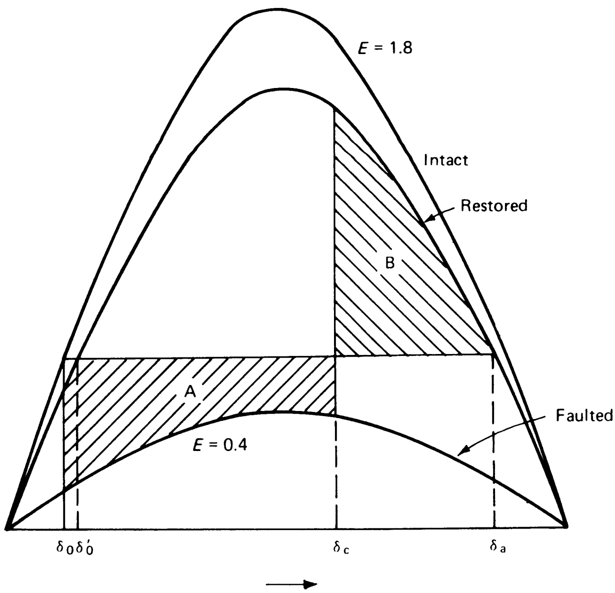


Figure 4.14 Critical clearing angle for transient stability based on equal-area criterion

Considering the component areas under the curves,

area A is

$$0.6\,\delta_c - 0.337 \times 0.6 - \int_{0.337}^{\delta_c} (0.4 \sin \delta)\, d\delta$$

$$= 0.6\delta_c - 0.202 + 0.4 \cos \delta_c - 0.378$$

area B is

$$\int_{\delta_c}^{2.73} (1.5 \sin \delta)\, d\delta - 0.6\,(\delta_a - \delta_c)$$

$$= 1.37 + 1.5 \cos \delta_c - 0.6\delta_a + 0.6\delta_c$$

When area A = area B

$$0.4 \cos \delta_c - 1.5 \cos \delta_c = 1.37 - 1.638 + 0.378 + 0.202$$

$$\cos \delta_c = -0.288$$

$$\delta_c = 106.75^\circ$$

From the above analysis it will be seen that high system impedance between synchronous power stations, for given system loads, will mean higher values of the operating load angles between the stations and hence a reduced transient stability margin. This is a factor which must be considered in the design of power systems, for example, one in which a large hydro station feeds power over a long distance very-high-voltage circuit, into a metropolitan area. (In such cases there are also serious problems of voltage control and automatic protection.) A typical swing curve for such a system is shown in Figure 4.15.

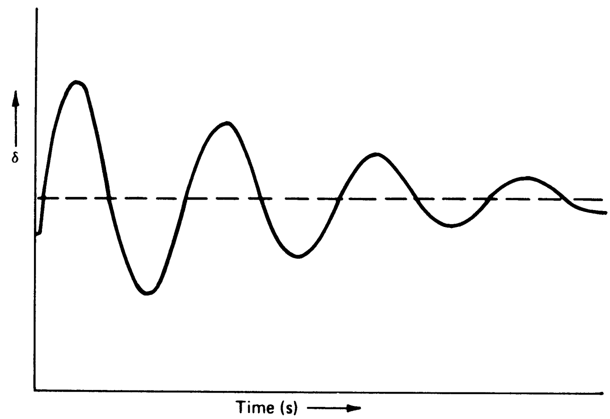


Figure 4.15 Typical swing curve

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5

Generalised machines

5.1 THE GENERALISED MACHINE CONCEPT

It has been indicated in Chapter 2 that analytically all machines are basically similar and that, in general, the difference between a.c. machines and d.c. machines arises from the observer's reference frame (e.g. the commutator in d.c. machines and the slip rings in a.c. machines). Of course there are many essential differences in design and construction, which make the performance characteristics of one machine so different from those of another. However, when it comes to analysing the performance of a particular machine, especially during transient operation, it is convenient to start from a generalised equation.

A definite trend towards a generalised machine concept started in the early 1930s. Kron[2] went further than anybody else in this field, and visualised the performance of all electrical machines as a locus traced out by electromechanical variables in an n-dimensional non-Euclidean space, with electrical and mechanical co-ordinates. He also derived what he called a *primitive machine*, a machine of which all others were particular cases.

Recently some authors have suggested that with the computer to do all the work of calculation, it is unnecessary to go into these generalised concepts and that each machine can be analysed independently without much difficulty.

The present authors do not share this view. Although it is true that the digital computer considerably simplifies calculations, the performance equations have to be derived for individual cases. It is a distinct advantage, therefore, that a general machine equation be written down once and for all. Connection matrices, which relate the currents in the primitive machine to those in the machine to be analysed, can be written by simple observation and fed into the computer, which will then produce the performance characteristics of any given machine. The practical validity of mathematical models derived from generalised machine theory, for industrial machines, with all the nonlinear phenomena associated with them, has been established by C. V. Jones in reference 1.

5.2 THE PRIMITIVE MACHINE

It is rather difficult to visualise that an induction motor or a synchronous machine can be traced to the same ancestry as a d.c. machine but, in spite of their physical differences, their performance characteristics can all be derived from those of the primitive machine. This is achieved by matrix transformation.[2] Some of the rigorous mathematical theory is omitted here, since our present object is to show physically or graphically that synchronous, induction, or d.c. machines can be analysed in considerable detail, by quite simple basic matrix operations.

Let us take the case of a two-pole synchronous machine shown in Figure 5.1. It has a 3-phase armature winding on the stator, connected to a load or to a 3-phase supply. The rotor field winding has d.c. excitation. If it is an 'inverted' synchronous machine, with the 3-phase armature on the rotor and d.c. field structure on the stator, there is little difference so far as electrical analysis is concerned.

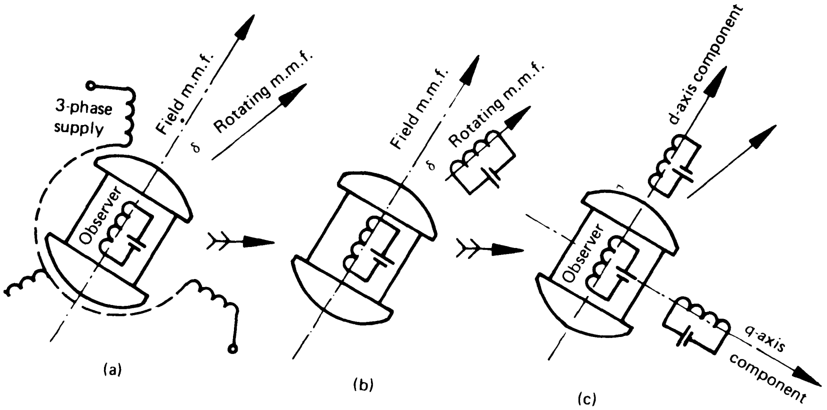

Figure 5.1 Synchronous machine as viewed from the rotor

If the observer 'sits on one of the stator windings', the mutual inductance between the stator and the rotor coils changes as the rotor rotates. Also, if the air gap is non-uniform (as is the case with salient-pole machines) the self-inductance varies with varying reluctance of the magnetic path. Hence the impedance becomes a function of the rotor position, which makes the equations rather complicated.

Let us then look at it in a different manner. We know that 3-phase voltage fed to the 3-phase winding of the stator produces a magnetic field of constant magnitude, rotating at synchronous speed. An observer 'sitting on the *rotor*' will see an m.m.f. of constant magnitude, stationary with respect to him, when the rotor also rotates synchronously.

So far as the observer is concerned there are now two stationary m.m.f., one produced by the d.c. excitation of the field and the other in the armature, displaced from the field m.m.f. by the load angle δ. This armature m.m.f. could well have been produced by two fictitious windings, one situated in line with the field poles in the direct axis and another on the quadrature axis, each carrying direct current of constant magnitude.

To this observer, the synchronous machine would appear as in Figure 5.1 (c). The two fictitious coils on the armature are generally known as the direct-axis and the quadrature-axis windings, since one is along the pole axis and the other at 90° (electrical) with respect to it. They rotate with the rotor. Hence there is no movement between the rotor and the *fictitious* stator coils. All of the self and mutual inductances of these windings are now constant. Also, the observer 'on the field coil' can see a constant air gap along the direct and quadrature axes and the reluctance of the magnetic path in this reference system is independent of time. This results in linear differential equations for the electrical performance and gives impedance equations which have constant terms. The analysis is thus considerably more straightforward. This reference frame is historically associated with the names of Blondel, Doherty, Nickle, and Park and has been widely used in synchronous machine analysis and power system calculations for three or four decades.[3–6]

Now let us consider an 'inverted' synchronous machine shown in Figure 5.2. The field winding is fixed on the stator and, as before, we refer to this as the direct axis. The armature windings on the rotor carry polyphase balanced currents which produce a magnetic field rotating synchronously *with respect to the rotating armature*, that is, the armature m.m.f. wave is standing still in space, with components again along the direct and quadrature axes. This same situation would obtain if there were a commutator on the armature, with two pairs of diametrically placed brushes, one pair in the direct axis and one pair in the quadrature axis.

Let us try to clarify this concept a little further. Consider a d.c. generator. From the direction of the field flux and the direction of rotation of the armature

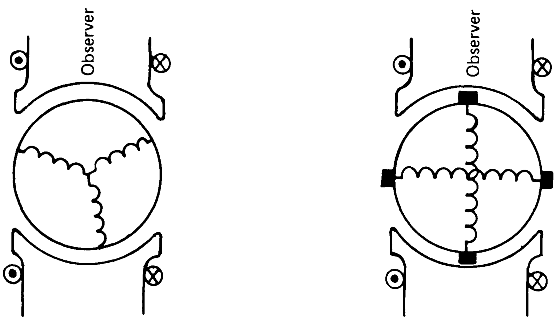

Figure 5.2 Inverted synchronous machine, observer on the stator

conductors we can easily obtain, using Fleming's right hand rule, the direction of the voltage generated in each conductor as in Figure 5.3(b). The varying sizes of the dots and crosses on the diagram indicate the varying magnitudes of the voltages. It is obvious that for the conditions shown, the conductors on the upper half plane of the brushes will carry current into the plane of the paper and those below, out of the paper. The situation can be represented by the current-carrying coil shown in section. The direction of the flux produced by this 'coil' is along the brush axis, and during steady-state operation this flux has a constant magnitude. In other words, the effect of the armature conductors rotating in the presence of a magnetic field may be represented by a fictitious, pseudo-stationary coil connecting two brushes on the commutator, which produces a constant m.m.f. along the brush axis.

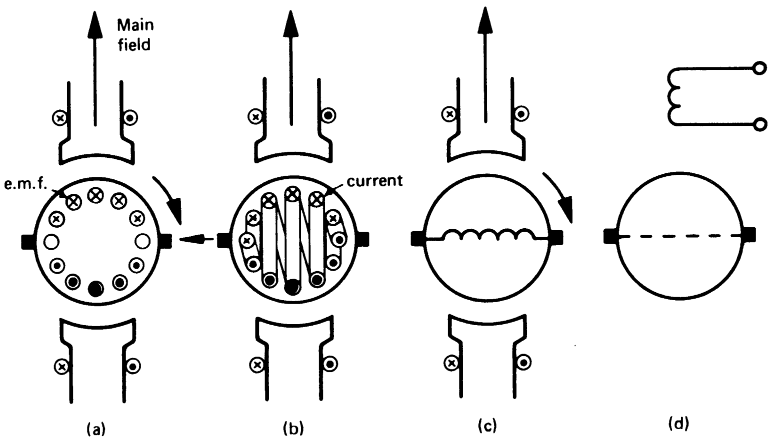

Figure 5.3 Representing the armature winding in a d.c. machine by an equivalent coil

We call this coil fictitious for obvious reasons, and pseudo-stationary because it appears to be stationary to the observer, but, unlike a stationary coil, a direct voltage is generated in it across the brushes, due to the rotation of the armature conductors. The actual rotation is not observed at the brushes because of the action of the commutator. Similarly we can consider the two armature m.m.f. components in the synchronous machine, along the direct and quadrature axes respectively, to be generated by two orthogonal fictitious coils and brushes along these axes.

If there were two field windings on the synchronous machines, one in the pole axis (direct axis) and the other in the interpolar (quadrature) axis there would be another field coil in the q-axis and the resulting machine would be as shown in

Figure 5.4(a). This machine, which looks very much like a crossfield machine (amplidyne, metadyne, etc.) actually represents a synchronous machine with two-axis excitation. It is also what Kron called the 'primitive machine'. All other machines can be represented as special cases of the primitive machine.

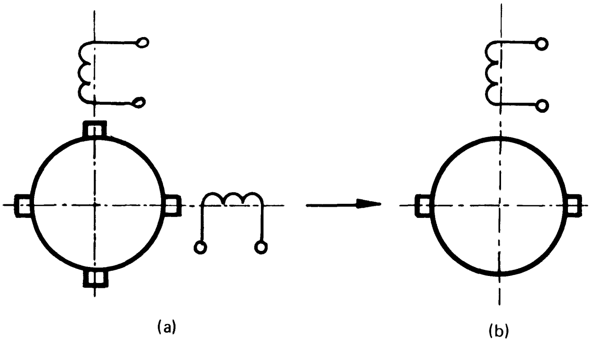

Figure 5.4 (a) 4-coil primitive machine and (b) separately excited d.c. machine

5.3 THE INDUCTION MOTOR AND THE PRIMITIVE MACHINE

The comparison of a synchronous machine with an equivalent 'primitive' or a commutator machine is relatively straightforward because in this case we consider that the rotor rotates at synchronous speed. We shall now show that an induction motor can also be represented by the same form of primitive machine. (The analysis of an induction motor by Stanley,[7] gives an excellent description of the use of stationary reference axes in this context.)

We shall consider the case of a 2-pole 3-phase induction motor. We have seen previously that balanced 3-phase voltages applied to the stator windings of this machine produce a magnetic field of constant magnitude, rotating at synchronous speed in space. The rotor rotates at a speed $\omega = \omega_s(1 - s)$ mechanical radian/second. In this case $\omega_s = 2\pi f$, where f is the frequency of the supply to the stator. The frequency of the rotor currents is sf.

If the observer 'sits on the rotor', the m.m.f. vector produced by the rotor currents rotates at an angular velocity $\omega' = 2\pi s f$ mechanical radian/second with respect to his position. To an observer on the stator, the rotor m.m.f. will appear to rotate at a frequency $\omega' + 2\pi f(1 - s) = 2\pi f$. Since the stator m.m.f. also rotates at an angular frequency $2\pi f$, the stator and rotor m.m.f. waves are again locked and stationary *with respect to each other*, while rotating synchronously at angular velocity ω_s in space. We can consider the primitive machine with stationary axes on both stator and rotor as a model for the induction motor, as in Figure 5.5(a). The identical orthogonal direct and quadrature axis stator coils will be fed by currents I sin ωt and I cos ωt respectively, the resultant

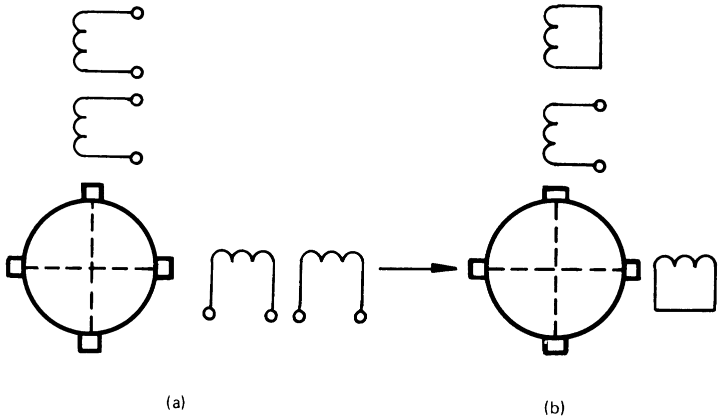

Figure 5.5 (a) 6-coil primitive machine and (b) synchronous machine with amortisseur windings

m.m.f. wave rotating at synchronous angular velocity $\omega_s = 2\pi f$. The rotor axes now carry orthogonal fictitious direct and quadrature coils with voltage and current induced by the stator at supply frequency but now these fictitious coils also have voltages *generated* in them due to the rotation of the air gap field. The induced and generated voltages add to give the physical current and voltage at slip frequency under balanced conditions, as we shall see in Section 5.9. In this reference frame the self and mutual inductances of the stator and rotor coils are all now constant and not time dependent.

5.3.1 Slip-ring reference axes

Along these reference axes, the rotor observer has continuous access to a rotating coil through the slip rings, representing a situation in which he 'sits on the rotating coil'. In a commonly used hybrid reference system, the observer measures the voltages and current of the stator coils from direct axes fixed to one stator phase, with an identical quadrature axis coil. He measures the voltage and current in the rotor coils through the slip rings. In this system the self inductances and resistances on the stator and the mutual inductances between stator phases are all constants. In this case, however, the mutual inductances between rotor and stator coils are sinusoidal functions of the angle of rotation of the rotor.

5.4 PRIMITIVE FORM OF VARIOUS MACHINES

The primitive machine which has been described in Section 5.2 has only four coils. Depending on the type of machine we wish to investigate, we can remove or add some coils (mathematically we would make the appropriate changes in the number of rows and columns of the matrix equations). This will be discussed further in Section 5.5.

Let us, for example, take a separately excited d.c. machine. This would have only two coils (Figure 5.4b) one along the direct-axis stator (coil ds) and the other on the quadrature axis of the rotor (coil qr). We simply 'erase' the coils dr and qs.

If we have a compound wound d.c. machine, there are two (field) coils on the direct axis of the stator, one of which is in series with the coil on the quadrature axis of the rotor (the armature). The coils dr and qs are again absent. If interpoles are provided on the machine to assist commutation, these take the form of coils in the quadrature axis, also in series with the armature. The coil qs is then retained in the equations, with its resistance and self and mutual inductance terms.

In a similar manner, a synchronous machine with amortisseur windings would have five coils, as shown in Figure 5.5 (b). We simply add two extra coils, one along the d-axis (stator) ds1 and the other along the q-axis (stator) qs1, and these are short-circuited. These two windings represent the amortisseur windings which may be either actual damper bars, as in large salient pole machines, or their equivalent due to eddy currents in a solid-iron rotor.

So far we have added coils only on the stator axes, but additional coils can just as easily be placed on the rotor axes. Let us take the example of an induction motor with a double cage. The rotors of these machines have two cages instead of one to facilitate starting, and can be represented by a primitive machine as shown in Figure 5.6. So instead of four coils, a double-cage induction motor would need six coils for its representation. In this manner it may be shown that all machines can be represented by one primitive machine, by adding or omitting certain windings.

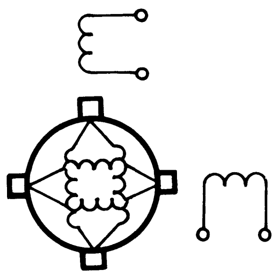

Figure 5.6 Primitive form of a double-cage induction motor

It is now required to obtain the voltage/torque equation for the primitive machine. The next step will be to show how the performance equations of different machines may be obtained from the primitive-machine equations.

In obtaining the primitive-machine equations, most authors use the rigorous 3-phase to 2-phase transformation followed by slip-ring-to-commutator matrix transformation. We shall adopt a more straightforward method and derive the voltage and torque equations directly.

5.5 PRIMITIVE MACHINE EQUATIONS ALONG STATIONARY AXES

Assume that there are two magnetic fields set up by the stator, along the direct and quadrature axes respectively. These fields are produced by the contributions from the windings along each axis. The magnetic fields are time-invariant when all the coils carry direct current, but that is a very special case (for instance, a synchronous machine during steady-state operation). We shall consider the most general case in which the fields are functions of time.

The voltage induced in a stator coil will be $-\mathrm{d}\psi/\mathrm{d}t$ (to be called the transformer voltage) and the resistance drop in voltage is Ri, so that the applied voltage has to overcome both these. Hence, $V = Ri + \mathrm{d}\psi/\mathrm{d}t$ for a stator coil.

For the armature coils, the situation is a little different. A fluctuating magnetic field will produce a transformer voltage as before, but there will also be generated voltages in the rotor conductors when they rotate with respect to the magnetic field. Hence the equation for the applied voltage in a rotor coil will contain the terms

$$\begin{array}{llll} \mathbf{V} & = \mathbf{Ri} & + \mathrm{d}\boldsymbol{\psi}/\mathrm{d}t & + \mathbf{B}p\theta \\ \text{app.voltage} & = \text{res.} & + \text{transformer} & + \text{generated} \\ & \quad\text{drop} & \quad\text{voltage} & \quad\text{voltage} \end{array} \tag{5.1}$$

Here $\mathbf{B}p\theta$ is the generated voltage, which depends on the speed of the rotor and the flux density B of the magnetic field in which the coils rotate. While calculating the transformer voltage, we assume that the conductors are stationary and the flux is changing. We then consider the flux to be constant and the conductors rotating with respect to it, in order to obtain the generated voltage.

5.5.1 Flux linkages

Let us again take a very general case, in which there are two coils on each axis of the stator, and one coil each on the direct and quadrature axes of the armature. This is represented in Figure 5.5(a). All these coils may not be necessary for a particular machine, in which case the appropriate coils are simply omitted and the equations are modified by deleting the corresponding rows and columns in the matrix equations.

In selecting a nomenclature for the various coils and their parameters, the following method has been adopted. The letters d and q obviously stand for direct and quadrature axes, s for stator coils, r for rotor coils; M_{d1} is the mutual inductance between coil 1 on the stator and the rotor coil along the direct axis, M_{d12} the mutual inductance between coils 1 and 2 in the stator along the direct axis, and so on.

Let us now write down the flux linkages of each of these coils. Each coil will embrace all the flux produced by itself, which is given by the product of its self-inductance and the current carried by the coil. It will also embrace fluxes produced by coils along its own axis. In this case, the product of the mutual inductance and the currents carried by the other coils determines the flux linkages. For instance, if we take the coil ds1, it will link all the flux lines produced by itself, so its own flux-linkage will be $L_{ds1}i_{ds1}$. Some flux lines produced by dr will not link ds1 (these constitute the leakage flux), but the majority will do so. Since these lines link both ds1 and dr they constitute the mutual flux given by $M_{d1}i_{dr}$ where M_{d1} is the mutual inductance between these coils. The mutual flux may either reinforce the flux produced by the coil ds1 or oppose it, and we have to use either a positive or negative sign to represent this situation. It is convenient to assume that the fluxes aid each other and proceed on that basis. When the equations are solved, some of the currents are likely to turn out to be negative. This indicates that the fluxes produced by these currents have directions opposite to those originally assumed.

On the basis of the above discussion, we can now write the flux-linkage equations for the coils of the primitive machine (Figure 5.5a)

$$\psi_{dr} = L_{dr}i_{dr} + M_{d1}i_{ds1} + M_{d2}i_{ds2} \tag{5.2a}$$

$$\psi_{qr} = L_{qr}i_{qr} + M_{q1}i_{qs1} + M_{q2}i_{qs2} \tag{5.2b}$$

$$\psi_{ds1} = L_{ds1}i_{ds1} + M_{d1}i_{dr} + M_{d12}i_{ds2} \tag{5.2c}$$

$$\psi_{qs1} = L_{qs1}i_{qs1} + M_{q1}i_{qr} + M_{q12}i_{qs2} \tag{5.2d}$$

$$\psi_{ds2} = L_{ds2}i_{ds2} + M_{d2}i_{dr} + M_{d12}i_{ds1} \tag{5.2e}$$

$$\psi_{qs2} = L_{qs2}i_{qs2} + M_{q2}i_{qr} + M_{q12}i_{qs1} \tag{5.2f}$$

5.6 STATOR AND ROTOR VOLTAGE EQUATIONS

If V_{ds1} is the voltage applied to the coil ds1, this has to overcome the resistance drop and the voltage induced in coil ds1 due to change in the flux-linkage (the transformer voltage). Since this coil is stationary in space, there cannot be any generated voltage in it.
Hence

$$V_{ds1} = R_{ds1}i_{ds1} + \frac{d\psi_{ds1}}{dt} \tag{5.3}$$

or

$$V_{ds1} = R_{ds1}i_{ds1} + L_{ds1}pi_{ds1} + M_{d1}pi_{dr} + M_{d12}pi_{ds2} \tag{5.4a}$$

In a similar manner, we can obtain the voltage equations for all the other stator coils. These are

$$V_{ds2} = R_{ds2}i_{ds2} + L_{ds2}pi_{ds2} + M_{d2}pi_{dr} + M_{d12}pi_{ds1} \tag{5.4b}$$

$$V_{qs1} = R_{qs1}i_{qs1} + L_{qs1}pi_{qs1} + M_{q1}pi_{qr} + M_{q12}pi_{qs2} \tag{5.4c}$$

$$V_{qs2} = R_{qs2}i_{qs2} + L_{qs2}pi_{qs2} + M_{q2}pi_{qr} + M_{q12}pi_{qs1} \tag{5.4d}$$

As we have already seen, the rotor voltages have a generated voltage component (in addition to the induced voltage and resistance drop) due to the rotation of the armature conductors. The voltage generated along the d-axis brushes is clearly due to the flux along the q-axis, but before we can write the equation for the rotor coils, it is necessary to ascertain whether or not the generated voltage has the same sign as the transformer voltage. A simple graphical method may be used once again to determine the signs of the voltages.

Let us consider the coil dr, and assume that the coils ds1 and ds2 are one-turn coils which carry current in the directions indicated in Figure 5.7, such that the fluxes will be in the direction shown. Since the flux produced and linked by coil dr (namely $L_{dr}i_{dr}$) is assumed to aid the fluxes produced by the stator coils, the direction of i_{dr} due to the impressed voltage V_{dr} is as indicated. When the d-axis flux changes with time a transformer voltage will be induced in coil dr in a direction that tends to oppose the applied voltage and produces flux in the direction shown in Figure 5.7(b). We can, for the sake of convenience, represent the direction of the voltage by the direction of this flux.

Let us now turn to the generated voltage in coil dr. The generated voltage across the d-axis brushes is produced by the q-axis flux. The assumed direction of the q-axis flux and rotation are as shown in Figure 5.7(c). Using Fleming's right-hand rule for generated voltage (or back e.m.f.), we obtain the direction of the generated voltage shown in Figure 5.7(d) [comparing Figure 5.7(b) and 5.7(d) we can see that the generated and transformer voltages in coil dr are in the same direction, for the given direction of rotation]. If we repeat the same exercise for the coil qr, we can see that the generated and transformer voltages are in opposite directions. [In Figure 5.7(b) and 5.7(d) the dots and crosses appear on the same side of the rotor. In Figure 5.7(f) and 5.7(h) they appear on opposite sides]. There may be, however, a *time* phase difference between the induced and generated voltages.

When the flux wave is sinusoidally distributed round the air gap, equivalent inductance coefficients giving rise to *induced* and *generated* voltages in a rotating coil, are identical. These voltages are, however, produced along different axes. With non-sinusoidal flux distribution, as in a d.c. machine, the generated voltage coefficients M_d', M_q', L_d', L_q' are not respectively equal to the induced voltage coefficients M_d, M_q, L_d, L_q. With d.c. machines the generated voltage coefficients can, of course, be measured directly by simple tests. With sinusoidal flux distribution, as in most a.c. machines, the generated and induced voltage coefficients can be assumed to be equal.

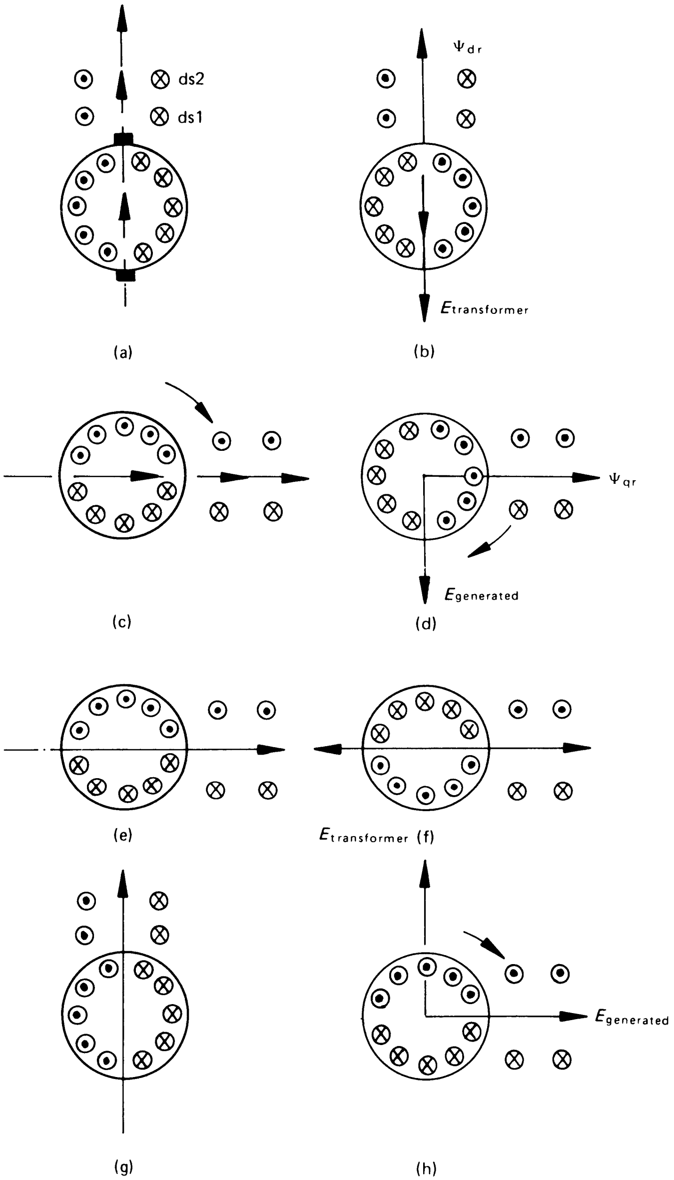

Figure 5.7 Directions of transformer and generated voltages in a primitive machine

The expressions for the applied voltages V_{dr} and V_{qr} are therefore:

$$V_{dr} = R_{dr}i_{dr} + \frac{d\psi_{d2}}{dt} + B_{qr}p\theta \tag{5.5a}$$

and

$$V_{qr} = R_{qr}i_{qr} + \frac{d\psi_{qr}}{dt} - B_{dr}p\theta \tag{5.5b}$$

where the flux *density* terms are

$$B_{qr} = L'_{qr}i_{qr} + M'_{q1}i_{qs1} + M'_{q2}i_{qs2} \tag{5.6a}$$

$$B_{dr} = L'_{dr}i_{dr} + M'_{d1}i_{ds1} + M'_{d2}i_{ds2} \tag{5.6b}$$

We can now expand equation (5.5) by substituting the values of ψ and B, giving

$$\begin{aligned} V_{dr} = {} & R_{dr}i_{dr} + L_{dr}pi_{dr} + M_{d1}pi_{ds1} + M_{d2}pi_{ds2} \\ & + L'_{qr}i_{qr}p\theta + M'_{q1}i_{qs1}p\theta + M'_{q2}i_{qs2}p\theta \end{aligned} \tag{5.7a}$$

$$\begin{aligned} V_{qr} = {} & R_{qr}i_{qr} + L_{qr}pi_{q2} + M_{q1}pi_{qs1} + M_{q2}pi_{qs2} \\ & - L'_{dr}i_{dr}p\theta - M'_{d1}i_{ds1}p\theta - M'_{d2}i_{ds2}p\theta \end{aligned} \tag{5.7b}$$

It will be noted that $L_{dr} \neq L_{qr}$, especially when we are dealing with salient-pole machines, in which the magnetic path along the d axis is different from that along the q axis.

We can now collect all the voltage equations and arrange them in a matrix form as shown in equation (5.8) (see p. 128).

The voltage equations for the machine then have the form

$$[\mathbf{V}] = [\mathbf{Z}][\mathbf{i}] \tag{5.9a}$$

or

$$[\mathbf{V}] = [\mathbf{R}][\mathbf{i}] + [\mathbf{L}]p[\mathbf{i}] + [\mathbf{G}][\mathbf{i}]p\theta \tag{5.9b}$$

or

$$\mathbf{V} = \mathbf{Ri} + \mathbf{L}p\mathbf{i} + \mathbf{Gi}p\theta \tag{5.9c}$$

where

$$\mathbf{R} = \begin{array}{c|cccccc|} & \text{ds1} & \text{ds2} & \text{dr} & \text{qr} & \text{qs1} & \text{qs2} \\ \hline \text{ds1} & R_{ds1} & & & & & \\ \text{ds2} & & R_{ds2} & & & & \\ \text{dr} & & & R_r & & & \\ \text{qr} & & & & R_r & & \\ \text{qs1} & & & & & R_{qs1} & \\ \text{qs2} & & & & & & R_{qs2} \\ \hline \end{array} \tag{5.10}$$

$$
\begin{array}{c}
\\
\left[\begin{array}{c} V_{ds1} \\ V_{ds2} \\ V_{dr} \\ V_{qr} \\ V_{qs1} \\ V_{qs2} \end{array}\right]
\end{array}
=
\begin{array}{c}
\\
\begin{array}{c} ds1 \\ ds2 \\ dr \\ qr \\ qs1 \\ qs2 \end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \quad ds1 \quad & \quad ds2 \quad & \quad dr \quad & \quad qr \quad & \quad qs1 \quad & \quad qs2 \quad \end{array} \\
\left[\begin{array}{cccccc}
R_{ds1}+L_{ds1}p & M_{d12}p & M_{d1}p & & & \\
M_{d12}p & R_{ds2}+L_{ds2}p & M_{d2}p & & & \\
M_{d1}p & M_{d2}p & R_r+L_{dr}p & L'_{qr}p\theta & M'_{q1}p\theta & M'_{q2}p\theta \\
-M'_{d1}p\theta & -M'_{d2}p\theta & -L'_{dr}p\theta & R_r+L_{qr}p & M_{q1}p & M_{q2}p \\
 & & & M_{q1}p & R_{qs}+L_{qs1}p & M_{q12}p \\
 & & & M_{q2}p & M_{q12}p & R_{qs2}+L_{qs2}p
\end{array}\right]
\end{array}
\cdot
\begin{array}{c}
\\
\left[\begin{array}{c} i_{ds1} \\ i_{ds2} \\ i_{dr} \\ i_{qr} \\ i_{qs1} \\ i_{qs2} \end{array}\right]
\end{array}
\qquad (5.8)
$$

$$\mathbf{L} = \begin{array}{cc} & \begin{array}{cccccc} \mathrm{ds1} & \mathrm{ds2} & \mathrm{dr} & \mathrm{qr} & \mathrm{qs1} & \mathrm{qs2} \end{array} \\ \begin{array}{c} \mathrm{ds1} \\ \mathrm{ds2} \\ \mathrm{dr} \\ \mathrm{qr} \\ \mathrm{qs1} \\ \mathrm{qs2} \end{array} & \begin{bmatrix} L_{\mathrm{ds1}} & M_{\mathrm{d12}} & M_{\mathrm{d1}} & & & \\ M_{\mathrm{d12}} & L_{\mathrm{ds2}} & M_{\mathrm{d2}} & & & \\ M_{\mathrm{d1}} & M_{\mathrm{d2}} & L_{\mathrm{dr}} & & & \\ & & & L_{\mathrm{qr}} & M_{\mathrm{q1}} & M_{\mathrm{q2}} \\ & & & M_{\mathrm{q1}} & L_{\mathrm{qs1}} & M_{\mathrm{q12}} \\ & & & M_{\mathrm{q2}} & M_{\mathrm{q12}} & L_{\mathrm{qs2}} \end{bmatrix} \end{array} \quad (5.11)$$

$$\mathbf{G} = \begin{array}{cc} & \begin{array}{cccccc} \mathrm{ds1} & \mathrm{ds2} & \mathrm{dr} & \mathrm{qr} & \mathrm{qs1} & \mathrm{qs2} \end{array} \\ \begin{array}{c} \mathrm{ds1} \\ \mathrm{ds2} \\ \mathrm{dr} \\ \mathrm{qr} \\ \mathrm{qs1} \\ \mathrm{qs2} \end{array} & \begin{bmatrix} & & & & & \\ & & & & & \\ & & & L'_{\mathrm{qr}} & M'_{\mathrm{q1}} & M'_{\mathrm{q2}} \\ -M'_{\mathrm{d1}} & -M'_{\mathrm{d2}} & -L'_{\mathrm{dr}} & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array} \quad (5.12)$$

and $\mathbf{Z} = \mathbf{R} + \mathbf{L}p + \mathbf{G}p\theta$ gives the impedance matrix. The matrix equation which has been derived above refers to a primitive machine having six coils, four on the stator and two on the rotor. A typical example of such a case is a dual-excitation synchronous machine with amortisseur windings.[8] If, however, we have a machine with fewer coils we can proceed as follows. Consider an induction motor. It has a 3-phase winding on the stator which can be reduced to an equivalent coil along each stator axis, and similarly coils along the rotor axes, with the resultant configuration as in Figure 5.4(a). There are no coils corresponding to ds2 and qs2 because an induction motor has a laminated stator structure. The rotor of a single-cage induction motor can be represented by two coils, so there are no windings corresponding to amortisseurs. The rows and columns corresponding to ds2 and qs2 are eliminated, and the reduced matrix is

$$
\begin{bmatrix} V_{ds1} \\ V_{dr} \\ V_{qr} \\ V_{qs1} \end{bmatrix}
=
\begin{array}{c|cccc}
 & ds1 & dr & qr & qs1 \\
\hline
ds1 & R_{ds1}+L_{ds1}p & M_{d1}p & & \\
dr & M_{d1}p & R_r+L_{dr}p & L_{qr}p\theta & M_{q1}p\theta \\
qr & -M_{d1}p\theta & -L_{dr}p\theta & R_r+L_{qr}p & M_{q1}p \\
qs1 & & & M_{q1}p & R_{qs1}+L_{qs1}p
\end{array}
\cdot
\begin{bmatrix} i_{ds1} \\ i_{dr} \\ i_{qr} \\ i_{qs1} \end{bmatrix}
\tag{5.13}
$$

5.6.1 The torque equation

The performance of an electrical machine is also specified by the torque equation. In the case of a motor, electrical power is applied to the machine terminals, and the electromagnetic flux and currents interact to produce torque, which drives the motor.

Taking the condensed form of voltage equation, we have

$$\mathbf{V} = \mathbf{Z}\mathbf{i}$$

or

$$\mathbf{V} = \mathbf{R}\mathbf{i} + \mathbf{L}p\mathbf{i} + \mathbf{G}\mathbf{i}p\theta$$

We pre-multiply this by $\mathbf{i}_t^*$ (the transpose of the complex conjugate of currents $\mathbf{i}$) and the product gives the power input to the motor as

$$\mathbf{i}_t^*\mathbf{V} = \mathbf{i}_t^*\mathbf{R}\mathbf{i} + \mathbf{i}_t^*\mathbf{L}p\mathbf{i} + \mathbf{i}_t^*\mathbf{G}\mathbf{i}p\theta \tag{5.14}$$

Here

(1) $\mathbf{i}_t^*\mathbf{R}\mathbf{i}$ is dissipated as copper loss

(2) $\mathbf{i}_t^*\mathbf{L}p\mathbf{i}$ is the rate of change of stored magnetic energy

(since $\frac{d}{dt}(\frac{1}{2}\mathbf{L}\mathbf{i}^2) = \mathbf{i}\mathbf{L}p\mathbf{i}$ if $\mathbf{L}$ is constant).

The input power first provides items (1) and (2), and the remainder appears as mechanical power at the output shaft of the machine.

Hence
$$P_m = \mathbf{i}_t^*\mathbf{G}\mathbf{i}p\theta \tag{5.15}$$

Since $P_m = \omega T = Tp\theta$,

$$T_o = \mathbf{i}_t^*\mathbf{G}\mathbf{i} \mp \mathbf{i}_t^*\mathbf{B} \tag{5.16}$$

where T_o is the output torque for motor operation, and $\mathbf{G}$ is the torque matrix. In the last term $\mathbf{B}$ represents flux density in the air gap. For generator operation, the situation is reversed. Mechanical input power is now converted into electrical output power and the rotor accelerates until a steady state is reached. If T_i represents the mechanical input torque, then during steady-state conditions the sum of input and output power must be zero, provided that we neglect the loss. That is

$$T_i + T_o = 0 \tag{5.17}$$

or

$$T_i = -T_o = -\mathbf{i}_t^* \mathbf{G}\mathbf{i} \tag{5.18}$$

When the rotor accelerates and the various losses are taken into account

$$T_i + T_o = Jp^2 + R_F p\theta \tag{5.19}$$

where R_F is a mechanical coefficient representing dissipation due to friction and windage and J is the moment of inertia of the rotating mass (kg m^2). The complete dynamic equation is, therefore,

$$T_i = Jp^2\theta + R_F p\theta - \mathbf{i}_t^* \mathbf{G}\mathbf{i} \tag{5.20}$$

We can now combine the voltage and torque equations and write the complete motional impedance matrix. It should be emphasised that the convention which we have adopted is that the voltages and torque in the left hand column are all *impressed* values.

$$\begin{array}{c} \\ \text{elect.} \\ \text{mech.} \end{array}
\begin{array}{c} \\ \left[\begin{array}{c} \mathbf{V} \\ T_i \end{array}\right] \end{array}
\begin{array}{c} \\ = \end{array}
\begin{array}{cc} \text{elect.} & \text{mech.} \\ \multicolumn{2}{c}{\left[\begin{array}{cc} \mathbf{R}+\mathbf{L}p & \mathbf{G}\mathbf{i} \\ -\mathbf{i}_t^* \mathbf{G} & Jp + R_F \end{array}\right]} \end{array}
\begin{array}{c} \\ \cdot \left[\begin{array}{c} \mathbf{i} \\ \omega \end{array}\right] \end{array}
\tag{5.21}$$

For a primitive machine with four coils, the expanded voltage/torque equation is:

$$\left[\begin{array}{c} V_{ds1} \\ V_{dr} \\ V_{qr} \\ V_{qs1} \\ T_i \end{array}\right]
=
\begin{array}{c|ccccc}
 & ds1 & dr & qr & qs1 & s \\
\hline
ds1 & R_{ds1}+L_{ds1}p & M_{d1}p & & & \\
dr & M_{d1}p & R_r + L_{dr}p & & & L'_{qr}i_{qr} + M'_{q1}i_{qs1} \\
qr & & & R_r + L_{qr}p & M_{q1}p & -M'_{d1}i_{ds1} - L'_{dr}i_{dr} \\
qs1 & & & M_{q1}p & R_r + L_{qs1}p & \\
s & i_{qr}M'_{d1} & -i_{qr}L'_{dr} & i_{dr}L'_{qr} & i_{dr}M'_{q1} & Jp + R_F
\end{array}
\cdot
\left[\begin{array}{c} i_{ds1} \\ i_{dr} \\ i_{qr} \\ i_{qs1} \\ p\theta \end{array}\right]
\tag{5.22}$$

The sign conventions used by various authors vary so widely that they result in a great deal of confusion. These differences are particularly noticeable in synchronous machine analysis. There are clearly two major groups—the majority following Park's conventions and the others using Kron's—in the equations for synchronous generators. In this section we shall show that by making some minor changes in the generalised equations (5.8) we can arrive at Park's original equations, and thus hope to avoid any confusion.

First, we must remember that the equations we have derived are applicable either to motors or generators, whereas Park's equations are derived specifically for generators. In equations (5.8) *all* voltages are impressed, whereas in a synchronous generator the voltages in the field windings are *impressed* and those in the armature coils are *generated* voltages. We can, however, change equations (5.8) to represent a generator simply by reversing the components of the armature current, viz i_{dr} and i_{qr}. The directions of the stator field currents will be unchanged because in both motor and generator operation these currents are produced by applied voltages (of course in synchronous machines ds2 and qs2 represent amortisseur windings which are fictitious short-circuited coils, with no applied voltages).

There is yet another difference between the conventions of Park and Kron. Synchronous machine engineers normally assume a clockwise rotation of the rotor carrying the field winding. If we invert the synchronous machine, putting the armature on the rotor and the field on the stator, as in Kron's primitive machine, the corresponding direction of rotation of the armature is counter clockwise. Equations (5.8) were derived on the assumption that the armature rotates in a clockwise direction. Hence in Park's equations the term $p\theta$ would be replaced by $-p\theta$. If we incorporate these changes, i.e. change the signs of i_{dr}, i_{qr}, and $p\theta$, Park's equations follow from those of Kron.

5.7 TRANSFORMATION OF REFERENCE FRAMES[1]

Having obtained the equations for a six-coil primitive machine, we have to find a method of using these to investigate specific machines. In other words, we have to find a method of transforming the performance equations of the primitive into those of a specific machine.

If we consider a synchronous machine with amortisseur windings for example, we simply remove the column and row qs2 and we have the necessary equations. For a turbo-generator with dual excitation,[8] no change is required in the primitive machine equations.

If, however, we take a compound wound d.c. machine, we proceed as follows.

First we represent the machine with its coils in their correct positions. It should be remembered that the flux produced by the compounding coil (which carries the armature or load current) is set up along the same axis as the main field. Hence the correct representation is as shown in Figure 5.8 with both stator coils on the d axis.

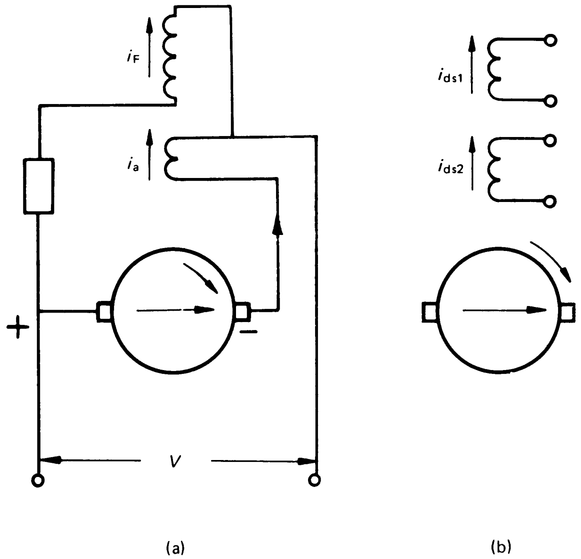

Figure 5.8 Compound-wound machine and its primitive representation

There are only three coils in this machine, ds1, ds2, and qr. The matrix rows and columns corresponding to qs1, qs2, and dr are now removed leaving the equations for a primitive form of this machine. The equations are not yet those for the compound-wound d.c. machine in which the coils ds1, ds2, and qr are connected as shown in Figure 5.8. The connection constraints have to be introduced in the transformation to the circuits of the compound-wound d.c. machine.

We shall derive the equations in Section 5.8 but first we must look in some detail at the techniques and implications of matrix transformations.

5.7.1 Matrix transformations

In Figure 5.9 let X_1 and X_2 represent a Cartesian coordinate system. Let a vector be represented by OA. This can be represented by its components x_1 and x_2 along the axes X_1 and X_2 respectively, and can also be represented in polar coordinates. Let the length of the vector be r, at an angle θ with respect to the X_1 axis. Now

$$x_1 = r\cos\theta \tag{5.23a}$$

$$x_2 = r\sin\theta \tag{5.23b}$$

If the vector undergoes a small increment we can write

$$\Delta x_1 = -r(\sin\theta)\Delta\theta + (\cos\theta)\Delta r \tag{5.24a}$$

$$\Delta x_2 = r(\cos\theta)\Delta\theta + (\sin\theta)\Delta r \tag{5.24b}$$

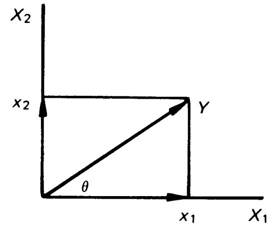

Figure 5.9 Geometrical representation of passive transformation

or in matrix form

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} \cdot \begin{bmatrix} r\Delta\theta \\ \Delta r \end{bmatrix} \tag{5.25}$$

Here we can see that the matrix

$$\begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

transforms the $\Delta\mathbf{x}$ quantities in the X_1X_2 coordinate system to coordinate variables in the $\mathbf{r}$, $\boldsymbol{\theta}$ system. In other words, it acts as the connection between the two systems and is therefore called the connection matrix. In an electromechanical system we take charge q and rotor displacement θ as basic coordinates. If we wish to change a three-phase system to a two-phase system we can relate the currents (i.e. derivative of the charge-coordinates) in one system to the currents in the other with the help of a connection matrix. In the geometrical example quoted, the invariance of the length of the vector is implicit. In the latter case the power is invariant.

Let us take another simple geometrical example to illustrate the principle of active transformation (Figure 5.10). If we have a three-dimensional space represented by Cartesian coordinates X_1, X_2, X_3, the movement of a particle in this space can be obtained by measuring its displacement Δx_1, Δx_2, and Δx_3 along three coordinates. If $\Delta x_1, \Delta x_2$, and Δx_3 are free to assume random values then the particle moves in a random fashion, its net displacement being given by

$$\sqrt{\Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2}.$$

Now let us introduce a constraint to this movement namely that $\Delta x_1 = \Delta x_2$. If the particle starts from say the origin and moves, subject to this constraint, it

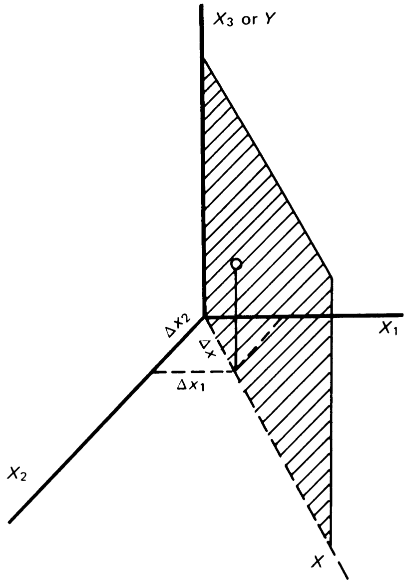

Figure 5.10 Geometrical representation of active transformation

can be seen that it moves along a plane—a two-dimensional space—through the X_3-axis, making an angle of 45° to both X_1 and X_2. We can now describe the motion of the particle on a simple $X\,Y$ plane. In order to transform the equation of motion from the general 3-dimensional space to the particular 2-dimensional space given by the constraint we can write the following relationships

$$\Delta x_1 = \frac{\Delta x}{\sqrt{2}} \tag{5.26a}$$

$$\Delta x_2 = \frac{\Delta x}{\sqrt{2}} \tag{5.26b}$$

$$\Delta x_3 = \Delta y \tag{5.26c}$$

or

$$\begin{array}{|c|}\hline \Delta x_1 \\ \hline \Delta x_2 \\ \hline \Delta x_3 \\ \hline \end{array} = \begin{array}{|c|c|}\hline \frac{1}{\sqrt{2}} & \\ \hline \frac{1}{\sqrt{2}} & \\ \hline & 1 \\ \hline \end{array} \begin{array}{|c|}\hline \Delta x \\ \hline \Delta y \\ \hline \end{array} \tag{5.27}$$

The connection matrix, therefore, relates the general or the primitive system to a system subject to specific constraints.

We shall now proceed to find an electrical analogue of such transformations.

5.7.2 An electrical circuit

Let us take a simple two-loop network (Figure 5.11). We can divide the network into its circuit elements or the 'primitive' elements.

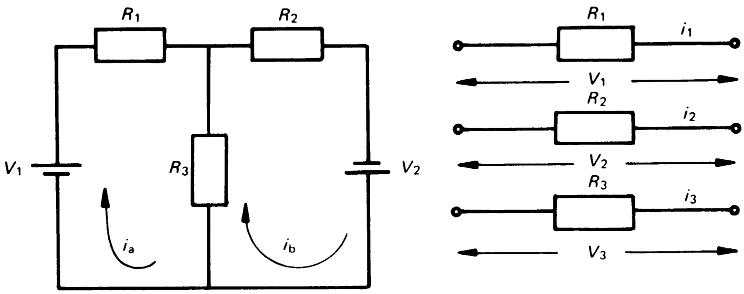

Figure 5.11 Simple two-loop network and its primitive elements

The equations describing the response of these elements to the applied voltages follow from Ohm's law, viz,

$$V_1 = R_1 i_1 \tag{5.28a}$$

$$V_2 = R_2 i_2 \tag{5.28b}$$

$$V_3 = R_3 i_3 \tag{5.28c}$$

In matrix forms these equations can be arranged as follows:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} R_1 & & \\ & R_2 & \\ & & R_3 \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \tag{5.29}$$

These are quite general equations.

If we now join the elements to represent the given network, we introduce certain constraints, as follows:

$$i_1 = i_a$$

$$i_2 = i_b$$

$$i_3 = i_a - i_b$$

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} \mathrm{a} & \mathrm{b} \end{array} \\ \begin{bmatrix} 1 & \\ & 1 \\ 1 & -1 \end{bmatrix} \end{array} \begin{bmatrix} i_a \\ i_b \end{bmatrix} \tag{5.30}$$

The 2 × 3 matrix is the 'connection' between the two systems, the primitive (or the old) and the new, and gives the connection matrix **C**. The manner in which this matrix is used is as follows:

$$\mathbf{i}^{\text{old}} = \mathbf{C}^{\text{old}}_{\text{new}}\,\mathbf{i}^{\text{new}} \tag{5.31}$$

or

$$\mathbf{i} = \mathbf{C}\mathbf{i}' \tag{5.32}$$

If we take the tranpose of both sides, we have $\mathbf{i}_t = \mathbf{i}'_t\mathbf{C}_t$. Taking the complex conjugate of both sides, $\mathbf{i}^*_t = (\mathbf{i}'_t)^*\mathbf{C}^*_t$. We shall now proceed to obtain the impedance matrix for the constrained system, namely our given network. The condition of power invariance is postulated. In other words, the power dissipated in the connected network must equal the power dissipated in the primitive system. The active and reactive vector power in an electrical system is given by

$$\begin{aligned} P &= \mathbf{i}^*_t\mathbf{Z}\mathbf{i} \\ &= (\mathbf{i}^*_t\mathbf{R}\mathbf{i}) \pm \mathrm{j}(\mathbf{i}^*_t\mathbf{X}\mathbf{i}) \\ &= P \pm \mathrm{j}Q \end{aligned} \tag{5.33}$$

Here $\mathbf{X} = \omega\mathbf{L}$

By substitution, the power equation becomes

$$P = (\mathbf{i}'_t)^*\,(\mathbf{C}^*_t\mathbf{Z}\mathbf{C})\mathbf{i}' \tag{5.34}$$

or,

$$P = (\mathbf{i}'_t)^*\mathbf{Z}'\mathbf{i}' \tag{5.35}$$

where

$$\mathbf{Z}' = \mathbf{C}^*_t\mathbf{Z}\mathbf{C} \tag{5.36}$$

In order to relate the voltages **V** in the old (primitive) network to **V**′ in the new system, we write

$$\mathbf{V} = \mathbf{Z}\mathbf{i} \tag{5.37}$$

$$\mathbf{V}' = \mathbf{Z}'\mathbf{i}' = \mathbf{C}^*_t\mathbf{Z}\mathbf{C}\mathbf{i}' = \mathbf{C}^*_t\mathbf{Z}\mathbf{i} = \mathbf{C}^*_t\mathbf{V} \tag{5.38}$$

Hence if we relate one system to the other—keeping power as an invariant—the following relations hold

$$(1)\quad \mathbf{i} = \mathbf{C}\mathbf{i}' \tag{5.32}$$

$$(2)\quad \mathbf{V}' = \mathbf{C}_t^*\mathbf{V} \tag{5.38}$$

$$(3)\quad \mathbf{Z}' = \mathbf{C}_t^*\mathbf{Z}\mathbf{C} \tag{5.36}$$

If we deal with d.c. quantities the equations become

$$(1)\quad \mathbf{i} = \mathbf{C}\mathbf{i}'$$

$$(2)\quad \mathbf{V}' = \mathbf{C}_t\mathbf{V}$$

$$(3)\quad \mathbf{Z}' = \mathbf{C}_t\mathbf{Z}\mathbf{C}$$

In the problem in question, let us employ this technique of transformation

$$\mathbf{Z}' = \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline a & 1 & & 1 \\ b & & 1 & -1 \\ \hline \end{array} \cdot \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & R_1 & & \\ 2 & & R_2 & \\ 3 & & & R_3 \\ \hline \end{array} \cdot \begin{array}{c|cc|} & a & b \\ \hline 1 & 1 & \\ 2 & & 1 \\ 3 & 1 & -1 \\ \hline \end{array}$$

$$= \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline a & 1 & & 1 \\ b & & 1 & -1 \\ \hline \end{array} \begin{array}{c|cc|} & & \\ \hline 1 & R_1 & \\ 2 & & R_2 \\ 3 & R_3 & -R_3 \\ \hline \end{array} = \begin{array}{c|cc|} & & \\ \hline a & R_1+R_3 & -R_3 \\ b & -R_3 & R_2+R_3 \\ \hline \end{array} \tag{5.39}$$

$$\mathbf{V}' = \mathbf{C}_t\mathbf{V}$$

$$\begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline a & 1 & & 1 \\ b & & 1 & -1 \\ \hline \end{array} \begin{array}{c|c|} & \\ \hline 1 & V_1 \\ 2 & V_2 \\ 3 & 0 \\ \hline \end{array} = \begin{array}{c|c|} & \\ \hline a & V_a \\ b & V_b \\ \hline \end{array} \tag{5.40}$$

The equation for the network is therefore

$$\begin{array}{c} \\ a \\ b \end{array}\begin{array}{c} \\ \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \end{array} = \begin{array}{c} \\ a \\ b \end{array}\begin{array}{cc} a & b \\ \multicolumn{2}{c}{\begin{bmatrix} R_1+R_3 & -R_3 \\ -R_3 & R_2+R_3 \end{bmatrix}} \end{array} \cdot \begin{bmatrix} i_a \\ i_b \end{bmatrix} \tag{5.41}$$

These trivial equations can of course be directly obtained by loop analysis but the method of impedance transformation has been used here to illustrate the technique.

5.8 THE D.C. MACHINE

We are now in a position to derive the equations for performance of a given machine from those of the primitive machine. Let us again choose the compound-wound generator (Figure 5.8a). The primitive form is represented in Figure 5.8(b). Obviously the primitive or unconnected form of the machine has three coils and hence we eliminate rows and columns for dr and qs from equations (5.13).

The connection matrix is obtained as follows.

Equating the coil currents in both machines,

$$\begin{bmatrix} i_{ds1} \\ i_{ds2} \\ i_{qr} \end{bmatrix} = \begin{array}{c} \\ ds1 \\ ds2 \\ qr \end{array}\begin{array}{cc} f & a \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & \\ & 1 \\ & 1 \end{bmatrix}} \end{array} \cdot \begin{bmatrix} i_f \\ i_a \end{bmatrix} \tag{5.42}$$

In these equations i_f and i_a represent the field and armature currents in the actual machine. Obviously we are introducing a constraint to the primitive machine, i.e. $i_{ds2} = i_a$. We do not consider the number of turns in each winding because we have taken the *measured* values of the various resistances and inductances. The primitive machine impedance matrix is

$$\begin{array}{c} \\ ds1 \\ ds2 \\ qr \end{array}\begin{array}{ccc} ds1 & ds2 & qr \\ \multicolumn{3}{c}{\begin{bmatrix} R_x + R_{ds1} + L_{ds1}p & M_{d12}p & \\ M_{d12}p & R_{ds2} + L_{ds2}p & \\ -M'_{d1}p\theta & -M'_{d2}p\theta & R + L_{qr}p \end{bmatrix}} \end{array} \tag{5.43}$$

In this equation R_x represents a control resistance in series with the field winding.

The mutual inductance between the coils ds1 and ds2 is designated by M_{d12}. The mutual inductance between the field and the rotor quadrature axis, which is responsible for the generated voltage in the coil qr, is M'_{d1}.

We can now make use of the connection matrix for the impedance transformation $\mathbf{Z}' = \mathbf{C}_t\mathbf{Z}\mathbf{C}$, where

$$\mathbf{C}_t = \begin{array}{c} \\ f \\ a \end{array} \begin{array}{ccc} ds1 & ds2 & qr \\ \hline 1 & & \\ & 1 & 1 \end{array}$$

The multiplication is carried out as shown below:

$$\begin{array}{c|ccc} & ds1 & ds2 & qr \\ \hline f & 1 & & \\ a & & 1 & 1 \end{array} \quad \begin{array}{c|ccc} & ds1 & ds2 & qr \\ \hline ds1 & R_x + R_{ds1} + L_{ds1}p & M_{d12}p & \\ ds2 & M_{d12}p & R_{ds2} + L_{ds2}p & \\ qr & -M'_{d1}p\theta & -M'_{d2}p\theta & R_r + L_{qr}p \end{array} \quad . \quad \begin{array}{c|cc} & f & a \\ \hline ds1 & 1 & \\ ds2 & & 1 \\ qr & & 1 \end{array} \tag{5.44}$$

or

$$\begin{array}{c|ccc} & ds1 & ds2 & qr \\ \hline f & 1 & & \\ a & & 1 & 1 \end{array} \quad \begin{array}{c|cc} & f & a \\ \hline ds1 & R_x + R_{ds1} + L_{ds1}p & M_{d12}p \\ ds2 & M_{d12}p & R_{ds2} + L_{ds2}p \\ qr & -M'_{d1}p\theta & R_r + L_{qr}p - M'_{d2}p\theta \end{array} \tag{5.45}$$

giving

$$\mathbf{Z} = \begin{array}{c|cc} & f & a \\ \hline f & R_x + R_{ds1} + L_{ds1}p & M_{d12}p \\ a & M_{d12}p - M'_{d1}p\theta & \begin{array}{l} (R_r + R_{ds2}) \\ \quad + (L_{qr} + L_{ds2})p \\ \quad - M'_{d2}p\theta \end{array} \end{array} \tag{5.46}$$

The 3 × 3 impedance matrix has now been reduced to a 2 × 2 impedance matrix. (In terms of mechanics, one could say that 3 degrees of freedom have been reduced to 2 degrees of freedom because of the constraint mentioned earlier.)

The voltage equation is transformed as follows:

$$\mathbf{C}_t\mathbf{V} = \begin{array}{c|ccc} & \mathrm{ds1} & \mathrm{ds2} & \mathrm{qr} \\ \hline \mathrm{f} & 1 & & \\ \mathrm{a} & & 1 & 1 \end{array} \cdot \begin{array}{c|c} \mathrm{ds1} & V_{\mathrm{ds1}} \\ \mathrm{ds2} & V_{\mathrm{ds2}} \\ \mathrm{qr} & V_{\mathrm{qr}} \end{array}$$

$$= \begin{array}{c|c} \mathrm{f} & V_{\mathrm{ds1}} \\ \mathrm{a} & V_{\mathrm{ds2}} + V_{\mathrm{qr}} \end{array} \tag{5.47}$$

The values of V_{ds1} and $(V_{\mathrm{ds2}} + V_{\mathrm{qr}})$ are known. These are equal to the applied voltage V. Hence the final voltage equation is

$$\begin{bmatrix} V \\ V \end{bmatrix} = \begin{array}{c|cc} & \mathrm{f} & \mathrm{a} \\ \hline \mathrm{f} & R_{\mathrm{x}} + R_{\mathrm{ds1}} + L_{\mathrm{ds1}}p & M_{\mathrm{d12}}p \\ \mathrm{a} & M_{\mathrm{d12}}p - M'_{\mathrm{d1}}p\theta & (R_{\mathrm{r}} + R_{\mathrm{ds2}}) + (L_{\mathrm{qr}} + L_{\mathrm{ds2}})p - M'_{\mathrm{d2}}p\theta \end{array} \cdot \begin{bmatrix} i_{\mathrm{f}} \\ i_{\mathrm{a}} \end{bmatrix} \tag{5.48}$$

This is the transient voltage equation.

For steady-state operation the rate of change of current in this machine is zero, and the operator $p = \mathrm{d}/\mathrm{d}t = 0$.

The steady-state equation for the field current is therefore,

$$V = (R_{\mathrm{x}} + R_{\mathrm{ds}})i_{\mathrm{f}} \tag{5.49}$$

and for the armature:

$$V = -M'_{\mathrm{d1}}p\theta i_{\mathrm{f}} + (R_{\mathrm{r}} + R_{\mathrm{ds}})i_{\mathrm{a}} - M'_{\mathrm{d2}}p\theta i_{\mathrm{a}} \tag{5.50}$$

where $-M'_{\mathrm{d1}}p\theta i_{\mathrm{f}}$ is the generated e.m.f. E due to the main field. In a motor this is the back e.m.f. E_{b} and

$$V = E_{\mathrm{b}} + (R_{\mathrm{r}} + R_{\mathrm{ds}} - \omega M'_{\mathrm{d2}})i_{\mathrm{a}} \tag{5.51}$$

where $\omega = p\theta$ = angular velocity. The term $(-M'_{d2}p\theta i_a)$ gives the series compounding effect.

This equation can be very easily derived directly, from observation. When the machine runs as a generator i_a becomes negative.

The use of the transformation and matrix multiplication to derive the simple equation given above is like using a sledge hammer to crack a nut. However, when we have to predict the transient and steady state response of more difficult machines and complex interconnected systems it is immensely helpful to follow this approach. It should be pointed out that we do not need to perform all the matrix manipulation every time we analyse the performance of a machine. This the computer can do. A general primitive machine impedance equation may be stored (it may be an 8×8 matrix or larger). Instructions are then given to eliminate some rows and columns, depending on the primitive form of the machine to be analysed. The connection matrix is provided, together with instructions for the matrix multiplication $\mathbf{C}_t\mathbf{ZC}$. The computer carries out all the manipulations required and prints out the final answers. Very large systems with many interconnected machines can be handled in this way.

5.9 THE INDUCTION MOTOR

Let us now consider the example of an induction motor in more detail. We have already seen in Section 5.4 that in order to obtain the primitive form of an induction motor (Figure 5.4a) we remove rows and columns corresponding to ds2 and qs2 in equation (5.8). The connection matrix is obviously a unit matrix since the coils are not interconnected and no constraints are imposed. Equation (5.13) was obtained as the transient equation of an induction motor and can be used for calculating the transient currents during starting or sudden change in load.

However, we shall derive the performance equations during normal or steady-state operation with the primary objective of showing how the standard equations of an induction motor and its equivalent circuit (Figure 2.37) follow from the primitive machine equation. We shall do this in several steps in order to clarify the various concepts involved, but in actual computation the results may be obtained directly.

Step 1: We must recall that the currents in the real or fictitious stator and rotor coils are both at supply frequency in the original machine when the supply is to the stator, so we replace the operator p by $j\omega_1$ where $\omega_1 = 2\pi f$, f being the supply frequency. The term $p\theta$ in the equations for the rotor circuit represents the angular velocity of rotation of the rotor. If $p\theta = \omega_r$ and ω_1 stands for the synchronous speed, then obviously

$$\omega_r = \omega_1(1-s) \tag{5.52}$$

Hence equation (5.13) takes the form

$$
\begin{array}{c}
V_{ds1} \\ V_{dr} \\ V_{qr} \\ V_{qs1}
\end{array}
=
\begin{array}{c|cccc|}
 & ds1 & dr & qr & qs1 \\
\hline
ds1 & R_{ds1}+j\omega_1 L_{ds1} & j\omega_1 M_{d1} & & \\
dr & j\omega_1 M_{d1} & R_r + j\omega_1 L_{dr} & \omega_1(1-s)M_{q1} & \omega_1(1-s)M_{q1} \\
qr & -\omega_1(1-s)M_{d1} & -\omega_1(1-s)L_{dr} & R_r + j\omega_1 L_{qr} & j\omega_1 M_{q1} \\
qs1 & & & j\omega_1 M_{q1} & R_{qs1}+j\omega_1 L_{qs1} \\
\hline
\end{array}
\begin{array}{c}
i_{ds1} \\ i_{dr} \\ i_{qr} \\ i_{qs1}
\end{array}
\tag{5.53}
$$

Step 2: The coils ds1 and qs1 are identical, having been obtained from transformation of three similar stator coils which are symmetrically distributed. Coils dr and qr are also identical in every respect because the air gap in an induction motor is uniform and these are also derived from a symmetrical three phase rotor winding or its equivalent in a cage rotor. Hence the following relations obtain

$$
\begin{aligned}
R_{ds1} &= R_{qs1} = R_1, \qquad R_r = R_2 \\
L_{ds1} &= L_{qs1} = L_1, \qquad L_{dr} = L_{qr} = L_2 \\
M_{d1} &= M_{q1} = M \\
M'_{d1} &= M_{d1} = M \\
M'_{q1} &= M_{q1} = M
\end{aligned}
$$

The last relations are true because we can legitimately assume that the flux wave is sinusoidally distributed in space and hence the coefficients of mutual inductance for transformer and generated voltages are the same. Replacing the inductance terms by their equivalent reactances at frequency f hertz

$$
\begin{array}{c}
V_{ds1} \\ V_{dr} \\ V_{qr} \\ V_{qs1}
\end{array}
=
\begin{array}{c|cccc|}
 & ds1 & dr & qr & qs1 \\
\hline
ds1 & R_1 + jX_1 & jX_m & & \\
dr & jX_m & R_2 + jX_2 & (1-s)X_2 & (1-s)X_m \\
qr & -(1-s)X_m & -(1-s)X_2 & R_2 + jX_2 & jX_m \\
qs1 & & & jX_m & R_1 + jX_1 \\
\hline
\end{array}
\begin{array}{c}
i_{ds1} \\ i_{dr} \\ i_{qr} \\ i_{qs1}
\end{array}
\tag{5.54}
$$

Step 3: The coils ds1 and qs1 will carry supply-frequency currents, such that the net m.m.f. they produce rotates at synchronous speed. If the coil ds1 carries a current $I_1 \cos \omega_1 t$, then during balanced operation, the coil qs1 should carry a current $I_1 \sin \omega_1 t$, so that the net m.m.f. produced by these two coils (which are identical and have the same number of turns N_1) is $N_1 (I_1^2 \sin^2 \omega_1 t + I_1^2 \cos \omega_1 t)^{\frac{1}{2}} = N_1 I_1$. This resultant m.m.f. vector is constant and rotates with an angular velocity ω_1. If the current i_{ds1} is represented by a vector, I_1, then the current i_{qs1} should also be $|I_1|$ lagging the former by $\pi/2$ or equal to $-\mathrm{j}I_1$. In a similar manner we can say that if the coil dr carries a current I_2, then the coil qr carries a current $-\mathrm{j}I_2$. If the voltage applied to coil ds1 is V_1, then the voltage across qs1 must be $-\mathrm{j}V_1$ since the two coils have identical impedance and the current in the former leads the current in the latter by $\pi/2$. The rotor voltages are zero since the rotor coils in an induction motor are short-circuited. Equation (5.54) now becomes

$$\begin{bmatrix} V_1 \\ 0 \\ 0 \\ -\mathrm{j}V_1 \end{bmatrix} \begin{array}{c} \mathrm{ds1} \\ = \mathrm{dr} \\ \mathrm{qr} \\ \mathrm{qs1} \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{ds1} & \mathrm{dr} & \mathrm{qr} & \mathrm{qs1} \end{array} \\ \begin{bmatrix} R_1+\mathrm{j}X_1 & \mathrm{j}X_{\mathrm{m}} & & \\ \mathrm{j}X_{\mathrm{m}} & R_2+jX_2 & (1-s)X_2 & (1-s)X_{\mathrm{m}} \\ -(1-s)X_{\mathrm{m}} & -(1-s)X_2 & R_2+\mathrm{j}X_2 & jX_{\mathrm{m}} \\ & & \mathrm{j}X_{\mathrm{m}} & R_1+\mathrm{j}X_1 \end{bmatrix} \end{array} \cdot \begin{bmatrix} I_1 \\ I_2 \\ -\mathrm{j}I_2 \\ -\mathrm{j}I_1 \end{bmatrix} \tag{5.55}$$

If we take the equation for coil dr, we can write

$$\begin{aligned} 0 &= \mathrm{j}X_{\mathrm{m}}I_1 - \mathrm{j}I_1(1-s)X_{\mathrm{m}} + (R_2+\mathrm{j}X_2)I_2 - \mathrm{j}(1-s)X_2 I_2 \\ &= \mathrm{j}I_1 sX_{\mathrm{m}} + (R_2 + \mathrm{j}sX_2)I_2 \\ &= \mathrm{j}I_1 X_{\mathrm{m}} + \left[\frac{R_2}{s} + \mathrm{j}X_2\right] I_2 \end{aligned} \tag{5.56}$$

Similarly the equation for coil qr reduces to

$$0 = \mathrm{j}I_1 X_{\mathrm{m}} + \left[\frac{R_2}{s} + \mathrm{j}X_2\right] I_2 \tag{5.57}$$

Since the equations for ds and qs are the same, and also those for coils dr and qr, we can choose one equation from the stator and one from the rotor and replace the 4×4 matrix equation by a 2×2 matrix equation, and write it as

$$\begin{array}{c} \\ \\ \end{array}\begin{bmatrix} V_1 \\ 0 \end{bmatrix} = \begin{array}{c|cc} & \text{ds1} & \text{dr} \\ \hline \text{ds1} & R_1 + jX_1 & jX_m \\ \text{dr} & jX_m & \dfrac{R_2}{s} + jX_2 \end{array} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{5.58}$$

$$\begin{bmatrix} V_1 \\ 0 \end{bmatrix} = \begin{array}{c|cc} & \text{ds1} & \text{dr} \\ \hline \text{ds1} & R_1 + jX_1 & jX_m \\ \text{dr} & jX_m & \dfrac{(R_2 + jX_2) + R_2(1-s)}{s} \end{array} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{5.59}$$

These equations give the steady state performance of the induction motor. They are very similar to those derived for a transformer (equations 2.44) and we have already seen how from those we could construct an equivalent circuit. In a similar manner equations (5.59) can be modified to give the equivalent circuit of an induction motor. The resistances and reactances of the rotor will then be primed quantities, to account for the turns-ratio of the stator and rotor windings, giving the equivalent circuit shown in Figure 2.37(b). Since direct measurement of rotor parameters is not possible in a cage rotor induction motor, it is standard practice to obtain the total referred values $(R_1 + R_2')$ and $(x_1 + x_2')$ by locked rotor and no-load tests, the equivalent of short circuit and open circuit tests in a transformer. Measurement techniques are described in reference 1.

It is important at this stage to investigate the relationship between the voltages, currents, and impedances in this fictitious stationary-axis primitive of the induction motor and the corresponding parameters of an actual 3-phase induction motor, in terms of its line or phase values.

To begin with, let us consider the voltages and currents in the stator. The power input to the stator of the primitive machine is

$$P_s = V_{ds1}^* I_{ds1} + V_{ds2}^* I_{ds2} \tag{5.60}$$

For balanced operation of the induction motor, we can write (as discussed in step 3)

$$P_s = 2\, V_1^* I_1 \tag{5.61}$$

However, we know that the power input to the stator of a 3-phase induction motor in terms of phase voltage and phase current is

$$P_s = 3 V_p^* I_p \tag{5.62}$$

Since power is invariant,

$$2V_1^* I_1 = 3V_p^* I_p \tag{5.63}$$

Next, we equate the copper-loss in the stator windings in the 3-phase machine and its 'commutator-type' equivalent

$$3I_p^2 R_1 = 2I_1^2 R_1, \tag{5.64}$$

assuming that the resistance $R_{ds1} = R_1$ which is the resistance per phase of the 3-phase induction motor. From this equivalence, we can deduce that

$$I_1 = \sqrt{\frac{3}{2}} I_p \tag{5.65}$$

It follows clearly from the power equation that

$$V_1 = \sqrt{\frac{3}{2}} V_p \tag{5.66}$$

The same relationships hold for the rotor currents and voltages.

In equation (5.55) we can use the phase voltages and currents without introducing any error because the factor $\sqrt{3/2}$ is cancelled. However, we have to be cautious in calculating the torque from the equation $\mathbf{i}_t^* \mathbf{G} \mathbf{i}$ because both $\mathbf{i}_t$ and $\mathbf{i}$ obtained in terms of phase current are $\sqrt{3/2}$ times their equivalents in the commutator primitive machine, and the latter product results in a quantity which is 2/3 of the actual torque. The values per phase are equal in both machines.

The values of resistances and reactances are the same as those that are obtained from standard tests on a 3-phase induction motor. A detailed analysis and discussion of the relationships of the measured values of a 3-phase machine and its 3-phase commutator equivalent is given in reference 1.

5.10 REPRESENTATION OF THE TRANSIENT EQUATIONS IN STATE-VARIABLE FORM

We have obtained transient equations for various machines in the form $\mathbf{V} = \mathbf{Z}\mathbf{i} = (\mathbf{R} + \mathbf{L}p + \mathbf{G}p\theta)\mathbf{i}$, where each term is a matrix. The solution for currents $\mathbf{i}$ is obviously

$$\mathbf{i} = \mathbf{Z}^{-1} \mathbf{V} \tag{5.67}$$

From the point of view of actual computation the above form presents difficulties because $\mathbf{Z}$ contains p-terms and the solution is not easy when a large number of variables is involved. It is very convenient to obtain a solution for such a multi-variable system when the equation is expressed in state-variable form. We can write the voltage equation

$$\mathbf{L}p\mathbf{i} = \mathbf{V} - \mathbf{R}\mathbf{i} - \mathbf{G}\mathbf{i}p\theta$$

or

$$p\mathbf{i} = \mathbf{L}^{-1}\mathbf{V} - \mathbf{L}^{-1}(\mathbf{R} + \mathbf{G}p\theta)\mathbf{i} \tag{5.68}$$

For a machine with three coils, for instance, the voltage equations can be written for computer solution

$$p\begin{bmatrix} i_{ds} \\ i_{dr} \\ i_{qr} \end{bmatrix} = \begin{bmatrix} L_{ds1} & M_d & \\ M_d & L_{dr} & \\ & & L_{qr} \end{bmatrix}^{-1} \begin{bmatrix} V_{ds1} \\ V_{dr} \\ V_{qr} \end{bmatrix}$$

$$- \begin{bmatrix} L_{ds1} & M_d & \\ M_d & L_{dr} & \\ & & L_{qr} \end{bmatrix}^{-1} \begin{bmatrix} R_{ds1} & & \\ & R_r & L_{qr}\omega \\ -\omega M_d & -\omega L_{dr} & R_r \end{bmatrix} \begin{bmatrix} L_{ds} \\ i_{dr} \\ i_{qr} \end{bmatrix} \tag{5.69}$$

This equation is clearly of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{5.70}$$

where

$$\begin{aligned} \dot{\mathbf{x}} &= p\mathbf{i} \\ \mathbf{A} &= -(\mathbf{L}^{-1}\mathbf{R} + \mathbf{L}^{-1}\mathbf{G}p\theta) \\ \mathbf{B}\mathbf{u} &= \mathbf{L}^{-1}\mathbf{V} \end{aligned}$$

In this case the matrices **A** and **B** have constant terms if the angular velocity $p\theta$ does not change when currents and voltages change under sudden transient conditions. This assumption is not true in general but it is valid in many cases. For correct analysis, it is necessary to take into account changes in the rotor velocity, and we have to consider the torque equation also. In this case we take the complete equation (5.22). It is, however, convenient to express the **Gi** term as flux density **B**. Equation (5.21) then becomes

$$\begin{bmatrix} \mathbf{V} \\ T_i \end{bmatrix} = \begin{bmatrix} \mathbf{R} + \mathbf{L}p & \mathbf{B} \\ -\mathbf{B} & Jp + R_F \end{bmatrix} \cdot \begin{bmatrix} \mathbf{i} \\ \omega \end{bmatrix} \tag{5.71}$$

or

$$\begin{array}{|c|}\hline \mathbf{V} \\ \hline T_{\mathrm{i}} \\ \hline \end{array} = \begin{array}{|c|c|}\hline \mathbf{R} & \\ \hline & R_{\mathrm{F}} \\ \hline \end{array} \cdot \begin{array}{|c|}\hline \mathbf{i} \\ \hline \omega \\ \hline \end{array} + \begin{array}{|c|c|}\hline \mathbf{L} & \\ \hline & J \\ \hline \end{array}\, p \begin{array}{|c|}\hline \mathbf{i} \\ \hline \omega \\ \hline \end{array} + \begin{array}{|c|c|}\hline & \mathbf{B} \\ \hline -\mathbf{B} & \\ \hline \end{array} \cdot \begin{array}{|c|}\hline \mathbf{i} \\ \hline \omega \\ \hline \end{array} \tag{5.72}$$

giving

$$p \begin{array}{|c|}\hline \mathbf{i} \\ \hline \omega \\ \hline \end{array} = - \begin{array}{|c|c|}\hline \mathbf{L} & \\ \hline & J \\ \hline \end{array}^{-1} \cdot \begin{array}{|c|c|}\hline \mathbf{R} & \mathbf{B} \\ \hline -\mathbf{B} & R_{\mathrm{F}} \\ \hline \end{array} \cdot \begin{array}{|c|}\hline \mathbf{i} \\ \hline \omega \\ \hline \end{array} + \begin{array}{|c|c|}\hline \mathbf{L} & \\ \hline & J \\ \hline \end{array}^{-1} \cdot \begin{array}{|c|}\hline \mathbf{V} \\ \hline T_{\mathrm{i}} \\ \hline \end{array} \tag{5.73}$$

The state-variable x is once again given by

$$\begin{array}{|c|}\hline \mathbf{x} \\ \hline \end{array} = \begin{array}{|c|}\hline \mathbf{i} \\ \hline \omega \\ \hline \end{array} \tag{5.74}$$

and

$$\mathbf{A}(t) = \begin{array}{|c|c|}\hline \mathbf{L} & \\ \hline & J \\ \hline \end{array}^{-1} \cdot \begin{array}{|c|c|}\hline \mathbf{R} & \mathbf{B} \\ \hline -\mathbf{B} & R_{\mathrm{F}} \\ \hline \end{array} \tag{5.75}$$

$$\mathbf{Bu}(t) = \begin{array}{|c|c|}\hline \mathbf{L} & \\ \hline & J \\ \hline \end{array}^{-1} \cdot \begin{array}{|c|}\hline \mathbf{V} \\ \hline T_{\mathrm{i}} \\ \hline \end{array} \tag{5.76}$$

and the state equation becomes

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{Bu}(t) \tag{5.77}$$

In general the matrix **A** is time-dependent, and in the equations above **A** is a function of **i**. The solution must therefore be obtained numerically.

5.11 NONLINEARITIES IN MACHINE EQUATIONS

In this chapter it has been stated that the performance equations of most machines may be derived from the generalised primitive machine equations. In so doing some basic assumptions have been implied. These were not discussed earlier lest the main issue be obscured. However, it is essential that they should be emphasised at this stage, and their implications studied in some detail. The various factors to be discussed in this section are the effects of

(1) saturation
(2) commutation
(3) space harmonics

5.11.1 Saturation

As has been pointed out in Section 2.5 the magnetic path becomes saturated in almost all machines, and as a result the current and the flux it produces do not have a linear relationship. The effects of saturation are particularly evident in certain types of machines, such as the shunt generator (where operation depends largely on saturation) and some cross-field machines (amplidyne, metadyne, etc). The effect of saturation on a.c. machines is also important under certain conditions, but it is not always necessary to take this into account. The inductance and torque matrices have been assumed to have constant coefficients in Park's axes, but they are no longer constant when we consider saturation, their values depending on various currents. During disturbances the currents may vary over wide ranges. In computation we have to start with a certain set of inductances and then alter their values in short time steps as the calculation proceeds, according to the saturation characteristic of the magnetic path.

If we consider a separately-excited d.c. machine (Figure 5.4b) we notice that there is only one coil, namely the field coil, along the d axis. Also, saturation is greater in the d axis than the q axis, which has a large air gap along its path. We can therefore plot the variation of $M_{\mathrm{fd}} (= M_{\mathrm{d}})$ with the field current. This can be obtained from the magnetisation or open circuit characteristics. However, if we consider a compound-wound d.c. machine (Figure 5.8a), we can see that the additional winding along the stator d axis, which carries the armature current (i_{q}), either boosts or bucks the main field, i.e. it either saturates the d axis further or reduces the saturation effect. In computing the effect of saturation on the reactance M_{d}, we would also have to include the effects of the armature current.

In a synchronous machine, in addition to the main field, the d axis has two other windings, ds2 and dr. The value of i_{dr} depends on M_{d}, and the value of M_{d} depends on i_{dr}, so it appears to be a vicious circle.

We shall consider two methods of including the effects of saturation in machine equations.

5.11.1(i) *The Frölich equation*

The Frölich equation[3] has been found to be reasonably accurate in taking into account the effects of saturation in an iron circuit. The flux/current relationship is given by the equation

$$\Phi = \frac{ai}{b+i} + ci \tag{5.78}$$

where Φ is the flux set up in a coil by the current i.

The constants are suitably adjusted to fit the saturation characteristics of a machine coil.

An alternative method may be illustrated by the example of a synchronous machine. We can take a common mutual inductance M_d for all three coils ds1, ds2, and dr, if we refer to a common winding and

$$L_{ds1} = l_1 + M_d \tag{5.79a}$$

$$L_{ds2} = l_2 + M_d \tag{5.79b}$$

$$L_{dr} = l_r + M_d \tag{5.79c}$$

We can reasonably assume that the leakage reactances are not affected by saturation and thus only M_d changes. The net flux linkage in the d axis magnetic path is

$$\psi_{dr} = M_d i_{ds1} + M_d i_{ds2} + M_d i_{dr} \tag{5.80}$$

if we exclude the leakage inductance.

The easiest way to take saturation into account would therefore be as follows:

(a) plot the magnetisation characteristic
(b) plot the change of M_d with the field current
(c) fit the curve by a polynomial and store it in a computer
(d) obtain i_f starting with the unsaturated value of M_d
(e) use the polynomial to obtain the saturated value of M_d from i_f (with the new value of i_f, a new M_d can be obtained, and the convergence is very fast)
(f) use the saturated value of M_d in the motional impedance matrix (equation 5.22)

In a similar manner, the effect of saturation in the q axis may be taken into account, but this is negligible and can usually be omitted.

However, this procedure of updating M_d consumes a great deal of computer time, and the additional accuracy achieved thereby may not always be worthwhile. If we consider small oscillations for instance, it is unnecessary to keep on altering the value of M_d with the small change in currents. Instead, we can obtain the saturated value of M_d under the initial operating conditions and use this value as a constant, with little loss in accuracy.

Further comments on the effects of saturation will be made in the appropriate contexts.

5.11.2 Commutation

The phenomenon of physical commutation is complex, and we have considered it only in an ideal form, to indicate how a commutator differs from slip rings so far as the observer is concerned. However, in reality we find that as each commutator segment moves away from the brushes to make way for the next, two consecutive segments are inevitably short-circuited by the brushes and as a result two coils of the armature are short-circuited at that instant. Secondly, the commutator segments are finite in number and hence stay in contact with the brushes for a finite period of time. In Section 2.9 where we considered only a split ring, it is stated that the observer moves only half way round the air gap with a coil side (Figure 2.10) in contrast to moving all the way in slip ring machines. If the number of commutator segments is large the movement of the observer is correspondingly reduced. In the limit, if we had an infinite number of segments the observer would be stationary. But, since the number is actually finite, the observer undergoes a very small oscillation, as it were, and as a result a high frequency ripple of low amplitude is observed. A detailed analysis of commutation and its effects on the commutator primitive machine will be found in reference 1, where it is shown that Kron's equations are sufficiently accurate for most practical purposes because the period of commutation is very short. Commutation is, of course, a very important consideration in the *design* of the machine.[9]

5.11.3 Space harmonics

In most studies relating to synchronous machines, it is assumed that the flux wave is sinusoidally distributed in space, as shown in Figure 2.28. This assumption is reasonable for induction motors and cylindrical-rotor synchronous machines of good design. It is not always a valid assumption, however, and the effects of space harmonics, which produce voltage time-harmonics, electrical noise and parasitic torques, must in certain cases be taken into account. The analysis, however, is complex and is outside the scope of this book.

5.12 MACHINE TORQUE EXPRESSIONS

Before we apply the general theory to industrial machines, it is clearly desirable to show that the torque equation $T = \mathbf{i}^*\mathbf{G}\mathbf{i}$ gives the familiar classical expressions in particular cases. We illustrate this with three examples.

5.12.1 The separately excited d.c. motor

The armature voltage equation is

$$V = E + I_a R_a \tag{5.81}$$

where V is the voltage impressed across the armature, E is the back e.m.f. generated by rotation of the armature winding in the air-gap flux, I_a is the total armature current, and R_a is the resistance of the armature and brushes from the two input terminals to the armature. (The latter definitions obviate the consideration of the parallel paths through the armature.)

The armature power input is

$$VI_a = EI_a + R_a I_a^2 \tag{5.82}$$

The last term on the right-hand side is the armature copper loss and the term EI_a gives the machine generated power output. The latter term may be written in terms of the air gap flux density B set up by the field current i_{ds} with the concept of the 'rotational mutual inductances' M'_d where

$$B = i_{ds} M'_d \tag{5.83}$$

The parameter M'_d can be measured by routine tests on the machine. Then

$$E = Bp\theta = i_{ds} M'_d p\theta \tag{5.84}$$

The power output is

$$EI_a = i_{ds} M'_d I_a p\theta \quad \text{(W)} \tag{5.85}$$

and the torque is given by

$$T = \frac{EI_a}{\omega_r} = i_{ds} M'_d I_a \tag{5.86}$$

The generalised machine impressed torque is given by

$$T_{\text{generated}} = -\mathbf{i}^*\mathbf{G}\mathbf{i} \tag{5.87}$$

$$\text{where } \mathbf{i} = \begin{array}{cccc} \text{ds} & \text{dr} & \text{qr} & \text{qs} \\ \hline \multicolumn{1}{|c|}{i_{ds}} & \multicolumn{1}{c|}{i_{dr}} & \multicolumn{1}{c|}{i_{qr}} & \multicolumn{1}{c|}{i_{qs}} \\ \hline \end{array} \tag{5.88}$$

and

$$\mathbf{G} = \begin{array}{c|cccc|} & \text{ds} & \text{dr} & \text{qr} & \text{qs} \\ \hline \text{ds} & & & & \\ \text{dr} & & & L'_{\text{qr}} & M'_{\text{q}} \\ \text{qr} & -M'_{\text{d}} & -L'_{\text{d}} & & \\ \text{qs} & & & & \\ \hline \end{array} \tag{5.89}$$

The current vector for the separately excited d.c. motor is

$$\mathbf{i} = \begin{array}{|cc|} \text{ds} & \text{qr} \\ \hline i_{\text{ds}} & i_{\text{qr}} \\ \hline \end{array} \tag{5.90}$$

The torque matrix contains only one term

$$\mathbf{G} = \begin{array}{c|cc|} & \text{ds} & \text{qr} \\ \hline \text{ds} & & \\ \text{qr} & -M'_{\text{d}} & \\ \hline \end{array} \tag{5.91}$$

and the product $(-\mathbf{i}^*\mathbf{G}\mathbf{i})$ gives

$$i_{\text{ds}}M'_{\text{d}}i_{\text{qr}} = i_{\text{ds}}M'_{\text{d}}I_{\text{a}} = EI_{\text{a}}/\omega_{\text{r}}$$

as before.

5.12.2 The induction motor

It is required to show that the torque matrix equation reduces to $(R_{\text{r}}i_{\text{r}}^2)/s$ where R_{r} is the rotor phase resistance, i_{r} is the rotor phase current, and s is the slip.

From the referred values on the equivalent circuit (Figure 2.37) it can be seen that the rotor copper loss is given by the real part of Ei_{r}^* where E in this case is the phase voltage induced in the rotor at standstill.

Here we see from the equivalent circuit, that

$$E = (i_{\text{stator}} - i_{\text{rotor}})\text{j}X_{\text{m}} \tag{5.92}$$

and thus the torque (in synchronous watts)

$$T = \text{Re}\,[(i_{\text{stator}} - i_{\text{rotor}})]\text{j}X_{\text{m}}i^*_{\text{rotor}} = R_{\text{r}}(i_{\text{r}})^2/s \tag{5.93}$$

Assume that, with respect to the vector E,

$$i_{\text{stator}} = i_1 - \mathrm{j}i_2 \quad \text{and} \quad i_{\text{rotor}} = i_3 - \mathrm{j}i_4$$

Then

$$T = \operatorname{Re}[\{(i_1 - \mathrm{j}i_2) - (i_3 - \mathrm{j}i_4)\}\mathrm{j}X_{\mathrm{m}}(i_3 - \mathrm{j}i_4)^*] \tag{5.94}$$

or

$$\begin{aligned} T &= -i_1 X_{\mathrm{m}} i_4 + i_2 X_{\mathrm{m}} i_3 \\ &= \operatorname{Re}(i_{\text{stator}})\mathrm{j}X_{\mathrm{m}}(i^*_{\text{rotor}}) \\ &= Ei^*_{\text{rotor}} = \text{rotor power input} \\ &= (i_{\text{rotor}})^2 R_{\mathrm{r}}/s = (\text{rotor copper loss})/\text{slip} \end{aligned} \tag{5.95}$$

In terms of the generalised machine matrices,

$$\begin{aligned} T &= -\mathbf{i}^*\mathbf{G}\mathbf{i} \\ &= -i^*_{\mathrm{qr}} M_{\mathrm{d}} i_{\mathrm{ds}} + i^*_{\mathrm{dr}} M_{\mathrm{q}} i_{\mathrm{qs}} \end{aligned} \tag{5.96}$$

The torque per phase is therefore

$$T_{\mathrm{ph}} = -i^*_{\mathrm{qr}}\, M\, i_{\mathrm{ds}} \quad (\mathrm{N\ m}) \tag{5.97}$$

In synchronous watts ($\omega M_{\mathrm{d}} = X_{\mathrm{m}}$)

$$\begin{aligned} T_{\mathrm{ph}} &= i^*_{\mathrm{qr}} X_{\mathrm{m}} i_{\mathrm{ds}} = E^* i_{\text{rotor}} \quad (\text{synchronous W}) \\ &= (\text{rotor copper loss})/\text{slip} \end{aligned} \tag{5.98}$$

Alternatively: The matrix equation (per phase) for the induction motor is

$$\begin{bmatrix} V \\ 0 \end{bmatrix} = \begin{array}{c|cc} & \mathrm{ds} & \mathrm{dr} \\ \hline \mathrm{ds} & R_{\mathrm{ds}} + \mathrm{j}X_{\mathrm{ds}} & \mathrm{j}X_{\mathrm{m}} \\ \mathrm{dr} & \mathrm{j}sX_{\mathrm{m}} & R_{\mathrm{dr}} + \mathrm{j}sX_{\mathrm{dr}} \end{array} \cdot \begin{bmatrix} i_{\mathrm{ds}} \\ i_{\mathrm{dr}} \end{bmatrix} \tag{5.99}$$

From which,

$$\begin{bmatrix} i_{\mathrm{ds}} \\ i_{\mathrm{dr}} \end{bmatrix} = \frac{1}{D} \begin{array}{c|cc} & \mathrm{ds} & \mathrm{dr} \\ \hline \mathrm{ds} & Z_{\mathrm{dr}} & -\mathrm{j}X_{\mathrm{m}} \\ \mathrm{dr} & -\mathrm{j}sX_{\mathrm{m}} & Z_{\mathrm{ds}} \end{array} \cdot \begin{bmatrix} V \\ 0 \end{bmatrix} \tag{5.100}$$

where D is the determinant $(Z_{\mathrm{ds}} Z_{\mathrm{dr}} + s X_{\mathrm{m}}^2)$
Thus

$$i_{\mathrm{ds}} = \frac{1}{D}(Z_{\mathrm{dr}} V) \ . \quad i_{\mathrm{dr}} = \mathrm{j}sX_{\mathrm{m}} V/D \tag{5.101}$$

Again

$$T_{\text{ph}} = -i_{\text{ds}} \text{j} X_{\text{m}} i^*_{\text{dr}} \quad \text{(synchronous W)} \tag{5.102}$$

giving

$$T_{\text{ph (input)}} = -\left[\frac{sX^2_{\text{m}} R_{\text{r}}}{DD^*}\right] V^2 \tag{5.103}$$

Expansion gives

$$T_{\text{ph (generated)}} = \frac{R_{\text{r}}}{s}(i_{\text{dr}} i^*_{\text{dr}}) = (i_{\text{dr}})^2 R/s \quad \text{(synchronous W)} \tag{5.104}$$

Deducting the rotor copper loss, the generated shaft torque (synchronous W) is

$$\begin{aligned} &T_{\text{ph (gen.)}} - (i_{\text{dr}})^2 R_{\text{r}} \\ &= (i_{\text{dr}})^2 \left[\frac{R_{\text{r}}}{s} - R_{\text{r}}\right] = i^2_{\text{dr}} R_{\text{r}}\left(\frac{1}{s} - 1\right) \end{aligned} \tag{5.105}$$

5.12.3 The synchronous machine

In Kron's system with impressed armature and field quantities

$$\begin{array}{c} \\ \text{ds} \\ \text{dr} \\ \text{qr} \end{array} \begin{array}{c} \\ \begin{bmatrix} V_{\text{ds}} \\ V_{\text{dr}} \\ V_{\text{qr}} \end{bmatrix} \end{array} \begin{array}{c} \\ \text{ds} \\ = \text{dr} \\ \text{qr} \end{array} \begin{array}{ccc} \text{ds} & \text{dr} & \text{qr} \\ \left[\begin{matrix} R_{\text{ds}} + L_{\text{ds}} p \\ M_{\text{d}} p \\ -M_{\text{d}} p\theta \end{matrix}\right. & \begin{matrix} M_{\text{d}} p \\ R_{\text{dr}} + L_{\text{dr}} p \\ -L_{\text{dr}} p\theta \end{matrix} & \left.\begin{matrix} \\ L_{\text{qr}} p\theta \\ R_{\text{qr}} + L_{\text{qr}} p \end{matrix}\right] \end{array} \begin{array}{c} \\ \begin{bmatrix} i_{\text{ds}} \\ i_{\text{dr}} \\ i_{\text{qr}} \end{bmatrix} \end{array}$$

$$\mathbf{G} = \begin{array}{c|ccc} & \text{ds} & \text{dr} & \text{qr} \\ \hline \text{ds} & & & \\ \text{dr} & & & L_{\text{qr}} \\ \text{qr} & -M_{\text{d}} & -L_{\text{dr}} & \end{array}$$

The generated torque per phase is

$$\begin{aligned} T_{\text{ph}} &= -\mathbf{iGi} \\ &= i_{\text{qr}} M_{\text{d}} i_{\text{ds}} + i_{\text{qr}} L_{\text{dr}} i_{\text{dr}} - i_{\text{dr}} L_{\text{qr}} i_{\text{qr}} \\ &= i_{\text{qr}}(M_{\text{d}} i_{\text{ds}} + L_{\text{dr}} i_{\text{dr}}) - i_{\text{dr}}(L_{\text{qr}} i_{\text{qr}}) \\ &= \psi_{\text{d}} i_{\text{qr}} - \psi_{\text{q}} i_{\text{dr}} \end{aligned} \tag{5.106}$$

In Park's convention, as we have seen earlier, the concepts of generated and impressed quantities differ from those of Kron, but the theories and equations are otherwise identical.

5.13 REFERENCES

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6

Electrical machine dynamics continued

6.1 INTRODUCTION

In Chapter 4 we have seen that the general dynamical analysis of a machine involves the determination of the initial steady state conditions of operation, the magnitude, nature, and duration of the disturbing forces, and the calculation of the dynamical response. It is thus necessary to have detailed information about the parameters of the machine and the nature of nonlinear functions and saturation effects associated with them.

In the investigation of electrical machine dynamics in industrial systems, however, certain problems arise. It may be difficult to determine the necessary parameters for the plant, for example in the case of amortisseur windings, which are closed cages of copper bars, it is virtually impossible to measure accurately the resistance or the self and mutual inductances. As we shall see later, these parameters and others may be combined to give reasonable overall 'transient' and 'sub-transient' reactances. A serious computational difficulty may arise due simply to the combination of the nonlinear aspects of the analysis, the size of the system, and the number of machines involved. It may also be difficult to obtain stable solutions on the computer because of numerical ill-conditioning in the terms of the electrodynamical equations.

Some of these problems are discussed below, in studies of multi-machine systems.

6.2 BASIC MACHINE ELECTRODYNAMICS

The general electrical and mechanical equations of the primitive machine, along stationary (real or fictitious commutator) reference axes, may be written

$$\text{impressed voltage } \mathbf{V} = \mathbf{Ri} + \mathbf{L}p\mathbf{i} + \mathbf{G}\mathbf{i}p\theta \tag{6.1}$$

$$\text{impressed torque } T = Jp^2\theta + R_{\mathrm{F}}p\theta - \mathbf{i}^*\mathbf{G}\mathbf{i} \tag{6.2}$$

where the asterisk denotes the conjugate of complex quantities. These can be incorporated into one matrix equation

$$
\begin{bmatrix} V_{ds} \\ V_{dr} \\ V_{qr} \\ V_{qs} \\ T \end{bmatrix}
=
\begin{array}{c|ccccc}
 & \mathrm{ds} & \mathrm{ar} & \mathrm{qr} & \mathrm{qs} & \mathrm{s} \\
\hline
\mathrm{ds} & R_{ds}+L_{ds}p & M_d p & & & \\
\mathrm{dr} & M_d p & R_{dr}+L_{dr}p & & & L_{qr}i_{qr}+M_q i_{qs} \\
\mathrm{qr} & & & R_{qr}+L_{qr}p & M_q p & -M_d i_{ds} - L_{dr}i_{dr} \\
\mathrm{qs} & & & M_q p & R_{qs}+L_{qs}p & \\
\mathrm{s} & -M_d i_{qr} & -L_{dr}i_{qr} & L_{qr}i_{dr} & M_q i_{dr} & Jp+R_F
\end{array}
\cdot
\begin{bmatrix} i_{ds} \\ i_{dr} \\ i_{qr} \\ i_{qs} \\ \omega_r \end{bmatrix}
\tag{6.3}
$$

where $\omega_r = p\theta$, the angular velocity of the rotor (mechanical rad/s).

In the last row, the initial steady state currents are determined by solution of the matrix equations. If the speed is not constant but the values of the impressed quantities in the left-hand column matrix are known, we proceed iteratively. At each interval in time a new value of ω_r is available and the calculation can be programmed to give the solution step by step. Convergence is usually rapid.

6.3 INTERCONNECTED MACHINES

In industrial processes electrical machines may be used singly, in groups of identical machines, or in groups of different machines interconnected to achieve some required overall operating characteristics. A simple example of this is the use of a pilot exciter and main exciter to provide controlled and stable field current for a large alternator in a power station, all machine rotors being on one shaft. In steel production, groups of machines may operate in parallel or in controlled sequence.

In the machines which we have investigated in Chapters 4 and 5 and in the foregoing analysis of torque equations, the choice of d and q axes was obvious. However, where machines of different types are interconnected, we must consider carefully the choice of reference axes. For example, the armature voltage and torque equations of synchronous machines are usually expressed along Park's axes, namely in a direct axis along the rotor field pole and a quadrature axis lying between the field poles. In the ideal synchronous machine in steady state operation, all values of voltage, current, and torque along these axes are constant with respect to time, and the operator $p(= \mathrm{d}/\mathrm{d}t)$ in the equations becomes zero. With the induction motor the usual reference frame is fixed to the stator with the direct axis along one phase and the quadrature axis orthogonal to this position.[3] The rotor axes are also fixed in this position in space. Along these axes, in steady state operation, the p operator in the equations becomes $j\omega$, where $\omega = 2\pi f$. Obviously if these machines are operating together, it is essential that we select reference axes such that the operator p has the same steady-state and transient significance for all time dependent behaviour. A reference frame for the induction motor, which rotates uniformly with the synchronous flux wave gives equations of the same form as those of Park for the synchronous machine and a complex arrangement of such machines may then be treated mathematically as a single coupled dynamical system.

In simple cases of a few coupled machines it may be possible in a computer programme to treat each machine separately with its own frame of reference, the computer making the adjustments in values when transformations between the reference axes become necessary as the machines interact. An example using different reference axes for each of two electrically connected machines is given in the following section.

6.4 THE ALTERNATOR/INDUCTION MOTOR

In this application we shall look at the dynamical performance of a single alternator feeding an induction motor. We shall consider first the two machines operating in isolation from any other supply. We shall then look at the case in which the alternator is also synchronised with the grid and synchronism is suddenly lost, leaving the alternator to supply only the motor.

Figure 6.1 shows the circuit for laboratory experiments described by Gorley[1] to investigate the overall dynamical response of the two machines. The

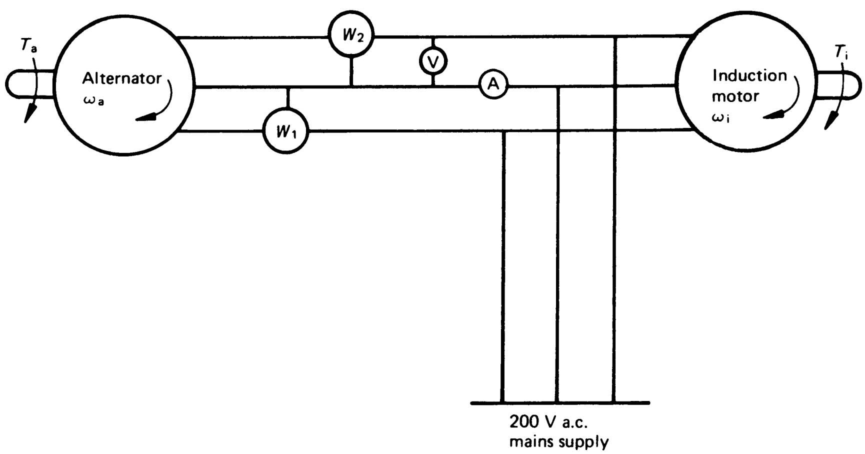


Figure 6.1 Two-machine laboratory set. ω_a = tachometer measuring the alternator speed, ω_i = tachometer measuring the induction motor speed, T_a = alternator driving torque and T_i = induction motor driving torque

alternator is a 3-phase 5 kVA 200 V 4-pole machine, with the field coils on the rotor, driven by a 230 V d.c. shunt motor. The latter has some additional resistance inserted in the supply cable in order to produce a speed–torque characteristic similar to that of a steam turbine. The induction motor is a 3-phase 3.5 kVA 200 V 4-pole machine with a shaft load provided by a 3 kVA dynamometer. The machine parameters are given in Table 6.1, in per-unit values at a base of 200 V, 10 A, 3.464 kVA.

Table 6.1 Laboratory machine parameters
(Per-unit. Base 200 V, 3.464 kVA)

Alternator		
R_f	0.225	field resistance
L_f	6.74	field inductance
M_f	5.48	mutual inductance field/armature
R_a	0.026	armature resistance
L_d	0.510	direct axis self inductance
L_q	0.382	quadrature axis self inductance
x_d	0.51	direct axis synchronous reactance
x_d'	0.184	direct axis transient reactance
x_d''	0.076	direct axis sub-transient reactance
J_a	1.22	moment of inertia, including drive (kg m^2)
R_{s1}	0.04	frictional coefficient (N m/rad)
Induction motor		
R_1	0.043 5	stator resistance
R_2	0.001 275	rotor resistance
X_1	2.28	stator reactance
X_2	0.072 4	rotor reactance
X_m	0.39	mutual reactance stator/rotor
J_i	0.154 6	moment of inertia (kg m^2)
R_{s2}	0.17	frictional coefficient (N m/rad)

In this example the computer programme is arranged to calculate the machine quantities with respect to the individual reference system for each machine. Park's equations are used for the alternator. In this system the 3-phase armature quantities are resolved along direct axes in line with the rotor field poles and quadrature axes at right angles to these. The d and q axes are fixed to the field structure and therefore rotate, accelerate, or oscillate along with the rotor. In balanced steady state operation the rotating d and q axis quantities are constant in magnitude and in the steady-state equations, terms which are functions of the time rate of change of current or flux become zero. That is, in the matrix equations, all of those terms which are functions of the p-operator ($= d/dt$) become zero in the steady state.

The induction motor is analysed along similar axes, but in this case the direct axis is fixed along the axis of one of the phases and the quadrature axis is orthogonal to this (see Stanley[3]). The resulting equations are similar in form to those of the alternator (or any other machine expressed in d, q axes). However, in this case the machine flux is rotating at synchronous speed with respect to the reference axes and consequently the motor current, voltage, and flux along the d and q axes will be oscillating at mains frequency. Hence in the induction motor matrix equations, the p-operator becomes $j\omega$ in the steady state, where ω is the synchronous angular velocity in electrical radians per second. In this system, the induction motor d and q axis quantities have the same magnitude as the phase quantities.

In the present case, therefore, the alternator and induction motor will be treated quite separately and the current, voltage, and torque for each machine will be calculated with respect to its own reference system. This method is used here because the analysis requires iterative computation in any case to determine the current, speed, and torque of each machine, with quantities such as reactance dependent at each stage upon the frequency and hence upon the speed of each machine.

The technique is to assume that over a small interval of time the induction motor terminal voltage is constant. The induction motor equations are solved and the current, torque, and speed are calculated. The corresponding output current from the alternator is then assumed to be constant over the next interval and new values of voltage, speed, and frequency are calculated for both machines, using the known speed–torque characteristic of the shunt motor which drives the alternator. Appropriate changes are then made in the induction motor equations and the process is repeated.

After resolution of the phase quantities along d and q axes, the two machines have the form shown in Figure 6.2 and the matrix form of the transient equations for each machine can be written down directly from the equations of the primitive machine.

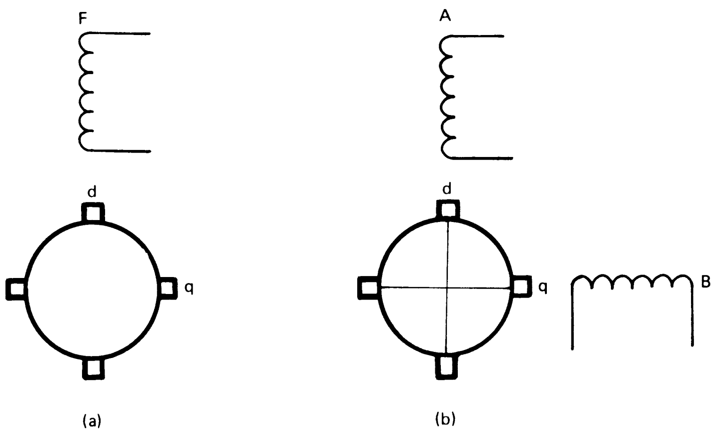


Figure 6.2 Models used to represent the two laboratory machines

6.4.1 Primitive machine

$$
\begin{array}{c}
\\
\begin{bmatrix} V_{ds} \\ V_{dr} \\ V_{qr} \\ V_{qs} \end{bmatrix}
\end{array}
=
\begin{array}{c|cccc}
 & ds & dr & qr & qs \\
\hline
ds & R_{ds}+L_{ds}p & M_d p & & \\
dr & M_d p & R_{dr}+L_{dr}p & L_{qr}p\theta & M_q p\theta \\
qr & -M_d p\theta & -L_{dr}p\theta & R_{qr}+L_{qr}p & M_q \\
qs & & & M_q & R_{qs}+L_{qs}p
\end{array}
\cdot
\begin{array}{c}
\\
\begin{bmatrix} i_{ds} \\ i_{dr} \\ i_{qr} \\ i_{qs} \end{bmatrix}
\end{array}
\tag{6.4}
$$

$$\text{Torque } T = \mathbf{i}^*\mathbf{G}\mathbf{i} = \mathbf{i}^*\mathbf{B} \tag{6.5}$$

$$
\text{where } \mathbf{G} =
\begin{array}{c|cccc}
 & ds & dr & qr & qs \\
\hline
ds & & & & \\
dr & & & L_{qr} & M_q \\
qr & -M_d & -L_{dr} & & \\
qs & & & &
\end{array}
\tag{6.6}
$$

6.4.2 The alternator

Using Kron's 'impressed voltage' convention

$$
\begin{array}{c}
\\
\text{imp.} \\ \text{imp.} \\ \text{imp.}
\end{array}
\begin{array}{c}
\\
\begin{bmatrix} V_f \\ V_d \\ V_q \end{bmatrix}
\end{array}
=
\begin{array}{c|ccc}
 & ds & dr & qr \\
\hline
ds & R_f+L_f p & M_d p & \\
dr & M_d p & R_a+L_d p & L_q p\theta \\
qr & -M_d p\theta & -L_d p\theta & R_a+L_q p
\end{array}
\cdot
\begin{array}{c}
\\
\begin{bmatrix} i_f \\ i_d \\ i_q \end{bmatrix}
\end{array}
\tag{6.7}
$$

or Park's convention

$$\text{impressed } V_f = R_f i_f + p\psi_f \tag{6.8}$$

$$\text{generated } V_d = -R_a i_d - p\psi_d - \psi_q p\theta \tag{6.9}$$

$$\text{generated } V_q = -R_a i_q - p\psi_q + \psi_d p\theta \tag{6.10}$$

$$\text{generated torque} = \psi_d i_q - \psi_q i_d \tag{6.11}$$

where

$$\psi_f = L_f i_f + M_d i_d$$
$$\psi_d = L_d i_d + M_d i_f$$
$$\psi_q = L_q i_q$$

6.4.3 The alternator in steady state ($p = 0$)

Kron's convention

$$\begin{array}{ll} \text{imp.} & \boxed{V_f} \\ \text{imp.} & \boxed{V_d} \\ \text{imp.} & \boxed{V_q} \end{array} = \begin{array}{c|c|c|c} & \text{ds} & \text{dr} & \text{qr} \\ \hline \text{ds} & R_f & & \\ \text{dr} & & R_a & L_q p\theta \\ \text{qr} & -M_d p\theta & -L_d p\theta & R_a \end{array} \cdot \begin{array}{|c|} \hline i_f \\ i_d \\ i_q \\ \hline \end{array} \tag{6.12}$$

or (Park's convention)

$$\text{imp.} \quad V_f = R_f i_f \tag{6.13}$$
$$\text{gen.} \quad V_d = -R_a i_d - \psi_q p\theta \tag{6.14}$$
$$\text{gen.} \quad V_q = -R_a i_q + \psi_d p\theta \tag{6.15}$$

6.4.4 The induction motor

The matrix equation (6.4) may be used as it stands for transient analysis of the induction motor in d and q axes. The steady state equations are

$$\begin{array}{ll} \text{stator} & \boxed{V_s} \\ \text{rotor} & \boxed{0} \end{array} = \begin{array}{c|c|c} & \text{ds} & \text{dr} \\ \hline \text{ds} & R_s + jX_s & jX_m \\ \text{dr} & jsX_m & R_r + jsX_r \end{array} \cdot \begin{array}{|c|} \hline i_s \\ i_r \\ \hline \end{array} \tag{6.16}$$

when the motor is running with the rotor short-circuited.

The voltage and current terms are now at the supply frequency (as they are in the equivalent circuit) and at the electrical angular velocity $\omega = 2\pi f$. The generated torque in equivalent synchronous watts is given by the real part of

$$T = \mathbf{i}^* \omega \mathbf{G}\, \mathbf{i}$$

where **i*** is the complex conjugate of **i** as in Section 6.2.

In the dynamical analysis of the laboratory machines, one fact simplifies the problem, namely, the induction motor windings have resistance values which are large compared to the inductances, and the electrical time constants of the motor are short compared to the time of response following mechanical disturbances. In this case, therefore, the steady-state equations of the induction motor could be used to investigate acceleration effects and general electromechanical performance. However, in the case of the synchronous machine, with the same duration of disturbances, because of somewhat longer time constants the transient equations are required. The alternator equations are expressed with the field quantities eliminated as below, giving the armature short-circuit 'transient reactances'. The equivalent circuits for the alternator windings with these inductances are as shown in Figure 6.3. The equations for electromechanical transient performance are therefore as follows.

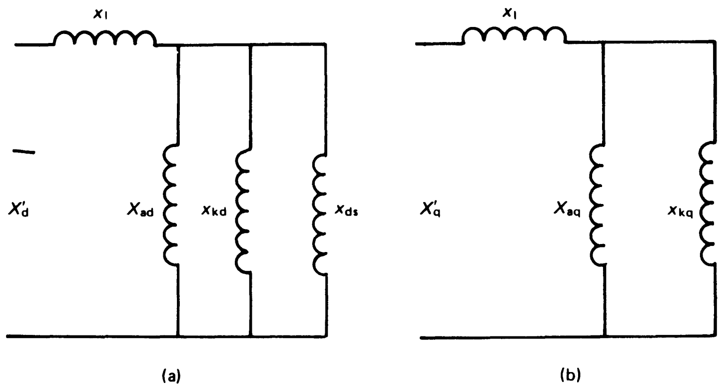


Figure 6.3 Alternator transient reactances. (a) Direct axis and (b) quadrature axis

6.4.5 Alternator equations for transient performance

$$\begin{bmatrix} V_f \\ V_d \\ V_q \\ T_a \end{bmatrix} = \begin{array}{cc} & \begin{array}{cccc} \mathrm{ds} & \mathrm{d} & \mathrm{q} & \mathrm{f} \end{array} \\ \begin{array}{c} \mathrm{ds} \\ \mathrm{d} \\ \mathrm{q} \\ \mathrm{f} \end{array} & \begin{bmatrix} R_f + L_f p & M_d p & & \\ M_d p & R_a + L_d p & L_q p\theta & \\ -M_d p & -L_d p\theta & R_a + L_q p & \\ M_d i_q & -L_q i_q & L_d i_d & J_a p + R_{Fa} \end{bmatrix} \end{array} \cdot \begin{bmatrix} i_f \\ i_d \\ i_q \\ \omega \end{bmatrix} \tag{6.17}$$

where R_{Fa} is the friction coefficient and J_a is the moment of inertia of the alternator and its driving motor.

Elimination of the field current gives the short-circuit currents in terms of transient (or operational) inductances,

$$\begin{array}{c} \\ \begin{bmatrix} V_d - GpV_f \\ V_q - Gp\theta V_f \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{array}{c} \mathrm{d} \\ = \\ \mathrm{q} \end{array} \end{array} \begin{array}{cc} \mathrm{d} & \mathrm{q} \\ \multicolumn{2}{c}{\begin{bmatrix} R_a - L'_d p & L'_q p\theta \\ -L'_d p\theta & -R_a - L'_q p \end{bmatrix}} \end{array} \cdot \begin{array}{c} \\ \begin{bmatrix} i_d \\ i_q \end{bmatrix} \end{array} \tag{6.18}$$

where $G = \dfrac{M_d}{R_f + L_f p}$

$$L'_d = L_d - \frac{M_d^2 p}{R_f + L_f p}$$

$$L'_q = L_q$$

These expressions assume that there are no amortisseur windings. When the impedance matrix contains rows and columns for the parameters for closed amortisseur windings, these can be eliminated along with the field quantities to give the 'subtransient' operational reactances x''_d and x''_q. The transient and subtransient reactances can be measured directly by short-circuit tests on the alternator. In Appendix 3 we see how the self and mutual inductances for the machine in the d and q axes can be obtained by computation, using simple equivalent circuits for the direct and quadrature axis windings, together with the measured values of x' and x'' in each axis.

6.4.6 Equations for dynamical response of the induction motor

The equations giving the dynamical response of the induction motor can now be expressed in terms of variations in the value of slip s, in the normal steady state equations.

$$\begin{array}{c} \\ \begin{bmatrix} V_s \\ 0 \\ T_m \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{array}{c} \mathrm{ds} \\ = \mathrm{dr} \\ \mathrm{s} \end{array} \end{array} \begin{array}{ccc} \mathrm{ds} & \mathrm{dr} & \mathrm{s} \\ \multicolumn{3}{c}{\begin{bmatrix} R_s + \mathrm{j}Xs & \mathrm{j}X_m & \\ \mathrm{j}sX_m & R_r + \mathrm{j}sX_r & \\ & \dfrac{3R_r i_r}{s\omega_m} & J_m p + R_{Fm} \end{bmatrix}} \end{array} \cdot \begin{array}{c} \\ \begin{bmatrix} i_s \\ i_r \\ \omega_m \end{bmatrix} \end{array} \tag{6.19}$$

The instantaneous generated and impressed values of torque are then calculated for both machines and these, together with the inertia constants, are used to determine the consequent dynamical response in the step-by-step routine. The tests (a) to (d) below were carried out on the machines and the predicted and test results are given in Figures 6.4 to 6.7:

(a) sudden loss of 0.75 per unit shaft load on the induction motor
(b) sudden application of 0.75 per unit load to the induction motor shaft
(c) a 3-phase short-circuit of short time duration applied to the alternator
(d) sudden loss of synchronism, with the alternator supplying the induction motor and also importing a range of values of reactive power from the mains.

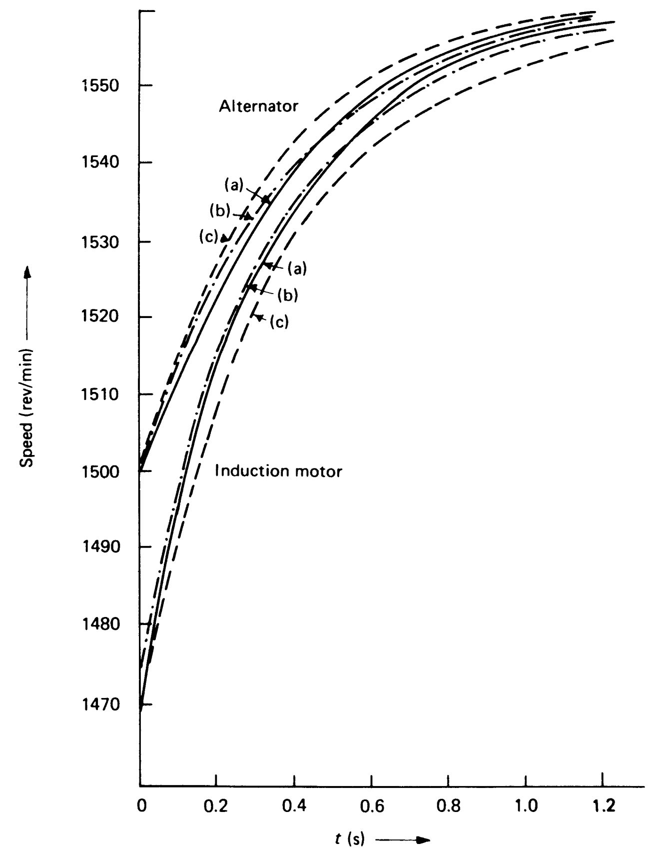


Figure 6.4 Actual and predicted variations in machine speeds following instantaneous loss of load. (a) Actual response, (b) predicted response, exact model and (c) predicted response, simple model

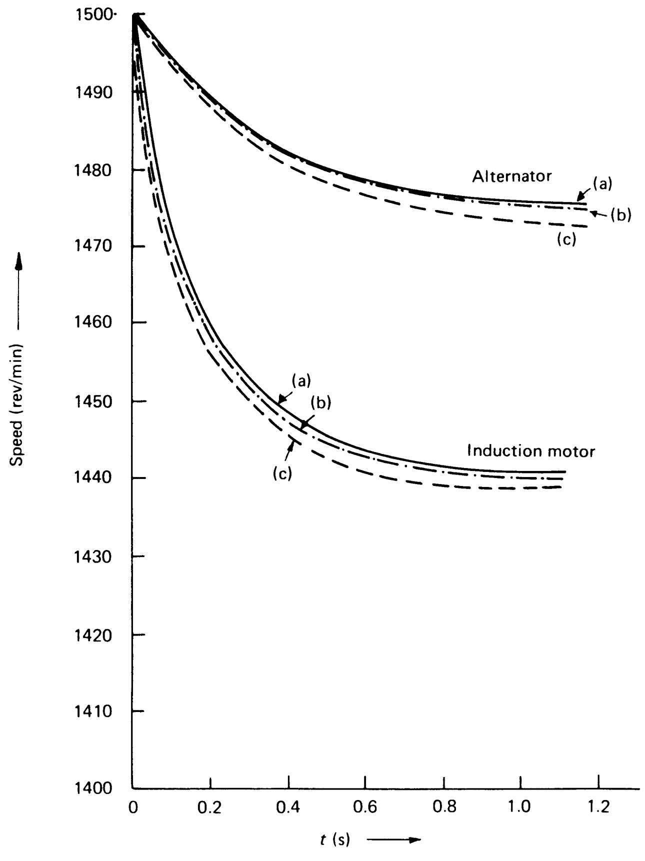


Figure 6.5 Actual and predicted variations of machine speed following the instantaneous loading of the induction motor. (a) Actual response, (b) predicted response, exact model and (c) predicted response, simple model

In tests (a) to (c) the machines were not connected to the mains. In test (d) the amount of *real power* (kW) fed to the mains had little effect on the dynamical behaviour following loss of synchronism—the reader should ask himself why this should be so.

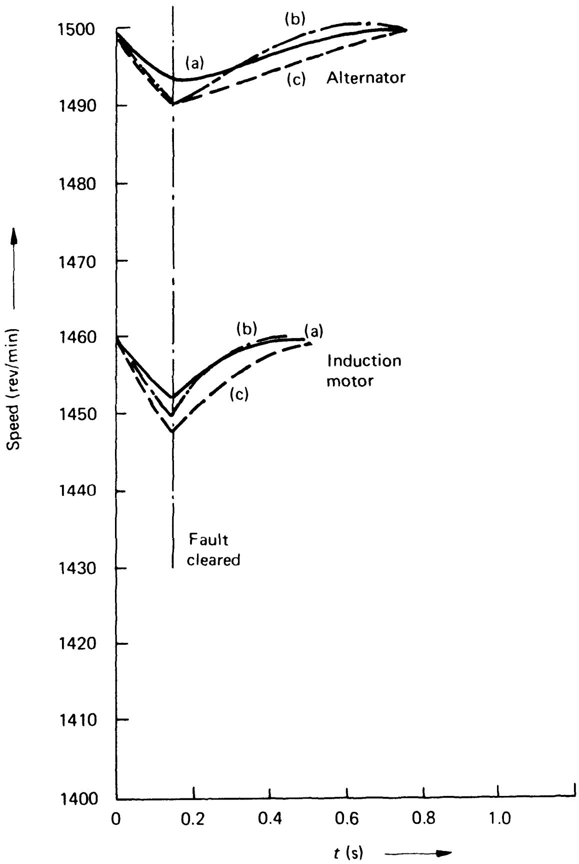


Figure 6.6 Actual and predicted variations in machine speeds following a three-phase short circuit. (a) Actual response, (b) predicted response, exact model and (c) predicted response, simple model

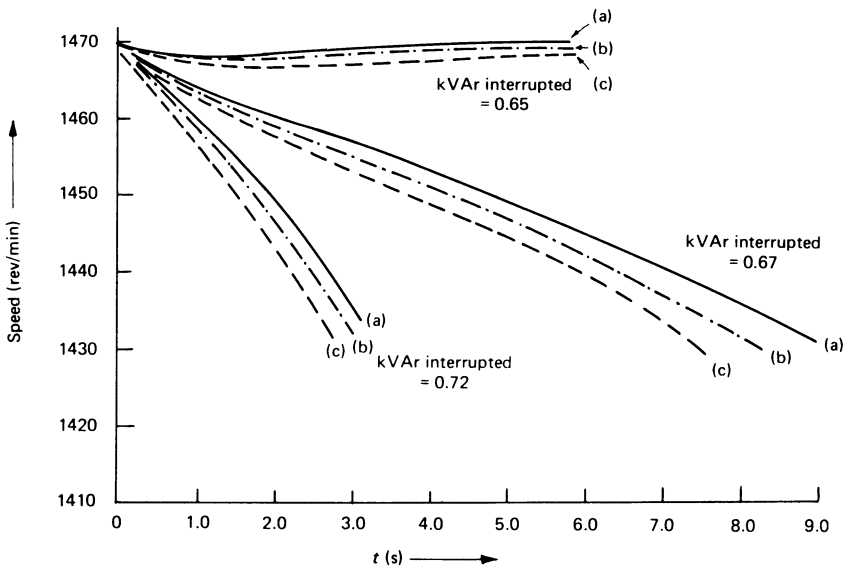


Figure 6.7 Simulation of induction motor speed changes for varying kVAr interruptions. (a) Actual response, (b) predicted response, exact model and (c) predicted response, simple model

The computer flow diagram is given in Figure 6.8. With the above matrix models this programme takes a few minutes of computing time for a response curve over a time of about 2 s. The simplified model shown in Figure 6.9 uses only the transient reactance x'_d for the alternator, and for the induction motor only the stator reactance and the rotor equivalent resistance. The results are still reasonably close and the computing time has been cut to about one sixth of the previous time.

6.5 INTERCONNECTED MACHINE C-MATRICES

For many purposes it is desirable to programme the overall dynamical behaviour of interconnected machines, using the combined equations of performance; for example when using the state A-matrices and in the study of stability conditions and eigenvalues.

If we must allow for the nonlinear magnetic saturation characteristics in the machines, the analysis is very complicated and lengthy computer programmes are required to carry out even the basic response calculations. Very often, however, we can assume that the machine iron characteristics are linear over suitable operating ranges, giving constant system parameters. In Section 6.6 we shall look at the dynamics of a system of several interconnected machines, in which we must give consideration to the choice of reference frame and to variations in the inductances during the transient response. In Chapter 7 we

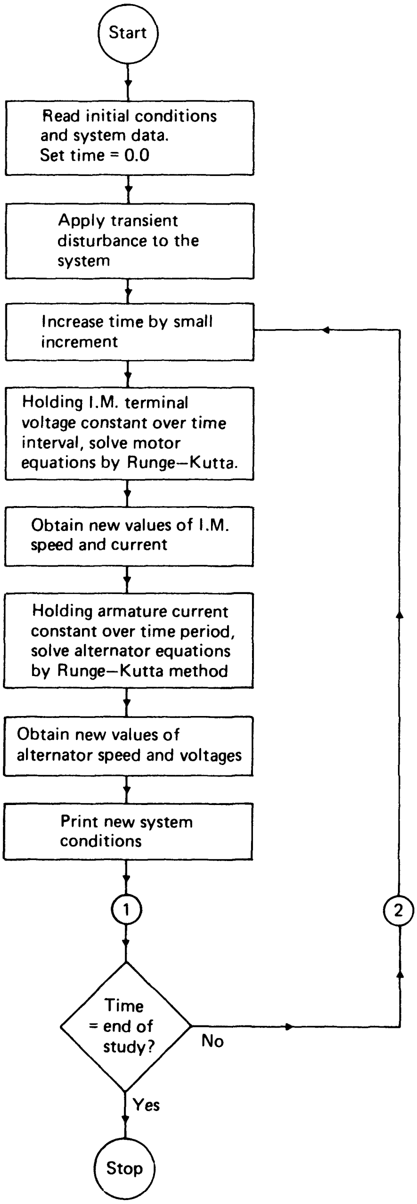


Figure 6.8 Flow chart for program 8

shall consider linear small oscillations of this system about a given operating state.

First, however, we give two simple linear examples to illustrate the principles of electrical interconnection, using the connecting **C**-matrices.

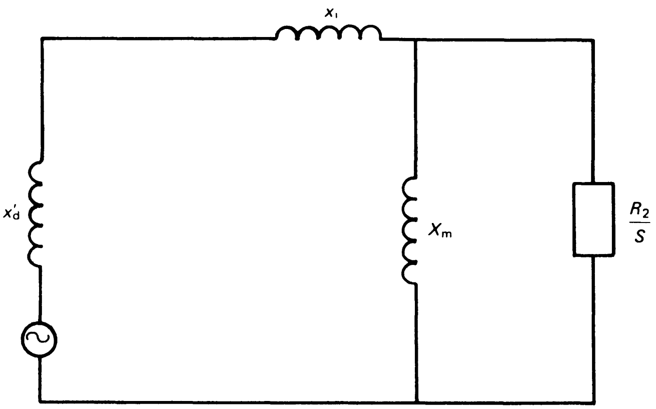


Figure 6.9 Simplified model of the laboratory set

6.5.1 The d.c. generator and motor

Figure 6.10 gives the circuit for the electrically coupled machines. We shall assume that the prime mover driving the shunt generator is very large and runs at constant speed whatever the load on the generator. The load/speed characteristic of the prime mover can be included in the computer programme very easily, as we shall see later. Considering impressed voltages on *each* machine, in accordance with Kron's convention, the equations for each machine are identical in form, namely,

(a) *generator*

$$\begin{bmatrix} V_{f1} \\ V_{a1} \end{bmatrix} = \begin{array}{c c} & \begin{array}{cc} \text{ds1} & \text{qr1} \end{array} \\ \begin{array}{c} \text{ds1} \\ \text{qr1} \end{array} & \begin{bmatrix} R_{f1} + L_{f1}p & \\ -M_{d1}p\theta & R_{a1} + L_{a1}p \end{bmatrix} \end{array} \cdot \begin{bmatrix} i_{f1} \\ i_{a1} \end{bmatrix} \tag{6.20}$$

or

$$V_1 = Z_1 i_1$$

(b) *motor*

$$\begin{bmatrix} V_{f2} \\ V_{a2} \end{bmatrix} = \begin{array}{c c} & \begin{array}{cc} \text{ds2} & \text{qr2} \end{array} \\ \begin{array}{c} \text{ds2} \\ \text{qr2} \end{array} & \begin{bmatrix} R_{f2} + L_{f2}p & \\ -M_{d2}p\theta & R_{a2} + L_{a2}p \end{bmatrix} \end{array} \cdot \begin{bmatrix} i_{f2} \\ i_{a2} \end{bmatrix} \tag{6.21}$$

$$V_2 = Z_2 i_2$$

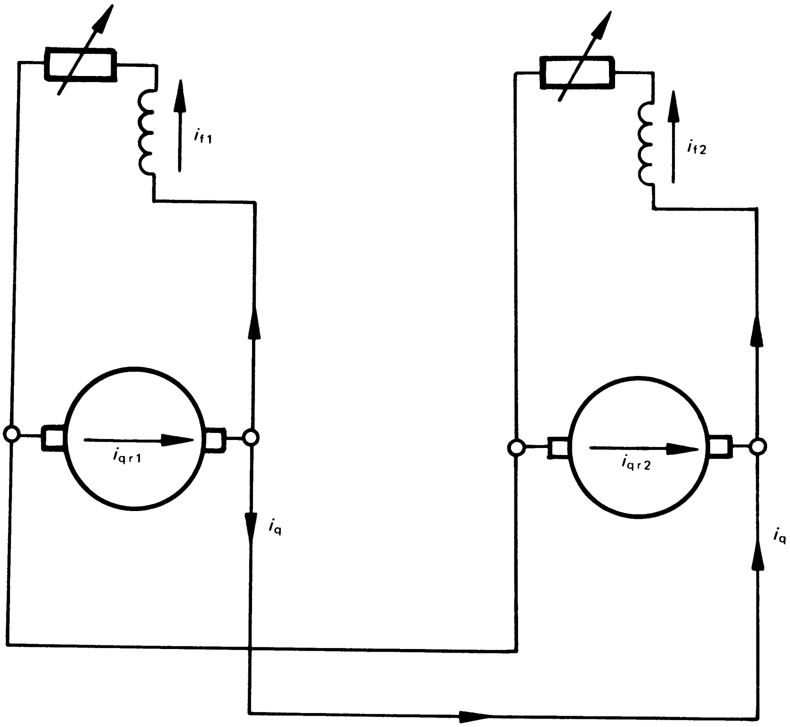


Figure 6.10 D.C. shunt generator/shunt motor

In these equations R_a is the resistance of the armature across the input terminals and R_f is the total resistance in the field circuit.

The C-matrix connecting the two machines is given by the current relationship

$$
\begin{aligned}
i_{ds1} &= i_{f1} \\
i_{qr1} &= i_{f1} + i_q \\
i_{ds2} &= i_{f2} \\
i_{qr2} &= i_{f2} - i_q
\end{aligned}
\qquad
\mathbf{C} = \begin{array}{c} \\ \text{ds1} \\ \text{qr1} \\ \text{ds2} \\ \text{qr2} \end{array}
\begin{array}{ccc} \text{f1} & \text{f2} & \text{q} \\ \hline 1 & & \\ 1 & & 1 \\ & 1 & \\ & +1 & -1 \end{array}
\tag{6.22}
$$

The combined matrix is given by

$$
\mathbf{Z}' = \mathbf{C}_t \mathbf{Z} \mathbf{C} \tag{6.23}
$$

where

$$
\mathbf{Z} = \begin{array}{c|cccc}
 & \mathrm{ds1} & \mathrm{qr1} & \mathrm{ds2} & \mathrm{qr2} \\
\hline
\mathrm{ds1} & R_{f1}+L_{f1}p & & & \\
\mathrm{qr1} & -M_{d1}p\theta & R_{a1}+L_{a1}p & & \\
\mathrm{ds2} & & & R_{f2}+L_{f2}p & \\
\mathrm{qr2} & & & -M_{d2}p\theta & R_{a2}+L_{a2}p
\end{array} \qquad (6.24)
$$

and

$$
\mathbf{Z}' = \begin{array}{c|ccc}
 & \mathrm{f1} & \mathrm{f2} & \mathrm{q} \\
\hline
\mathrm{f1} & R_{f1}+L_{f1}p - M_{d1}p\theta + R_{a1}+L_{a1}p & & R_{a1}+L_{a1}p \\
\mathrm{f2} & & R_{f2}+L_{f2}p - M_{d2}p\theta + R_{a2}+L_{a2}p & -R_{a2}-L_{a2}p \\
\mathrm{q} & -M_{d1}p\theta + R_{a1} + L_{a1}p & M_{d2}p\theta - R_{a2} - L_{a2}p & R_{a1}+L_{a1}p + R_{a2}+L_{a2}p
\end{array} \qquad (6.25)
$$

In the steady state, terms containing the operator p become zero and

$$
\mathbf{Z}'_{\text{s-state}} = \begin{array}{c|ccc}
 & \mathrm{f1} & \mathrm{f2} & \mathrm{q} \\
\hline
\mathrm{f1} & R_{f1}+R_{a1} - M_{d1}p\theta & & R_{a1} \\
\mathrm{f2} & & R_{f2}+R_{a2} - M_{d2}p\theta & -R_{a2} \\
\mathrm{q} & -M_{d1}p\theta + R_{a1} & M_{d2}p\theta - R_{a2} & R_{a1}+R_{a2}
\end{array} \qquad (6.26)
$$

The torque generated in each machine is given by **iGi** where, from matrix (6.25)

$$\mathbf{G} = \begin{array}{c|ccc} & \mathrm{f1} & \mathrm{f2} & \mathrm{q} \\ \hline \mathrm{f1} & -M_{d1} & & \\ \mathrm{f2} & & -M_{d2} & \\ \mathrm{q} & -M_{d1} & M_{d2} & \end{array} \tag{6.27}$$

and the generated torque is

$$T = -i_{f1}M_{d1}(i_q + i_{f1}) + i_{f2}M_{d2}(i_q - i_{f2}) \tag{6.28}$$

The first term is the torque generated by current in the generator, to oppose the torque impressed upon the shaft by the prime mover. The second term gives the torque generated by the motor to meet the load torque impressed upon the motor shaft.

The new voltage vector is given by $\mathbf{V}' = \mathbf{C}_t\mathbf{V}$ where

$$\mathbf{V} = \begin{array}{c|c|} \mathrm{ds1} & 0 \\ \mathrm{qr1} & V_{qr1} \\ \mathrm{ds2} & 0 \\ \mathrm{qr2} & V_{qr2} \end{array} \quad \text{and} \quad \mathbf{V}' = \begin{array}{c|c|} \mathrm{f1} & V_{qr1} \\ \mathrm{f2} & V_{qr2} \\ \mathrm{q} & V_{qr1} - V_{qr2} \end{array} = \begin{array}{c|c|} \mathrm{f1} & V_{f1} \\ \mathrm{f2} & V_{f2} \\ \mathrm{q} & V_q \end{array} \tag{6.29}$$

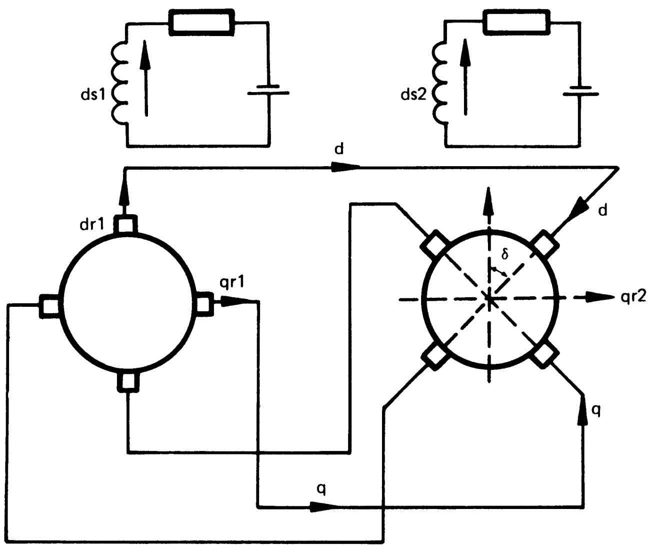


Figure 6.11 Alternator/synchronous motor

6.5.2 An alternator supplying a synchronous motor

The machines are as shown in Figure 6.11. In the present problem we consider that they operate in synchronism. Due to the shaft load on the motor, it operates with the armature direct and quadrature axes lagging those of the alternator by the load angle δ. Our assumption of synchronous operation implies that $p\delta$ is zero.

Both machines are, in fact, analysed with respect to the axes of the alternator, the motor equations being transformed back through the load angle δ. The transformation matrix is

$$\mathbf{C} = \begin{array}{c|c|c|c|c|} & \text{ds1} & \text{d}' & \text{q}' & \text{ds2} \\ \hline \text{ds1} & 1 & & & \\ \hline \text{dr1} & & 1 & & \\ \hline \text{qr1} & & & 1 & \\ \hline \text{ds2} & & & & 1 \\ \hline \text{dr2} & & -\cos\delta & \sin\delta & \\ \hline \text{qr2} & & -\sin\delta & -\cos\delta & \\ \hline \end{array} \tag{6.30}$$

The equations for the two machines are identical in form, namely (in Kron's notation)

$$\begin{array}{|c|} \hline V_{\mathrm{ds}} \\ \hline V_{\mathrm{dr}} \\ \hline V_{\mathrm{qr}} \\ \hline \end{array} \begin{array}{c} \text{ds} \\ =\text{dr} \\ \text{qr} \end{array} \begin{array}{|c|c|c|} \multicolumn{1}{c}{\text{ds}} & \multicolumn{1}{c}{\text{dr}} & \multicolumn{1}{c}{\text{qr}} \\ \hline R_{\mathrm{ds}} + L_{\mathrm{ds}}p & M_{\mathrm{d}}p & \\ \hline M_{\mathrm{d}}p & R_{\mathrm{dr}} + L_{\mathrm{dr}}p & L_{\mathrm{qr}}p\theta \\ \hline -M_{\mathrm{d}}p\theta & -L_{\mathrm{dr}}p\theta & R_{\mathrm{qr}} + L_{\mathrm{qr}}p \\ \hline \end{array} \cdot \begin{array}{|c|} \hline i_{\mathrm{ds}} \\ \hline i_{\mathrm{dr}} \\ \hline i_{\mathrm{qr}} \\ \hline \end{array} \tag{6.31}$$

The equation of transformation into the interconnected reference axes is

$$\begin{aligned} \mathbf{Z}' &= \mathbf{C}_{\mathrm{t}}\mathbf{Z}(p)\mathbf{C} \\ &= \mathbf{C}_{\mathrm{t}}\mathbf{Z}\mathbf{C} + \mathbf{C}_{\mathrm{t}}\mathbf{Z}(\partial\mathbf{C}/\partial\delta)p\delta \end{aligned} \tag{6.32}$$

Since $p\delta$ is zero

$$\mathbf{Z}' = \mathbf{C}_{\mathrm{t}}\mathbf{Z}\mathbf{C}$$

where

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_{\text{alt}} & \\ & \mathbf{Z}_{\text{motor}} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{Z}_1 & \\ & \mathbf{Z}_2 \end{bmatrix}$$

and $\mathbf{Z}'$ is

$$\begin{array}{c|c|c|c|c|}
 & \text{ds1} & \text{d}' & \text{q}' & \text{ds2} \\
\hline
\text{ds1} & R_{\text{ds1}} + L_{\text{ds1}}p & M_{\text{d1}}p & & \\
\hline
\text{d}' & M_{\text{d1}}p & \begin{array}{c} R_1 + R_2 + L_{\text{dr1}}p \\ + (L_{\text{dr2}}\cos^2\delta + L_{\text{qr2}}\sin^2\delta)p \\ + L'\sin\delta\cos\delta p\theta_2 \end{array} & \begin{array}{c} L'\sin\delta\cos\delta p + L_{\text{qr1}}p\theta_1 \\ + (L_{\text{qr2}}\cos^2\delta + L_{\text{dr2}}\sin^2\delta)p\theta_2 \end{array} & \begin{array}{c} -M_{\text{d2}}\cos\delta p \\ + M_{\text{d2}}\sin\delta p\theta_2 \end{array} \\
\hline
\text{q}' & -M_{\text{d1}}p\theta & \begin{array}{c} L'\sin\delta\cos\delta p - L_{\text{dr1}}p\theta \\ - (L_{\text{dr2}}\cos^2\delta + L_{\text{qr2}}\sin^2\delta)p\theta_2 \end{array} & \begin{array}{c} R_1 + R_2 + L_{\text{qr1}}p \\ (L_{\text{qr2}}\cos^2\delta + L_{\text{dr2}}\sin^2\delta)p \\ - L'\sin\delta\cos\delta p\theta_2 \end{array} & \begin{array}{c} M_{\text{d2}}\sin\delta p \\ + M_{\text{d2}}\cos\delta p\theta_2 \end{array} \\
\hline
\text{ds2} & & -M_{\text{d2}}\cos\delta p & M_{\text{d2}}\sin\delta p & R_{\text{ds2}} + L_{\text{d2}}p \\
\hline
\end{array} \tag{6.33}$$

$$\text{where } L' = L_{\text{qr2}} - L_{\text{dr2}}$$

In the steady state $p\theta_1 = p\theta_2 = \omega$ and coefficients of the operator p become zero. The steady state matrix equation is therefore

$$\begin{bmatrix} V_{\text{ds1}} \\ 0 \\ 0 \\ V_{\text{ds2}} \end{bmatrix} = \begin{array}{c|c|c|c|c|}
 & \text{ds1} & \text{d}' & \text{q}' & \text{ds2} \\
\hline
\text{ds1} & R_{\text{ds1}} & & & \\
\hline
\text{d}' & & \begin{array}{c} R_1 + R_2 \\ + X'\sin\delta\cos\delta \end{array} & \begin{array}{c} X_{\text{qr1}} \\ + X_{\text{qr2}}\cos^2\delta \\ + X_{\text{dr2}}\sin^2\delta \end{array} & X_{\text{md2}}\sin\delta \\
\hline
\text{q}' & X_{\text{md1}} & \begin{array}{c} -X_{\text{dr1}} \\ - X_{\text{dr2}}\cos^2\delta - X_{\text{qr2}}\sin^2\delta \end{array} & \begin{array}{c} R_1 + R_2 \\ - X'\sin\delta\cos\delta \end{array} & X_{\text{md2}}\cos\delta \\
\hline
\text{ds2} & & & & R_{\text{ds2}} \\
\hline
\end{array} \begin{bmatrix} i_{\text{ds1}} \\ i_{\text{d}} \\ i_{\text{q}} \\ i_{\text{ds2}} \end{bmatrix} \tag{6.34}$$

$$\text{where } X' = X_{\text{qr2}} - X_{\text{dr2}}$$

If the two machines are not in synchronism, the matrix (6.33) must be completed by the addition of the term

$$\mathbf{C}_t \mathbf{L} \frac{\partial \mathbf{C}}{\partial \delta} p\delta$$

which is

$$\begin{array}{c|c|c|c|c|}
 & \text{ds1} & \text{d}' & \text{q}' & \text{ds2} \\
\hline
\text{ds1} & & & & \\
\hline
\text{d}' & & L' \sin\delta \cos\delta \, p\delta & -(L_{dr2}\cos^2\delta + L_{qr2}\sin^2\delta)p\delta & \\
\hline
\text{q}' & & (L_{qr2}\cos^2\delta + L_{dr2}\sin^2\delta)p\delta & -L' \sin\delta \cos\delta \, p\delta & \\
\hline
\text{ds2} & & M_{d2}\sin\delta \, p\delta & M_{d2}\cos\delta \, p\delta & \\
\hline
\end{array} \tag{6.35}$$

The system torque is again given by $T = \mathbf{iGi}$ where the terms of the matrix $\mathbf{G}$ are the coefficients of $p\theta_1$ and $p\theta_2$ in matrix (6.33). The matrix multiplication gives

$$\begin{aligned} T = \psi_{d1} i_q - \psi_{q1} i_d + M_{d2} i_{d2}(i_d \sin\delta + i_q \cos\delta) + \tfrac{1}{2} i_d i_d L' \sin 2\delta \\ - \tfrac{1}{2} i_q i_q L' \sin 2\delta + i_d i_q L' \cos 2\delta \end{aligned} \tag{6.36}$$

where

$$\psi_{d1} = -M_{d1} i_{ds1} - L_{dr1} i_d$$
$$\psi_{q1} = -L_{qr1} i_q$$

The second-harmonic terms arise because of the saliency term $L' = L_{qr2} - L_{dr2}$. There are no second harmonic terms in the torque equation explicitly associated with L_{dr1} and L_{qr1} because the reference frame is fixed to these axes.

However, when i_d and i_q are expressed in terms of phase values, further expansion shows that there will be second harmonic torque terms associated with saliency in either machine.

6.6 THE WARD–LEONARD SYSTEM

One very familiar machine group is that forming the Ward–Leonard speed control system. This has now been superseded by static solid state control devices but it is still quite extensively used. For our present purposes it is a good example of an electrodynamical system and we shall use it to demonstrate the application of some of the features which are essential in the analysis of any interconnected machine group, however complicated. The student or engineer who is able to predict successfully the electromechanical transient response of a Ward–Leonard system in its nonlinear operating modes will be well equipped to handle most of the problems he may meet in electrical machine dynamics. This system does not, of course, involve conditions of parallel operation, or the special problems of machines operating continuously in synchronism and these will be treated separately later.

The machine system analysed by Main[2] is shown in Figure 6.12. In the conventional WL system an electrical motor, often an induction motor, drives a d.c. shunt or separately excited generator at constant speed. The output voltage of the WLG is variable and is controlled by a rheostat in the field circuit. The variable voltage output is applied to the armature of a separately excited d.c. motor (WLM) which operates with constant field current. The WLM can therefore be operated at variable speed, over a wide range, the speed being approximately proportional to the output voltage of the WLG and therefore controlled by the WLG field current.

In the system shown in Figure 6.12 it will be seen that the WLG and WLM have their field circuits fed from special exciter generators driven from the induction motor shaft. This method of excitation is not always used but it allows a certain degree of control refinement and provides a somewhat more reliable supply than some alternative sources. The arrangement, however, with its inherent response timelags does not enhance the electromechanical transient behaviour of the system following disturbances.

It is important to note that there is no electrical connection between the induction motor drive and other machines. This means that the electromechanical transient behaviour of the induction motor may be treated quite independently of the other machines, using the most convenient reference frame. The computed induction motor shaft torque may then be impressed as a known mechanical input to the WL generator/exciters which are all on the induction motor shaft. The speed of the induction motor will obviously be affected by the response of the rest of the system and an iterative computer programme is therefore required. The WL d.c. machines are clearly coupled electrically and an overall transient matrix is required for this part of the system, which will express the interactions of these machines. A connection matrix can be set up by inspection, as shown below, which will give the appropriate coupling of the equations for the individual machines.

In the investigation of conditions of small oscillations and overall hunting stability by eigenvalue analysis, given later, it is essential to have a single matrix for the whole system, including the induction motor. In this the p-operator must have the same significance for all machines, namely terms on which it operates should become zero in the steady state. In that case a special reference frame is required for the induction motor, in which the reference axes are stationary with respect to the synchronously rotating flux wave, in other words the reference axes also rotate synchronously. On reflection, this reference frame is similar to that in the d.c. machines, except that in the latter case the flux happens to be stationary in space and not rotating.

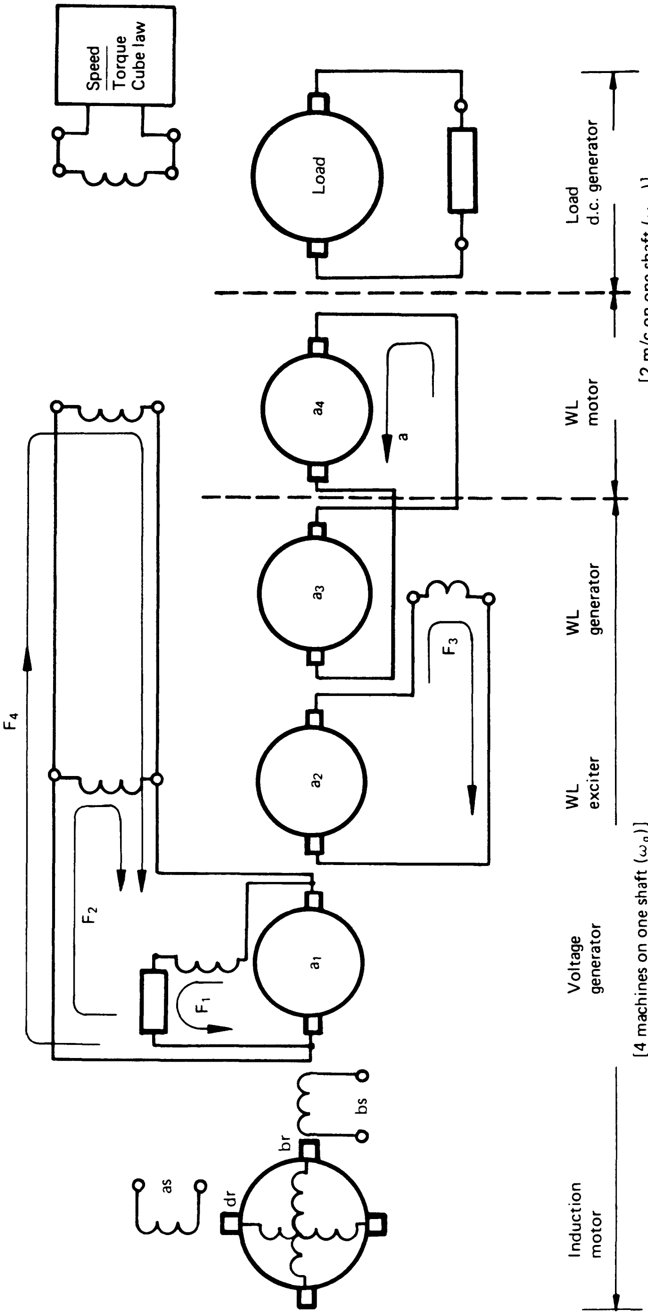


Figure 6.12 Ward–Leonard five-machine system

However, let us deal first with the steady-state and transient response of the system, in which the application of *different* reference frames to induction motor and d.c. machines will give a straightforward solution. The data for the machines in a laboratory system are given in Table 6.2 and the parameters are as shown in Table 6.3

Table 6.2 Ratings of the laboratory test machines

D.c. machines	
1. Ward–Leonard voltage generator	*A.c. induction motor*
10 kW	15 kW
230 V	415 V
43.5 A	30 A
1440 r.p.m.	1440 r.p.m.
2. Ward–Leonard generator exciter	
5 kW	
110 V	
45.5 A	
1440 r.p.m.	
3. Ward–Leonard generator	Dynamometer
10 kW	10 kW
230 V	100 V
43.5 A	100 A
1440 r.p.m.	1300 r.p.m.
4. Ward–Leonard motor	
10 kW	
230 V	
43.5 A	
1300 r.p.m.	

*Table 6.3 Parameters of the laboratory test machines**

D.c. machines

	machine 1	machine 2	machine 3	machine 4
R_f	18.1	13.0	16.25	11.8
R_a	0.05	0.037 5	0.0375	0.04
L_f	14.25	3.25	14.875	9.625
L_a	0.000 5	0.000 2675	0.000 837 5	0.000 912 5
M	0.238 75	0.052 375	0.227 5	0.18

Induction motor

R_s	0.035
R_r	0.032 5
L_s	0.014 5
L_r	0.011 4
M	0.012 4

* Per-unit values based on the induction motor rating, see Appendix 1.

Table 6.3 (continued)

Inertia constants

Induction motor/d.c. generator set = 0.22 kg m^2
Ward–Leonard motor/dynamometer set = 0.094 8 kg m^2

Frictional loss coefficients

Induction motor/d.c. generator set = 0.012 5 Nm/rad
Ward–Leonard motor/dynamometer set = 0.01 Nm/rad

6.6.1 The machine matrices

The simplest reference frame for the induction motor is that used by Stanley and Kron, with stationary direct and quadrature axes on the stator and rotor, and with the direct axis along the axis of one of the stator phases. The measured phase values of resistance and inductance can be used directly in the equations in this reference frame (test techniques are described in Appendix A2) and the transient impedance matrix becomes

$$\mathbf{Z} = \begin{array}{c|c|c|c|c|c|}
 & \mathrm{ds} & \mathrm{dr} & \mathrm{qr} & \mathrm{qs} & \mathrm{s} \\ \hline
\mathrm{ds} & R_s + L_s p & Mp & & & \\ \hline
\mathrm{dr} & Mp & R_r + L_r p & L_r p\theta & Mp\theta & \\ \hline
\mathrm{qr} & -Mp\theta & -L_r p\theta & R_r + L_r p & Mp & \\ \hline
\mathrm{qs} & & & Mp & R_s + L_s p & \\ \hline
\mathrm{s} & Mi_{qr} & L_r i_{qr} & -L_r i_{dr} & -Mi_{dr} & Jp + R_F \\ \hline
\end{array} \tag{6.37}$$

Since the induction motor structure is completely symmetrical about the axis of the shaft, the following substitutions have been made,

$$R_{ds} = R_{qs} = R_s$$
$$R_{dr} = R_{qr} + R_r$$
$$L_{ds} = L_{qs} = L_s$$
$$L_{dr} = L_{qr} = L_r$$
$$M_d = M_q = M$$

In steady state operation

$$p = \mathrm{j}\omega; \qquad p\theta = v\omega \text{ where } v = (1 - \text{slip})$$

$$i_{qs} = -\mathrm{j}i_{ds} \qquad V_{qs} = -\mathrm{j}V_{ds}$$

$$i_{qr} = -\mathrm{j}i_{dr} \qquad V_{qr} = -\mathrm{j}V_{dr}$$

The d.c. machines have the simple transient impedance matrix

$$\mathbf{Z} = \begin{array}{c} \\ \mathrm{F} \\ \mathrm{a} \\ \mathrm{s} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{F} & \mathrm{a} & \mathrm{s} \end{array} \\ \left[\begin{array}{ccc} R_f + L_f p & & \\ -Mp\theta & R_a + L_a p & \\ Mi_a & & Jp + R_F \end{array}\right] \end{array} \tag{6.38}$$

6.6.2 The system equations

The separate transient matrices may now be combined by a connection matrix **C**, which will give the coupling terms due to the electrical cross-connections between the d.c. machine field and armature windings. The first thing to note is that this interconnection matrix will not involve either time functions or functions of any of the machine variables, except the current flowing in the connections. The connection matrix, therefore will have terms which are plus or minus one or zero and the simple operation $\mathbf{C}_t\mathbf{Z}\mathbf{C}$ will give the overall matrix, where $\mathbf{Z}$ is the block diagonal sum of all the machine matrices. The induction motor matrix may in fact be included in this, since it is not electrically coupled to the other machines and retains its detached position in the new matrix.

To set up the **C**-matrix, current paths have been selected as in Figure 6.12. Any other set of nine independent meshes could have been chosen, coupling the twelve windings. The s-rows and columns give the speed relationships amongst the machines.

Equating current in the coils with the current flowing in the selected paths, we have

$$\begin{aligned} i_{f1} &= i_{f1} \\ i_{a1} &= i_{f1} + i_{f2} + i_{f4} \\ i_{f2} &= i_{f2} \\ i_a &= i_{f3} \\ &\text{etc.} \end{aligned} \tag{6.39}$$

The induction motor has electrical quantities impressed upon the stator, and generates mechanical shaft torque. The d.c. generators have impressed shaft torque, and generate electrical quantities. Therefore, in relation to *impressed* electrical quantities across the connection matrix, the direction of rotation of the d.c. machines will be negative relative to that of the induction motor. This is shown in the **C**-matrix. Another point to note is that all of the chosen current paths are closed, and while there are generated voltages in some of the circuits, given by terms such as $M_d i p\theta$, there are no externally impressed voltages, except those applied to the stator of the induction motor. Similarly, while there are generated torque components the only externally impressed torque is that impressed by the dynamometer load on the WL motor shaft. The torque impressed upon the shaft of the d.c. generators is, of course, generated by the induction motor—the sum of these two is represented in the sA column of the **C**-matrix.

The connection matrix therefore becomes as shown in equation (6.40).

C =

		F_1	F_2	F_3	F_4	a	as	ar	br	bs	sA	sB
	F_1	1										
VG	a1	1	1		1							
	s1										−1	
	F_2		1									
WLE	a2			1								
	s2										−1	
	F_3			1								
WLG	a3					1						
	s3										−1	
	F_4				1							
WLM	a4					−1						
	s4											1
	as						1					
	ar							1				
IM	br								1			
	bs									1		
	s5										1	

(6.40)

The operation $\mathbf{C}_t\mathbf{Z}\mathbf{C}$ now gives the overall transient impedance matrix $\mathbf{Z}'$. The new voltage matrix is given by $\mathbf{V}' = \mathbf{C}_t\mathbf{V}$ and the system equations are given in equation (6.41).

The moment of inertia J_g is that for the whole of the WL generator/induction motor shaft and J_m is that for the WL motor and load.

The steady state equations are obtained from the transient equation by making the appropriate substitutions for the p-operator, giving the equation shown in (6.42).

6.6.3 Nonlinearities

In all electrical machines there are several causes of nonlinear behaviour. In the context of machine dynamics some of these are more serious than others. Among the nonlinearities which may often be neglected in computing dynamical response are eddy currents and magnetic hysteresis effects in the iron circuits, and transient commutation effects. Three of the more important nonlinearities which must, in most cases, be considered, are as follows.

6.6.3(i) *The effect of mechanical friction*

The power loss due to friction is $R_F p\theta$ where R_F is the frictional loss coefficient. This is not normally a constant quantity but is a (weak) function of the angular velocity of the rotor. Values of R_F appropriate to the most important speed ranges can be determined by testing.

6.6.3(ii) *Magnetic saturation in the iron*

Examination of the open-circuit generated voltage curve for a d.c. machine will show that the rotational generated voltage $(Mi_{ds}p\theta)$ is not directly proportional to i_{ds}—in other words the rotational inductance M is a function of the field current. This can be approximated by the formulae given by Frölich, one of which gives generated voltage (see also equation 5.78)

$$V = \frac{a i_{ds}\omega_r}{b + i_{ds}} \tag{6.43}$$

and since

$$V = Mi_{ds}\omega_r$$

the rotational inductance

$$M = \frac{a}{b + i_{ds}} \tag{6.44}$$

where a and b are constants which can be determined by a curve-fitting procedure. The computer can be programmed to use the correct value of M for different values of i_{ds}. Where the field current undergoes continuous changes with time, as in the case of electromechanical oscillations or transients, an

$$
\begin{bmatrix}0\\0\\0\\0\\0\\V_{\mathrm{as}}\\0\\0\\V_{\mathrm{bs}}\\0\\0\end{bmatrix}
=
\begin{array}{c|ccccccccccc|}
 & \mathrm{F1} & \mathrm{F2} & \mathrm{F3} & \mathrm{F4} & \mathrm{a} & \mathrm{as} & \mathrm{ar} & \mathrm{br} & \mathrm{bs} & \mathrm{s1} & \mathrm{s2}\\
\hline
\mathrm{F1} & \begin{matrix}R_{\mathrm{f1}}+R_{\mathrm{a1}}\\+(L_{\mathrm{f1}}+L_{\mathrm{a1}})p\\-M_1p\theta_{\mathrm{g}}\end{matrix} & R_{\mathrm{a1}}+L_{\mathrm{a1}}p & & R_{\mathrm{a1}}+L_{\mathrm{a1}}p & & & & & & & \\
\mathrm{F2} & \begin{matrix}R_{\mathrm{a1}}+L_{\mathrm{a1}}p\\-M_1p\theta_{\mathrm{g}}\end{matrix} & \begin{matrix}R_{\mathrm{a1}}+R_{\mathrm{f2}}\\+(L_{\mathrm{f2}}+L_{\mathrm{a1}})p\end{matrix} & & R_{\mathrm{a1}}+L_{\mathrm{a1}}p & & & & & & & \\
\mathrm{F3} & & -M_2p\theta_{\mathrm{g}} & \begin{matrix}R_{\mathrm{a2}}+R_{\mathrm{f3}}\\+(L_{\mathrm{a2}}+L_{\mathrm{f3}})p\end{matrix} & & & & & & & & \\
\mathrm{F4} & \begin{matrix}R_{\mathrm{a1}}+L_{\mathrm{a1}}p\\-M_1p\theta_{\mathrm{g}}\end{matrix} & R_{\mathrm{a1}}+L_{\mathrm{a1}}p & & \begin{matrix}R_{\mathrm{a1}}+R_{\mathrm{f4}}\\+(L_{\mathrm{a1}}+L_{\mathrm{f4}})p\end{matrix} & & & & & & & \\
\mathrm{a} & & & -M_3p\theta_{\mathrm{g}} & M_4p\theta_{\mathrm{m}} & \begin{matrix}R_{\mathrm{a3}}+R_{\mathrm{a4}}\\+(L_{\mathrm{a3}}+L_{\mathrm{a4}})p\end{matrix} & & & & & & \\
\mathrm{as} & & & & & & R_{\mathrm{a}}+L_{\mathrm{s}}p & Mp & & & & \\
\mathrm{ar} & & & & & & Mp & R_{\mathrm{r}}+L_{\mathrm{r}}p & L_{\mathrm{r}}p\theta_{\mathrm{g}} & Mp\theta_{\mathrm{g}} & & \\
\mathrm{br} & & & & & & -Mp\theta_{\mathrm{g}} & -L_{\mathrm{r}}p\theta_{\mathrm{g}} & R_{\mathrm{r}}+L_{\mathrm{r}}p & Mp & & \\
\mathrm{bs} & & & & & & & & Mp & R_{\mathrm{s}}+L_{\mathrm{s}}p & & \\
\mathrm{s1} & -M_1(i_{\mathrm{f1}}+i_{\mathrm{f2}}+i_{\mathrm{f4}}) & -M_2i_{\mathrm{f3}} & -M_3i_{\mathrm{a}} & & & Mi_{\mathrm{br}} & L_{\mathrm{r}}i_{\mathrm{br}} & -L_{\mathrm{r}}i_{\mathrm{ar}} & -Mi_{\mathrm{ar}} & \begin{matrix}R_{\mathrm{g}}p\\+J_{\mathrm{g}}p^2\end{matrix} & \\
\mathrm{s2} & & & & M_4i_{\mathrm{a}} & & & & & & & \begin{matrix}R_{\mathrm{m}}p\\+J_{\mathrm{m}}p^2\end{matrix}\\
\hline
\end{array}
\cdot
\begin{bmatrix}i_{\mathrm{f1}}\\i_{\mathrm{f2}}\\i_{\mathrm{f3}}\\i_{\mathrm{f4}}\\i_{\mathrm{a}}\\i_{\mathrm{as}}\\i_{\mathrm{ar}}\\i_{\mathrm{br}}\\i_{\mathrm{bs}}\\\theta_1\\\theta_2\end{bmatrix}
\qquad (6.41)
$$

$$
\begin{array}{|c|}
\hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline V_{as} \\ \hline 0 \\ \hline 0 \\ \hline V_{bs} \\ \hline
\end{array}
=
\begin{array}{c}
\\ \mathrm{F1} \\ \mathrm{F2} \\ \mathrm{F3} \\ \mathrm{F4} \\ \mathrm{a} \\ \mathrm{as} \\ \mathrm{ar} \\ \mathrm{br} \\ \mathrm{bs}
\end{array}
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\multicolumn{1}{c}{\mathrm{F1}} & \multicolumn{1}{c}{\mathrm{F2}} & \multicolumn{1}{c}{\mathrm{F3}} & \multicolumn{1}{c}{\mathrm{F4}} & \multicolumn{1}{c}{\mathrm{a}} & \multicolumn{1}{c}{\mathrm{as}} & \multicolumn{1}{c}{\mathrm{ar}} & \multicolumn{1}{c}{\mathrm{br}} & \multicolumn{1}{c}{\mathrm{bs}} \\
\hline
R_{f1}+R_{a1}-M_1\omega_g & R_{a1} & & R_{a1} & & & & & \\ \hline
R_{a1}-M_1\omega_g & R_{a1}+R_{f2} & & R_{a1} & & & & & \\ \hline
 & -M_2\omega_g & R_{a2}+R_{f3} & & & & & & \\ \hline
R_{a1}-M_1\omega_g & R_{a1} & R_{a1}+R_{f4} & & & & & & \\ \hline
 & & -M_3\omega_g & M_4\omega_m & R_{a3}+R_{a4} & & & & \\ \hline
 & & & & & R_s & jX_m & & \\ \hline
 & & & & & jX_m & R_r & vX_r & vX_m \\ \hline
 & & & & & -vX_m & -vX_r & R_r & jX_m \\ \hline
 & & & & & & & jX_m & R_s \\ \hline
\end{array}
\cdot
\begin{array}{|c|}
\hline i_{f1} \\ \hline i_{f2} \\ \hline i_{f3} \\ \hline i_{f4} \\ \hline i_a \\ \hline i_{as} \\ \hline i_{ar} \\ \hline i_{br} \\ \hline i_{bs} \\ \hline
\end{array}
\qquad (6.42)
$$

iterative programme can be used which in the present case converges step by step quite rapidly to the correct values of the system parameters and the corrected values of the current, voltage, and speed of the machine as the calculation proceeds. Variable self-inductance can of course be handled in the same way.

6.6.3(iii) *The load characteristic*

The load on the shaft of the WL motor was assumed to be a fan, in which the required torque input varied as the cube of the speed ($T = k\omega_r^3$). There is no difficulty in including the simple speed–torque curve in the programme. In the laboratory the fan load was simulated by a dynamometer machine. A speed signal from a tachogenerator was fed to an electronic function generator, which in turn controlled the field current of the dynamometer. In transient tests, the equipment had a satisfactorily short time constant compared to the mechanical transient response time.

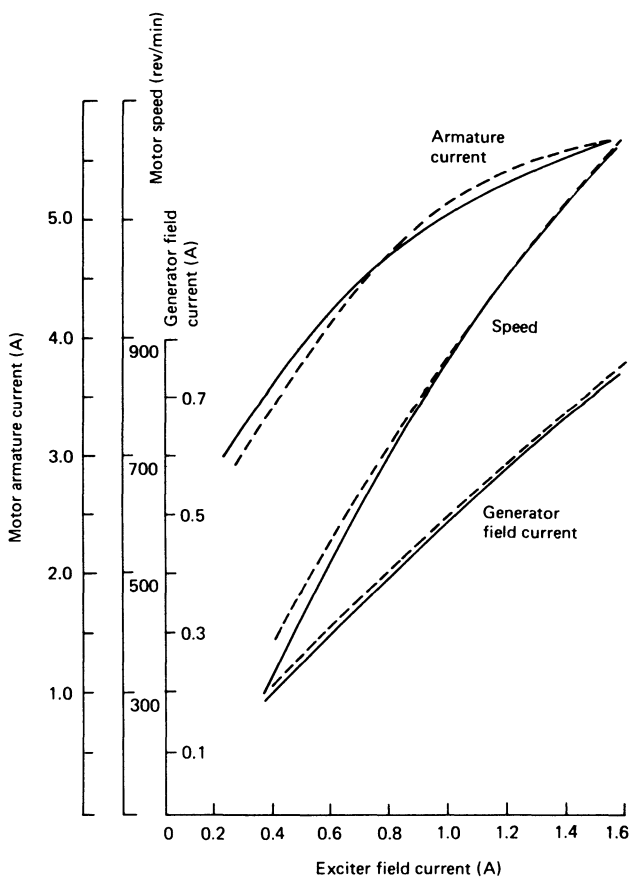


Figure 6.13 No-load characteristics—test and---computed

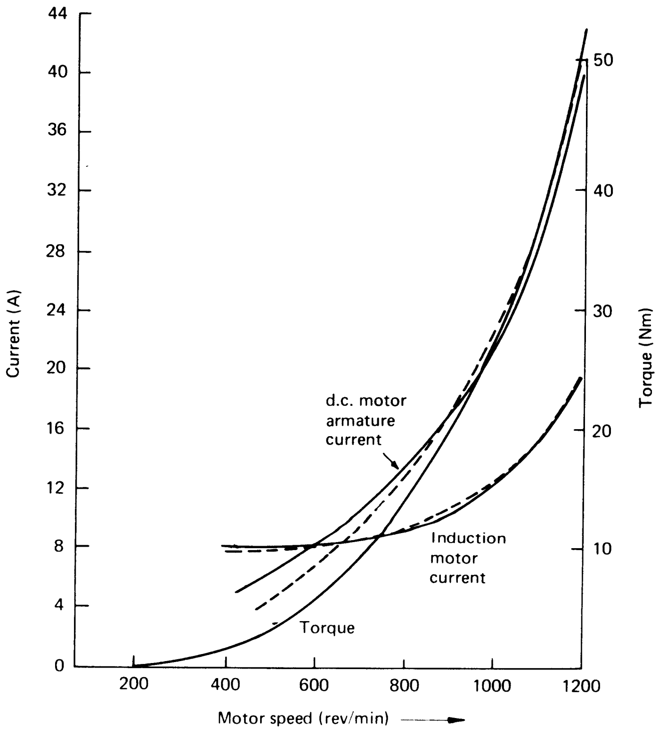


Figure 6.14 Motor currents and d.c. motor torque as a function of Ward–Leonard motor speed—test and---computed

6.6.4 Steady-state tests

To check the validity of the mathematical model for the WL system, steady-state tests were carried out in the laboratory. Some of the results are shown in Figures 6.13, 6.14, and 6.15, together with curves calculated using the step-by-step computer programme. A flow diagram for this is given in Figure 6.16. There is satisfactory agreement between the computed and test curves.

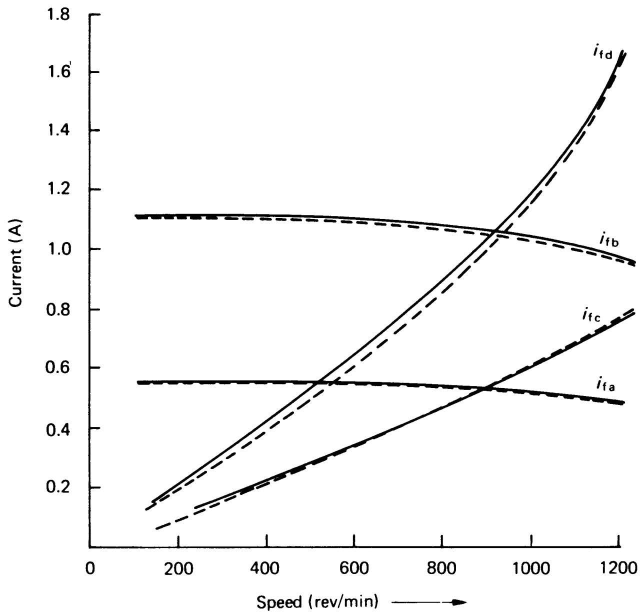


Figure 6.15 D.C. machine field currents as a function of Ward–Leonard motor speed—test,---computed, i_{fa} field current of voltage generator, i_{fb} field current of generator exciter, i_{fc} field current of generator and i_{fd} field current of Ward–Leonard motor

The system was then subjected to transient disturbances and the results were compared with those predicted by the computer model. This analysis is described in Section 6.6.5.

6.6.5 Transient analysis

To test the transient performance of the WL machines, sudden changes were made in various parts of the system and the behaviour measured, for example the dynamic response following instantaneous loss of load, sudden changes in load, and faults affecting the supply to the induction motor, etc.

Accurate simulation of the machines involves the nonlinearities mentioned in the previous section. The computer programme[2] is therefore organised to calculate the transient response by incremental time equations derived from the overall transient equation (6.41). A small time interval Δt is therefore chosen, during which all of the variable quantities in the equations are held constant. A Runge–Kutta iteration is then applied to each equation in turn as the appropriate quantities are computed over the set of equations. The computation then proceeds to the next time interval, using the new values in the equations and recalculating in a similar manner.

Consider the first row of the transient equation

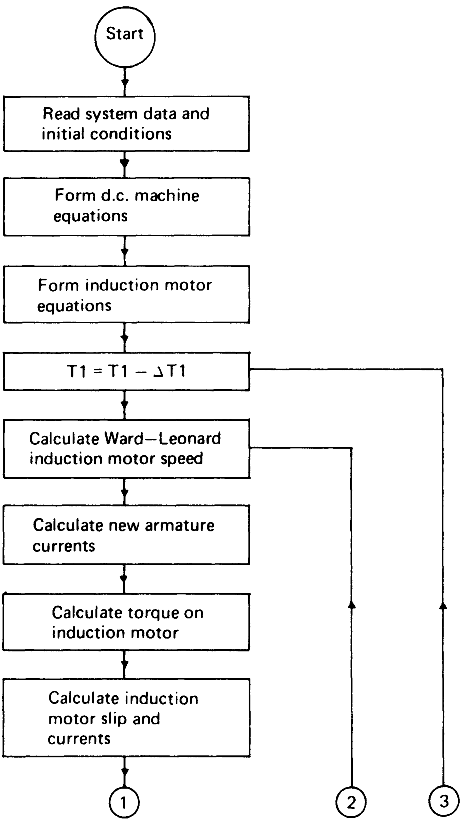
$$0 = \left\{ R_{f1} + R_{a1} + (L_{f1} + L_{a1})p - M_1 p\theta_g \right\} i_{f1} + (R_{a1} + L_{a1}p)i_{f2}$$
$$+ (R_{a1} + L_{a1}p)i_{f4} \qquad (6.45)$$

This can be rewritten

$$-pi_f = \{(R_{a1} + R_{f1} - M_1 p\theta_g)i_{f1} + (R_{a1} + L_{a1}p)i_{f2}$$
$$+ (R_{a1} + L_{a1}p)i_{f4}\}/(L_{a1} + L_{f1}) \qquad (6.46)$$

If the field current i_{f1} of machine 1 is the quantity of interest in equation (6.45), as in the first test described below, the operator p can be replaced by $\Delta i_{f1}/\Delta t$ to give

$$-\frac{\Delta i_{f1}}{\Delta t} = \left\{ (R_{a1} + R_{f1} - M_1 p\theta_g)i_{f1} + R_{a1} i_{f2} + R_{a1} i_{f4} \right\} / (L_{a1} + L_{f1}) \qquad (6.47)$$



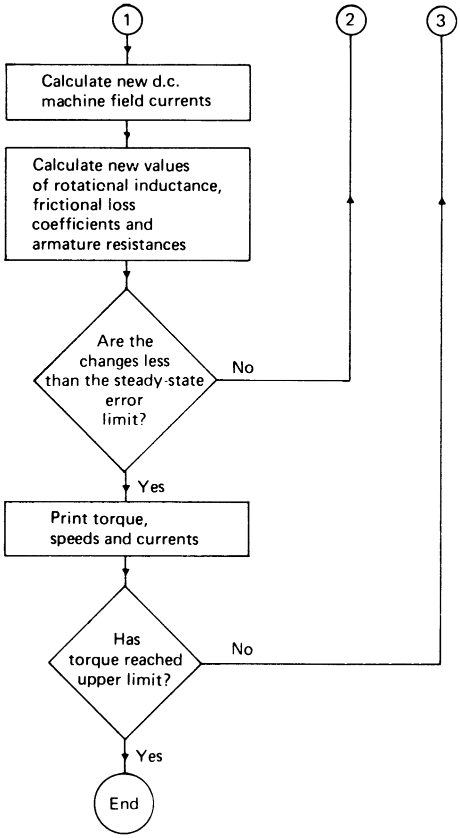


Figure 6.16 Flow chart for program 1

The other incremental time equations for the system are

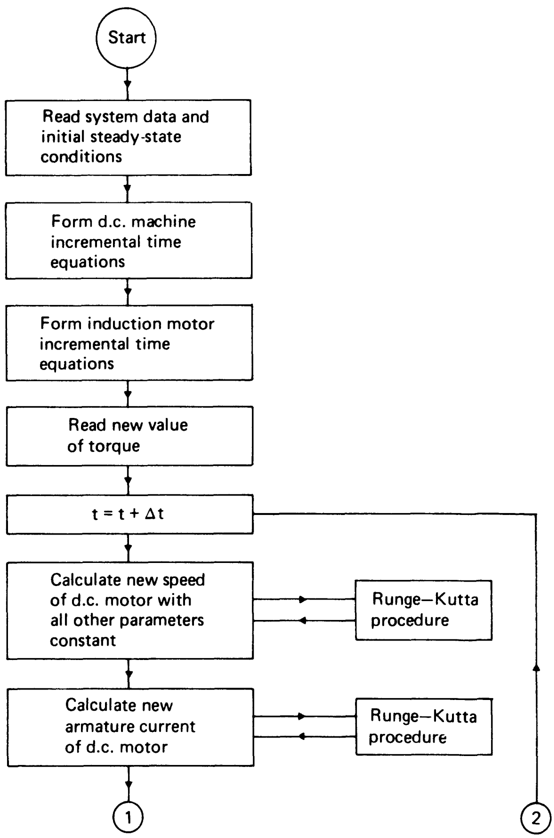
$$
\begin{aligned}
-\frac{\Delta i_{f2}}{\Delta t} &= \left\{(R_{a1} - M_1\omega_g)i_{f1} + (R_{a1} + R_{f2})i_{f2} + R_{a1}i_{f4}\right\}/(L_{a1} + L_{f2}) \\
-\frac{\Delta i_{f3}}{\Delta t} &= \left\{-M_2\omega_g i_{f2} + (R_{a2} + R_{f3})i_{f3}\right\}/(L_{a2} + L_{f3}) \\
-\frac{\Delta i_{f4}}{\Delta t} &= \left\{(R_{a1} - M_1\omega_g)i_{f1} + R_{a1}i_{f2} + (R_{a1} + R_{f4})i_{f4}\right\}/(L_{a1} + L_{f4}) \qquad (6.48) \\
-\frac{\Delta i_a}{\Delta t} &= \left\{-M_3\omega_g i_{f3} + M_4\omega_m i_{f4} + (R_{a3} + R_{a4})i_a\right\}/(L_{a3} + L_{a4}) \\
-\frac{\Delta \omega_g}{\Delta t} &= \left\{T_e - (i_{f1} + i_{f2} + i_{f4})M_1 i_{f1} - M_2 i_{f3} i_{f2} - M_3 i_a i_{f3} - R_g\omega_g\right\}/J_g \\
-\frac{\Delta \omega_m}{\Delta t} &= \left\{-T_L + M_4 i_a i_{f4} + R_m\omega_m\right\}/J_m
\end{aligned}
$$

The Runge–Kutta iteration is applied to each equation in turn. The flow chart for this programme is shown in Figure 6.17 to predict the transient response of the system to a step change in load and the computed and test results are given in Figure 6.18.

6.6.6 Laboratory tests

6.6.6(i) *Instantaneous loss of load*

The WL set is gradually loaded, at a suitable speed, with the fan load simulated as described above. To produce the effect of loss of load, a switch in the load dynamometer field circuit is opened and the voltage, current, and speed recorded until stable conditions are reached. In the computation, the steady-state quantities at this load and speed are calculated, then the appropriate sudden change is made in the circuit equations and the Runge–Kutta iterative procedure is applied. The results of the test and the response predicted by the computer programme are shown in Figure 6.19.



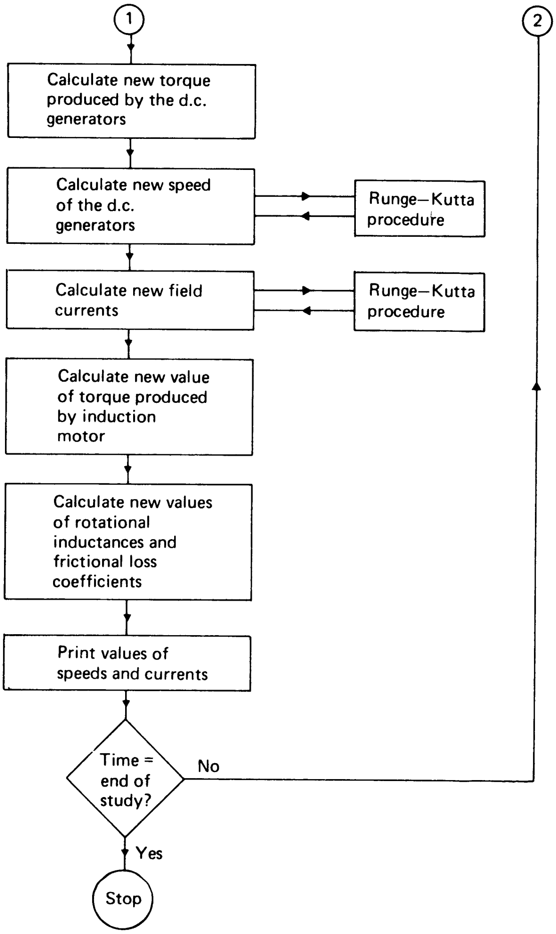


Figure 6.17 Flow chart for program 2

6.6.6(ii) *Step change in load*

This can be produced on test by a sudden change in the value of the load resistance across the output of the dynamometer. In the computer programme the incremental changes are now complicated by the cube-law characteristic of the load. In this test the field time-constant of the dynamometer fan simulator also has a significant effect on the response. The results are shown in Figure 6.18.

6.6.6(iii) *Faults*

The effect of partial short-circuit at the induction motor supply is shown in Figure 6.20(b) with the fault removed after one second. The flow diagram for this calculation is shown in Figure 6.21.

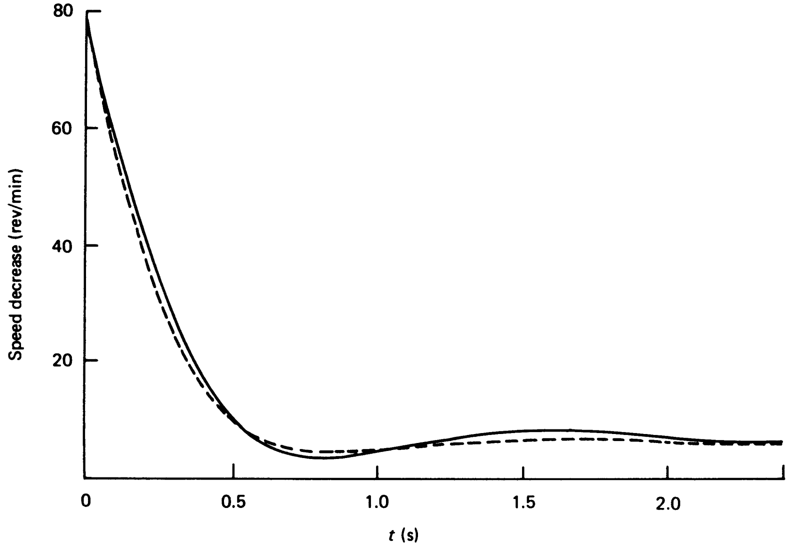


Figure 6.18 Response of Ward–Leonard system induction motor to an instantaneous increase in fan load.—test and---computed

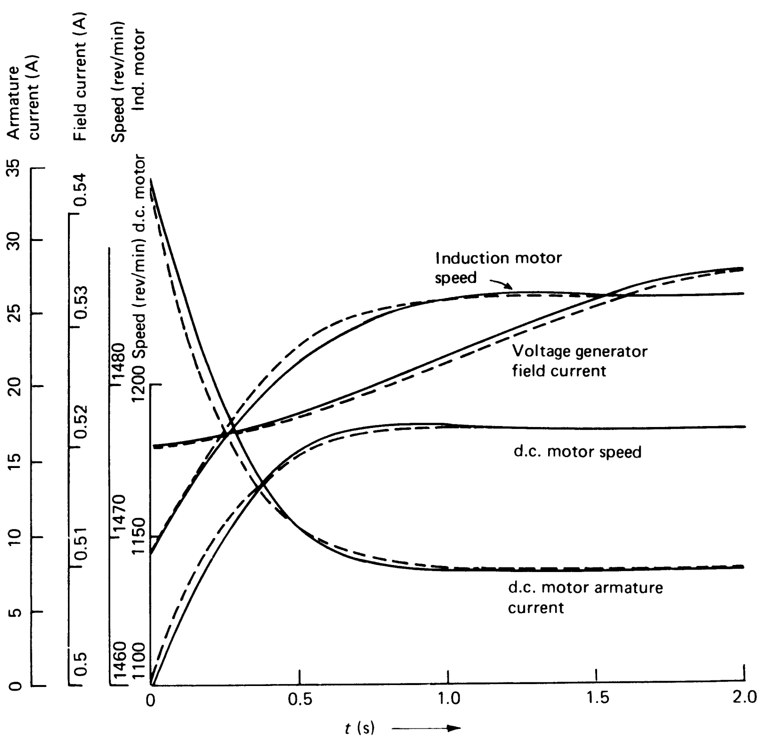


Figure 6.19 Ward–Leonard system response to instantaneous loss of fan load.—test and---computed

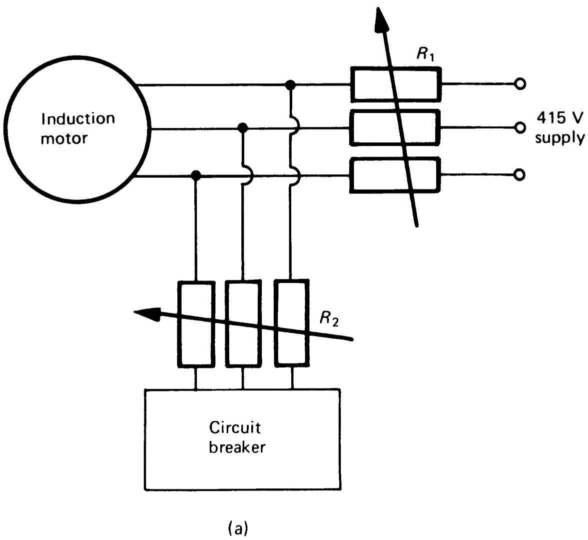


Figure 6.20(a) Arrangement for applying a three-phase fault to the induction motor supply

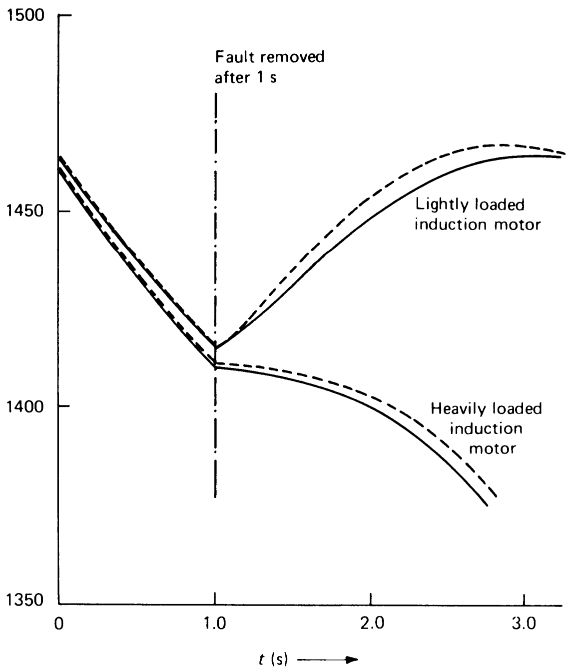
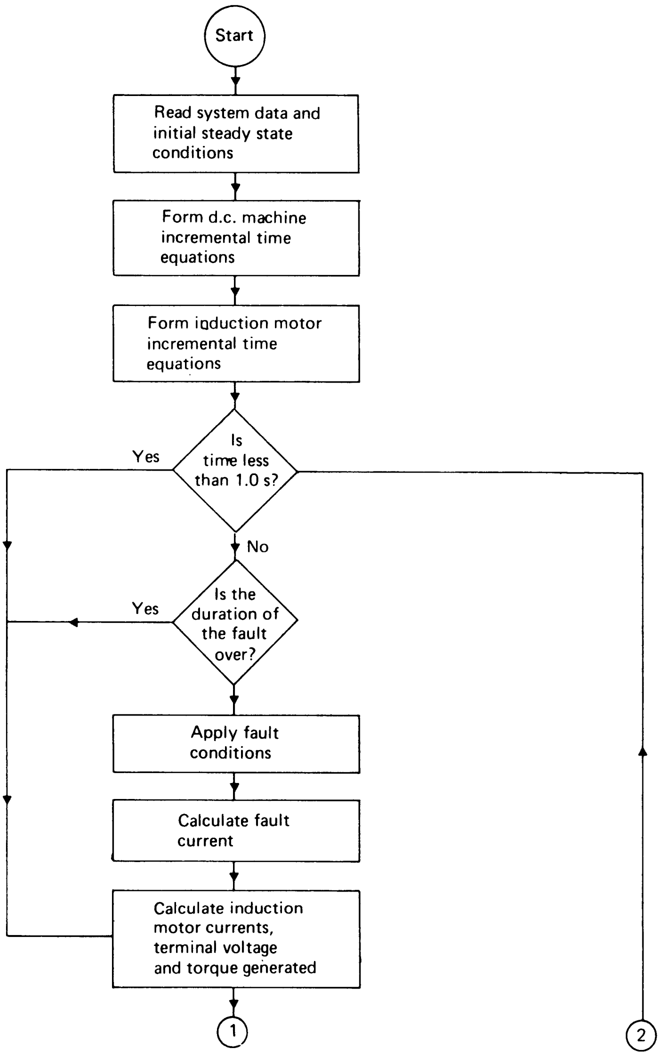


Figure 6.20 (b) Induction motor speed following a three-phase short circuit of the supply.—test and ---computed

6.7 POWER SYSTEM DYNAMICS

In Section 4.3.2 the transient stability of synchronous machines has been discussed, with a description of the equal-area criterion of stability, from which can be determined the critical synchronous angle of swing or departure of the rotor from the steady state position; for instance immediately following a fault on the system. Solution of the nonlinear second order differential equation (4.50) will then give the time available for the automatic circuit breakers to clear the fault.

In Section (6.4) we looked at the transient dynamical behaviour of an alternator and induction motor. The relationship between these two problems



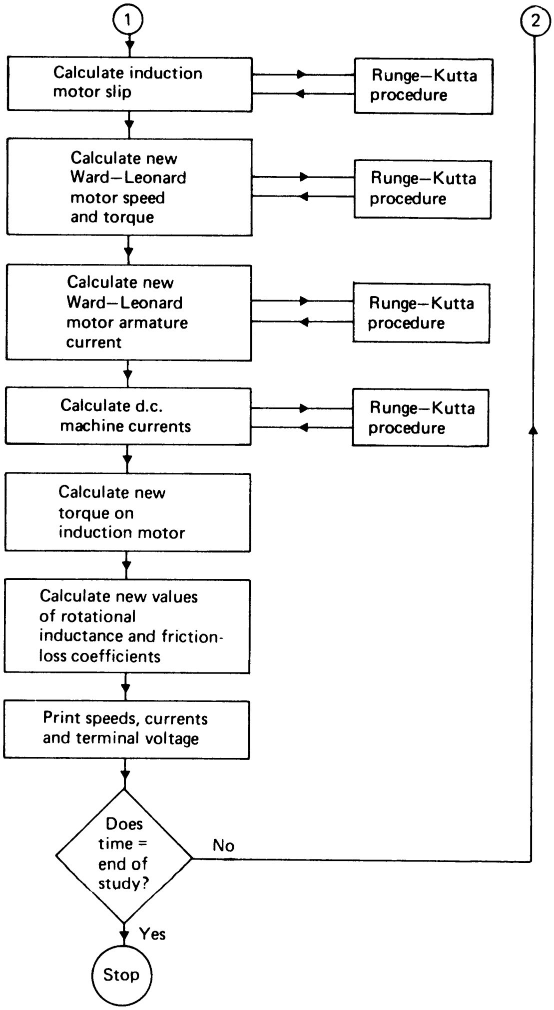


Figure 6.21 Flow chart for program 3

has been investigated (by the Central Electricity Generating Board) by tests on the national grid system. Alternator swing causes rapid change in system voltage and frequency, and these changes, of course, affect the performance of large induction motors. Thus in some cases, although the alternator may have adequate transient stability, the induction motors, because of their particular voltage–speed–torque characteristics may not recover from the effects of transient voltage dip and the consequent increase in slip, and they may simply stall. This could be a serious problem for the supply authority.

The dynamical response of the turbo-alternators and induction motors can be determined exactly as for the laboratory machines previously discussed. The problem in general is now much larger, however, for several reasons.

(1) There will usually be many generating stations, lines, and load points on the system.
(2) The analysis must include the effects of different kinds of faults namely, phase-phase etc., and these may occur anywhere in the system.
(3) Disturbances may be caused by routine switching operations outside or inside the stations.
(4) The system response will be strongly affected by the action of fast acting voltage regulators and turbine governors.
(5) The initial loading of the system network and machines will have an important affect on the dynamical behaviour.

The effects of regulators and governors will be discussed in Chapters 7 and 8. A typical sequence of events might be as follows:

(i) a system short-circuit occurs, say about 60 km away from the station;
(ii) the output current from the station increases, but this is now largely reactive fault current;
(iii) the *real* power output (MW) from the alternator drops and the turbines accelerate, with transient increases in frequency;
(iv) the station voltage falls;
(v) the fault is cleared;
(vi) the induction motors have seen a double effect, namely a transient drop in voltage and a transient increase in slip;
(vii) the induction motors will also accelerate and then recover normal running along with the alternator speed changes when the fault is cleared;
(viii) or the induction motors will fail to recover from the higher slip condition even when the fault is cleared and they will then stall.

A similar sequence arises if certain system switching operations are carried out which affect the load on the station. Inadvertent voltage drop may be caused, or even loss of synchronism with the grid.

Two important aspects of the operation of any power system are the generation and transmission of reactive MV A (or megavars, MVAr). These have considerable effect on the voltage since the system reactances are high compared to the resistance values. If a feeder which operates in parallel with others should be tripped when it is heavily loaded with reactive MV A, there will almost certainly be considerable voltage drop at some points on the system. Even the operation of switching parallel feeders in or out of circuit can cause considerable voltage fluctuations if the feeders carry significant amounts of reactive power — this is partly due to the normal time-constants of the circuits involved. To investigate this dynamical problem on the power system, all of the parameters are required, together with the load and voltage details needed for an initial load-flow computation. This will be carried out by an iterative programme such as the Gauss–Seidel method. When the load flow is known, the alternators can be represented by their transient reactances and the loads by equivalent impedances. A computer programme can now be arranged to give the system current, voltage, power, motor torques, and speed and the frequency change (if any) for the duration of the disturbance. It should be noted that, particularly when loss of synchronism with the grid occurs, or when the system frequency change is significant, this will also require an iterative programme. The motor and alternator torque and speed cannot be calculated unless the system reactances are known. These values, of course, depend on the frequency which is determined by the alternator speed, which depends on the speed–torque characteristics of the turbines. The sequential computation continues for the duration of the transient response which will persist even after the cause of the disturbance is cleared.

There are other complications. One of these is that the reactance of each alternator changes continuously from a low value of transient reactance towards a much higher value, approaching the synchronous reactance, during the disturbance and the computation must allow for this.

However, even with these and other nonlinear difficulties, it is found that in practice the iterative solutions normally converge quite rapidly.

6.8 SMALL PERTURBATIONS (HUNTING)

The effects of small perturbations will be taken up in greater detail in Chapters 7 and 8 but we shall introduce the subject here and look at mechanical analogies for the electromechanical systems which we shall meet. Hunting is a sustained state of sinusoidal perturbation in speed, about the normal constant mean value. The hunting characteristics of machines or systems differ from those in the steady state or under transient conditions.

Let us consider first an induction motor and then a synchronous motor, and examine very briefly the effects of a small disturbance to the motion of each.

The speed–torque curve of the induction motor is as shown in Figure 6.22. If the load on the shaft increases by a small amount for a short time, the speed will drop and then return to the original value, as in Figure 6.23. This behaviour is analogous to that of a mechanical system shown in Figure 2.6. In terms of the speed this is a first order system described by the simple equation

$$J\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\omega_r) + D\Delta\omega_r = 0 \tag{6.49}$$

where D is a damping coefficient. Also,

$$J\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\omega_r) = \Delta T_{\mathrm{elect.}} \tag{6.50}$$

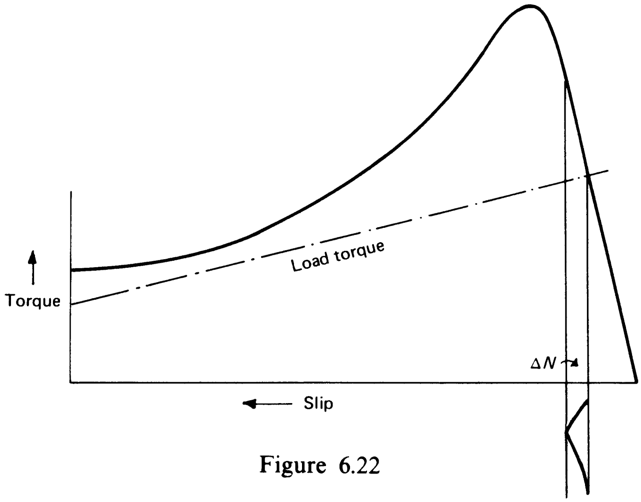


Figure 6.22

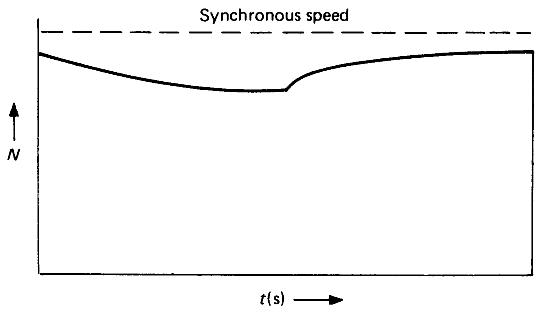


Figure 6.23

and

$$D = -\frac{\Delta T_e}{\Delta \omega_r} \tag{6.51}$$

Thus D is constant in the linear range of the speed–torque curve. It is also independent of ω_r and equation (6.49) is a linear differential equation. Speed changes in an induction motor are invariably damped as we can see by comparing the machine equation (6.50) with that for the spring loaded mechanical system. In the latter,

$$m\frac{d^2}{dt^2}(\Delta x) + D\frac{d}{dt}(\Delta x) + K\Delta x = 0 \tag{6.52}$$

The analogous equation for a series RLC electrical circuit is

$$L\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\Delta q)+R\frac{\mathrm{d}}{\mathrm{d}t}(\Delta q)+\frac{\Delta q}{C}=0 \tag{6.53}$$

In these two systems, if

$$\left.\begin{array}{l}\dfrac{R}{2L}>\dfrac{1}{LC}\\ \dfrac{D}{m}>\dfrac{K}{m}\end{array}\right\}\ \text{the system is damped and does not oscillate} \tag{6.54}$$

$$\left.\begin{array}{l}\dfrac{R}{2L}<\dfrac{1}{LC}\\ \dfrac{D}{m}<\dfrac{K}{m}\end{array}\right\}\ \text{the system oscillates} \tag{6.55}$$

In the case of an induction motor subjected to a change in speed, the displacement or spring coefficient K is nonexistent. We shall investigate this further in Chapter 7. With a synchronous machine there is a very strong displacement term which causes synchronising power to flow. Disturbances of short duration, acting with the rotor inertia, will cause oscillation about the mean speed, as in Figure 6.24, the synchronous load angle also undergoing sinusoidal oscillation about the steady state position. Referring to equation (4.44) we have

$$\Delta(Mp^2\delta)=\Delta P_{\text{input}}-\Delta\left(\frac{EV}{X}\sin\delta\right) \tag{6.56}$$

Since in the state of free oscillation, the impressed power ΔP_{input} is zero,

$$Mp^2(\Delta\delta)+\left[\frac{EV}{X}\cos\delta\right]\Delta\delta=0 \tag{6.57}$$

or

$$Mp^2(\Delta\delta)+P_{\mathrm{s}}\Delta\delta=0 \tag{6.58}$$

where P_{s} is the synchronising power coefficient.

In terms of torque

$$Jp^2(\Delta\delta)+T_{\mathrm{s}}\Delta\delta=0 \tag{6.59}$$

If we assume that $\Delta\delta$ is small, then

$$\frac{EV}{X}\cos\delta\approx\frac{EV}{X}\cos(\delta\pm\Delta\delta)\approx\text{constant} \tag{6.60}$$

and equation (6.57) is then a linear differential equation and the synchronising power (or torque) coefficient is constant.

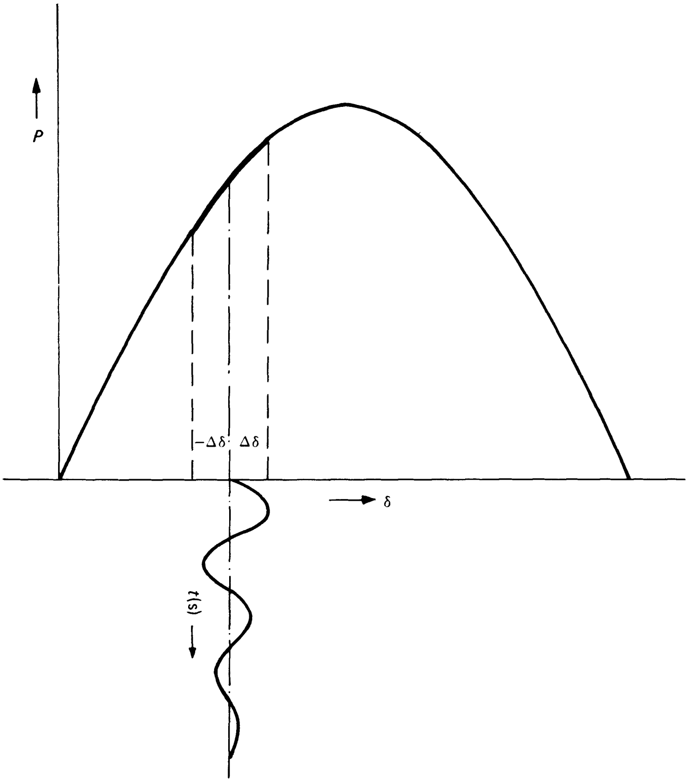


Figure 6.24

If we include the effect of damping in the machine, equations (6.58) and (6.59) become

$$Mp^2\Delta\delta + P_{\mathrm{D}}p\Delta\delta + P_{\mathrm{S}}\Delta\delta = 0 \tag{6.61}$$

or

$$Jp^2\Delta\delta + T_{\mathrm{D}}p\Delta\delta + T_{\mathrm{S}}\Delta\delta = 0 \tag{6.62}$$

6.9 REFERENCES

1. Gorley, J. S., 'Application of Diakoptics to Power System Analysis', *PhD Thesis*, University of Liverpool (1972).
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3. Stanley, H. C., 'An Analysis of the Induction Machine', *Trans. A.I.E.E.*, **57**, 751 (1938).

7

Small oscillations

7.1 INTRODUCTION

The problem of small oscillations is encountered mainly with machines in which there is some form of feedback. This may consist of signals generated within the machine inherently, due to the configuration of windings, as in an amplidyne cross-field generator, or externally, for example, in the form of a control system, or a voltage regulator on a d.c. generator. In a large power system, with synchronous generators interconnected through long lines and transformers, or with mismatched voltage regulators, dynamic instability may be a problem. In the case of long lines the feedback signals arise from the effect of the generated current flowing from the machine into the external circuit and the inductive reaction of such current upon the field circuit, upon which the generated voltage and current depend. Synchronous or other types of motor with feedback are prone to small oscillations or hunting. These small oscillations may be (1) forced, or (2) self-excited, as in many physical systems. A diesel alternator may be subjected to forced oscillations due to cyclic variations in torque from the prime mover. It is essential therefore to determine the natural frequencies of the system and to see if any are close to the frequency of impressed cyclic variation.

Self-excited oscillations are fairly common in power systems. Usually two kinds of self-excited oscillations are encountered. One is a high-frequency electrical oscillation, to which the machine rotors, because of their high inertia, cannot respond. In our present study we are concerned with machine dynamics and therefore these high-frequency oscillations are not discussed. However, the analysis of the complete system using, for example, eigenvalue techniques, will indicate whether or not such frequencies will be present. We are primarily concerned here with low-frequency oscillations (a few hertz) to which the rotors can respond. These are electromechanical in nature.

Since the oscillations are of small amplitude, the system equations may be linearised about the operating points. This makes it fairly easy to evolve criteria which will tell us whether the machines are dynamically stable or not, using methods such as the following:

(1) the Nyquist criterion
(2) Routh–Hurwitz criterion
(3) computation of the eigenvalues
(4) computation of the synchronising and damping torque coefficients
(5) *D*-partition techniques
(6) computation of rotor swing with time—the so-called swing curves

In the case of transient (nonlinear) instability, the standard approach is to compute the swing curves. This method, however, is unnecessarily costly and time-consuming in the case of small (linear) oscillations and methods 2, 3, 4, and 5 are all more widely used in dynamic stability studies. In the following sections the most useful of these methods, namely 2, 3, and 4, will be discussed in some detail. The damping and synchronising torque coefficients not only indicate whether or not a system is stable, they clearly indicate the extent of the stability/instability region and provide physical concepts which are not obtained by other methods.

7.2 SYNCHRONOUS MACHINE EQUATION DURING SMALL OSCILLATIONS

Having analysed earlier (Section 6.8) the physical forces which control the kinetics of the rotor in an electrical machine, we are in a position to write down the dynamic equation during small oscillations as

$$Jp^2\Delta\theta + T_{de}p\Delta\theta + R_Fp\Delta\theta + T_S\Delta\theta = \Delta T_i \tag{7.1}$$

where $T_S\Delta\theta$ = synchronising torque

$T_{de}p\Delta\theta$ = electrical damping torque

$R_Fp\Delta\theta$ = mechanical damping torque

$Jp^2\Delta\theta$ = rotational torque due to the inertia of the rotor

ΔT_i = change in input/output torque

(We know from mechanical systems that frictional *force* is proportional to velocity, and *torque* due to friction is proportional to angular velocity.)

When the oscillations are self-excited $\Delta T_i = 0$ and hence

$$Jp^2\Delta\theta + (T_{de} + R_F)p\Delta\theta + T_S\Delta\theta = 0 \tag{7.2}$$

and the conditions for stability are as follows:

(a) if $T_S = 0$, the machine does not oscillate, but loses synchronism
(b) if $(T_{de} + R_F) = 0$ oscillations begin and are self-sustained

(c) $(T_{de} + R_F) < 0$ oscillations build up in magnitude and the machine loses synchronism

(d) $(T_{de} + R_F) > 0$ there may or may not be oscillations but in either case the system will be stable if $T_S > 0$

These conditions are shown in Figure 7.1.

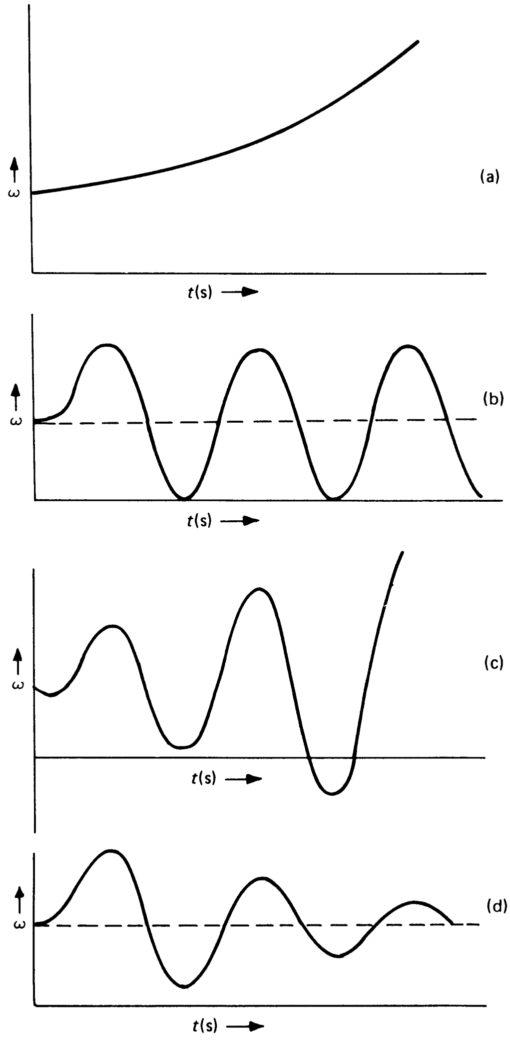

Figure 7.1 Dynamic response of a synchronous machine. (a) Non-oscillatory increase of rotor angle ($T_s = 0$), (b) sustained oscillations ($T_{de} + R_F = 0$), (c) negatively damped oscillations ($T_{de} + R_F < 0$) and (d) damped oscillations ($T_{de} + R_F > 0$)

We shall now see how in general we can write the torque equation in terms of fluxes and current and obtain expressions for T_{de} and T_S.

7.3 GENERAL EQUATIONS FOR SMALL OSCILLATIONS

In Section 6.2 the transient equation for torque was derived

$$Jp^2\theta + R_F p\theta - \mathbf{i}_t^* \mathbf{G}\mathbf{i} = T_{\text{input}} \tag{7.3}$$

For small oscillations, we take increments in θ, $\mathbf{i}$, and T_i, and may write

$$Jp^2\Delta\theta + R_F p\Delta\theta - \mathbf{i}_t^* \mathbf{G}\Delta\mathbf{i} - \Delta\mathbf{i}_t^* \mathbf{G}\mathbf{i} - \mathbf{i}_t^* \frac{\partial \mathbf{G}}{\partial\theta}\Delta\theta\,\mathbf{i} = \Delta T_i \tag{7.4}$$

In machines with stationary axes (e.g. commutator axes, real or fictitious) $\mathbf{G}$ is composed of constant terms, and hence $\partial\mathbf{G}/\partial\theta = 0$.

Rearranging equation (7.4), we have

$$Jp^2\Delta\theta + R_F p\Delta\theta - \mathbf{i}_t^*(\mathbf{G} + \mathbf{G}_t)\Delta\mathbf{i} = \Delta T_i \tag{7.5}$$

In order to compute the components of the vector $\Delta\mathbf{i}$ we have to turn to the voltage equation,

$$\mathbf{V} = \mathbf{R}\mathbf{i} + \mathbf{L}p\mathbf{i} + \mathbf{G}\mathbf{i}p\theta \tag{7.6}$$

If the rotor moves through $\Delta\theta$, the voltages and currents change and

$$\begin{aligned} \mathbf{V} + \Delta\mathbf{V} = {} & \mathbf{R}(\mathbf{i} + \Delta\mathbf{i}) + \mathbf{L}p(\mathbf{i} + \Delta\mathbf{i}) + \mathbf{G}(\mathbf{i} + \Delta\mathbf{i})p\theta \\ & + \mathbf{G}\mathbf{i}p(\Delta\theta) \end{aligned} \tag{7.7}$$

Subtracting equation (7.6) from equation (7.7)

$$\Delta\mathbf{V} = \mathbf{R}\Delta\mathbf{i} + \mathbf{L}p\Delta\mathbf{i} + \mathbf{G}p\theta\Delta\mathbf{i} + \mathbf{G}\mathbf{i}p(\Delta\theta)$$

and

$$\Delta T_i = Jp^2\Delta\theta + R_F p\Delta\theta - \Delta\mathbf{i}_t^*(\mathbf{G} + \mathbf{G}_t)\mathbf{i} \tag{7.8}$$

We can combine these two equations to give a common voltage/torque equation written in matrix form. Currents $\mathbf{i}$ and angular velocity $p\theta$ are represented by $\mathbf{i}_0$ and $p\theta_0$, their steady-state values,

$$\begin{array}{|c|}\hline \Delta\mathbf{V} \\ \hline \Delta T_i \\ \hline \end{array} = \begin{array}{|c|c|}\hline \mathbf{R} + \mathbf{L}p + \mathbf{G}p\theta_0 & \mathbf{G}\mathbf{i}_0 \\ \hline -\mathbf{i}_0^*(\mathbf{G} + \mathbf{G}_t) & Jp + R_F \\ \hline \end{array} \cdot \begin{array}{|c|}\hline \Delta\mathbf{i} \\ \hline p\Delta\theta \\ \hline \end{array} \tag{7.9}$$

R, **L**, **G**, $\Delta\mathbf{V}$, $\Delta\mathbf{i}$ represent matrices. Equation (7.9) in an expanded form for a four-coil primitive machine is

$$
\begin{bmatrix} \Delta V_{ds} \\ \Delta V_{dr} \\ \Delta V_{qr} \\ \Delta V_{qs} \\ \Delta T_i \end{bmatrix}
=
\begin{array}{c|ccccc}
 & \text{ds} & \text{dr} & \text{qr} & \text{qs} & \text{s} \\
\hline
\text{ds} & R_{ds}+L_{ds}p & M_d p & & & \\
\text{dr} & M_d p & R_r + L_{dr}p & L_{qr}p\theta & M_q p\theta & L_{qr}i_{qr} + M_q i_{qs} \\
\text{qr} & -M_d p\theta & -L_{dr}p\theta & R_r + L_{qr}p & M_q p & -M_d i_{ds} - L_{dr}i_{dr} \\
\text{qs} & & & M_q p & R_{qs}+L_{qs}p & \\
\text{s} & -i^{*}_{qr}M_d & -i^{*}_{qs}M_q - (L_{qr}-L_{dr})i^{*}_{qr} & -i^{*}_{ds}M_d - i^{*}_{dr}(L_{qr}-L_{dr}) & -i^{*}_{dr}M_q & Jp + R_F
\end{array}
\cdot
\begin{bmatrix} \Delta i_{ds} \\ \Delta i_{dr} \\ \Delta i_{qr} \\ \Delta i_{qs} \\ p\Delta\theta \end{bmatrix}
\tag{7.10}
$$

7.4 REPRESENTATION OF THE OSCILLATION EQUATIONS IN STATE-VARIABLE FORM

If we intend to determine the stability of an electrical machine during small oscillations by computing the damping and synchronising torque coefficients, it is sufficient to use the voltage/torque equation (7.10). However, this form of representation is not useful when the Routh–Hurwitz criterion is to be applied or eigenvalues are to be computed. For these cases, it is particularly convenient to express the performance equations in the state-variable form.

An investigation of synchronous machine equations and parameters in state variable form will be found in reference 1.

Equation (7.10) can be written with matrix elements

$$
\begin{bmatrix} \Delta\mathbf{V} \\ \Delta T_i \end{bmatrix}
=
\begin{bmatrix} \overline{\mathbf{L}} & \\ & J \end{bmatrix}
p
\begin{bmatrix} \Delta\mathbf{i} \\ \Delta\omega \end{bmatrix}
+
\begin{bmatrix} \mathbf{R} + \mathbf{G}p\theta_0 & \mathbf{G}\mathbf{i}_0 \\ -\mathbf{i}_0^{*}(\mathbf{G}+\mathbf{G}_t) & R_F \end{bmatrix}
\cdot
\begin{bmatrix} \Delta\mathbf{i} \\ \Delta\omega \end{bmatrix}
\tag{7.11}
$$

We have added the term R_F, to account for any mechanical friction that may be presented and $p\Delta\theta$ has been replaced by $\Delta\omega$. The above equation can be rearranged and written in a condensed form

$$p\Delta\mathbf{x} = -\mathbf{L}^{-1}(\overline{\mathbf{A}})\Delta\mathbf{x} + (\mathbf{L})^{-1}\Delta\mathbf{f} \tag{7.12}$$

where

$$\Delta\mathbf{x} = \begin{array}{|c|}\hline \Delta\mathbf{i} \\ \hline \Delta\omega \\ \hline \end{array}$$

$$\Delta\mathbf{f} = \begin{array}{|c|}\hline \Delta\mathbf{V} \\ \hline \Delta T_{\mathrm{i}} \\ \hline \end{array}$$

$$\mathbf{L} = \begin{array}{|c|c|}\hline \bar{\mathbf{L}} & \\ \hline & J \\ \hline \end{array}$$

$$\bar{\mathbf{A}} = \begin{array}{|c|c|}\hline \mathbf{R} + \mathbf{G}p\theta_0 & \mathbf{G}\mathbf{i}_0 \\ \hline -\mathbf{i}_0^*(\mathbf{G} + \mathbf{G}_{\mathrm{t}}) & R_{\mathrm{F}} \\ \hline \end{array}$$

Equation (7.12) is of the state variable form

$$\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{u} \tag{7.13}$$

The characteristic equation is

$$\det\,(s\mathbf{I} - \mathbf{A}) = 0$$

where $\mathbf{I}$ is a unit matrix and s is the Laplace operator — in our equations this is the operator $p = \mathrm{d}/\mathrm{d}t$. Obtaining an expression for the determinant is a tedious task. We shall use a simple method which can be used to evaluate the coefficients of the characteristic equation with the help of a digital computer. In the general case our operator p will be replaced with the usual operator s which may be a complex quantity with real and imaginary components.

7.4.1 Bocher's formula

If a state variable matrix equation is expressed

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

the nth order polynomial representing its characteristic equation is given by $|(s\mathbf{I} - \mathbf{A})| = 0$ which is

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots a_n = 0 \tag{7.14}$$

s being a complex variable. For determinants of higher order than three

Bocher's method, described below, is found to be more efficient.

The coefficients $a_0, a_1, a_2 \cdots a_n$ are given by

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= -T_1 \\
a_2 &= -\tfrac{1}{2}(a_1 T_1 + T_2) \\
a_3 &= -\tfrac{1}{3}(a_2 T_1 + a_1 T_2 + T_3) \\
a_n &= -\frac{1}{n}(a_{n-1} T_1 + a_{n-2} T_2 + \cdots a_1 T_{n-1} + T_n)
\end{aligned} \tag{7.15}$$

where T_1 is the trace of the square matrix $\mathbf{A}$ (the sum of the diagonal elements of a square matrix is called its trace, and T_n is the trace of the matrix obtained by multiplying $\mathbf{A}$ by itself n times).

Example: Let us consider a very simple example:

$$\mathbf{x} = \mathbf{A}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & -4 \\ 1 & -3 \end{bmatrix}$$

The characteristic equation of the system is

$$\det(s\mathbf{I} - \mathbf{A}) = \det\left[s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -4 \\ 1 & -3 \end{bmatrix} \right]$$

$\mathbf{I}$ is a unit matrix, i.e. a matrix with all the diagonal terms unity and the off diagonal terms as zero.

$$\det(s\mathbf{I} - \mathbf{A}) = \begin{vmatrix} s & 4 \\ -1 & s+3 \end{vmatrix}$$

giving the characteristic equation

$$s^2 + 3s + 4 = 0 \tag{7.16}$$

Alternatively, using Bocher's formula we have

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= -T_1 = 3 \\
a_2 &= -\tfrac{1}{2}(a_1 T_1 + T_2)
\end{aligned}$$

T_2 is the sum of the diagonal terms of matrix

$$\begin{bmatrix} 0 & -4 \\ 1 & -3 \end{bmatrix} \times \begin{bmatrix} 0 & -4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ -3 & 5 \end{bmatrix}$$

and is equal to 1.

$$a_2 = -\tfrac{1}{2}(-3 \times 3 + 1) = 4$$

which also gives equation (7.16).

The advantages of Bocher's formula are not, of course, apparent in such a simple case. Standard computer programmes are available to work out the coefficients of the polynomial for higher degree expressions.

7.5 THE CHARACTERISTIC EQUATION OF A D.C. GENERATOR WITH FEEDBACK CONTROL

In the earlier equations, we have not considered the effects of feedback control. Here we consider a simple example of a d.c. generator with a feedback signal from its terminal voltage. The present example is given with the object of indicating how the state-variable equations can be formed when there is a feedback control system (Figure 7.2).

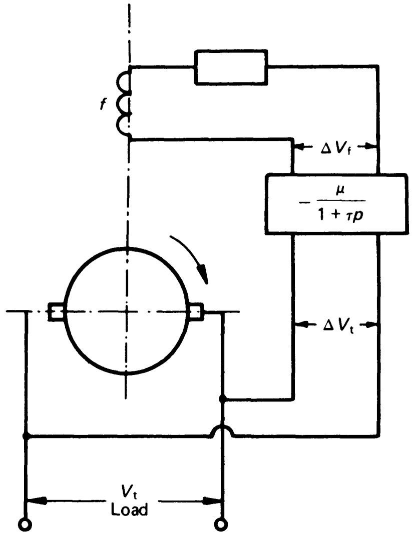

Figure 7.2 D.C. generator with a voltage feedback control

The transient equation for the system is

$$
\begin{array}{c}
\\
\begin{array}{|c|}\hline V_{\mathrm{f}} \\ \hline V_{\mathrm{q}} \\ \hline \end{array}
\end{array}
=
\begin{array}{c|c|c|}
 & \mathrm{f} & \mathrm{qr} \\ \hline
\mathrm{f} & R_{\mathrm{f}}+L_{\mathrm{f}}p & \\ \hline
\mathrm{qr} & M_{\mathrm{d}}p\theta & R+L_{\mathrm{q}}p \\ \hline
\end{array}
\cdot
\begin{array}{c}
\\
\begin{array}{|c|}\hline i_{\mathrm{f}} \\ \hline i_{\mathrm{q}} \\ \hline \end{array}
\end{array}
\tag{7.17}
$$

The voltage/torque equation for small oscillations is

$$
\begin{array}{c}
\\
\begin{array}{|c|}\hline \Delta V_{\mathrm{f}} \\ \hline \Delta V_{\mathrm{q}} \\ \hline \Delta T_{\mathrm{i}} \\ \hline \end{array}
\end{array}
=
\begin{array}{c|c|c|c|}
 & \mathrm{f} & \mathrm{qr} & \mathrm{s} \\ \hline
\mathrm{f} & R_{\mathrm{f}}+L_{\mathrm{f}}p & & \\ \hline
\mathrm{qr} & -M_{\mathrm{d}}p\theta & R+L_{\mathrm{q}}p & -M_{\mathrm{d}}i_{\mathrm{f}} \\ \hline
\mathrm{s} & M_{\mathrm{d}}i_{\mathrm{q}} & M_{\mathrm{d}}i_{\mathrm{f}} & Jp+R_{\mathrm{F}} \\ \hline
\end{array}
\cdot
\begin{array}{c}
\\
\begin{array}{|c|}\hline \Delta i_{\mathrm{f}} \\ \hline \Delta i_{\mathrm{q}} \\ \hline \Delta\omega \\ \hline \end{array}
\end{array}
\tag{7.18}
$$

The method of obtaining the transient voltage equation is illustrated earlier (Section 5.8). The increment in torque is

$$
\Delta\mathbf{i}_{\mathrm{t}}(\mathbf{G}+\mathbf{G}_{\mathrm{t}})\mathbf{i} =
\begin{array}{|c|c|}\hline \Delta i_{\mathrm{f}} & \Delta i_{\mathrm{q}} \\ \hline \end{array}
\cdot
\begin{array}{|c|c|}\hline & -M_{\mathrm{d}} \\ \hline -M_{\mathrm{d}} & \\ \hline \end{array}
\cdot
\begin{array}{|c|}\hline i_{\mathrm{f}} \\ \hline i_{\mathrm{q}} \\ \hline \end{array}
\tag{7.19}
$$

or $\qquad \Delta\mathbf{i}_{\mathrm{t}}(\mathbf{G}+\mathbf{G}_{\mathrm{t}})\mathbf{i} = -(\Delta i_{\mathrm{f}} M_{\mathrm{d}} i_{\mathrm{q}} + \Delta i_{\mathrm{q}} M_{\mathrm{d}} i_{\mathrm{f}})$

Since $\qquad \Delta T_{\mathrm{i}} = Jp\Delta\omega + R_{\mathrm{F}}\Delta\omega - \Delta\mathbf{i}_{\mathrm{t}}(\mathbf{G}+\mathbf{G}_{\mathrm{t}})\mathbf{i}$

we can fill the row s in equation (7.18) quite easily.

Now
$$\Delta V_{\mathrm{f}} = -\frac{\mu}{1+\tau p}\Delta V_{\mathrm{t}} \tag{7.20}$$

where $\qquad V_{\mathrm{t}} = V_{\mathrm{qr}}$

also $\qquad V_{\mathrm{qr}} = -Z_{\mathrm{L}} i_{\mathrm{L}} = -Z_{\mathrm{L}} i_{\mathrm{qr}}$

$$\Delta V_{\mathrm{qr}} = -Z_{\mathrm{L}}\Delta i_{\mathrm{qr}} \tag{7.21}$$

In order to include the relation (7.20) in equation (7.18) we write

$$(1+\tau p)\Delta V_{\mathrm{f}} = -\mu\Delta V_{\mathrm{t}}$$

or

$$
\begin{aligned}
p\Delta V_{\mathrm{f}} &= -\frac{\mu}{\tau}\Delta V_{\mathrm{t}} - \frac{\Delta V_{\mathrm{f}}}{\tau} \\
&= \frac{\mu}{\tau}(R_{\mathrm{L}} + L_{\mathrm{L}}p)\Delta i_{\mathrm{qr}} - \frac{\Delta V_{\mathrm{f}}}{\tau}
\end{aligned}
\tag{7.22}
$$

or

$$
p\Delta V_{\mathrm{f}} - \frac{\mu}{\tau}L_{\mathrm{L}}p\Delta i_{\mathrm{qr}} = +\frac{\mu R_{\mathrm{L}}}{\tau}\Delta i_{\mathrm{qr}} - \frac{\Delta V_{\mathrm{f}}}{\tau} \tag{7.23}
$$

We can now expand equation (7.18) to take into account the voltage feedback and after some rearrangement express it in state-variable form as:

$$
\begin{bmatrix}
1 & & -\frac{\mu}{\tau}L_{\mathrm{L}} & \\
 & L_{\mathrm{f}} & & \\
 & & L_{\mathrm{T}} & \\
 & & & J
\end{bmatrix}
p
\begin{bmatrix}
\Delta V_{\mathrm{f}} \\ \Delta_{\mathrm{f}} \\ \Delta i_{\mathrm{qr}} \\ \Delta\omega
\end{bmatrix}
=
\begin{bmatrix}
-\frac{1}{\tau} & & \frac{\mu R_{\mathrm{L}}}{\tau} & \\
1 & -R_{\mathrm{f}} & & \\
 & M_{\mathrm{d}}p\theta_0 & -R_{\mathrm{T}} & M_{\mathrm{d}}i_{\mathrm{f0}} \\
 & -M_{\mathrm{d}}i_{\mathrm{q0}} & -M_{\mathrm{d}}i_{\mathrm{f0}} & R_{\mathrm{F}}
\end{bmatrix}
\cdot
\begin{bmatrix}
\Delta V_{\mathrm{f}} \\ \Delta i_{\mathrm{f}} \\ \Delta i_{\mathrm{qr}} \\ \Delta\omega
\end{bmatrix}
\tag{7.24}
$$

where $L_{\mathrm{T}} = L_{\mathrm{L}} + L_{\mathrm{qr}}$

and $R_{\mathrm{T}} = R_{\mathrm{L}} + R$

and i_{q0}, i_{f0}, and $p\theta_0$ represent their respective initial values.
The **A** matrix in equation (7.13) is as shown in equation (7.25). It is quite straightforward to work out the characteristic equation using Bocher's formula. The matrix inversion and multiplication which have been shown above are, in fact, incorporated in standard computer programmes. All that is necessary is to feed the data for the **L**- and **A**-matrices to the computer, which then calculates the coefficients.

7.6 DAMPING AND SYNCHRONISING TORQUES IN STABILITY ANALYSIS

We have seen that the torque equation of the rotor of an electrical machine during hunting is

$$
Jp^2\Delta\theta + T_{\mathrm{D}}p\Delta\theta + T_{\mathrm{S}}\Delta\theta = 0 \tag{7.26}
$$

where T_{D} and T_{S} are damping and synchronising torque coefficients. In

$$\begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & \dfrac{1}{L_{\mathrm{f}}} & & \\ \hline -\dfrac{\mu L_{\mathrm{L}}}{\tau L_{\mathrm{T}}} & & \dfrac{1}{L_{\mathrm{T}}} & \\ \hline & & & \dfrac{1}{J} \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|}\hline -\dfrac{1}{\tau} & & \dfrac{\mu R_{\mathrm{L}}}{\tau} & \\ \hline 1 & -R_{\mathrm{f}} & & \\ \hline & M_{\mathrm{d}}p\theta_0 & -R_{\mathrm{T}} & M_{\mathrm{d}}i_{\mathrm{f}0} \\ \hline & -M_{\mathrm{d}}i_{\mathrm{q}0} & -M_{\mathrm{d}}i_{\mathrm{f}0} & R_{\mathrm{F}} \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline -\dfrac{1}{\tau} & & \dfrac{\mu R_{\mathrm{L}}}{\tau} & \\ \hline \dfrac{1}{L_{\mathrm{f}}} & \dfrac{-R_{\mathrm{f}}}{L_{\mathrm{f}}} & & \\ \hline \dfrac{\mu L_{\mathrm{L}}}{\tau^2 L_{\mathrm{T}}} & \dfrac{M_{\mathrm{d}}p\theta_0}{L_{\mathrm{T}}} & \begin{array}{c}\dfrac{-R_{\mathrm{T}}}{L_{\mathrm{T}}} \\ -\dfrac{\mu^2 L_{\mathrm{L}} R_{\mathrm{L}}}{\tau^2 L_{\mathrm{T}}}\end{array} & \dfrac{M_{\mathrm{d}}i_{\mathrm{f}0}}{L_{\mathrm{T}}} \\ \hline & \dfrac{-M_{\mathrm{d}}i_{\mathrm{q}0}}{J} & \dfrac{-M_{\mathrm{d}}i_{\mathrm{f}0}}{J} & \dfrac{R_{\mathrm{F}}}{J} \\ \hline \end{array} \qquad (7.25)$$

terms of power this equation may be written

$$J\omega p^2\Delta\theta + P_{\mathrm{D}}p\Delta\theta + P_{\mathrm{S}}\Delta\theta = 0 \qquad (7.27)$$

where P_{D} and P_{S} are damping and synchronising power coefficients. Expressed in per-unit values, based on a selected machine rating G, the power per unit will be the same as the torque per unit and the hunting equation of torque is derived as follows. The machine rated torque T and the power base are related by the equation $T = G/\omega$. Thus in per-unit torque values, equation (7.26) becomes

$$\frac{Jp^2\Delta\theta}{T} + \frac{T_{\mathrm{D}}}{T}p\Delta\theta + \frac{T_{\mathrm{S}}}{T}\Delta\theta = 0$$

$$\frac{Jp^2\Delta\theta}{G/\omega} + T_{\mathrm{D}}(\mathrm{p.u.})\, p\Delta\theta + T_{\mathrm{S}}(\mathrm{p.u.})\Delta\theta = 0$$

or

$$\frac{J\omega p^2\Delta\theta}{G} + T_{\mathrm{D}}(\mathrm{p.u.})p\Delta\theta + T_{\mathrm{S}}(\mathrm{p.u.})\Delta\theta = 0$$

or

$$M(\mathrm{p.u.})p^2\Delta\theta + T_{\mathrm{D}}(\mathrm{p.u.})p\Delta\theta + T_{\mathrm{S}}(\mathrm{p.u.})\Delta\theta = 0$$

where

$$M = J\omega$$

Equation (7.26) is usually written simply as

$$Mp^2\Delta\theta + T_{\mathrm{D}}p\Delta\theta + T_{\mathrm{S}}\Delta\theta = 0$$

the use of per-unit values being implied. In this equation the damping torque coefficient T_{D} includes damping from both electrical and purely mechanical sources.

Having obtained the characteristic equation of a machine, the roots, or eigenvalues, can be computed by standard procedures. If the roots have no positive real parts, the system is stable. However, when the roots are obtained it may be found that one of the roots has a small negative real part and a low value of imaginary part. This root will predominantly influence the rotor dynamics. This will indicate that the machine will oscillate at an angular frequency determined by the imaginary part of the root and the rate of decay of the oscillations will depend on the magnitude of the real part. If a voltage regulator is present, roots with relatively small imaginary parts will be present which will also affect the dynamics of the rotor. The effect of the other roots, with faster decay rates (higher negative real values) will be apparent only at the initial stage and these will not affect the small oscillation behaviour over the longer period given by the dominant root. Roots with large values of imaginary part will not affect the rotor dynamics because the rotor, due to its inertia, will not respond to high frequency oscillations. Hence we can often assume that there will be virtually one dominant root which will determine the machine kinetics during small oscillation

$$s = -\alpha + jh\omega$$

It will be shown later that for all practical purposes we can take $s = p = jh\omega$ for steady state oscillation and proceed with our calculation of the damping and synchronising torque coefficient.

The procedure is as follows:

(a) calculate the steady-state values of various voltages and currents
(b) substitute $p = jh\omega$ in the motional impedance matrix wherever p occurs in equation (7.10)
(c) compute $\Delta\mathbf{i}$
(d) compute $\Delta\mathbf{i}_t(\mathbf{G}+\mathbf{G}_t)\mathbf{i}_0$
(e) $T_S\Delta\theta = -\text{Real } \Delta\mathbf{i}_t(\mathbf{G}+\mathbf{G}_t)\mathbf{i}_0$
(f) $hT_D\Delta\theta = -\text{Imaginary } \Delta\mathbf{i}_t(\mathbf{G}+\mathbf{G}_t)\mathbf{i}_0 + R_F h\omega\Delta\theta$

Relations (e) and (f) follow as soon as we substitute $p = jh\omega$ in equation (7.26) and compare it with equation (7.5), having assumed $\Delta T_i = 0$ since we are considering self-oscillation and no governors are present. Equation (7.26) becomes

$$(-Mh^2\omega^2 + jh\omega T_D + T_S)\Delta\theta = 0 \tag{7.28}$$

The important question that arises at this stage is, what value of $h\omega$ should we use for p? If we have already formed the characteristic equation, we know whether or not the system is stable. If we are further interested in learning the extent of stability or instability, a study of the eigenvalues gives us sufficient information. Hence, the calculation of T_S and T_D seems superfluous once the eigenvalues have been computed. However, for a single machine on a large system there is a very direct method which gives the value of $jh\omega$ without forming or solving the characteristic equation.

Solving equation (7.26) we have

$$\theta = A\varepsilon^{(-\alpha + jh\omega)t} + B\varepsilon^{(-\alpha - jh\omega)t} \tag{7.29}$$

where

$$\alpha = T_D/2M$$

and

$$h\omega = \sqrt{\frac{T_S}{M} - \frac{T_D^2}{4M^2}}$$

So strictly speaking we should put

$$p = -\frac{T_D}{2M} + j\sqrt{\frac{T_S}{M} - \frac{T_D^2}{4M^2}}$$

and not $p = jh\omega$ as we have proposed. However, a typical calculation has shown that the error involved in putting $p = jh\omega$ is less than 0.13 %. Therefore we can use a simple iterative technique to obtain the value of $h\omega$. We know from experience that the hunting frequency is usually in the range 1–4 Hz, which makes $h = 0.02$–0.08. We can take $\omega = 1$ in this convergent computation. Hence we can compute T_S and T_D assuming $h = 0.01$ (say) and then update its value from the relation

$$h = \sqrt{\frac{T_S}{M} - \left(\frac{T_D}{2M}\right)^2} \approx \sqrt{\frac{T_S}{M}} \tag{7.30}$$

and use it again, until the error is within a specified limit. A computer flow diagram is given in Figure 7.3. In cases where T_S has a very low value, approaching zero, a starting value of $h = 0.01$ will probably give faulty results. It is therefore advisable to begin with a much lower value, say $h = 0.000\,5$. If the value of T_S is found to be negative the computation step $h = \sqrt{T_S/M}$ is omitted (since this would result in programme failure). Obviously the dynamical system is not stable for this operating condition. For certain combinations of active and reactive power T_S becomes zero and the machine will pull out of synchronism. The hunting frequency clearly has no meaning in such a case. However, if with the small initial value of h the term T_S is found to be positive, h is increased in large steps. The step-length may then be adjusted. For example, if for $h = 0.15$, $h < \sqrt{T_S/M}$ and for $h = 0.16$, $h > \sqrt{T_S/M}$ then small intermediate steps can be used. In practice it is found that the iterative computation of h between

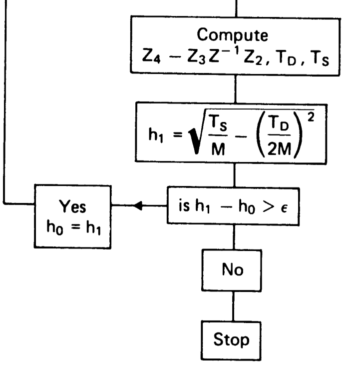

Figure 7.3 Flow chart for computing hunting frequency

equation (7.30) and the matrix equation (7.10) is straightforward when reasonable precautions are taken.

7.6.1 An experimental system

Tests were conducted on a laboratory machine to determine the parameters. The test methods and the conversion into an appropriate 'per-unit' system are given in Appendices 1 and 2.

The experimental machine was of the laboratory universal type, operating as a 3-phase synchronous motor driving a dynamometer load. The machine was synchronised with the supply mains (50 Hz) and the field was energised from a separate d.c. supply.

Measured parameters and conversion to per-unit values

Supply voltage = 200 V r.m.s. line to line
Supply frequency = 50 Hz

Electrical

Alternator 1.732 kVA 3000 r.p.m.
Field reactance $X_f = 812\,\Omega$
Direct axis reactance $X_{dr} = 180\,\Omega$
Quadrature axis syn. reactance $X_{qr} = 176\,\Omega$
Direct axis mutual reactance $X_{md} = 173\,\Omega$
Armature resistance $R = 3\,\Omega$
Field resistance $R_f = 6.2\,\Omega$

Mechanical

Moment of inertia = 0.062 kg m^2
Coefficient of friction $R_F = 0.002\,15$ N m/rad
(calculated by retardation test)

These parameters are now converted into per-unit values

Per-unit values

Base voltage = 200 V line to line voltage
Base current = 5 A line current
Base VA = 1732
Base impedance = 69 Ω
Base impedance for the field circuit = 312 Ω (see Appendix 1)

$V_0 = 1$ p.u.	$X_{md} = 2.5$ p.u.
$X_d = 2.6$ p.u.	$X_{mq} = 2.45$ p.u.
$X_q = 2.56$ p.u.	$R_a = 0.043$ p.u.
$X_f = 2.6$ p.u.	$R_f = 0.02$ p.u.

The following observations were made

(a) Continuous reduction in the field excitation current increased the synchronous load angle until finally the machine pulled out of synchronism.
(b) Increase in resistance in the armature circuit of the motor initiated self-oscillation. If the resistance were too great the oscillations built up and synchronism was lost.
(c) With lower values of armature resistance and high field resistance, any oscillations which were initiated were soon damped out.
(d) Low field resistances also made the machine more oscillatory.

7.6.2 Computation of the synchronising and damping torque coefficients

The apparent differences in the conventions used by engineers in formulating equations for synchronous machines have been discussed. We shall therefore use Park's sign convention for synchronous machines. For all the other cases, the sign convention used in deriving the generalised equation will be followed.

The transient voltage/torque equation for a synchronous alternator, expressed in the Park convention becomes

$$\begin{bmatrix} V_f \\ V_{dr} \\ V_{qr} \\ T_i \end{bmatrix} = \begin{array}{c|cccc} & \text{f} & \text{dr} & \text{qr} & \text{s} \\ \hline \text{f} & R_f + L_f p & -M_d p & & \\ \text{dr} & M_d p & -(R + L_{dr} p) & L_{qr} p\theta & \\ \text{qr} & M_d p\theta & -L_{dr} p\theta & -(R + L_{qr} p) & \\ \text{s} & -i_{qr} M_d & (L_{dr} - L_{qr}) i_{qr} & \begin{matrix} -M_d i_f \\ +(L_{dr} - L_{qr}) i_{dr} \end{matrix} & Mp^2 + R_F p \end{array} \cdot \begin{bmatrix} i_f \\ i_{dr} \\ i_{qr} \\ \theta \end{bmatrix} \tag{7.31}$$

The following points should be noted

(1) Since the experimental machine had laminated stator and rotor and no damper windings, coils ds2 and qs2 are absent, and hence rows and columns corresponding to these coils are omitted from the matrix equation.
(2) The coil ds has been designated f because this coil is generally the field winding.

(3) In order to conform with Park's convention, the signs of i_{dr} and i_{qr} have been reversed and $p\theta$ has been replaced by $-p\theta$ (Section 5.6.1).

7.6.2(i) *Computation of steady-state current*

If we refer to the vector diagram for a synchronous alternator on an infinite busbar (Figure 2.33) we see that

$$V_{dr} = V\sin\delta$$
$$V_{qr} = V\cos\delta$$

For motor operation, the angle δ becomes negative, so that

$$V_{dr} = -V\sin\delta$$
$$V_{qr} = V\cos\delta$$

In equation (7.31), if we put $p = \mathrm{d}/\mathrm{d}t = 0$ for steady-state operation the following voltage equations result

$$\begin{array}{c|cc|ccc|c|} & & & \text{f} & \text{dr} & \text{qr} & \\ \hline V_f & & \text{f} & R_f & & & i_f \\ -V\sin\delta_0 & = & \text{dr} & & -R & L_{qr}p\theta & \cdot\; i_{dr} \\ V\cos\delta_0 & & \text{qr} & M_d p\theta & -L_{dr}p\theta & -R & i_{qr} \\ \hline \end{array} \qquad (7.32)$$

The term $M_d p\theta\, i_f$ represents the generated voltage E (Figure 2.33). The terms $L_{qr}p\theta$, etc., are replaced by their corresponding values of reactance. The steady-state currents on solving equation (7.31) are

$$i_{f0} = \frac{E}{X_{md}}$$

$$i_{d0} = \frac{RV\sin\delta_0 - X_{qr}(V\cos\delta_0 - E)}{R^2 + X_{dr}X_{qr}} \qquad (7.33)$$

$$i_{q0} = \frac{X_{dr}V\sin\delta_0 - R(V\cos\delta_0 - E)}{R^2 + X_{dr}X_{qr}}$$

7.6.2(ii) *Hunting equation*

The equation for small oscillations is

$$
\begin{bmatrix} \Delta V_f \\ (-V\cos\delta_0)\Delta\delta \\ (-V\sin\delta_0)\Delta\delta \\ \Delta T_i \end{bmatrix}
=
\begin{array}{cc}
 & \begin{array}{cccc} \mathrm{f} & \mathrm{dr} & \mathrm{qr} & \mathrm{s} \end{array} \\
\begin{array}{c} \mathrm{f} \\ \mathrm{dr} \\ \mathrm{qr} \\ \mathrm{s} \end{array} &
\begin{bmatrix}
R_f + L_f p & -M_d p & & \\
M_d p & -(R + L_{dr} p) & L_{qr} p\theta & L_{qr} i_{q0} p \\
M_d p\theta & -L_{dr} p\theta & -(R + L_{qr} p) & -L_{dr} i_{d0} p + M_d i_f p \\
-M_d i_{q0} & (L_{dr} - L_{qr}) i_{q0} & (L_{dr} - L_{qr}) i_{d0} - M_d i_{f0} & Mp^2 + R_F p
\end{bmatrix}
\end{array}
\begin{bmatrix} \Delta i_f \\ \Delta i_{dr} \\ \Delta i_{qr} \\ \Delta\theta \end{bmatrix}
\tag{7.34}
$$

If there is no feedback to the field winding, $\Delta V_f = 0$ and for self oscillations $\Delta T_i = 0$. We also note that for motor operation $\theta = \omega_0 t - \delta$, and since $\omega_0 t$ is a uniform function of time $\Delta\theta = -\Delta\delta$.

We can transfer the left-hand column over to the right-hand side, and on substituting $p = jh\omega$, the following equation results

$$
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
=
\begin{array}{cc}
 & \begin{array}{cccc} \mathrm{f} & \mathrm{dr} & \mathrm{qr} & \mathrm{s} \end{array} \\
\begin{array}{c} \mathrm{f} \\ \mathrm{dr} \\ \mathrm{qr} \\ \mathrm{s} \end{array} &
\begin{bmatrix}
R_f + jX_f h & -jX_{md} h & & \\
jX_{md} h & -R - jX_{dr} h & X_{qr} & jX_{qr} i_{q0} h - V\cos\delta_0 \\
-X_{md} & -X_{dr} & -R - jX_{qr} h & -jX_{dr} i_{d0} h + jX_{md} i_f h - V\sin\delta_0 \\
-X_{md} i_{q0} & (X_{dr} - X_{qr}) i_{q0} & (X_{dr} - X_{qr}) i_{d0} - X_{md} i_{f0} & -Mh^2\omega^2 + jh\omega R_F
\end{bmatrix}
\end{array}
\cdot
\begin{bmatrix} \Delta i_f \\ \Delta i_{dr} \\ \Delta i_{qr} \\ \Delta\theta \end{bmatrix}
\tag{7.35}
$$

This equation is of the form

$$
\begin{bmatrix} 0 \\ 0 \end{bmatrix}
=
\begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \mathbf{Z}_3 & \mathbf{Z}_4 \end{bmatrix}
\cdot
\begin{bmatrix} \Delta \mathbf{i} \\ \Delta\theta \end{bmatrix}
$$

or

$$0 = \mathbf{Z}\Delta\mathbf{x}$$

where $\mathbf{Z}$ is the motional impedance matrix.
The torque equation is thus of the form

$$(\mathbf{Z}_4 - \mathbf{Z}_3\mathbf{Z}_1^{-1}\mathbf{Z}_2)\Delta\theta = 0 \tag{7.36}$$

which gives

$$-h^2\omega^2 M + \mathrm{j}h\omega R_\mathrm{F} - \Delta\mathbf{i}_\mathrm{t}(\mathbf{G}+\mathbf{G}_\mathrm{t})\mathbf{i}_0 = 0 \tag{7.37}$$

The coefficient of electrical damping torque is

$$T_\mathrm{de} = -\text{Imaginary }(\mathbf{Z}_3\mathbf{Z}_1^{-1}\mathbf{Z}_2)/h \tag{7.38}$$

and the coefficient of synchronising torque is

$$T_\mathrm{S} = -\text{Real }(\mathbf{Z}_3\mathbf{Z}_1^{-1}\mathbf{Z}_2) \tag{7.39}$$

The equation is usually written in terms of new constants

$$-h^2\bar{M} + \mathrm{j}h\bar{R}_\mathrm{F} - \Delta\mathbf{i}_\mathrm{t}(\mathbf{G}+\mathbf{G}_\mathrm{t})\mathbf{i}_0 = 0 \tag{7.40}$$

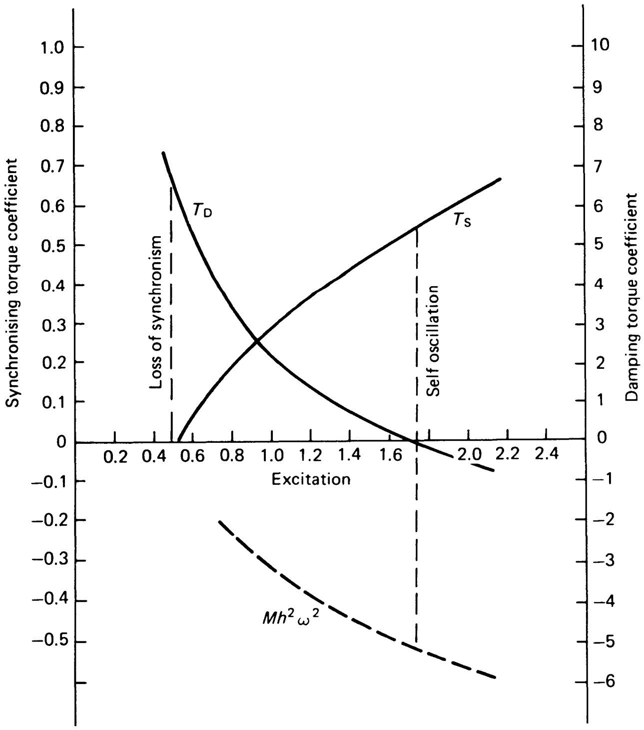

Figure 7.4 Variation of damping and synchronising torque coefficients with field excitation ($V_\mathrm{t} = 1.0$, $X_\mathrm{d} = 2.6$, $X_\mathrm{q} = 2.56$, $X_\mathrm{md} = 2.5$, $R_\mathrm{f} = 0.103$, $R_\mathrm{a} = 0.415$, all in per-unit values)

where $\overline{M} = \omega^2 M$ and $\overline{R_F} = \omega R_F$ but in the literature the bar is often omitted.

The damping and synchronising torque coefficients computed for the experiments have been plotted in Figures 7.4 and 7.5.

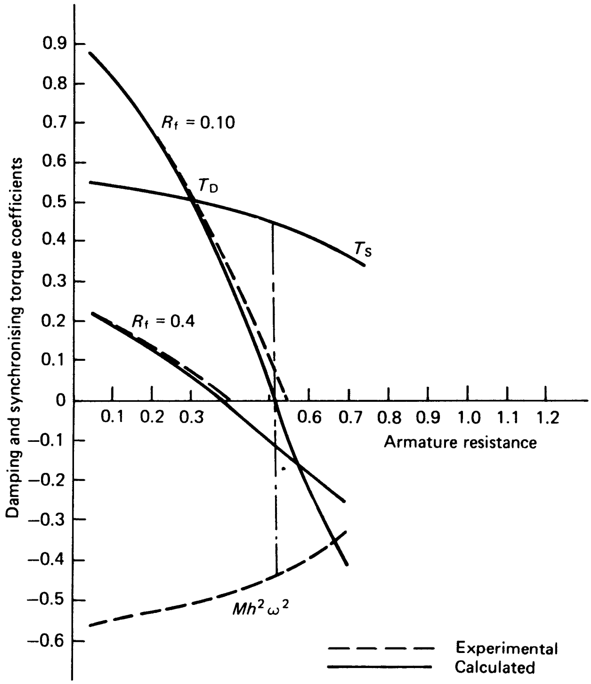

Figure 7.5 Variation of damping and synchronising torque coefficients with armature resistance (variable hunting frequency).----test and—computed ($V_t = 1.0, E = 1.52, X_d = 2.6, X_q = 2.56, X_{md} = 2.5, X_f = 2.56$)

On studying these figures, we can confirm some earlier conclusions.

(a) As the armature resistance (armature + line) is increased both synchronising and damping torque coefficients decrease. The rate of decrease of T_D is much faster than that of T_S and consequently the machine starts self oscillation beyond a certain value of line resistance, the oscillations build up, and the machine loses synchronism. It should be noted, however, that if the line resistance is very high, then the damping torque coefficient again becomes positive. Experimentally, however, this cannot be easily confirmed.

(b) The synchronising torque is almost a mirror-image of the inertia torque, as might be expected, and this goes to confirm that the equation $T_S = Mh^2\omega^2$ is valid.

(c) As the field excitation is increased, synchronising torque increases (Figure 7.4). This is easily explained in physical terms, when we consider that as the field excitation is increased the field magnet is strengthened and the 'tie' between the stator and the rotor magnets grows stronger. This is equivalent to increasing the spring constant K. If the spring constant is increased, the frequency of oscillation is obviously increased.

(d) When the excitation is so low that the synchronising torque approaches zero, the machine loses synchronism without exciting any oscillations (Figure 7.4).

(e) As we increase the load on the synchronous machine the load angle δ increases. The damping torque increases correspondingly and the synchronising torque is reduced (Figure 7.6). This observation might lead to the conclusion that if a synchronous machine is stable at no load it will be stable at higher loads (as long as positive synchronising power is available). This conclusion is not valid, however, for synchronous machines when damper windings (amortisseur windings) are present. The amortisseur windings, which contribute much of the positive damping torque, are most effective when the load angle is small. As a result, increasing load tends to reduce the positive damping power available. However, a synchronous machine which is fitted with amortisseur winding, or damper bars, will always be more stable than a machine which is not so fitted.

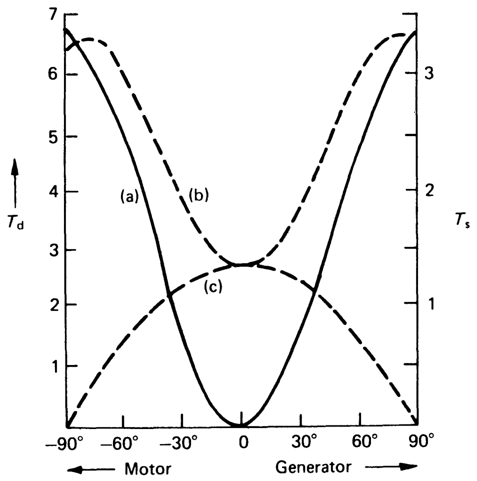

Figure 7.6 Variation of damping and synchronising torque coefficients with load angle. (a) Damping torque coefficient, (b) synchronising torque coefficient ($R_f = 0.001$ per unit) and (c) same as (b) ($R_f = 0.1$)

(f) Quite often the synchronising torque coefficient (which is analogous to a spring constant) is defined as

$$\frac{\mathrm{d}P}{\mathrm{d}\delta} = \frac{EV\cos\delta}{X_\mathrm{d}} = P_\mathrm{S} \tag{7.41}$$

but this is a gross approximation, for the following reasons.

(i) This equation neglects the effects of armature resistance and saliency.
(ii) It is assumed that the vector E does not change, which is true only if the field resistance is very large indeed (Figure 7.6 curve C).[2] If the resistance is low the cosine relation in equation (7.41) is not valid since the field current will undergo changes and E is no longer independent of $\Delta\delta$.

(g) The effect of field resistance on T_de is shown in Figure 7.7. It will be seen in

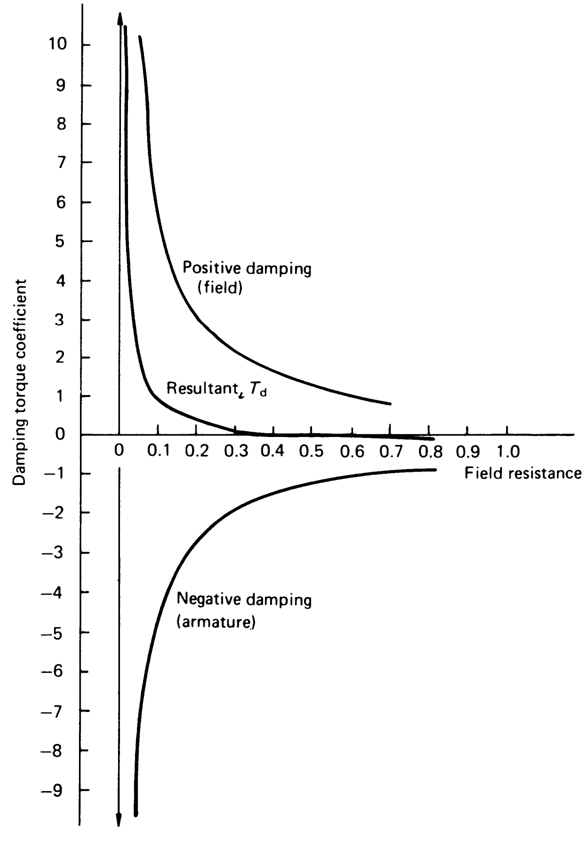

Figure 7.7 Variation of positive and negative damping torque components with field resistance ($V_\mathrm{t} = 1.0$, $E = 1.52$, $X_\mathrm{d} = 2.6$, $X_\mathrm{q} = 2.56$, $X_\mathrm{md} = 2.5$, $R_\mathrm{a} = 0.34$)

Chapter 8 that the damping contributed by the field winding is proportional to $(\Delta i_f)^2 R_f$ when Δi_f is the oscillatory current set up in the field winding during small oscillation. If the field resistance R_f is small, then Δi_f will be large but $(\Delta i_f)^2 R_f$ will also be small. On the other hand if R_f is large, then Δi_f will be small and again $(\Delta i_f)^2 R_f$ will be small. There is a critical value of R_f for which the term $(\Delta i_f)^2 R_f$ and therefore the total damping will be maximum.

From these discussions we can see that the coefficients T_{de} and T_S not only provide information about the stability of a machine, they also explain the underlying reasons for stability.

7.6.3 Effect of a voltage regulator

The effect of a continuously acting voltage regulator on the steady-state stability of a synchronous machine has been studied in detail by various authors during the past two decades.[3-5].

It has also been recognised, however, that the action of the voltage regulator does not necessarily improve dynamic stability and may very often contribute negative damping, thereby making the machine more prone to hunting. As mentioned earlier, a regulator having a large gain and a low time constant would be desirable for transient stability of a synchronous machine, but like any feedback loop with a large gain and a low time constant, this tends to make the system more oscillatory. In modern thyristor type exciters with very low time constants it has therefore been necessary to introduce stabilising signals to prevent hunting. As DeMello and Concordia[6] have pointed out, it is extremely difficult to design a universal stabilising function for such a system since the hunting characteristics of a regulated synchronous machine depend critically on the various parameters of the machine and regulator and on the real and reactive power the machine has to deliver.

In this section, we shall show how the earlier equations will have to be modified to take into account the presence of a voltage regulator. Having done this, stability boundaries obtained by using the Routh–Hurwitz criterion will be presented. It will then be shown that the same stability boundary will result from the computation of synchronising and damping torque coefficients.

The system under study is an alternator connected to an infinite busbar through a tie line of which the impedance is $R_t + j\omega L_t$. The generator is provided with amortisseur windings.

The excitation control scheme is described by

$$\Delta V_f = \frac{\mu_r(V_{ref} - V_m) + \mu_1 p\delta + \mu_2 p^2 \delta}{(1 + \tau_r p)} \tag{7.42}$$

where V_m is the voltage at the machine terminals.

The two additional signals, proportional to rotor excursion velocity and acceleration can be introduced. By putting $\mu_1 = \mu_2 = 0$, we can reduce this to a single-time-constant regulator. A voltage regulator incorporating a stabilising circuit, will have a transfer function

$$g(p) = \frac{\Delta V_f}{\Delta V_m} = \frac{-\mu_r(1 + \tau_s p)}{1 + (\tau_e + \tau_s \mu_s)p + \tau_e \tau_s p^2} \tag{7.43}$$

where

$$\mu_r = \text{regulator gain}$$
$$= (\text{exciter gain}) \times (\text{amplifier gain}) \times (\text{converter gain})$$
$$= \mu_{ex} \times \mu_a \times \mu_c$$
$$\mu_s = \text{overall stabiliser gain}$$
$$= 1 + \mu_{ex}\mu_{st}$$

It should be pointed out that the overall regulator loop gain is μ_r' where

$$-\mu_r' = \frac{\Delta E}{\Delta V_m} = \frac{\Delta V_f X_{md}}{\Delta V_m R_f} \tag{7.44}$$

Hence

$$-\mu_r = \mu_r' \frac{R_f}{X_{md}} \tag{7.45}$$

(This is obvious, because ΔV_f represents the applied voltage to the field circuit and causes a change $\Delta i_f = \Delta V_f / R_f$ in the field circuit. The change in the excitation voltage $\Delta E = X_{md} \Delta i_f$.)

7.6.3(i) *Modification of hunting equations*

Equation (7.35) will need two extra rows and columns to take into consideration the presence of the amortisseur windings and the equation of the field circuit will have to be modified to account for the a.v.r.

We know that at steady state,

$$V_m^2 = V_{dm}^2 + V_{qm}^2 \tag{7.46}$$

where V_{dm} and V_{qm} are the d and q axis components of the terminal voltage of the machine. By taking small increments and neglecting second order effects

$$\Delta V_m = \frac{V_{dm}}{V_m}(\Delta V_{dm}) + \frac{V_{qm}}{V_m}(\Delta V_{qm})$$
$$= V_{dm}'(\Delta V_{dm}) + V_{qm}'(\Delta V_{qm}) \tag{7.47}$$

V_{dm} and V_{qm} can easily be obtained from equation (7.31). If we measure the voltage at the machine terminals only, we have

$$V_{dm} = M_d p i_f - (R + L_{dr} p) i_{dr} + L_{qr} p\theta i_{qr} \tag{7.48}$$

If we measure the d-axis component of the busbar voltage, the equation is obviously modified to

$$V_{db} = M_d p i_f - [(R + R_t) + (L_{dr} + L_t)p] i_{dr} + (L_{qr} + L_t) p\theta i_{qr} \tag{7.49}$$

or

$$V_{db} + R_t i_{dr} + L_t p i_{dr} - L_t p\theta i_{qr} = V_{dm} \tag{7.50}$$

Thus

$$\Delta V_{dm} = \Delta V_{dt} + R_t \Delta i_{dr} + L_t p \Delta i_{dr} - L_t p\theta \Delta i_{qr} - L_t i_{qr} p \Delta\theta \tag{7.51}$$

where

$$\Delta V_{db} = \Delta(V \sin\delta_b) = V\cos\delta_b \Delta\delta_b = V\cos\delta_b \Delta\theta \tag{7.52}$$

In a similar manner ΔV_{qm} may be obtained.

The field equation is now modified as shown. We have seen earlier that this equation is

$$\Delta V_f = (R_f + L_f p)\Delta i_f + M_d p \Delta i_{dr} + M_{kd} p \Delta i_{kd} \tag{7.53}$$

or

$$g(p)\Delta V_m = (R_f + L_f p)\Delta i_f + M_d p \Delta i_{dr} + M_{kd} p \Delta i_{kd} \tag{7.54}$$

Using equations (7.47) ΔV_m, expressed in terms of Δi_{dr}, Δi_{qr}, and $p\Delta\theta$, is substituted. After rearranging the terms we have the final equation for alternator operation (δ is positive and $\Delta\theta = \Delta\delta$) which is shown in equation (7.55) where

$$A_1 \doteq g(p)\frac{V_{dm}}{V_m}[-jhX_t - R_t + X_t]$$

$$A_2 = g(p)\frac{V_{qm}}{V_m}[-jhX_t - R_t - X_t]$$

$$A_3 = g(p)\left[\frac{V_{dm}}{V_m} X_t i_{q0} - \frac{V_{qm}}{V_m} X_t i_{d0}\right] - g(p)V\left[\frac{V_{dm}}{V_m}\cos\delta_b - \frac{V_{qm}}{V_m}\sin\delta_b\right]$$

Equation (7.55) is then solved iteratively, as described in Section 6.6.3 and T_S and T_{de} are calculated.

If the Routh–Hurwitz criterion is to be used for determining stability, then the operator p should not be replaced by $jh\omega$. The voltage–torque equations are rearranged in the state variable form and fed into a digital computer to generate the characteristic equations, for different values of regulator gain and

$$
\begin{array}{|c|}
\hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline
\end{array}
\;
\begin{array}{c}
\mathrm{f} \\ \mathrm{dr} \\ =\mathrm{qr} \\ \mathrm{kd} \\ \mathrm{kq} \\ \mathrm{s}
\end{array}
\;
\begin{array}{c|c|c|c|c|c|}
\mathrm{f} & \mathrm{dr} & \mathrm{qr} & \mathrm{kd} & \mathrm{kq} & \mathrm{s} \\
\hline
R_{\mathrm{f}} + \mathrm{j}hX_{\mathrm{f}} & -\mathrm{j}hX_{\mathrm{md}} + A_1 & A_2 & \mathrm{j}hX_{\mathrm{mkd}} & & A_3 \\
\hline
\mathrm{j}hX_{\mathrm{md}} & -R_{\mathrm{e}} - \mathrm{j}hX_{\mathrm{de}} & X_{\mathrm{qe}} & \mathrm{j}hX_{\mathrm{mkd}} & X_{\mathrm{mkq}} & \mathrm{j}X_{\mathrm{qe}} i_{\mathrm{q}0} h - V\cos\delta_{\mathrm{b}} \\
\hline
-X_{\mathrm{md}} & -X_{\mathrm{de}} & -R_{\mathrm{e}} - \mathrm{j}hX_{\mathrm{qe}} & X_{\mathrm{mkd}} & \mathrm{j}hX_{\mathrm{mkq}} & -\mathrm{j}X_{\mathrm{de}} i_{\mathrm{d}0} h + \mathrm{j}X_{\mathrm{md}} i_{\mathrm{f}} h + V\sin\delta_{\mathrm{b}} \\
\hline
\mathrm{j}hX_{\mathrm{mkd}} & \mathrm{j}hX_{\mathrm{mkd}} & & R_{\mathrm{kd}} + \mathrm{j}X_{\mathrm{kd}} & & \\
\hline
 & & \mathrm{j}hX_{\mathrm{mkq}} & & R_{\mathrm{kq}} + \mathrm{j}X_{\mathrm{kq}} & \\
\hline
-X_{\mathrm{md}} i_{\mathrm{q}0} & (X_{\mathrm{dr}} - X_{\mathrm{qr}}) i_{\mathrm{q}0} & (X_{\mathrm{dr}} - X_{\mathrm{qr}}) i_{\mathrm{d}0} - X_{\mathrm{md}} i_{\mathrm{f}0} & & & -Mh^2 + \mathrm{j}hR_{\mathrm{F}} \\
\hline
\end{array}
\;
\begin{array}{|c|}
\hline \Delta i_{\mathrm{f}} \\ \hline \Delta i_{\mathrm{dr}} \\ \hline \Delta i_{\mathrm{qr}} \\ \hline \Delta i_{\mathrm{kd}} \\ \hline \Delta i_{\mathrm{kq}} \\ \hline \Delta\theta \\ \hline
\end{array}
\qquad (7.55)
$$

different values of real and reactive power. For each characteristic equation, the Routh–Hurwitz criterion, which is built into the programme, is applied to test the stability of the system.

On using the parameters listed below, stability boundaries were obtained for different combinations of real and reactive power.

Table 7.1 List of machine parameters

Parameters	*Per-unit*
X_d	1.596
X_q	1.57
X_{md}	1.457
X_{mq}	1.43
R_a	0.002 04
X_{fd}	1.598
X_{fq}	1.598
R_{fd}	0.000 85
R_{fq}	0.000 85
R_{kd}	0.009
R_{kq}	0.009
X_{kd}	1.51
X_{kq}	1.51
H	3.06

One of the severe limitations of applying the Routh criterion is said to be that it only predicts whether a system is stable or unstable and no more. This is not entirely correct as will be seen in the section that follows, although computation of T_{de} and T_S provides much more information.

Figure 7.8 represents a typical stability boundary obtained using Routh's criterion. The results have been tested on a microalternator fitted with an a.v.r.

On the curve in Figure 7.8 for which the reactive power $Q = Q_1$ for example, from a to b, the rotor pulls out of synchronism and no oscillation occurs. At b, critically damped oscillation occurs, i.e. if the rotor is displaced it returns to its original position asymptotically, without overshoot. From this point on $T_S > 0$. From b to c damped oscillation occurs and the amount of damping in the system is reduced as μ_r is increased until at c, self-oscillation occurs. These oscillations are of constant amplitude and may be initiated by marginal disturbance in the system. From c onwards, the oscillations can no longer be damped out and the rotor oscillates until synchronism is lost. From this point on $T_{de} < 0$.

The discontinuity (point d) itself has marginal stability where T_{de} and T_S both tend to zero.

Damping and synchronising torque coefficients were computed using the iterative technique described earlier. In this computation if T_S came out to be negative, the next higher value of μ was automatically selected. Figures 7.8(a) and (b) give the stability boundaries for a proportional a.v.r.

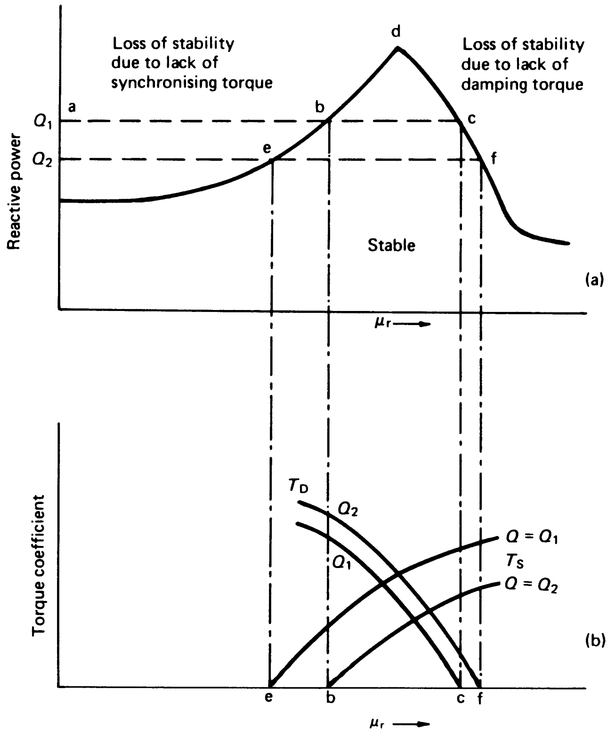

Figure 7.8 Interpretation of the stability boundary of a synchronous generator determined by using Routh–Hurwitz criterion

This analysis shows that stability boundaries plotted on the basis of the Routh–Hurwitz criterion provide more than 'yes' or 'no' answers if interpreted properly.

The obvious conclusions from the results presented above are:

(1) beyond a certain level of negative Q, neither small, nor large gains of a.v.r. are desirable;

(2) the loss of stability at very large values of gain is due to T_{de} becoming negative and experiments indicate that loss of stability then occurs with undamped oscillations of increasing amplitude.

Further conclusions regarding the effects of an a.v.r. for different operating conditions are discussed in Chapter 8.

7.7 HUNTING ANALYSIS OF INTERCONNECTED MACHINES

7.7.1 Ward–Leonard machines

The five-machine Ward–Leonard set described in Chapter 6 will be used as an example of the application of hunting theory to an interconnected machine system. The same closed paths through the network will be considered as in the transient analysis. However, in the hunting analysis the matrix of equations of small oscillations will be considered as a whole. The reference frame for the induction motor must therefore be such that the operator p has the same significance for induction motor quantities as it has for quantities in the direct current machines. In the latter case the operator p becomes zero in the steady state and consequently terms such as Mpi also become zero. In the case of the induction motor, the normal fixed reference axes are such that p becomes $j\omega$ in steady state. It is therefore necessary to transform the induction machine equations into a reference frame which rotates synchronously with the flux wave, as shown below. The hunting impedance matrices for the machines will be set up and these will be combined into an overall motional impedance for the system, from which the oscillatory states can be predicted. The computed results will be compared with figures measured on a laboratory set. The machine parameters are those given in Table 6.3. The motional impedance matrix is of the general form

$$\begin{bmatrix} \Delta \mathbf{V} \\ \Delta T \end{bmatrix} = \begin{bmatrix} \mathbf{R} + \mathbf{L}p + \mathbf{G}p\theta & \mathbf{G}\mathbf{i}_0 p \\ -\mathbf{i}_0(\mathbf{G}+\mathbf{G}_t) & Jp^2 + R_F p \end{bmatrix} \cdot \begin{bmatrix} \Delta \mathbf{i} \\ \Delta\theta \end{bmatrix} \qquad (7.56)$$

or

$$\Delta \mathbf{f} = \mathbf{Z} \Delta \dot{\mathbf{x}}$$

For the d.c. machine the transient electromechanical impedance matrix expressed along field and armature axes is

$$\mathbf{Z} = \begin{array}{c c} & \begin{array}{ccc} \mathrm{F} & \mathrm{a} & \mathrm{s} \end{array} \\ \begin{array}{c} \mathrm{F} \\ \mathrm{a} \\ \mathrm{s} \end{array} & \begin{bmatrix} R_f + L_f p & & \\ -Mp\theta & R_a + L_a p & -i_f Mp \\ i_a M & i_f M & Jp^2 + R_F p \end{bmatrix} \end{array} \qquad (7.57)$$

In steady state, p becomes zero. Under hunting conditions

$$\begin{aligned} p &= jh\omega \\ p\theta &= \omega_r \qquad & h &= \frac{f_{\text{hunting}}}{f_{\text{synch.}}} \\ \omega L_f &= X_f \\ & & \omega_r &= v\omega = \text{rotor angular velocity} \\ \omega L_a &= X_a \\ & & \omega_r M &= vX_m \\ \omega M &= X_m \end{aligned}$$

and the hunting matrix for the d.c. machine is therefore

$$\mathbf{Z}_1 = \begin{array}{c} \\ \text{F} \\ \text{a} \\ \text{s} \end{array} \begin{array}{c|c|c|c|} & \text{F} & \text{a} & \text{s} \\ \hline & R_f + jhX_f & & \\ \hline & -vX_m & R_a + jhX_a & -hi_f X_m \\ \hline & i_a M & i_f M & \begin{matrix} jh\omega R_F \\ -h^2\omega^2 J \end{matrix} \\ \hline \end{array} \tag{7.58}$$

In the balanced 3-phase induction motor, resolved along direct and quadrature axes, the following relationships may be assumed

$$\begin{aligned} L_{dr} &= L_{qr} = L_r \qquad & R_{dr} &= R_{qr} = R_r \\ L_{ds} &= L_{qs} = L_s \qquad & R_{ds} &= R_{qs} = R_s \\ M_d &= M_q = M \end{aligned} \tag{7.59}$$

Let a and b denote orthogonal axes rotating synchronously with the flux wave, on both stator and rotor as in Figure 7.9. The transformation matrix from axes d and q, is

$$\mathbf{C} = \begin{array}{c} \\ \text{ds} \\ \text{dr} \\ \text{qr} \\ \text{qs} \\ \text{s} \end{array} \begin{array}{c|c|c|c|c|c|} & \text{as} & \text{ar} & \text{br} & \text{bs} & \text{s} \\ \hline & \cos\alpha & & & -\sin\alpha & \\ \hline & & \cos\alpha & -\sin\alpha & & \\ \hline & & \sin\alpha & \cos\alpha & & \\ \hline & \sin\alpha & & & \cos\alpha & \\ \hline & & & & & 1 \\ \hline \end{array} \tag{7.60}$$

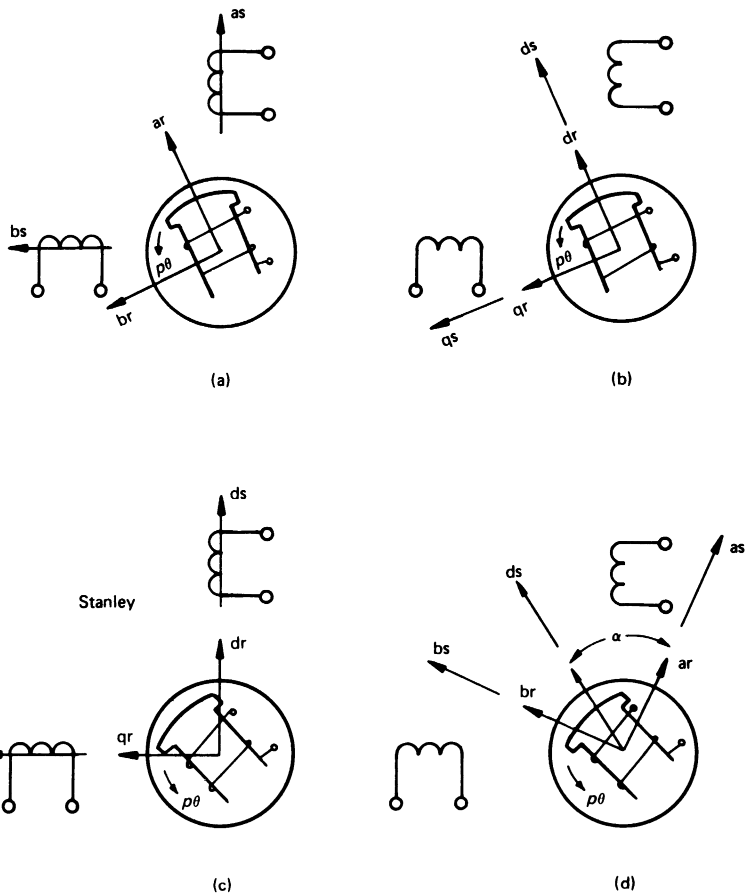

Figure 7.9 Reference frames used in electrical machine analysis. (a) Slip-ring axes, (b) Park's axes, (c) stationary dq axes and (d) freely rotating axes

The matrix **C** is now a function of the axis position-angle α and the transformed impedance matrix is given by

$$\mathbf{Z}_2' = \mathbf{C}_t\mathbf{Z}_2\mathbf{C} + \mathbf{C}_t\mathbf{L}\frac{\partial \mathbf{C}}{\partial \alpha}p\alpha \qquad (7.61)$$

where **L** is the inductance matrix (shown in equation 5.53).

The new current and voltage vectors are given by

$$\mathbf{i} = \mathbf{C}\mathbf{i}'$$

$$\mathbf{V}' = \mathbf{C}_t\mathbf{V}$$

The rotor slip speed $p\theta_s = p\alpha - p\theta = (1 - v)\omega$.

In this reference frame the rotor appears to rotate backwards at the angular velocity of slip. The transformed motional impedance matrix for the induction motor in the rotating frame is

$$
\mathbf{Z}_2' = \begin{array}{c} \\ \mathrm{as} \\ \mathrm{ar} \\ \mathrm{br} \\ \mathrm{bs} \\ \mathrm{s} \end{array}
\begin{array}{ccccc}
\mathrm{as} & \mathrm{ar} & \mathrm{br} & \mathrm{bs} & \mathrm{s} \\
\hline
R_s + L_s p & Mp & Mp\alpha & -L_s p\alpha & \\
Mp & R_r + L_r p & -L_r p\theta_s & -Mp\theta_s & (i_{br}L_r + i_{bs}M)p \\
Mp\theta_s & L_r p\theta_s & R_r + L_r p & Mp & -(i_{as}M + i_{ar}L_r)p \\
L_s p\alpha & Mp\alpha & Mp & R_s + L_s p & \\
i_{br}M & -i_{bs}M & i_{as}M & -i_{ar}M & R_F p + Jp^2
\end{array}
\tag{7.62}
$$

In the steady hunting condition

$$
\begin{aligned}
p &= \mathrm{j}h\omega \\
p\alpha &= \omega \\
p\theta_s &= s\omega \\
\omega L_s &= X_s \\
\omega L_r &= X_r \\
\omega M &= X_m
\end{aligned}
$$

and for steady hunting

$$
\mathbf{Z}_2' = \begin{array}{c} \\ \mathrm{as} \\ \mathrm{ar} \\ \mathrm{br} \\ \mathrm{bs} \\ \mathrm{s} \end{array}
\begin{array}{ccccc}
\mathrm{as} & \mathrm{ar} & \mathrm{br} & \mathrm{bs} & \mathrm{s} \\
\hline
R_s + \mathrm{j}hX_s & \mathrm{j}hX_m & -X_m & -X_s & \\
\mathrm{j}hX_m & R_r + \mathrm{j}hX_r & -sX_r & -sX_m & \mathrm{j}h(i_{br}X_r + i_{bs}X_m) \\
sX_m & sX_r & R_r + \mathrm{j}hX_r & \mathrm{j}hX_m & -\mathrm{j}h(i_{as}X_m + i_{ar}X_r) \\
X_s & X_m & \mathrm{j}hX_m & R_s + \mathrm{j}hX_s & \\
i_{br} & -i_{bs} & i_{as}M & -i_{ar}M & \mathrm{j}h\omega R_F - h^2\omega^2 J
\end{array}
\tag{7.63}
$$

The steady-state currents in the row and column s are calculated from the steady-state matrix. This may be obtained from the original transient impedance matrix in the rotating frame. Under steady load conditions the operator p becomes zero, $p\alpha = \omega$ and $p\theta_s = s\omega$ and the steady-state impedance matrix is

$$\mathbf{Z}' = \begin{array}{c|c|c|c|c|} & \text{as} & \text{ar} & \text{br} & \text{bs} \\ \hline \text{as} & R_s & & -X_m & -X_s \\ \hline \text{ar} & & R_r & -sX_r & -sX_m \\ \hline \text{br} & sX_m & sX_r & R_r & \\ \hline \text{bs} & X_s & X_m & & R_s \\ \hline \end{array} \tag{7.64}$$

The impressed steady-state voltage vector is known and hence the currents can be calculated.

7.7.2 Interconnected machines

The **C**-matrix which was used in Chapter 6 to connect the transient impedances of the machines may now be used to interconnect the hunting motional impedance matrices. Since the interconnecting paths are all relatively stationary the interconnection is carried out by the simple operation $\mathbf{C}_t\mathbf{Z}\mathbf{C}$ where, in this case, the matrix $\mathbf{Z}$ is the block diagonal sum of all the individual machine motional impedance matrices.

During self-excited oscillations, initiated, for example, by an impulsive disturbance applied to the load dynamometer, the impressed hunting voltage and torque may be considered to be zero, since the hunting investigation is restricted to the subsequent condition of oscillation. The motional equation is therefore

$$0 = \mathbf{Z}'\Delta\mathbf{x}' \tag{7.65}$$

where

$$\Delta\mathbf{x}'_t = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \text{F}_1 & \text{F}_2 & \text{F}_3 & \text{F}_4 & \text{a} & \text{as} & \text{ar} & \text{br} & \text{bs} & \text{sA} & \text{sB} \\ \hline i_{f1} & i_{f2} & i_{f3} & i_{f4} & i_a & i_{as} & i_{ar} & i_{br} & i_{bs} & \Delta\theta_g & \Delta\theta_m \\ \hline \end{array} \tag{7.66}$$

However, in order to obtain some degree of control over the hunting conditions, a voltage regulator was installed. This was energised by a signal obtained from a tachogenerator on the shaft of the Ward–Leonard motor. This signal was compared with a reference voltage and the difference acted as input to a power unit which provided current to the field of the voltage-generator machine

(machine field F_1). In response to voltage changes at the tachogenerator, corresponding to speed changes at the Ward–Leonard motor, the voltage regulator acted to adjust the output voltage of the Ward–Leonard generator to maintain constant speed at the Ward–Leonard motor. With the voltage regulator in circuit the field of the Ward–Leonard motor was separately excited. The appropriate coupling terms in the hunting matrix were omitted and the new form of $\mathbf{Z}'$ is shown in equation (7.67).

The regulator term is obtained by simple analysis as follows.

Let $\Delta p\theta$ be a change in the speed of the Ward–Leonard motor and ΔV_{f1} be the corresponding change in voltage applied to the voltage-generator field.

With negligible time delay in the regulator

$$\Delta V_{f1} = -\mu_0 \Delta p\theta \tag{7.68}$$

Where μ_0 is the overall gain of the feedback system. This may be written

$$\Delta V_{f1} = -\mu_r \mu_t \Delta p\theta \tag{7.69}$$

where μ_r is the regulator gain and μ_t the gain of the tachogenerator. With regulator response delay time-constant τ_s,

$$\frac{\Delta V_{f1}}{\Delta p\theta} = -\frac{\mu_r \mu_t}{(1+\tau_s p)} \tag{7.70}$$

However, the time constant of the voltage generator was measured at 0.7 s and the time constant of the regulator (τ_s) was 0.1 s, so that the delay could reasonably be ignored and the term to be substituted in the last column of row one of $\mathbf{Z}'$ was as shown.

7.7.3 Computation and test results

The hunting equation of torque has the general form

$$\Delta T = Jp^2\Delta\theta + R_F p\Delta\theta - \mathbf{i}_0(\mathbf{G}+\mathbf{G}_t)\Delta\mathbf{i} - \mathbf{i}_0\Delta\mathbf{G}\mathbf{i} \tag{7.71}$$

In the selected reference frames the inductance values are constant and $\Delta\mathbf{G}$ is zero. The electrically generated torque is therefore

$$\Delta T_e = \mathbf{i}_0\mathbf{G}\Delta\mathbf{i} + \Delta\mathbf{i}\mathbf{G}\mathbf{i}_0 \tag{7.72}$$

In the motional impedance matrix, after substituting $jh\omega$ for the operator p, the currents $\Delta\mathbf{i}$ may be eliminated and the hunting torque has the form

$$\Delta T = (T_S \pm j\,T_{de})\Delta\theta - h^2\omega^2 J\theta + jR_F h\omega\theta \tag{7.73}$$

In this expression

$$T_S = -\mathrm{Re}(\mathbf{i}_0\mathbf{G}\Delta\mathbf{i} + \Delta\mathbf{i}\mathbf{G}\mathbf{i}_0) \tag{7.74}$$

$$T_{de} = -\mathrm{Im}(\mathbf{i}_0\mathbf{G}\Delta\mathbf{i} + \Delta\mathbf{i}\mathbf{G}\mathbf{i}_0) \tag{7.75}$$

$\mathbf{Z}' =$ (7.67)

	F₁	F₂	F₃	F₄	a	as	ar	br	bs	s_A	s_B
F₁	$R_{f1}+R_{a1}$ $+jh_s(X_{f1}+X_{a1})$ $-M_1\omega_s$	$R_{a1}+$ $+jh_sX_{a1}$								$jh_si_{f1}X_{m1}$	$-(\mu_r\mu_i)p$
F₂	$R_{a1}+jh_sX_{a1}$ $-M_1\omega_s$	$R_{a1}+R_{f2}$ $+jh_s(X_{a1}+X_{f2})$								$jh_si_{f1}X_{m1}$	
F₃		$-M_2\omega_s$	$R_{a2}+R_{f3}$ $jh_s(X_{a2}+X_{f3})$							$jh_si_{f2}X_{m2}$	
F₄				R_{f4} $+jh_mX_{f4}$							
a			$-M_3\omega_s$	$M_4\omega_m$	$R_{a3}+R_{a4}$ $+j(h_sX_{a3}+h_mX_{a4})$					$jh_si_{f3}X_{m3}$	$jh_mi_{f4}X_{m4}$
as						R_s $+jh_sX_s$	jh_sX_m	$-X_m$	$-X_s$		
ar						jh_sX_m	R_r+ $+jh_sX_r$	$-sX_r$	$-sX_m$	$jh_s(i_{br}X_r$ $+i_{bs}X_m)$	
br						sX_m	sX_r	R_r $+jh_sX_r$	jh_sX_m	$-jh_s(i_{as}X_m$ $+i_{ar}X_r)$	
bs						X_s	X_m	jh_sX_m	$R_s+jh_sX_s$		
s_A	$-(2i_{f1}+i_{f2})M_1$	$-(i_{f1}M_1+$ $+i_{f3}M_2)$	$-(i_{f2}M_2$ $+i_aM_3)$	$-i_{f1}M_1$	$-i_{f3}M_3$	$i_{br}M$	$-i_{bs}M$	$i_{as}M$	$-i_{ar}M$	$-h_s^2\omega_s^2J_s$ $-jh_s\omega_sR_s$	
s_B				i_aM_4	$i_{f4}M_4$					$-h_m^2\omega_m^2J_m$ $+jh_m\omega_mR_m$	

This can be illustrated as follows. The equations of hunting may be expressed

$$\begin{array}{c|c|} & \\ \mathrm{e} & \Delta \mathbf{V} \\ \mathrm{m} & \Delta \mathbf{T} \end{array} = \begin{array}{c|c|c|} & \mathrm{e} & \mathrm{m} \\ \mathrm{e} & \boldsymbol{Z}_{\mathrm{a}} & \boldsymbol{Z}_{\mathrm{b}} \\ \mathrm{m} & \boldsymbol{Z}_{\mathrm{c}} & \boldsymbol{Z}_{\mathrm{d}} \end{array} \cdot \begin{array}{|c|} \\ \Delta \mathbf{i} \\ \Delta \boldsymbol{\theta} \end{array} \tag{7.76}$$

The impressed quantities are zero, and $\Delta \mathbf{i}$ may be eliminated to give

$$0 = (\boldsymbol{Z}_{\mathrm{d}} - \boldsymbol{Z}_{\mathrm{c}}\boldsymbol{Z}_{\mathrm{a}}^{-1}\boldsymbol{Z}_{\mathrm{b}})\Delta\boldsymbol{\theta} \tag{7.77}$$

In this expression $\boldsymbol{Z}_{\mathrm{d}}$ corresponds to the mechanical terms $(-Jh^2\omega^2 + \mathrm{j}R_{\mathrm{F}}h\omega)$. The real and imaginary parts of $\boldsymbol{Z}_{\mathrm{c}}\boldsymbol{Z}_{\mathrm{a}}^{-1}\boldsymbol{Z}_{\mathrm{b}}$ give the electrically generated T_{S} and the T_{de} respectively. A flow chart for this computation is shown in Figure 7.10 and an example of the calculated values of the damping and synchronising torques is shown in Figure 7.11. In general, d.c. machines contribute a large amount of damping torque to any machine system of which they are part. The WL machines will not normally show a tendency to hunt, except when negative damping is contributed by a voltage regulator. The induction motor is inherently heavily damped and also produces positive synchronising torque which in the Ward–Leonard system may be partly cancelled by negative synchronising torque produced by a regulator. The action of regulators and speed governors in machine dynamics is complex and is discussed further in Chapter 8.

Table 7.2 Steady-state values for the hunting tests (per unit)

	test 1	*test 2*	*test 3*
i_{f1}	0.007 7	0.007 6	0.010 1
i_{f2}	0.022	0.021	0.026 7
i_{f3}	0.011 3	0.010 8	0.014
i_{f4}	0.033 7	0.033 3	0.033 3
i_{a}	0.45	0.523	0.45
ω_{m}	0.423	0.453	0.523
ω_{g}	0.981	0.975	0.979

The steady-state test conditions are shown in Table 7.2. Under conditions of tests 1 and 2, the induction motor/Ward–Leonard generator and the Ward–Leonard motor had the same oscillation frequency. In test 3 these two dynamical sets exhibited the complex pattern shown in Figure 7.12. At certain operating points the oscillation frequency was the same for each set, at other operating points the hunting frequencies were different. In all cases the computed and test values of the hunting frequencies were close.

The predicted oscillatory behaviour of the system was investigated by determining the hunting frequencies from the system eigenvalues.

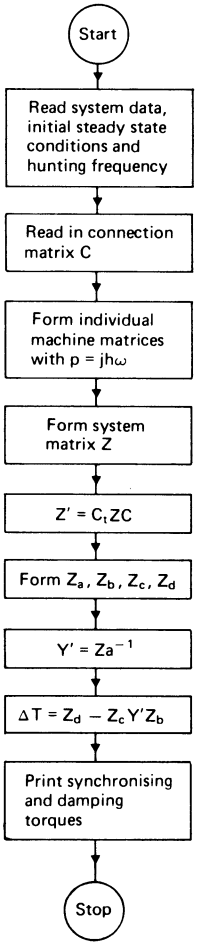

Figure 7.10 Flow chart for T_S and T_{de}

In the above analysis the overall hunting equation is of the form

$$0 = \bar{\mathbf{Z}}\Delta\bar{\mathbf{x}} \tag{7.78}$$

where $\bar{\mathbf{Z}}$ is the total system impedance matrix, the inductance terms being associated with the operator $p = \mathrm{d}/\mathrm{d}t$. Upon expansion, since $\bar{\mathbf{R}}$ and $\bar{\mathbf{L}}$ are constants, this equation may be written

$$(\bar{\mathbf{R}} + p\bar{\mathbf{L}})\Delta\bar{\mathbf{x}} = 0 \tag{7.79}$$

where $\bar{\mathbf{R}}$ and $\bar{\mathbf{L}}$ are square non-singular matrices and $\Delta\bar{\mathbf{x}}$ is the response vector. In terms of the state variable (column) vector $\Delta\bar{\mathbf{x}}$ and a square non-singular

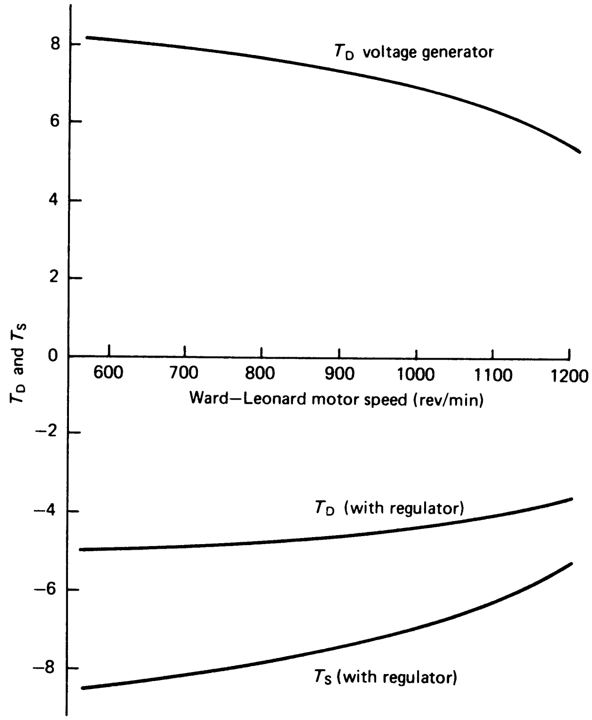

Figure 7.11 Variation of damping and synchronising torque with Ward–Leonard motor speed

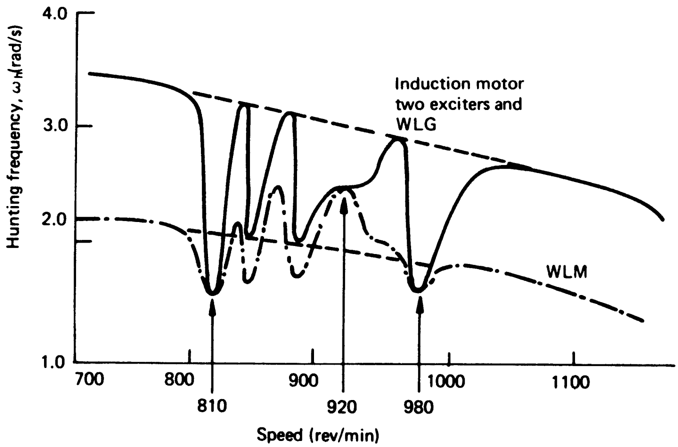

Figure 7.12 Hunting frequencies in induction-motor/Ward–Leonard generator and Ward–Leonard motor at different operating speeds

system matrix **A**, this may be written

$$p(\Delta\bar{\mathbf{x}}) = \mathbf{A}\Delta\bar{\mathbf{x}} \tag{7.80}$$

or

$$(p\mathbf{I} - \mathbf{A})\Delta\bar{\mathbf{x}} = 0 \tag{7.81}$$

where

$$\mathbf{A} \equiv -\overline{\mathbf{R}}(\overline{\mathbf{L}})^{-1}$$

The characteristic polynomial is given by expansion of the determinant $|s\mathbf{I} - \mathbf{A}|$ (Section 7.4) and the roots of the characteristic equation give the eigenvalues.

In computing the eigenvalues for the Ward–Leonard machine system, Bocher's formula was used to give the coefficients in the characteristic equation. The roots (in this case twelve) were obtained by application of standard iteration techniques. A typical computation is shown in Table 7.3.

For the conditions of test 1, for example, there are three significant pairs of complex roots (in the rows indicated by -1 in Table 7.3).

Table 7.3 Computed hunting frequencies for hunting test 1

Coefficients of the polynomial		
1.000 0		
−2.929 7E+2		
2.788 7E+4		
−2.968 5E+7		
1.031 8E+9		
1.963 1E+11		
4.980 0E+12		
5.958 1E+13		
2.901 6E+14		
6.066 3E+14		
5.629 5E+14		
−2.456 5E+15		
Roots of the polynomial (there are 6 pairs of roots)		
1.000	1.269 0	0.000 0
−1.000	−1.506 4	3.005 0
1.000	−5.814 5	−4.337 1E+1
−1.000	−1.030 7E+1	9.214 5
1.000	2.531 0E+2	1.424 2E+2
−1.000	−1.560 4E+1	3.136 2E+2

$$-1.5064 \pm j\,3.005$$
$$-10.307 \pm j\,9.2145$$
$$-15.504 \pm j\,313.62$$

The dominant root yields 3.005 rad/s or 0.478 Hz, the second yields 9.214 5 rad/s or 1.47 Hz, and the third gives oscillations at 50 Hz, the supply frequency. The corresponding test frequencies are seen from Figure 7.13 to be close, namely 0.466 Hz with a superimposed frequency 1.5 Hz for both the induction motor/WL generator mass and the WL motor.

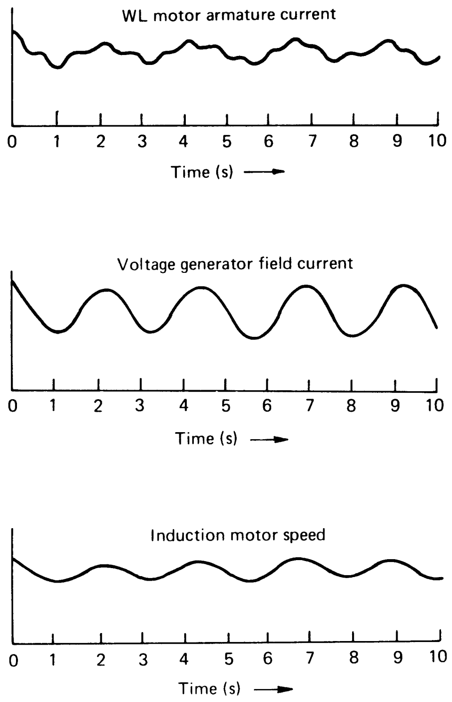

Figure 7.13 Traces of the ultra-violet recordings of hunting

7.8 REFERENCES

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8

Synchronous machine oscillations

8.1 INTRODUCTION*

We have seen that the dynamical stability of an electromechanical system is determined by the damping and synchronising torque coefficients and the inertia constants. In a mechanical system damping and spring constants can be easily visualised—damping, for instance arises due to friction. In an electrical system, mechanical friction constitutes a small part of the total damping, the main damping torque being of electrical origin. In trying to understand the electrically generated damping, we can tell intuitively that this is caused by power dissipation due to copper loss. In a machine, during steady-state operation, copper loss takes place continually and largely accounts for the power difference between input and output. If, however, the rotor begins to oscillate about its steady-state angular velocity, oscillating currents induced as a result, generate additional copper loss. For instance, if an oscillating current $\Delta I \sin(\omega_0 t + \alpha)$ in a winding is superimposed upon a 50 Hz steady-state current $I \sin \omega t$, where $\omega_0 = h\omega$, then the additional average copper loss is

$$\frac{1}{T}\int_0^T \{I \sin \omega t + \Delta I \sin(\omega_0 t + \alpha)\}^2 R\,\mathrm{d}t - \frac{1}{T}\int_0^T (I \sin \omega t)^2 R\,\mathrm{d}t$$

$$= \frac{\Delta I^2 R}{2} = (\Delta I_{\mathrm{rms}})^2 R \tag{8.1}$$

where the integration period is the common repetition time for the two

* In this chapter we retain the subscripts ds for the synchronous machine field quantities and use subscripts f and b for components of current, voltage, and impedance in forward and backward reference axes.

oscillatory currents. This additional copper loss appears to be the only dissipation (neglecting mechanical damping) which may suppress the rotor oscillation and bring it back to its normal uniform angular velocity. This argument, however, cannot explain why the rotor oscillations may sometimes build up, indicating the presence of *negative damping*—apparently produced by copper loss which is always positive. We shall see presently that the physical nature of damping is more complicated than that.

Efforts to establish a correlation between the damping coefficient T_d and the additional oscillatory copper loss, in the case of an experimental synchronous machine, using Park's reference frame, turned out to be futile. In a laboratory experiment, for example, the quantity $h\omega T_d$ gave a value $T_d = 0.388\,41$ per unit and $(\Delta I^2 R)/h$ gave a figure 1.131 37 (here, $\omega = 2\pi f$ is unity, as shown in Appendix A1). Clearly there is a fundamental error in equating these two expressions. It was not, in fact, until Gabriel Kron examined the problem in 1952 that the anomaly was clarified.[2] Although Kron developed his theory of hunting using tensor analysis,[1] we shall endeavour to clarify the picture with aid of a straightforward vector diagram.

8.2 ABSOLUTE AND APPARENT CHANGES IN CURRENT AND VOLTAGE VECTORS

Figure 8.1 shows a somewhat simplified vector diagram of current in a synchronous machine, during small oscillation of the rotor. In steady state operation, the total armature current wave represented by the vector OM rotates synchronously at constant angular velocity, along with the generator field, to which Park's direct and quadrature axes are fixed. The diagram shows also orthogonal axes α and β which are aligned along Park's d and q axes in the steady state. During small oscillation of the rotor, Park's axes also oscillate about the synchronous steady state position and hence in these axes, no field oscillation is observed directly. The field structure and with it, Park's axes, will also oscillate with respect to the freely rotating axes α and β.

Now consider a small change MN in the amplitude of the current vector OM during the time in which the rotor moves forward through the small angle $\Delta\lambda$. The direct axis component of armature current at the beginning of the oscillation is OA and at its forward limit is OC. The increment of current in Park's direct axis is therefore BC, where $|OA| = |OB|$. Now this is obviously an artificial increase in current as far as armature copper loss is concerned, since it contains a component CE which arises solely by the change in rotor position while the change MN is taking place. In the freely rotating synchronous reference frame $\alpha\beta$ which does not oscillate, the current increment in the axis α is AD which will produce copper loss in the equivalent armature coils. The corresponding components along the respective quadrature axes are also shown on the diagram. The components AD and A′D′ in the α and β axes do yield

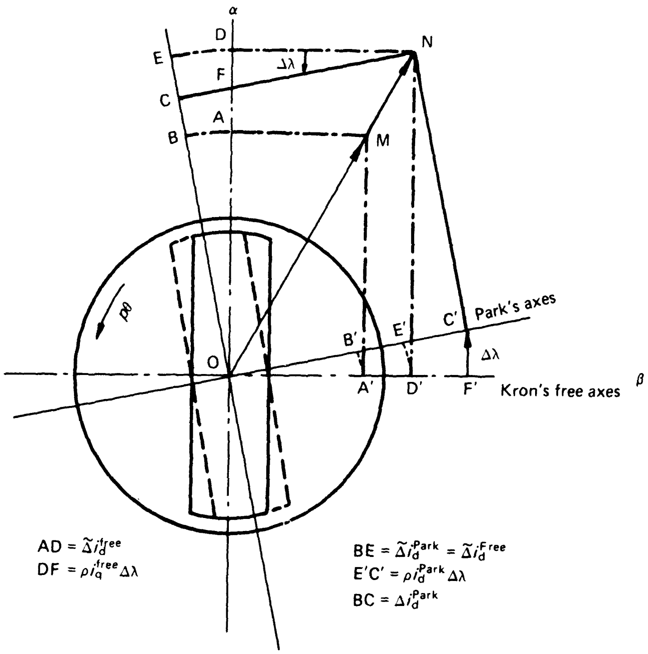


Figure 8.1 Changes in armature current and m.m.f. as seen from Park's and Kron's free axes

hunting copper loss which gives directly the damping torque during small oscillation. We shall call the components AD and A′D′ 'absolute' increments $\tilde{\Delta} i_\alpha = \mathrm{AD}$, $\tilde{\Delta} i_\beta = \mathrm{A'D'}$ and it can be seen from the diagram that the following relations obtain

$$\mathrm{AD} = \mathrm{BE} = \mathrm{BC} + \mathrm{CE} = \mathrm{BC} - i_q \Delta\lambda$$

$$\mathrm{A'D'} = \mathrm{B'E'} = \mathrm{B'C'} + \mathrm{C'E'} = \mathrm{B'C'} - i_d \Delta\lambda$$

$$\tilde{\Delta}\mathbf{i}^{\mathrm{Free}} = \tilde{\Delta}\mathbf{i}^{\mathrm{Park}} = \Delta\mathbf{i}^{\mathrm{Park}} + \boldsymbol{\rho}\,\mathbf{i}^{\mathrm{Park}} \Delta\lambda \tag{8.2}$$

giving

$$\tilde{\Delta} i_\alpha = \tilde{\Delta} i_d = \Delta i_d - i_q \Delta\lambda \tag{8.3a}$$

$$\tilde{\Delta} i_\beta = \tilde{\Delta} i_q = \Delta i_q + i_d \Delta\lambda \tag{8.3b}$$

Here $\boldsymbol{\rho}$ is a rotation operator

$$\boldsymbol{\rho} = \begin{array}{c|cc} & \mathrm{d} & \mathrm{q} \\ \hline \mathrm{d} & & -1 \\ \mathrm{q} & 1 & \end{array} \tag{8.4}$$

In matrix form

$$
\begin{array}{c} \\ d \\ q \end{array}
\begin{array}{c} \\ \left[\begin{array}{c} \tilde{\Delta i}_d \\ \tilde{\Delta i}_q \end{array}\right] \end{array}
=
\begin{array}{c} \\ d \\ q \end{array}
\begin{array}{c} \\ \left[\begin{array}{c} \Delta i_d \\ \Delta i_q \end{array}\right] \end{array}
+
\begin{array}{c} \\ d \\ q \end{array}
\begin{array}{c} \begin{array}{cc} \;d\; & \;q\; \end{array} \\ \left[\begin{array}{cc} & -1 \\ 1 & \end{array}\right] \end{array}
\cdot
\begin{array}{c} \\ \left[\begin{array}{c} i_d \Delta\lambda \\ i_q \Delta\lambda \end{array}\right] \end{array}
\tag{8.5}
$$

In examining the relationships given by Figure 8.1 it will be remembered that the rotor excursion is considered to be small and therefore second order effects are neglected, for example

$$(i_d + \Delta i_d)\Delta\lambda = i_d \Delta\lambda \qquad \text{etc.} \tag{8.6}$$

Since the hunting behaviour of the synchronous machine would appear to have a simpler form in a freely rotating reference frame, we shall examine the necessary matrix transformations in the following section.

8.3 TRANSFORMATION TO KRON'S FREELY ROTATING REFERENCE AXES

First we shall restate some of the expressions for the primitive machine which we have derived in Chapter 6. In direct and quadrature axes the voltage and torque equations can be expressed in terms of resistance and inductance matrices and a motional inductance matrix **G** which we call the torque matrix. The machine equations then have the transient form

$$\text{voltage} \quad \mathbf{V} = \mathbf{R}\mathbf{i} + \mathbf{L}p\mathbf{i} + \mathbf{G}p\theta \tag{8.7}$$

$$\text{torque} \quad T = -\mathbf{i}^*\mathbf{B} = -\mathbf{i}^*\mathbf{G}\mathbf{i} \tag{8.8}$$

For small oscillations these equations give

$$\Delta\mathbf{V} = (\mathbf{R} + \mathbf{L}p + \mathbf{G}p\theta)\Delta\mathbf{i} + \mathbf{G}\mathbf{i}\Delta(p\theta) \tag{8.9}$$

$$\Delta T = -(\mathbf{i}^*\mathbf{G}\Delta\mathbf{i} + \Delta\mathbf{i}^*\mathbf{G}\mathbf{i}) + Jp(\Delta p\theta) \tag{8.10}$$

(Here we omit a term $\mathbf{i}^*\Delta\mathbf{G}\mathbf{i}$ since in d and q axes the terms of the torque matrix **G** are constant.)

In terms of general 'motional impedance' matrix $\mathbf{Z}$

$$\Delta\mathbf{V} = \mathbf{Z}(\Delta\mathbf{i} + \Delta\boldsymbol{\theta})$$

$$
\begin{array}{c|c|}
 & \\
\hline
\text{elect.} & \Delta\mathbf{V} \\
\hline
\text{mech.} & \Delta T \\
\hline
\end{array}
=
\begin{array}{|c|c|}
\multicolumn{1}{c}{\text{elect.}} & \multicolumn{1}{c}{\text{mech.}} \\
\hline
\mathbf{R}+\mathbf{L}p+\mathbf{G}p\theta & \mathbf{G}\mathbf{i} \\
\hline
-\mathbf{i}^{*}(\mathbf{G}+\mathbf{G}_{\mathrm{t}}) & Jp^{2} \\
\hline
\end{array}
\cdot
\begin{array}{|c|}
\hline
\Delta\mathbf{i} \\
\hline
\Delta\theta \\
\hline
\end{array}
\tag{8.11}
$$

which expands to[1] the form given in equation (8.12).

It should be noted here that $\Delta p\theta = p\Delta\theta$ and $p^2\Delta\theta = p(\Delta p\theta)$.

The hunting equations of other types of machine in any allowable reference axes can be obtained by transformation of the voltage and current vectors and the motional impedance matrix $\boldsymbol{\mathcal{Z}}$ in equation (8.12) by the appropriate connection $\mathbf{C}$-matrix. This is obtained by the equation $\mathbf{i} = \mathbf{C}\mathbf{i}'$ relating the steady state current in the primitive machine coils to the current in the coils of the derived machine—as shown in previous chapters. Some of the elements of the connection matrix $\mathbf{C}$ may be functions of a time-varying variable—usually a rotational angle in machine analysis—and the law of transformation of $\boldsymbol{\mathcal{Z}}$ is therefore quite complicated, namely[1]

$$
\begin{aligned}
\boldsymbol{\mathcal{Z}}(\Delta\mathbf{i}' + \Delta\boldsymbol{\theta}') &= \mathbf{C}_{\mathrm{t}}\boldsymbol{\mathcal{Z}}\mathbf{C}\Delta\mathbf{i}' + \mathbf{C}_{\mathrm{t}}\boldsymbol{\mathcal{Z}}\mathbf{C}\Delta\bar{\boldsymbol{\theta}}' + \mathbf{C}_{\mathrm{t}}\boldsymbol{\mathcal{Z}}\Delta\mathbf{C}\mathbf{i}' + \Delta\mathbf{C}_{\mathrm{t}}\boldsymbol{\mathcal{Z}}\mathbf{C}\mathbf{i}' \\
&= \mathbf{C}_{\mathrm{t}}\boldsymbol{\mathcal{Z}}\mathbf{C}(\Delta\mathbf{i}' + \Delta\bar{\boldsymbol{\theta}}') + \mathbf{C}_{\mathrm{t}}\boldsymbol{\mathcal{Z}}\frac{\partial\mathbf{C}}{\partial\boldsymbol{\theta}'}\mathbf{i}'\Delta\bar{\boldsymbol{\theta}}' \\
&\quad + \left[\frac{\partial\mathbf{C}_{\mathrm{t}}}{\partial\bar{\boldsymbol{\theta}}'}\mathbf{R}\mathbf{C}\mathbf{i}' + \frac{\partial\mathbf{C}_{\mathrm{t}}}{\partial\bar{\boldsymbol{\theta}}'}\mathbf{G}\mathbf{C}\mathbf{i}'p\theta\right]\Delta\bar{\boldsymbol{\theta}}'
\end{aligned}
\tag{8.13}
$$

In using these equations to derive the synchronous machine hunting equations in Kron's freely rotating axes, we first omit the stator quadrature axis field coil qs together with all its self and mutual parameters. The $\mathbf{C}$-matrix is obtained from the relationship between the values of current in the primitive machine d and q axis coils and those expressed along the free axes. We take the free axes at an arbitrary synchronous angle λ to the axes of Park in the steady state. From Figure 8.2 the current relationship is seen to be

$$
\begin{array}{c|c|}
\hline
\mathrm{ds} & i_{\mathrm{ds}} \\
\hline
\mathrm{dr} & i_{\mathrm{dr}} \\
\hline
\mathrm{qr} & i_{\mathrm{qr}} \\
\hline
\mathrm{s} & p\theta \\
\hline
\end{array}
=
\begin{array}{c|c|c|c|c|}
 & \multicolumn{1}{c}{\mathrm{ds}} & \multicolumn{1}{c}{\alpha} & \multicolumn{1}{c}{\beta} & \multicolumn{1}{c}{\mathrm{s}} \\
\hline
\mathrm{ds} & 1 & & & \\
\hline
\mathrm{dr} & & \cos\lambda & -\sin\lambda & \\
\hline
\mathrm{qr} & & \sin\lambda & \cos\lambda & \\
\hline
\mathrm{s} & & & & 1 \\
\hline
\end{array}
\cdot
\begin{array}{c|c|}
\hline
\mathrm{ds} & i_{\mathrm{ds}} \\
\hline
\alpha & i_{\alpha} \\
\hline
\beta & i_{\beta} \\
\hline
\mathrm{s} & p\theta \\
\hline
\end{array}
\tag{8.14}
$$

$$
\begin{array}{c}
\mathrm{ds}\\ \mathrm{dr}\\ \mathrm{qr}\\ \mathrm{qs}\\ \mathrm{s}
\end{array}
\begin{bmatrix}
\Delta V_{ds}\\ \Delta V_{dr}\\ \Delta V_{qr}\\ \Delta V_{qs}\\ \Delta T
\end{bmatrix}
=
\begin{array}{c}
\\ \mathrm{ds}\\ \mathrm{dr}\\ \mathrm{qr}\\ \mathrm{qs}\\ \mathrm{s}
\end{array}
\begin{array}{c}
\begin{array}{ccccc}
\mathrm{ds} & \mathrm{dr} & \mathrm{qr} & \mathrm{qs} & \mathrm{s}
\end{array}\\
\begin{bmatrix}
R_{ds}+L_{ds}p & M_d p & & & \\
M_d p & R_r+L_{dr}p & L_{qr}p\theta & M_q p\theta & M_q i_{qs}+L_{qr}i_{qr}\\
-M_d p\theta & -L_{dr}p\theta & R_r+L_{qr}p & M_q p & -M_d i_{ds}-L_{dr}i_{dr}\\
 & & M_q p & R_{qs}+L_{qs}p & \\
M_d i_{qr} & \begin{array}{l}-M_q i_{qs}\\ -i_{qr}(L_{qr}-L_{dr})\end{array} & \begin{array}{l}M_d i_{ds}\\ -i_{dr}(L_{qr}-L_{dr})\end{array} & -M_q i_{dr} & Jp
\end{bmatrix}
\end{array}
\begin{bmatrix}
\Delta i_{ds}\\ \Delta i_{dr}\\ \Delta i_{qr}\\ \Delta i_{qs}\\ \Delta p\theta
\end{bmatrix}
\quad (8.12)
$$

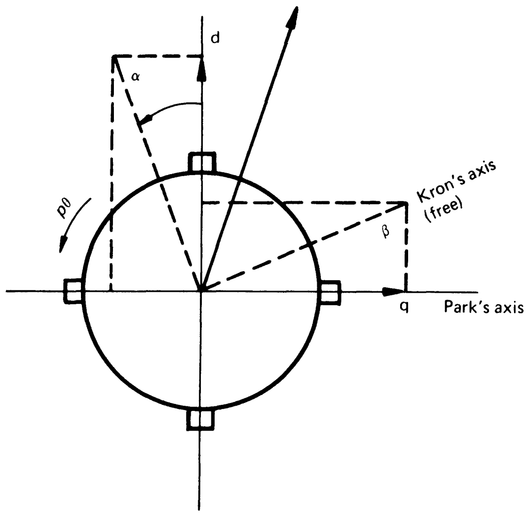


Figure 8.2 Transformation from Park's to Kron's free axes

and the **C**-matrix is therefore

$$\mathbf{C} = \begin{array}{c|cccc|} & \mathrm{ds} & \alpha & \beta & \mathrm{s} \\ \mathrm{ds} & 1 & & & \\ \mathrm{dr} & & \cos\delta & -\sin\delta & \\ \mathrm{qr} & & \sin\delta & \cos\delta & \\ \mathrm{s} & & & & 1 \end{array} \tag{8.15}$$

This **C**-matrix now operates upon the impedance matrix **Z** in equation (8.12) in accordance with the terms of equation (8.13). The operation $\Delta\mathbf{V}' = \mathbf{C}_t\Delta\mathbf{V}$ gives the derived voltage vector. In transforming the voltage vector it must be remembered that the total change in the voltage components will have two parts, one part being a magnitude change and the other due to the change in rotor angle $\Delta\delta$ with respect to its steady-state position. Thus, for example

$$\Delta\overline{V}_{dr} = \Delta V_{dr} + \frac{\partial V}{\partial\delta}\Delta\delta \tag{8.16}$$

$$
\begin{array}{c}
\\ \\
\left[\begin{array}{c} \Delta V_{\mathrm{ds}} \\ \Delta V_{\alpha} \\ \Delta V_{\beta} \\ \Delta T \end{array}\right]
\end{array}
\begin{array}{c}
\\ \\
\begin{array}{c} \mathrm{ds} \\ \alpha \\ \beta \\ \mathrm{s} \end{array}
\end{array}
=
\begin{array}{c}
\begin{array}{cccc} \mathrm{ds} & \alpha & \beta & \mathrm{s} \end{array} \\
\left[\begin{array}{c|c|c|c}
R_{\mathrm{ds}} + L_{\mathrm{ds}} p & M_{\mathrm{d}} \cos\lambda p & -M_{\mathrm{d}} \sin\lambda p & \begin{array}{c} -M_{\mathrm{d}} \sin\lambda i_{\alpha} p \\ -M_{\mathrm{d}} \cos\lambda i_{\beta} p \end{array} \\
\hline
\begin{array}{c} M_{\mathrm{d}} \cos\lambda p \\ -M_{\mathrm{d}} \sin\lambda p\theta \end{array} & \begin{array}{c} R_{\mathrm{dr}} + L_{\mathrm{dr}} \cos^2\lambda p \\ + L_{\mathrm{qr}} \sin^2\lambda p \\ + (L_{\mathrm{qr}} - L_{\mathrm{dr}}) \sin\lambda \cos\lambda p\theta \end{array} & \begin{array}{c} (L_{\mathrm{qr}} - L_{\mathrm{dr}}) \sin\lambda \cos\lambda p \\ + L_{\mathrm{dr}} \sin^2\lambda p\theta \\ + L_{\mathrm{qr}} \cos^2\lambda p\theta \end{array} & \begin{array}{c} 2(L_{\mathrm{qr}} - L_{\mathrm{dr}}) \cos\lambda \sin\lambda i_{\alpha} p\theta \\ + (L_{\mathrm{qr}} - L_{\mathrm{dr}})(\cos^2\lambda - \sin^2\lambda) i_{\beta} p \\ - i_{\mathrm{ds}} M_{\mathrm{d}} \sin\lambda p \\ - 2(L_{\mathrm{qr}} - L_{\mathrm{dr}}) \cos\lambda \sin\lambda i_{\beta} p\theta \\ + (L_{\mathrm{qr}} - L_{\mathrm{dr}})(\cos^2\lambda - \sin^2\lambda) i_{\alpha} p\theta \\ - M_{\mathrm{d}} \cos\lambda i_{\mathrm{ds}} p\theta \end{array} \\
\hline
\begin{array}{c} -M_{\mathrm{d}} \sin\lambda p \\ -M_{\mathrm{d}} \cos\lambda p\theta \end{array} & \begin{array}{c} (L_{\mathrm{qr}} - L_{\mathrm{dr}}) \sin\lambda \cos\lambda p \\ - L_{\mathrm{dr}} \cos^2\lambda p\theta \\ - L_{\mathrm{qr}} \sin^2\lambda p\theta \end{array} & \begin{array}{c} R_{\mathrm{qr}} + L_{\mathrm{qr}} \cos^2\lambda p \\ + L_{\mathrm{dr}} \sin^2\lambda p \\ - (L_{\mathrm{qr}} - L_{\mathrm{dr}}) \cos\lambda \sin\lambda p\theta \end{array} & \begin{array}{c} -2(L_{\mathrm{qr}} - L_{\mathrm{dr}}) \cos\lambda \sin\lambda i_{\beta} p \\ + (L_{\mathrm{qr}} - L_{\mathrm{dr}})(\cos^2\lambda - \sin^2\lambda) i_{\alpha} p \\ - i_{\mathrm{ds}} M_{\mathrm{d}} \cos\lambda p \\ - 2(L_{\mathrm{qr}} - L_{\mathrm{dr}}) \cos\lambda \sin\lambda i_{\alpha} p\theta \\ - (L_{\mathrm{qr}} - L_{\mathrm{dr}})(\cos^2\lambda - \sin^2\lambda) i_{\beta} p\theta \\ + M_{\mathrm{d}} i_{\mathrm{ds}} \sin\lambda p\theta \end{array} \\
\hline
\begin{array}{c} M_{\mathrm{d}} \sin\lambda i_{\alpha} \\ M_{\mathrm{d}} \cos\lambda i_{\beta} \end{array} & \begin{array}{c} i_{\mathrm{ds}} M_{\mathrm{d}} \sin\lambda \\ - (L_{\mathrm{qr}} - L_{\mathrm{dr}})(\cos^2\lambda - \sin^2\lambda) i_{\beta} \\ - 2(L_{\mathrm{qr}} - L_{\mathrm{dr}}) \cos\lambda \sin\lambda i_{\alpha} \end{array} & \begin{array}{c} i_{\mathrm{ds}} M_{\mathrm{d}} \cos\lambda \\ - (L_{\mathrm{qr}} - L_{\mathrm{dr}})(\cos^2\lambda - \sin^2\lambda) i_{\alpha} \\ + 2(L_{\mathrm{qr}} - L_{\mathrm{dr}}) \sin\lambda \cos\lambda i_{\beta} \end{array} & \begin{array}{c} J p^2 + M_{\mathrm{d}} \cos\lambda i_{\mathrm{ds}} i_{\alpha} \\ - M_{\mathrm{d}} \sin\lambda i_{\mathrm{ds}} i_{\beta} \\ - (L_{\mathrm{qr}} - L_{\mathrm{dr}})(\cos^2\lambda - \sin^2\lambda) i_{\alpha} i_{\alpha} \\ + (L_{\mathrm{qr}} - L_{\mathrm{dr}})(\cos^2\lambda - \sin^2\lambda) i_{\beta} i_{\beta} \\ + 4(L_{\mathrm{qr}} - L_{\mathrm{dr}}) \sin\lambda \cos\lambda i_{\alpha} i_{\alpha} \end{array}
\end{array}\right]
\end{array}
\begin{array}{c}
\\ \\
\begin{array}{c} \mathrm{ds} \\ \alpha \\ \beta \\ \mathrm{s} \end{array}
\end{array}
\begin{array}{c}
\\ \\
\left[\begin{array}{c} \Delta i_{\mathrm{ds}} \\ \Delta i_{\alpha} \\ \Delta i_{\beta} \\ \Delta\lambda \end{array}\right]
\end{array}
\qquad (8.17)
$$

In the synchronous machine, $V_{\mathrm{dr}} = V \sin\delta$ and $V_{\mathrm{qr}} = -V\cos\delta$. It is customary to take the second term on the right-hand side over into the $\Delta\theta'$ (or $\Delta\delta$) column of the impedance matrix.

The hunting equations in free axes, which are of course complicated by our arbitrary choice of the angle λ, now become as given in equation (8.17).

When the free axes are moved back into line with Park's axes in the steady state, the angle λ becomes zero and the hunting impedance matrix for the machine, expressed along non-oscillating reference axes on the armature, becomes

$$
\begin{array}{c|c|c|c|c|}
 & \mathrm{ds} & \alpha & \beta & \mathrm{s} \\
\hline
\mathrm{ds} & R_{\mathrm{ds}} + L_{\mathrm{ds}}p & M_{\mathrm{d}}p & & -M_{\mathrm{d}}i_{\beta}p \\
\hline
\alpha & M_{\mathrm{d}}p & R_{\mathrm{dr}} + L_{\mathrm{dr}}p & L_{\mathrm{qr}}p\theta & \begin{array}{c}(L_{\mathrm{qr}} - L_{\mathrm{dr}})i_{\beta}p \\ + (L_{\mathrm{qr}} - L_{\mathrm{dr}})i_{\alpha}p\theta \\ - M_{\mathrm{d}}i_{\mathrm{ds}}p\theta\end{array} \\
\hline
\beta & -M_{\mathrm{d}}p\theta & -L_{\mathrm{dr}}p\theta & R_{\mathrm{qr}} + L_{\mathrm{qr}}p & \begin{array}{c}(L_{\mathrm{qr}} - L_{\mathrm{dr}})i_{\alpha}p \\ - (L_{\mathrm{qr}} - L_{\mathrm{dr}})i_{\beta}p\theta \\ - M_{\mathrm{d}}i_{\mathrm{ds}}p\end{array} \\
\hline
\mathrm{s} & M_{\mathrm{d}}i_{\beta} & -(L_{\mathrm{qr}} - L_{\mathrm{dr}})i_{\beta} & \begin{array}{c}M_{\mathrm{d}}i_{\mathrm{ds}} \\ -(L_{\mathrm{qr}} - L_{\mathrm{dr}})i_{\alpha}\end{array} & \begin{array}{c}Jp^2 + M_{\mathrm{d}}i_{\mathrm{ds}}i_{\alpha} \\ - (L_{\mathrm{qr}} - L_{\mathrm{dr}})i_{\alpha}i_{\alpha} \\ + (L_{\mathrm{qr}} - L_{\mathrm{dr}})i_{\beta}i_{\beta}\end{array} \\
\hline
\end{array}
\tag{8.18}
$$

During steady hunting conditions, the operator p becomes $jh\omega$, where $h = f_{\mathrm{hunting}}/f_{\mathrm{synchronous}}$.

Also, the oscillations may be expressed in terms of either $\Delta\lambda$ or the small excursions in load angle $\Delta\delta$ since $\Delta\theta = \Delta\delta = \Delta\lambda$ and $\Delta(p\theta) = p(\Delta\theta) = p(\Delta\lambda) = p(\Delta\delta)$.

The hunting equations now become (putting $M = \omega J$)

(a) along Park's axes

$$
\begin{bmatrix} \Delta V_{ds} \\ \Delta V_{dr} \\ \Delta V_{qr} \\ \Delta T \end{bmatrix}
=
\begin{array}{c} \\ ds \\ dr \\ qr \\ s \end{array}
\begin{array}{c}
\begin{array}{cccc} \quad ds \quad & \quad dr \quad & \quad qr \quad & \quad s \quad \end{array} \\
\begin{bmatrix}
R_{ds}+jhX_{ds} & jhX_{md} & & \\
jhX_{md} & R_{dr}+jhX_{dr} & X_{qr} & jhi_{qr}X_{qr}-V\cos\delta \\
-X_{md} & -X_{dr} & R_{qr}+jhX_{qr} & -jh(i_{ds}X_{md}+i_{dr}X_{dr}) - V\sin\delta \\
i_{qr}X_{md} & i_{qr}(X_{dr}-X_{qr}) & i_{dr}(X_{dr}-X_{qr}) + i_{ds}X_{md} & -Mh^2\omega^2
\end{bmatrix}
\end{array}
\cdot
\begin{bmatrix} \Delta i_{ds} \\ \Delta i_{dr} \\ \Delta i_{qr} \\ \Delta\delta \end{bmatrix}
\tag{8.19}
$$

(b) along Kron's uniformly rotating axes

$$
\begin{bmatrix} \Delta V_{ds} \\ \Delta V_{\alpha} \\ \Delta V_{\beta} \\ \Delta T \end{bmatrix}
=
\begin{array}{c} \\ ds \\ \alpha \\ \beta \\ s \end{array}
\begin{array}{c}
\begin{array}{cccc} \quad ds \quad & \quad \alpha \quad & \quad \beta \quad & \quad s \quad \end{array} \\
\begin{bmatrix}
R_{ds}+jhX_{ds} & jhX_{md} & & -jhX_{md}i_\beta \\
jhX_{md} & R_{dr}+jhX_{dr} & X_{qr} & jh(X_{qr}-X_{dr})i_\beta + [(X_{qr}-X_{dr})i_\alpha - X_{md}i_{ds}] \\
-X_{md} & -X_{dr} & R_{qr}+jhX_{qr} & jh(X_{qr}-X_{dr})i_\alpha - (X_{qr}-X_{dr})i_\beta - jhX_{md}i_{ds} \\
X_{md}i_\beta & -(X_{qr}-X_{dr})i_\beta & X_{md}i_{ds} - (X_{qr}-X_{dr})i_\alpha & X_{md}i_{ds}i_\alpha - (X_{qr}-X_{dr})i_\alpha i_\alpha + (X_{qr}-X_{dr})i_\beta i_\beta - Mh^2\omega^2
\end{bmatrix}
\end{array}
\cdot
\begin{bmatrix} \Delta i_{ds} \\ \Delta i_{\alpha} \\ \Delta i_{\beta} \\ \Delta\delta \end{bmatrix}
\tag{8.20}
$$

The hunting equations in the free frame can, in fact, be derived from the equations in Park's axes, in a straightforward manner, using the substitutions

$$
\begin{aligned}
i_\alpha &= i_{ds} \\
i_\beta &= i_{qs} \\
\tilde{\Delta} i_\alpha &= \Delta i_\alpha = \tilde{\Delta} i_{dr} \\
\tilde{\Delta} i_\beta &= \Delta i_\beta = \tilde{\Delta} i_{qr} \\
\tilde{\Delta} i_{dr} &= \Delta i_{dr} - i_{qr}\Delta\delta \\
\tilde{\Delta} i_{qr} &= \Delta i_{qr} + i_{dr}\Delta\delta
\end{aligned}
\tag{8.21}
$$

In Park's axes the equation of generated hunting torque is

$$\Delta T = -\mathbf{i}^*\mathbf{G}\Delta\mathbf{i} - \Delta\mathbf{i}^*\mathbf{G}\mathbf{i} + Jp^2\Delta\delta \tag{8.22}$$

or

$$\Delta T = -\mathbf{i}^*(\mathbf{G} + \mathbf{G}_t)\Delta\mathbf{i} + Jp^2\Delta\delta \tag{8.23}$$

where the torque matrix is

$$\mathbf{G} = \begin{array}{c|c|c|c|} & \text{ds} & \text{dr} & \text{qr} \\ \hline \text{ds} & & & \\ \hline \text{dr} & & & L_{qr} \\ \hline \text{qr} & -M_d & -L_{dr} & \\ \hline \end{array} \tag{8.24}$$

In free axes, rotating uniformly at a synchronous angle λ with respect to Park's axes in the steady state, the equation of generated hunting torque is

$$\Delta T = -\mathbf{i}^*\mathbf{G}\Delta\mathbf{i} - \Delta\mathbf{i}^*\mathbf{G}\mathbf{i} - \mathbf{i}^*\frac{\partial\mathbf{G}}{\partial\lambda}\mathbf{i}\Delta\lambda + Jp^2\Delta\lambda \tag{8.25}$$

where

$$\mathbf{G} = \begin{array}{c|c|c|c|} & \text{ds} & \alpha & \beta \\ \hline \text{ds} & & & \\ \hline \alpha & -M_d\sin\lambda & (L_{qr} - L_{dr})\sin\lambda\cos\lambda & \begin{array}{l} L_{dr}\sin^2\lambda \\ +L_{qr}\cos^2\lambda \end{array} \\ \hline \beta & -M_d\cos\lambda & \begin{array}{l} -L_{dr}\cos^2\lambda \\ -L_{qr}\sin^2\lambda \end{array} & -(L_{qr} - L_{dr})\cos\lambda\sin\lambda \\ \hline \end{array} \tag{8.26}$$

and

$$\frac{\partial\mathbf{G}}{\partial\lambda} = \begin{array}{c|c|c|c|} & \text{ds} & \alpha & \beta \\ \hline \text{ds} & & & \\ \hline \alpha & -M_d\cos\lambda & (L_{qr} - L_{dr})(\cos^2\lambda - \sin^2\lambda) & -(L_{qr} - L_{dr})\cos\lambda\sin\lambda \\ \hline \beta & M_d\sin\lambda & (L_{qr} - L_{dr})\cos\lambda\sin\lambda & (L_{qr} - L_{dr})(\cos^2\lambda - \sin^2\lambda \\ \hline \end{array} \tag{8.27}$$

When the free axes are aligned with Park's axes in the steady state, the angle λ becomes zero and the torque matrices in the free frame become

$$\mathbf{G} = \begin{array}{c|c|c|c|} & \text{ds} & \alpha & \beta \\ \hline \text{ds} & & & \\ \hline \alpha & & & L_{\text{qr}} \\ \hline \beta & -M_{\text{d}} & -L_{\text{dr}} & \\ \hline \end{array} \tag{8.28}$$

as in Park's axes, and

$$\left.\frac{\partial \mathbf{G}}{\partial \lambda}\right|_{\lambda=0} = \begin{array}{c|c|c|c|} & \text{ds} & \alpha & \beta \\ \hline \text{ds} & & & \\ \hline \alpha & -M_{\text{d}} & (L_{\text{qr}} - L_{\text{dr}}) & \\ \hline \beta & & & -(L_{\text{qr}} - L_{\text{dr}}) \\ \hline \end{array} \tag{8.29}$$

It has already been pointed out that the angle of displacement may be expressed in terms of the change in load angle, in which case, the increment $\Delta\lambda$ with respect to the free frame angle may be replaced by $\Delta\delta$ and the hunting torque equation in that case becomes

$$\Delta T = -\mathbf{i}^*\mathbf{G}\Delta\mathbf{i} - \Delta\mathbf{i}^*\mathbf{G}\mathbf{i} - \mathbf{i}^*\left(\left.\frac{\partial \mathbf{G}}{\partial \lambda}\right|_{\lambda=0}\right)\mathbf{i}\Delta\delta + Jp^2\Delta\delta \tag{8.30}$$

8.4 EQUIVALENT NETWORKS

The motional impedance matrix derived above is not symmetrical and therefore cannot be represented by a stationary network in the axes (d, q or α, β). However, by transforming the variables to the forward and backward rotating axes, the unsymmetrical impedance matrix may be rendered symmetrical. The transformed symmetrical equations can then be represented by a stationary network containing R, L, and voltage and/or current sources.

For a three-phase induction motor during balanced operation, an equivalent network may be drawn easily because of the perfectly symmetrical nature of the system (as illustrated in Chapter 2). It is not, however, easy to draw an equivalent circuit of a single phase induction motor in a similar way. The standard practice is to draw an equivalent network in terms of forward and backward rotating frames of reference.

In this section we shall first show how to draw the equivalent network of a synchronous machine in steady state operation using f, b axes. In doing so, no distinction need be made between Park's frame of reference and that of Kron. The equations are identical when the rotor rotates synchronously and Kron's axes are made to coincide with those of Park. It is only when the rotor departs from synchronous speed that the difference may be seen. Park's axes being rigidly connected to the rotor follow the rotor displacements. Kron's axes continue to rotate synchronously. This point has been illustrated in Section 8.3.

The hunting network will be drawn in Park's axes and in those of Kron. An example will be given to show that the use of Kron's axes makes it possible to isolate the contributions of each winding to total damping and synchronising coefficients.[3]

The network representation is further pursued because it is then also possible to isolate the contributions made by a voltage regulator to the dynamic stability of the system. This technique isolates precisely and quantitatively the roles played by each component in the system in the dynamic performance of interconnected synchronous machines.

If we consider a three-winding primitive machine the voltage equations are

$$\begin{bmatrix} V_{ds} \\ V_{dr} \\ V_{qr} \end{bmatrix} = \begin{array}{c} \\ \mathrm{ds} \\ \mathrm{dr} \\ \mathrm{qr} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{ds} & \mathrm{dr} & \mathrm{qr} \end{array} \\ \begin{bmatrix} R_{ds} + L_{ds}p & M_d p & \\ M_d p & R_a + L_{dr}p & L_{qr}p \\ -M_d p\theta & -L_{dr}p\theta & R_a + L_{qr}p \end{bmatrix} \end{array} \cdot \begin{bmatrix} i_{ds} \\ i_{dr} \\ i_{qr} \end{bmatrix} \qquad (8.31)$$

Here $R_a = R_{dr} = R_{qr}$.

The impedance matrix is clearly not symmetrical about the main diagonal and this implies that the equations cannot be represented by a *stationary network* as they stand.

These equations, however, may be rendered symmetrical by a fairly simple transformation. If, instead of expressing the armature currents and voltages in d and q axes, we express them in forward and backward rotating axes, we shall see that the impedance matrix can be written in a symmetrical form.

Let us write

$$\begin{aligned} i_d &= \frac{1}{\sqrt{2}}(i_f + i_b) \\ i_q &= \frac{-j}{\sqrt{2}}(i_f - i_b) \\ V_d &= \frac{1}{\sqrt{2}}(V_f + V_b) \\ V_q &= \frac{j}{\sqrt{2}}(V_f - V_b) \end{aligned} \qquad (8.32)$$

Clearly, $P = V_d i_d + V_q i_q = V_f i_f + V_b i_b$ and the criterion of power invariance is not violated. Based on the above relationship, we can write

$$\begin{aligned} i_f &= \frac{1}{\sqrt{2}}(i_d + j i_q) \\ i_b &= \frac{1}{\sqrt{2}}(i_d - j i_q) \end{aligned} \tag{8.33}$$

and this relationship shows that although $|i_d| \neq |i_q|$, i_f and i_b have the same magnitude, one being the complex conjugate of the other. The armature currents i_d and i_q have been resolved into instantaneous symmetrical components by this transformation. So far as i_{ds} is concerned no such transformation is necessary. It would have been necessary if an additional field winding were present along the quadrature axes, as in a dual-excitation synchronous machine or an induction motor.

The connection matrix for the transformation is

$$\mathbf{C} = \frac{1}{\sqrt{2}} \begin{array}{c} \\ \mathrm{ds} \\ \mathrm{d} \\ \mathrm{q} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{ds} & \mathrm{f} & \mathrm{b} \end{array} \\ \begin{bmatrix} \sqrt{2} & & \\ & 1 & 1 \\ & -j & j \end{bmatrix} \end{array} \tag{8.34}$$

Let us now operate on the equations (8.31) using the relations

$$\begin{aligned} \mathbf{V}' &= \mathbf{C}_t^* \mathbf{V} \\ \mathbf{Z}' &= \mathbf{C}_t^* \mathbf{Z} \mathbf{C} \end{aligned} \tag{8.35}$$

and the resulting impedance matrix expressed in the f, b axes is

$$\mathbf{Z}' = \begin{array}{c} \\ \mathrm{ds} \\ \mathrm{f} \\ \mathrm{b} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{ds} & \mathrm{f} & \mathrm{b} \end{array} \\ \begin{bmatrix} R_{ds} + L_{ds}p & \dfrac{M_d}{\sqrt{2}}p & \dfrac{M_d}{\sqrt{2}}p \\ \dfrac{M_d}{\sqrt{2}}(p - jp\theta) & R_a + L_S(p - jp\theta) & -L_D(p - jp\theta) \\ \dfrac{M_d}{\sqrt{2}}(p + jp\theta) & -L_D(p + jp\theta) & R_a + L_S(p + jp\theta) \end{bmatrix} \end{array} \tag{8.36}$$

where

$$L_S = \frac{L_{dr} + L_{qr}}{2} \quad \text{and} \quad L_D = \frac{L_{qr} - L_{dr}}{2}$$

If we consider steady-state operation $p = j\omega$ and $p\theta = \omega_r$ the voltage equations in Park's reference frame, expressed in f, b axes are

$$\begin{array}{c} \\ \begin{bmatrix} V_{ds} \\ V_f \\ V_b \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{array}{c} ds \\ = f \\ b \end{array} \end{array} \begin{array}{c} \begin{array}{ccc} \quad ds \quad & \quad f \quad & \quad b \quad \end{array} \\ \begin{bmatrix} R_{ds} + jX_{ds} & \dfrac{jX_{md}}{\sqrt{2}} & \dfrac{jX_{md}}{\sqrt{2}} \\ \dfrac{jX_{md}}{\sqrt{2}}\left(1 - \dfrac{\omega_r}{\omega}\right) & R_a + jX_S\left(1 - \dfrac{\omega_r}{\omega}\right) & -jX_D\left(1 - \dfrac{\omega_r}{\omega}\right) \\ \dfrac{jX_{md}}{\sqrt{2}}\left(1 + \dfrac{\omega_r}{\omega}\right) & -jX_D\left(1 + \dfrac{\omega_r}{\omega}\right) & R_a + jX_S\left(1 + \dfrac{\omega_r}{\omega}\right) \end{bmatrix} \end{array} \begin{array}{c} \\ \cdot \begin{bmatrix} i_{ds} \\ i_f \\ i_b \end{bmatrix} \end{array}$$

(8.37)

This equation is not yet symmetrical but some minor manipulation will render it symmetrical straightaway. We shall also bear in mind that

$$X_{ds} = x_f + X_{md} \tag{8.38}$$

$$X_S = \frac{X_{qr} + X_{dr}}{2} = X_{dr} + \frac{X_{qr} - X_{dr}}{2} = x_l + X_{md} + \frac{X_{qr} - X_{dr}}{2} \tag{8.39}$$

where x_f and x_l are respectively the field and armature leakage reactances.

The final equations are

$$\begin{array}{c} \\ \begin{bmatrix} \sqrt{2}V_{ds} \\ \dfrac{\omega}{\omega - \omega_r}V_f \\ \dfrac{\omega}{\omega + \omega_r}V_b \end{bmatrix} \end{array} \begin{array}{c} \\ \begin{array}{c} ds \\ = f \\ b \end{array} \end{array} \begin{array}{c} \begin{array}{ccc} \quad ds \quad & \quad f \quad & \quad b \quad \end{array} \\ \begin{bmatrix} 2R_{ds} + 2j(x_l + X_{md}) & jX_{md} & jX_{md} \\ jX_{md} & \dfrac{(R_a)\omega}{\omega - \omega_r} + jX_s & -j\dfrac{X_q - X_d}{2} \\ jX_{md} & -j\dfrac{X_q - X_d}{2} & \dfrac{(R_a)\omega}{\omega + \omega_r} + jX_s \end{bmatrix} \end{array} \begin{array}{c} \\ \cdot \begin{bmatrix} \dfrac{i_{ds}}{\sqrt{2}} \\ i_f \\ i_b \end{bmatrix} \end{array} \tag{8.40}$$

If each row of this matrix equation is compared with the corresponding row in equation (8.37) they will be found to be identical. According to the per-unit convention followed in this chapter $\omega = 1$ and $\omega_r = p\theta = \beta\omega$.

Now the equation is clearly symmetrical about the diagonal and each row corresponds to each of the three loops in Figure 8.3. Since $X_{qr} < X_{dr}$, $\dfrac{X_q - X_d}{2}$ has been represented by a capacitor.

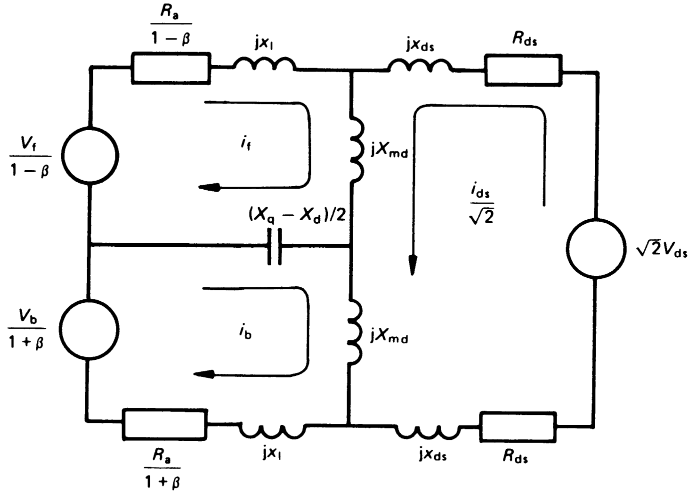


Figure 8.3 Electrical network representing matrix equation (8.40)

8.4.1 Hunting network in Park's axes

Let us consider the voltage equation in Park's axes during small oscillations,

$$\begin{bmatrix} \Delta V_{ds} \\ [V\cos\delta - B_{dr}]p\Delta\delta \\ [-V\sin\delta - B_{qr}]p\Delta\delta \end{bmatrix} = \begin{array}{c|ccc} & \text{ds} & \text{f} & \text{b} \\ \hline \text{ds} & R_{ds} + L_{ds}p & -M_d p & \\ \text{f} & M_d p & -R_a - L_{dr}p & L_{qr}p\theta \\ \text{b} & M_d p\theta & -L_{dr}p\theta & -R_a - L_{qr}p \end{array} \cdot \begin{bmatrix} \Delta i_{ds} \\ \Delta i_{dr} \\ \Delta i_{qr} \end{bmatrix} \tag{8.41}$$

where

$$B_{dr} = L_{qr}i_{qr}$$

and

$$B_{qr} = -M_d i_{ds} - L_{dr}i_{dr}$$

We can now perform transformations $\mathbf{C}_t^*\mathbf{ZC}$ and $\mathbf{C}_t^*\mathbf{V}$ on the impedance and the voltage matrices respectively. The $\mathbf{C}$ matrix is as given in equation (8.34). The resulting equation is

$$
\begin{bmatrix} \Delta V_{ds} \\ \dfrac{1}{\sqrt{2}} V\varepsilon^{-j\delta}\Delta\delta + B_f p\Delta\delta \\ \dfrac{1}{\sqrt{2}} V\varepsilon^{j\delta}\Delta\delta + B_b p\Delta\delta \end{bmatrix}
=
\begin{array}{c} \\ \mathrm{ds} \\ \mathrm{f} \\ \mathrm{b} \end{array}
\begin{array}{c}
\begin{array}{ccc} \mathrm{ds} & \mathrm{f} & \mathrm{b} \end{array} \\
\begin{bmatrix}
R_{ds} + L_{ds}p & -\dfrac{M_d p}{\sqrt{2}} & -\dfrac{M_d p}{\sqrt{2}} \\
\dfrac{M_d}{\sqrt{2}}(p + jp\theta) & -R_a - L_S(p + jp\theta) & -L_D(p + jp\theta) \\
\dfrac{M_d}{\sqrt{2}}(p - jp\theta) & -L_D(p - jp\theta) & -R_a - L_S(p - jp\theta)
\end{bmatrix}
\end{array}
\cdot
\begin{bmatrix} \Delta i_{ds} \\ \Delta i_f \\ \Delta i_b \end{bmatrix}
\qquad (8.42)
$$

$$
B_f = B_{dr} + jB_{qr} \qquad L_S = \frac{L_{dr} + L_{qr}}{2}
$$

$$
B_b = B_{dr} - jB_{qr} \qquad L_D = \frac{L_{dr} - L_{qr}}{2}
$$

For a synchronous machine $p\theta = \omega = 1$ and $p = jh\omega = jh$. The voltage terms and the impedance terms in f and b axes are divided by $(p + jp\theta)$ and $(p - jp\theta)$ respectively. The corresponding terms in the d, s axes are divided by h. The voltage ΔV_{ds} is multiplied by $\sqrt{2}$ and the current Δi_{ds} divided by $\sqrt{2}$. Appropriate adjustments are made to the impedance terms affected as a result, giving

$$
\begin{bmatrix} \sqrt{2}\dfrac{\Delta V_{ds}}{h} \\ -\left(\dfrac{1}{\sqrt{2}}\right)\dfrac{V\varepsilon^{-j\delta}}{h+1}\Delta\delta - \dfrac{jh}{h+1}B_f\Delta\delta \\ -\left(\dfrac{1}{\sqrt{2}}\right)\dfrac{V\varepsilon^{j\delta}}{h-1}\Delta\delta - \dfrac{jh}{h-1}B_b\Delta\delta \end{bmatrix}
=
\begin{array}{c} \\ \mathrm{ds} \\ \mathrm{f} \\ \mathrm{b} \end{array}
\begin{array}{c}
\begin{array}{ccc} \mathrm{ds} & \mathrm{f} & \mathrm{b} \end{array} \\
\begin{bmatrix}
2\left(\dfrac{R_{ds}}{h} + jX_{ds}\right) & -jX_{md} & -jX_{md} \\
-jX_{md} & \left(\dfrac{R_a}{h+1} + jX_S\right) & jX_D \\
-jX_{md} & jX_D & \left(\dfrac{R_a}{h-1} + jX_S\right)
\end{bmatrix}
\end{array}
\cdot
\begin{bmatrix} \dfrac{\Delta i_{ds}}{\sqrt{2}} \\ \Delta i_f \\ \Delta i_b \end{bmatrix}
$$

(8.43)

The inductance terms have been replaced by their corresponding reactances. This equation is symmetrical and may be represented by the stationary network shown in Figure 8.4(a).

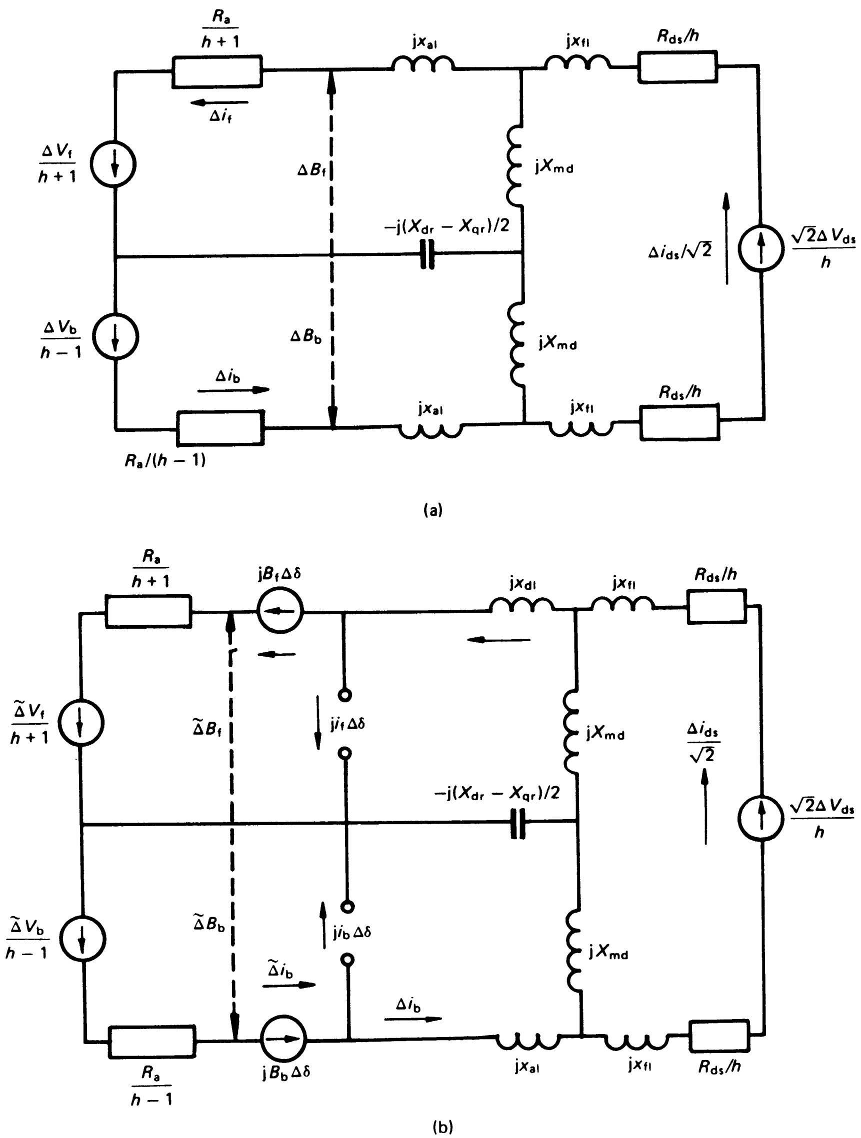


Figure 8.4 Hunting equivalent networks. (a) In Park's axes and (b) in Kron's axes

8.4.2 Hunting network in Kron's axes

To derive the hunting network in Kron's axes, we begin with the matrix equation (8.20) and subject it to the same operations.

In this section we shall give an example from a test conducted on a synchronous motor connected to an infinite busbar. We therefore use equations (8.20) in Kron's reference frame. Expressed in f, b axes the resulting equation is

$$
\begin{bmatrix}\Delta V_{ds}\\ \Delta V_{f}\\ \Delta V_{b}\\ \Delta T\end{bmatrix}
=
\begin{array}{cc}
 & \begin{array}{cccc} \mathrm{ds} & \mathrm{f} & \mathrm{b} & \mathrm{s}\end{array} \\
\begin{array}{c}\mathrm{ds}\\ \mathrm{f}\\ \mathrm{b}\\ \mathrm{s}\end{array} &
\begin{bmatrix}
R_f + jh\omega L_f & jh\omega\dfrac{M_d}{\sqrt{2}} & jh\omega\dfrac{M_d}{\sqrt{2}} & h\omega\dfrac{M_d}{\sqrt{2}}(i_f - i_b) \\
j(h+1)\dfrac{\omega M_d}{\sqrt{2}} & R_a + j(h+1)\omega L_S & -j(h+1)\omega L_D & jb_f(h+1)\omega \\
j(h-1)\dfrac{\omega M_d}{\sqrt{2}} & -j(h-1)\omega L_D & R_a + j(h-1)\omega L_S & jb_b(h-1)\omega \\
\dfrac{jM_d}{\sqrt{2}}(i_f - i_b) & -b_b & -b_f & -Jh^2\omega^2 + jb_f i_b + jb_b i_f
\end{bmatrix}
\end{array}
\cdot
\begin{bmatrix} i_{ds}\\ i_f\\ i_b\\ \Delta\lambda\end{bmatrix}
\tag{8.44}
$$

As usual $\omega = 1$. The terms b_f and b_b are defined below.

$$B_{dr} = L_{qr} i_{qr} \tag{8.45a}$$

$$B_{qr} = -M_d i_{ds} - L_{dr} i_{dr} \tag{8.45b}$$

$$b_{dr} = -B_{dr} + L_{dr} i_{qr} \tag{8.45c}$$

$$b_{qr} = -B_{qr} - L_{qr} i_{qr} \tag{8.45d}$$

$$B_f = -j\left(\frac{1}{\sqrt{2}} M_d i_{ds} + L_S i_f - L_D i_b\right) = (B_{dr} + jB_{qr})/\sqrt{2} \tag{8.46a}$$

$$B_b = j\left(\frac{1}{\sqrt{2}} M_d i_{ds} - L_D i_f + L_S i_b\right) = (B_{dr} - jB_{qr})/\sqrt{2} \tag{8.46b}$$

$$b_f = B_f + L_D i_b + L_S i_f = j\frac{1}{\sqrt{2}} M_d i_{ds} + 2jL_D i_b \tag{8.46c}$$

$$b_b = B_b - L_D i_f - L_S i_b = j\frac{1}{\sqrt{2}} M_d i_{ds} - 2jL_D i_f \tag{8.46d}$$

The hunting network is given in Figure 8.4(b). If we substitute into Park's equations the relationships (8.21), from which

$$\tilde{\Delta} i_\alpha = \Delta i_\alpha - \mathrm{j} i_\beta \Delta\delta \tag{8.47}$$

$$\tilde{\Delta} i_\beta = \Delta i_\beta + \mathrm{j} i_\alpha \Delta\delta \tag{8.48}$$

the equations in the freely rotating reference axes are given directly. The free-frame current equations in forward and backward axes, obtained from these are,

$$\tilde{\Delta} i_{\mathrm{f}} = \Delta i_{\mathrm{f}} - \mathrm{j} i_{\mathrm{f}} \Delta\delta \tag{8.49}$$

$$\tilde{\Delta} f_{\mathrm{b}} = \Delta i_{\mathrm{b}} + \mathrm{j} i_{\mathrm{b}} \Delta\delta \tag{8.50}$$

The following form of the hunting torque equation in the freely rotating reference frame results from direct substitution

$$T = -(\mathbf{i}^*\mathbf{G}\Delta\mathbf{i} + \Delta\mathbf{i}^*\mathbf{G}\mathbf{i} + \mathbf{i}^*\mathbf{G}\boldsymbol{\rho}\mathbf{i}\Delta\delta) + \mathbf{i}^*\mathbf{K}\mathbf{i}\Delta\delta + Jp^2\Delta\delta \tag{8.51}$$

where

$$\boldsymbol{\rho} = \begin{array}{c|ccc|} & \mathrm{ds} & \alpha & \beta \\ \hline \mathrm{ds} & & & \\ \alpha & & & -1 \\ \beta & & 1 & \\ \hline \end{array} \tag{8.52}$$

$$\mathbf{G} = \begin{array}{c|ccc|} & \mathrm{ds} & \alpha & \beta \\ \hline \mathrm{ds} & & & \\ \alpha & & & L_{\mathrm{qr}} \\ \beta & -M_{\mathrm{d}} & -L_{\mathrm{dr}} & \\ \hline \end{array} \tag{8.53}$$

$$\mathbf{G}\boldsymbol{\rho} = \begin{array}{c|ccc|} & \mathrm{ds} & \alpha & \beta \\ \hline \mathrm{ds} & & & \\ \alpha & & L_{\mathrm{qr}} & \\ \beta & & & L_{\mathrm{dr}} \\ \hline \end{array} \tag{8.54}$$

and

$$\mathbf{K} = \left(\frac{\partial \mathbf{G}}{\partial \lambda}\bigg|_{\lambda=0} - \mathbf{G}\rho\right) \qquad (8.55)$$

$$= \begin{array}{c|ccc} & \mathrm{ds} & \alpha & \beta \\ \hline \mathrm{ds} & & & \\ \alpha & -M_{\mathrm{d}} & -L_{\mathrm{dr}} & \\ \beta & & & -L_{\mathrm{qr}} \end{array} \qquad (8.56)$$

where

$$\frac{\partial \mathbf{G}}{\partial \lambda}\bigg|_{\lambda=0} = \begin{array}{c|ccc} & \mathrm{ds} & \alpha & \beta \\ \hline \mathrm{ds} & & & \\ \alpha & -M_{\mathrm{d}} & 2(L_{\mathrm{qr}} - L_{\mathrm{dr}}) & \\ \beta & & & -2(L_{\mathrm{qr}} - L_{\mathrm{dr}}) \end{array} \qquad (8.57)$$

In forward and backward components, the free-frame matrices become

$$\mathbf{G}'|_{\lambda=0} = \begin{array}{c|ccc} & \mathrm{ds} & \mathrm{f} & \mathrm{b} \\ \hline \mathrm{ds} & & & \\ \mathrm{f} & -\mathrm{j}\dfrac{M_{\mathrm{d}}}{2} & -\mathrm{j}L_{\mathrm{S}} & \mathrm{j}L_{\mathrm{D}} \\ \mathrm{b} & \mathrm{j}\dfrac{M_{\mathrm{d}}}{2} & -\mathrm{j}L_{\mathrm{D}} & \mathrm{j}L_{\mathrm{S}} \end{array} \qquad (8.58)$$

where

$$L_{\mathrm{S}} = \frac{L_{\mathrm{qr}} + L_{\mathrm{dr}}}{2} \quad \text{and} \quad L_{\mathrm{D}} = \frac{L_{\mathrm{qr}} - L_{\mathrm{dr}}}{2}$$

$$\frac{\partial \mathbf{G}'}{\partial \lambda}\bigg|_{\lambda=0} = \begin{array}{c|ccc} & \mathrm{ds} & \mathrm{f} & \mathrm{b} \\ \hline \mathrm{ds} & & & \\ \mathrm{f} & -\dfrac{M_{\mathrm{d}}}{\sqrt{2}} & & 2L_{\mathrm{D}} \\ \mathrm{b} & -\dfrac{M_{\mathrm{d}}}{\sqrt{2}} & 2L_{\mathrm{D}} & \end{array} \qquad (8.59)$$

$$
(\mathbf{G}\boldsymbol{\rho})' = \begin{array}{c|c|c|c|}
 & \mathrm{ds} & \mathrm{f} & \mathrm{b} \\ \hline
\mathrm{ds} & & & \\ \hline
\mathrm{f} & & L_{\mathrm{S}} & L_{\mathrm{D}} \\ \hline
\mathrm{b} & & L_{\mathrm{D}} & L_{\mathrm{S}} \\ \hline
\end{array}
\tag{8.60}
$$

$$
\mathbf{K}' = \left(\left.\frac{\partial \mathbf{G}'}{\partial \lambda}\right|_{\lambda=0} - (\mathbf{G}\boldsymbol{\rho})' \right)
\tag{8.61}
$$

$$
= \begin{array}{c|c|c|c|}
 & \mathrm{ds} & \mathrm{f} & \mathrm{b} \\ \hline
\mathrm{ds} & & & \\ \hline
\mathrm{f} & -\dfrac{M_{\mathrm{d}}}{\sqrt{2}} & -L_{\mathrm{S}} & L_{\mathrm{D}} \\ \hline
\mathrm{b} & -\dfrac{M_{\mathrm{d}}}{\sqrt{2}} & L_{\mathrm{D}} & -L_{\mathrm{S}} \\ \hline
\end{array}
\tag{8.62}
$$

These terms can all be found on the equivalent circuit. This form of the hunting equations enables us to identify the components of damping and synchronising torque associated with the rotor displacement, angular velocity, and angular acceleration as follows.

(i) Positive or negative damping torque, expressed in terms of the angular velocity $\Delta\omega$. This is given by the real part of

$$\mathbf{i}^* \mathbf{G} \Delta \mathbf{i} + \Delta \mathbf{i}^* \mathbf{G} \mathbf{i}$$

(ii) Synchronising torque in terms of the displacement $\Delta\delta$ is given by the real quantity

$$\mathbf{i}^* \mathbf{K} \mathbf{i}$$

This quantity is also given by the steady-state reactive power output of the machine (as a generator). It is positive when the generator operates at lagging power-factor and is negative if the machine is absorbing reactive power, that is, operating at leading power-factor.

(iii) Positive or negative synchronising torque, in terms of angular acceleration, this is given by the imaginary part of

$$\mathbf{i}^* \mathbf{G} \Delta \mathbf{i} + \Delta \mathbf{i}^* \mathbf{G} \mathbf{i}$$

(iv) Negative synchronising torque in terms of displacement, this is contributed by $\mathbf{i}^* \mathbf{G} \rho \mathbf{i}$

(v) Negative synchronising torque in terms of acceleration $p\Delta\omega$, contributed by $Jp\Delta\omega$

8.4.3 Numerical examples

The results given below were obtained in the laboratory for a small generalised machine operating as a two-pole 3-phase alternator. The machine was rated, in this operating mode, at 1.732 kV A, 5 A, 200 V, 50 Hz. The parameters were as follows

X_{dr}	X_{qr}	X_{md}	X_{ds}	x_{al}	x_{fl}	R_a	R_{ds}	h	δ
2.6	2.56	2.5	2.58	0.10	0.08	0.043 5	0.032 3	0.021 6	0.410 2

The reactance and resistance values are per-unit at 1.732 kV A base. The moment of inertia of the rotating mass of the generator and its driving motor was 0.062 2 kg m^2. The machine was synchronised with the mains supply, with power output 0.23 p.u., at power-factor 0.77 lagging. The excitation voltage E was 1.5 p.u., and the load angle δ was 0.410 2 radian.

Free oscillations were induced in the rotor speed by applying an impulsive load to the shaft. The oscillation frequency h was 1.08 Hz. Computed values of current are given in Table 8.1. To determine the damping and inertia torques, the

Table 8.1 Steady-state and oscillating currents

	Park's axes	*Kron's axes*
i_{dr}	−0.248 279 02	−0.248 279 02
i_{qr}	−0.122 464 41	−0.122 464 41
Δi_{ds}	−0.004 518 51 −j0.098 553 56	−0.004 518 51 −j0.098 553 56
Δi_f	0.132 688 57 −j0.168 595 12	—
Δi_b	0.152 657 63 +j0.299 650 57	—
$\tilde{\Delta i}_f$	—	0.0460 931 5 − j0.344 154 93
$\tilde{\Delta i}_b$	—	0.066 062 21 + j0.475 210 45

amplitude of the rotor oscillations was plotted on semi-log paper and the decay time constant was measured.

Then

$$T_{\mathrm{D}} = \frac{2\omega J}{\text{time constant}}$$

The inertia torque is given by $-Jh^2\omega^2$ and in free oscillations of the rotor, this is equal in magnitude to the electrically generated synchronising torque. The inertia torque is a synchronising component 'along the acceleration' and is the negative of the electrical synchronising torque 'along displacement $\Delta\delta$'. The total machine damping is the sum of electrical damping torque and mechanical damping due to friction and loss. The latter were measured in the form of a dissipation coefficient $R_{\mathrm{F}} = 0.15$ per unit.

The equivalent circuit for the machine was drawn, using the appropriate values of current. The computed loss values were

	free axes	*Park's axes*
field	$\dfrac{(\Delta i_{\mathrm{ds}})^2 R_{\mathrm{ds}}}{h}$	$\dfrac{(\Delta i_{\mathrm{ds}})^2 R_{\mathrm{ds}}}{h}$
armature f axes	$\dfrac{(\tilde{\Delta} i_{\mathrm{f}})^2 R_{\mathrm{a}}}{h+1}$	$\dfrac{(\Delta i_{\mathrm{f}})^2 R_{\mathrm{a}}}{h+1}$
armature b axes	$\dfrac{(\tilde{\Delta} i_{\mathrm{b}})^2 R_{\mathrm{a}}}{h-1}$	$\dfrac{(\Delta i_{\mathrm{b}})^2 R_{\mathrm{a}}}{h+1}$

In the freely rotating reference frame,

$$hT_{\mathrm{D}} = \frac{(\Delta i_{\mathrm{ds}})^2 R_{\mathrm{ds}}}{h} + \left\{\frac{(\tilde{\Delta} i_{\mathrm{f}})^2 R_{\mathrm{a}}}{h+1} + \frac{(\tilde{\Delta} i_{\mathrm{b}}) R_{\mathrm{a}}}{h-1}\right\} \tag{8.63}$$

The term $(\Delta i_{\mathrm{ds}})^2 R_{\mathrm{ds}}/h$ is always positive and is the damping contributed by the field circuit. The bracketed terms on the right-hand side give the armature damping. This is usually negative, since for most normal operating conditions $\Delta i_{\mathrm{f}} \cong \Delta i_{\mathrm{b}}$ and $|R_{\mathrm{a}}/(h-1)| > |R_{\mathrm{a}}/(h+1)|$ and the quantity $(h-1)$ is negative. The computed hunting copper loss in the equivalent circuits for free axes and Park's axes are given in Tables 8.2 and 8.3. Some computed and test results are given in Figures 8.5(a) and 8.5(b).

Conventional calculation of

$$T_{\mathrm{e}} = (T_{\mathrm{S}} + \mathrm{j}hT_{\mathrm{D}})\Delta\theta$$

from the matrix equations, gave

$$T_{\mathrm{S}} = 0.500\,544$$

$$T_{\mathrm{D}} = 0.388\,411\,6$$

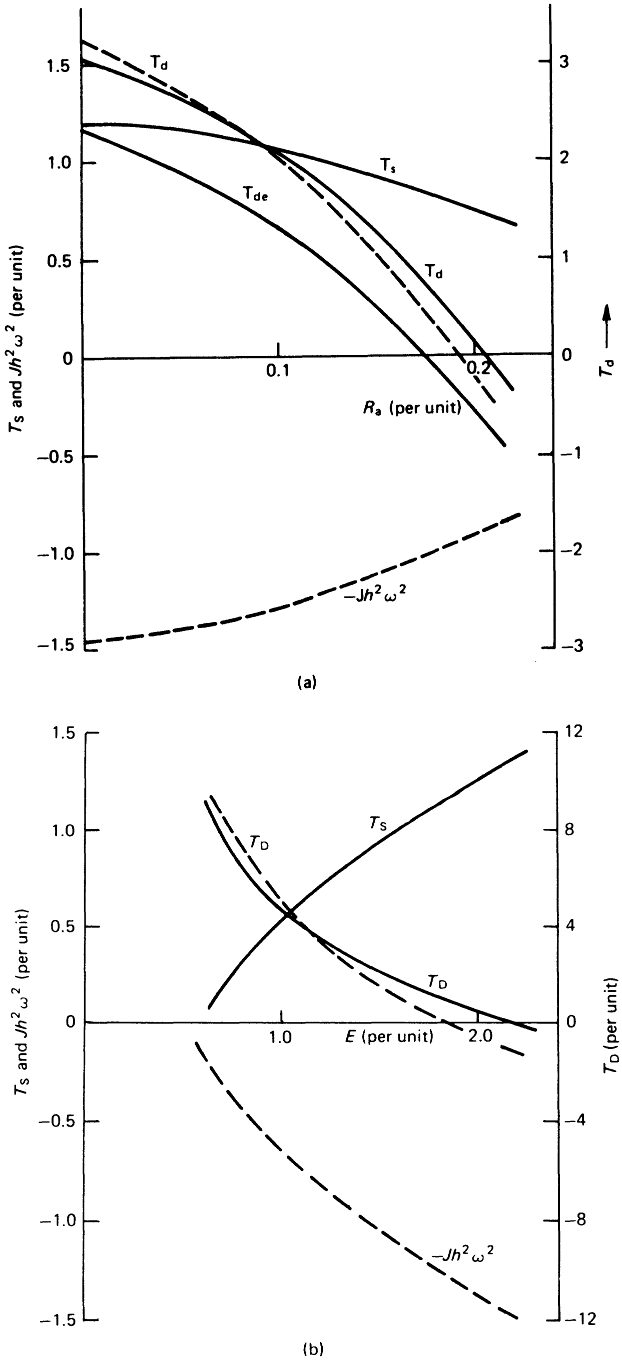


Figure 8.5 (a) Variation of T_s and T_d with line resistance;----test and—computed. (b) Variation of T_s and T_d with excitation; ----test and—computed

Table 8.2 Hunting copper loss and reactive power change in Park's axes

$\mathbf{i^*Ki}$	$= -0.178\,844\,14$
$\mathbf{i^*G\rho i}$	$= -0.196\,798\,31$
$\dfrac{(\Delta i_{ds})^2}{\sqrt{2}} x_{fl}$	$= 0.000\,778\,66$
$\left(\Delta i_f + \dfrac{\Delta i_{ds}}{\sqrt{2}}\right)^2 X_{md}$	$= 0.183\,868\,36$
$\left(\Delta i_b + \dfrac{\Delta i_{ds}}{\sqrt{2}}\right)^2 X_{md}$	$= 0.188\,054\,78$
$(\Delta i_f)^2 x_{al}$	$= 0.004\,603\,06$
$(\Delta i_b)^2 x_{al}$	$= 0.011\,309\,48$
$(\Delta i_f - \Delta i_b)^2 (X_{qr} - X_{dr})/\sqrt{2}$	$= -0.004\,403\,04$
$(\mathbf{i^*Ki} + \mathbf{i^*G\rho i}) + \sum (\Delta \mathbf{i})^2 \mathbf{X}$	$= 0.008\,568\,85$
$\dfrac{\dfrac{(\Delta i_{ds})^2}{\sqrt{2}} R_{ds}}{h}$	$= 0.048\,666\,11$
$\dfrac{(\Delta i_f)^2 R_a}{h+1}$	$= 0.015\,477\,20$
$\dfrac{(\Delta i_b)^2 R_a}{h-1}$	$= -0.039\,705\,73$
$\dfrac{\sum \text{copper loss}}{h}$	$= 1.131\,369\,08$

Table 8.3 Hunting copper loss and reactive power change in free axes

$\mathbf{i^*ki}$	$= -0.178\,844\,14$
$\mathbf{i^*G\rho i}$	$= -0.196\,798\,31$
$\dfrac{(\Delta i_{ds})^2}{\sqrt{2}} x_{fl}$	$= 0.000\,778\,66$
$\left(\tilde{\Delta} i_f + \dfrac{\Delta i_{ds}}{\sqrt{2}}\right)^2 X_{md}$	$= 0.432\,765\,27$
$\left(\tilde{\Delta} i_b + \dfrac{\Delta i_{ds}}{\sqrt{2}}\right)^2 X_{md}$	$= 0.421\,002\,09$
$(\tilde{\Delta} i_f)^2 x_{al}$	$= 0.012\,056\,72$
$(\tilde{\Delta} i_b)^2 x_{al}$	$= 0.023\,018\,92$
$(\tilde{\Delta} i_f - \tilde{\Delta} i_b)^2 (X_{qr} - X_{dr})/\sqrt{2}$	$= -0.013\,435\,17$
$(\mathbf{i^*ki} + \mathbf{i^*G\rho i}) + \sum (\tilde{\Delta} \mathbf{i})^2 \mathbf{X}$	$= 0.500\,544\,04^*$

Table 8.3 (continued)

$\dfrac{\dfrac{(\Delta i_{ds})^2}{\sqrt{2}} R_{ds}}{h}$	$= \quad 0.048\,666\,11$
$\dfrac{(\tilde{\Delta i}_f)^2 R_a}{h+1}$	$= \quad 0.040\,539\,18$
$\dfrac{(\tilde{\Delta i}_b)^2 R_a}{h-1}$	$= -0.080\,815\,60$
$\dfrac{\sum \text{copper loss}}{h}$	$= \quad 0.388\,411\,60$*

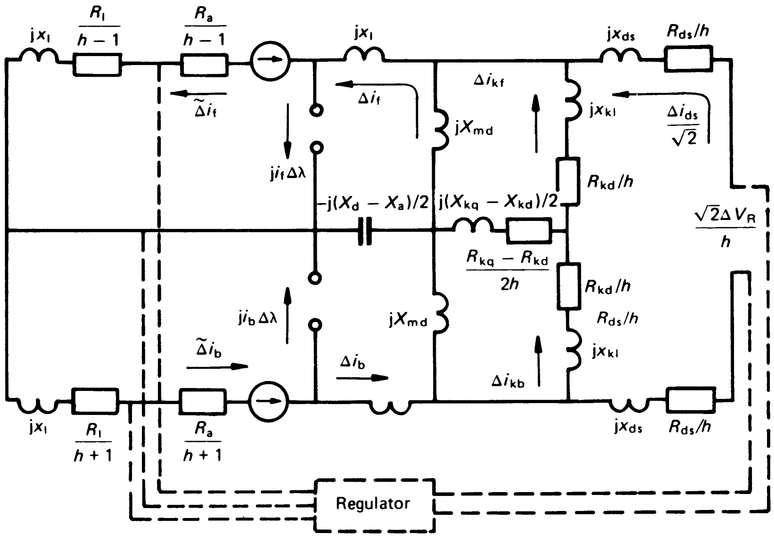
8.4.4 The effect of a voltage regulator

As we have seen previously, a voltage regulator on a power-system alternator will have a significant effect on the damping and synchronising characteristics of the machine. These effects can be studied very conveniently using the equivalent circuit technique.[4] The regulator feed-back circuit is simply added into the machine equivalent circuit, as shown originally by Kron and again in Figure 8.6(a). In the latter case it will be noticed that amortisseur damping windings have been included, with resistance R_k and reactance X_k, in both direct and quadrature axes. These terms in each axis form additional rows and columns in the hunting impedance matrices but their inclusion on the equivalent circuit is straightforward. The figures given below refer to a synchronous system which has been studied in the laboratory, consisting of a three-phase alternator with a voltage regulator, synchronised with the mains supply through an impedance in each phase. The machine parameters are given in Table 8.4.

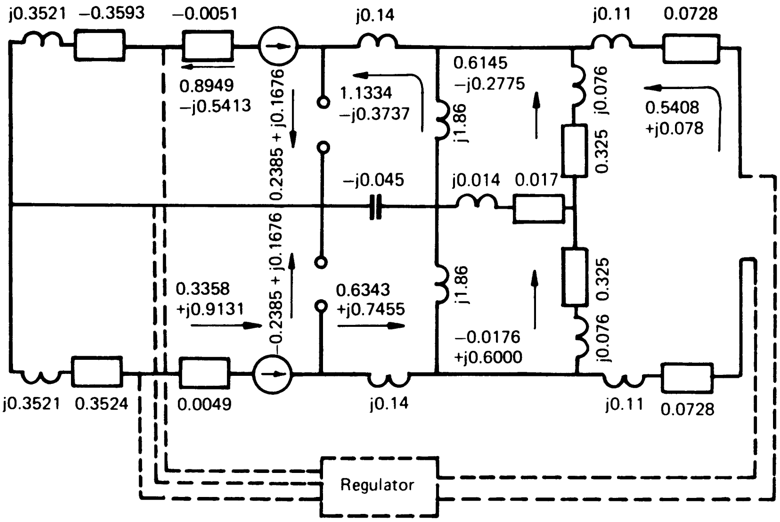
Table 8.4 Machine parameters, per unit

$X_d = 2.0$	$R_a = 0.005$
$X_q = 1.91$	$R_{ds} = 0.001\,5$
$X_{ds} = 1.97$	$R_{kd} = 0.007\,8$
$X_{mq} = 1.77$	$R_{kq} = 0.008\,4$
$X_{kd} = 1.94$	$H = 5.25$
$X_{kq} = 1.96$	$V = 1.0$
$X_{md} = 1.86$	$R_l = 0.352\,4$
$X_l = 0.35$	

In a computer programme designed to isolate the contribution of an automatic voltage regulator (AVR) to damping and synchronising torque, the following steps were followed.



(a)



(b)

Figure 8.6 (a) Hunting equivalent network in Kron's axes—including amortisseur winding and a voltage regulator, and (b) oscillating currents as observed from Kron's axes

8.4.4(i) *Damping torque*

The voltage regulator was represented by

$$E_{ds} = \frac{\mu(V_{ref} - V_{terminal})}{1 + \tau p} \tag{8.64}$$

where τ is the regulator time constant and μ is the gain.

The regulator term was added to the field circuit in the hunting equations for the machine and the overall damping and synchronising torque coefficients T_D and T_S were calculated. The equivalent circuit for the system was drawn as shown in Figures 8.6(a) and 8.6(b) in forward and backward axes, in the uniformly rotating reference frame. It will be remembered from the remarks in Chapter 6, that the hunting frequency $h\omega$ changes with variations in the system parameters and operating conditions. Therefore for any given conditions the hunting parameter h must be calculated by an iterative computation.

When h is known, all of the branch currents on the equivalent circuit can be determined. As in the previous cases, the copper loss in the machine windings was calculated. It was found that with the regulator in the circuit

$$T_D \neq \frac{1}{h}\sum\frac{(\tilde{\Delta i})^2 R}{h} \tag{8.65}$$

Here, T_D is computed in Park's axes and also in freely rotating axes and it is found to be an invariant scalar. From the equivalent circuit computation it is now found that

$$T_D = \frac{1}{h}\sum\frac{(\tilde{\Delta i})^2 R}{h} + T_{DR} \tag{8.66}$$

where

$$T_{DR} = R_e(\Delta V_{ds}^* \Delta i_{ds})/h^2 \tag{8.67}$$

This represents the change in the circuit hunting power contributed by the regulator and the equivalent circuit has now isolated the contribution of the regulator to the machine damping torque.

8.4.4(ii) *Synchronising torque*

Although $\Sigma\,(\tilde{\Delta \mathbf{i}})^2\mathbf{R}$ is invariant in different reference systems the quantity $\Sigma\,(\tilde{\Delta \mathbf{i}})^2\mathbf{X}$ is not, since the reactance values change in different reference axes while the resistances remain constant. With the AVR in circuit, the equation for the synchronising torque coefficient given by the equivalent circuit, becomes

$$T_S = [\sum(\tilde{\Delta \mathbf{i}})^2\mathbf{X} + \mathbf{i}_t^*\mathbf{G}\rho\mathbf{i} + \mathbf{i}_t^*\mathbf{K}\mathbf{i} + T_{SR}] \tag{8.68}$$

where

$$T_{SR} = I_m(\Delta V_{ds}^* \Delta i_{ds})/h \tag{8.69}$$

This is the regulator contribution to synchronising torque. Components of damping and synchronising torque coefficients, for various system and regulator conditions are shown in Figures 8.7(a), (b), (c), and (d).

As the analysis of damping and synchronising torque proceeds, the parameter

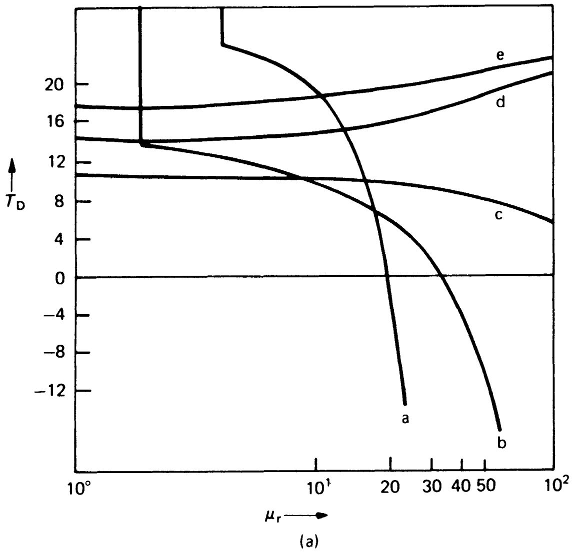


Figure 8.7 (a) Variation of damping torque coefficient with μ_v. $P = 0.4$, $\tau_r = 0.0$. a, $Q = -0.8$; b, $Q = -0.6$; c, $Q = -0.4$; d, $Q = -0.2$; e, $Q = 0.0$

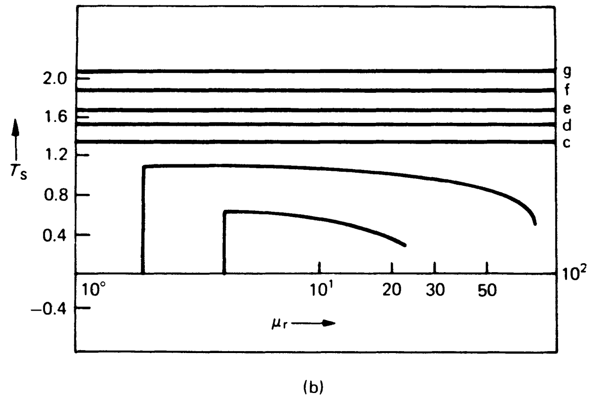


Figure 8.7 (b) Variation of synchronising torque coefficient with μ_v. $P = 0.8$, $\tau_r = 0.0$. a, $Q = -0.8$; b, $Q = -0.6$; c, $Q = -0.4$; d, $Q = -0.2$; e, $Q = 0.0$; f, $Q = 0.2$; g, $Q = 0.4$; h, $Q = 0.6$; i, $Q = 0.8$

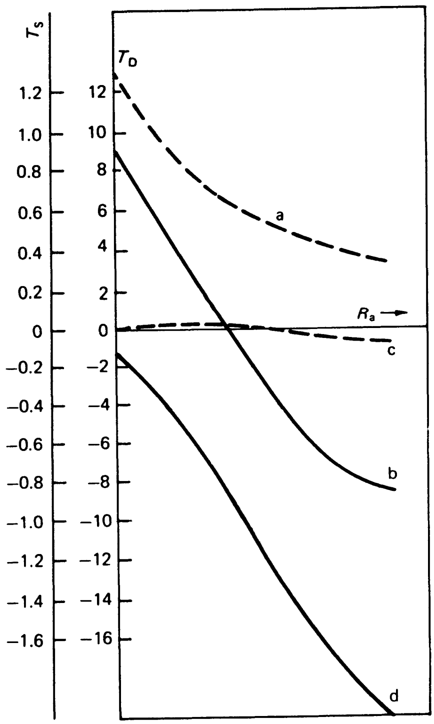
h is iteratively computed in a separate sub-routine. Figure 8.6(b) shows the equivalent circuit with $h = 0.021$.

These results of the application of Kron's theory of power system hunting, allied to state variable control theory demonstrate that power engineers now have available a technique which enables them to design into a system the required transient and hunting stability.

Mechanical speed governors can be treated by a technique similar to that used for the AVR on the equivalent circuit and the circuit models for several or many machines can be directly interconnected.[5]

The following points can be made from the analysis.

1. The transient and hunting stability of interconnected synchronous machines is closely associated with the parameters and response characteristics of the control equipment.



(c)

Figure 8.7 (c) Effects of line length on stability of the generator. $P = 0.4$, $Q = -0.4$, $\mu_r = 30.0$, $\tau_r = 0.1$ and $X_e = 10\Delta R_e$. a, total T_S; b, total T_D; c, regulator contribution to T_S; d, regulator contribution to T_D

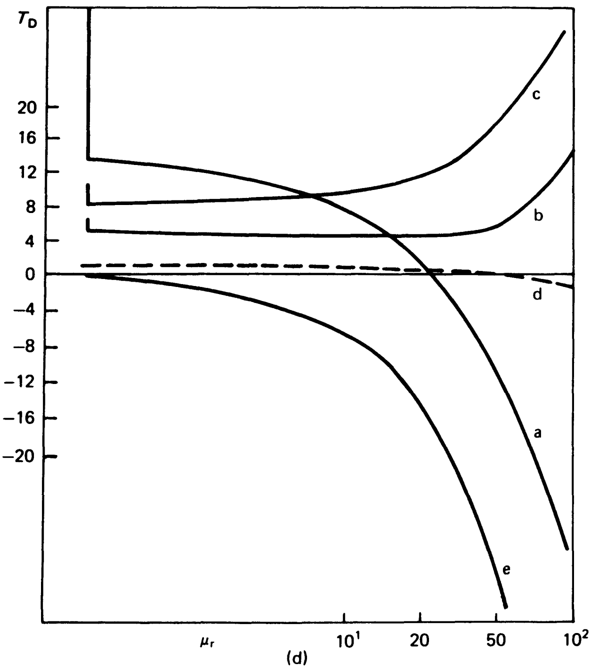


Figure 8.7 (d) Components of damping torque coefficient effects of μ_r. $P = 0.8$, $Q = -0.6$, $\tau_r = 0.0$. a, total T_D; b, from amortisseur; c, from field; d, from regulator

2. In such investigations, the whole interconnected system is involved and all plant connected to the system affects the dynamic stability to some extent.

3. It is now possible to examine directly the effect of the parameters of the armature windings, damper windings, and field circuits of the machines, including voltage regulators and speed governors, on the stability. This can be done in terms of the positive or negative damping and synchronising torque which these parts of the system contribute under any given conditions.

With Kron's hunting theory, presented here, investigation of the dynamical hunting stability of a large power system has been converted into a large-scale piece-wise linear electrical network problem.

8.5 REFERENCES

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Appendix 1
Per-unit notation

A1.1 TRANSFORMERS

In the analysis of electrical power circuits, the existence of transformers, or even transformer action between coupled coils raises the question of impedance values referred to one side or the other, related by the square of the turns ratio. In power system analysis particularly, this can be a tedious computational problem. The concepts of 'per-unit' values were introduced so that turns-ratio effects could be largely eliminated in routine calculations. The definitions are as follows (p.u. = per unit):

$$\text{Rated phase voltage} = 1.0\ \text{p.u.}$$

$$\text{Rated phase current} = 1.0\ \text{p.u.}$$

$$\text{Rated unit impedance } \tilde{Z}_{\text{ph}} = \frac{\text{rated phase voltage } V_{\text{ph}}}{\text{rated phase current } I_{\text{ph}}}$$

This gives the value in ohms which is to be taken as 1.0 per unit impedance.

Plant impedance Z (Ω) expressed in per-unit terms, is therefore

$$\frac{\text{plant impedance } Z_{\text{ph}}\,(\Omega)}{\text{unit impedance } \tilde{Z}_{\text{ph}}\,(\Omega)} \tag{A1.1}$$

Since the unit impedance $\tilde{Z}_{\text{ph}}$ involves rated voltage and rated current it is expressed with respect to the V A rating of the plant. Impedance values expressed with respect to the primary or secondary side of a transformer will be the same fraction of the corresponding value of $\tilde{Z}_{\text{ph}}$ on that side. The p.u. impedance is therefore the same on either side—as the following simple calculation shows.

Example 1

A 3 kV A 50 Hz single-phase 100/50 V transformer has leakage reactance 0.5 Ω on the primary side and 0.15 Ω on the secondary side. What are these values per-unit? Calculate the short-circuit current in both sides when rated voltage is applied to the primary side and the secondary terminals are short-circuited.

Solution

$$\text{Turns ratio } k = 100/50 = 2$$

$$k^2 = 4$$

$$\text{Rated current} = \frac{3000}{100} = 30 \text{ A primary}$$

$$= \frac{3000}{50} = 60 \text{ A secondary}$$

Reactance referred to primary side $= 0.5 + 0.15 \times 4 = 1.1\ \Omega$

Reactance referred to secondary side $= 0.15 + 0.5\,(1/4) = 0.275\ \Omega$

Primary short-circuit current $= 100/1.1 = 90.9$ A

Secondary short-circuit current $= 50/0.275 = 181.8$ A

Primary input short-circuit kVA $= 100 \times 90.9 \times 10^{-3} = 9.09$ kV A.

In per-unit terms

1.0 per-unit reactance on primary side $= 100/30 = 3.333\ \Omega$

1.0 per-unit reactance on secondary side $= 50/60 = 0.833\,3\ \Omega$

Transformer reactance p.u. referred to primary side $= 1.1/3.333 = 0.33$ p.u.

Transformer reactance p.u. referred to secondary side $= 0.275/0.833\,3 = 0.33$ p.u.

Thus the transformer has the same total reactance p.u., referred to either side.

$$\text{Primary short-circuit current p.u.} = \frac{\text{p.u. voltage}}{\text{p.u. reactance}} = \frac{1.0}{0.33} = 3.03 \text{ p.u.}$$

$$= 3.03 \times \text{rated current}$$

This applies to either side.

Primary short-circuit current $= 3.03 \times 30 = 90.9$ A

Secondary short-circuit $= 3.03 \times 60 = 181.8$ A

$$\begin{aligned}\text{Short-circuit kV A} &= \text{p.u. volts} \times \text{p.u. current on short circuit}\\ &= 1.0 \times 3.03 = 3.03 \text{ p.u.}\\ &= 3.03 \times 3 \text{ kV A} = 9.09 \text{ kV A}\end{aligned}$$

$$\text{Short-circuit kV A} = \frac{\text{rated kV A}}{\text{p.u. reactance}} = \frac{3}{0.33} = 9.09 \text{ kV A}$$

A1.2 FAULT CALCULATIONS

When impedance values occur in series or parallel, they must all be referred to the same V A rating. This is a direct proportionality and any V A rating can be taken as a common base.

Example 2

A 200 MV A 16.5 kV star-connected 3-phase alternator has synchronous reactance 1.5 p.u. Express this value in ohms.

Solution

$$\begin{aligned}\text{Line voltage} &= 16.5 \text{ kV}\\ \text{Phase voltage} &= 16.5/\sqrt{3} = 9.526 \text{ kV}\\ \text{Rated current} &= \frac{200 \times 10^3}{\sqrt{3} \times 16.5} = 6998 \text{ A}\end{aligned}$$

$$\text{Base (1.0 p.u.) reactance} = 9\,526/6\,998 = 1.36\ \Omega$$

$$\text{Thus machine reactance in ohms} = 1.5 \times 1.36 = 2.04\ \Omega$$

Check

$$\text{3-phase short-circuit current} = \frac{\text{phase voltage}}{\text{phase reactance}} = \frac{9\,526}{2.04} = 4\,669.6 \text{ A}$$

$$\text{Short-circuit MV A} = \sqrt{3} \times 16.5 \times 4.669\,6 = 133.3 \text{ MV A}$$

Alternatively

$$\text{Short-circuit current} = \frac{1.0}{1.5} \times \text{base current} = \frac{1.0}{1.5} \times 6\,998 = 4\,669 \text{ A}$$

$$\text{Short-circuit MV A} = \frac{\text{base MV A}}{\text{p.u. reactance}} = \frac{200}{1.5} = 133.3 \text{ MV A}$$

Example 3

A 10 MV A 6.6 kV 3-phase star-connected alternator, transient reactance 0.1 p.u., feeds a load through two cables in parallel. One cable is rated to carry 1000 A and has reactance 0.05 p.u. at its rated MV A. The other cable is rated 1500 A and has reactance per phase 0.2 Ω. Calculate the short-circuit current in each line at a three-phase fault near the load.

Solution

Take (say) 10 MV A base.
Cable A: 1000 A rating, reactance 0.05 p.u.

$$\text{Rating} = \sqrt{3} \times 6.6 \times 1000 \times 10^{-3} = 11.43 \text{ MV A}$$

Thus at 10 MV A base the per-unit reactance of this cable becomes

$$0.05 \times 10/11.43 = 0.0437 \text{ p.u.}$$

Cable B: 1500 A, reactance 0.2 Ω

$$\text{Rating} = \sqrt{3} \times 6.6 \times 1500 \times 10^{-3} = 17.15 \text{ MV A}$$

We now calculate the reactance of the cable per-unit at this MV A rating, as follows.

$$\text{Unit reactance at this rating} = \frac{\text{phase voltage}}{\text{phase current}} = \frac{3\,810}{1\,500} = 2.54\ \Omega$$

Thus the cable reactance p.u., is

$$0.2/2.54 = 0.0787 \text{ p.u. at } 17.15 \text{ MV A.}$$

Therefore at 10 MV A, cable B reactance is

$$\frac{0.078\,7}{17.15} \times 10 = 0.045\,9 \text{ p.u.}$$

The cable reactances can now be paralleled in the usual way

$$\frac{0.043\,7 \times 0.045\,9}{0.043\,7 + 0.045\,9} = 0.022\,4 \text{ p.u.}$$

This adds directly to the p.u. reactance of the generator, to give a total reactance p.u., at 10 MV A rating,

$$0.1 + 0.022\,4 = 0.122\,4 \text{ p.u.}$$

$$\text{Short-circuit MV A} = 10/0.122\,4 = 81.7 \text{ MV A}$$

$$\text{Short-circuit current} = \frac{81\,700}{\sqrt{3} \times 6.6} = 7\,147 \text{ A}$$

Alternatively

$$\text{At 10 MV A base, rated current} = \frac{10\,000}{\sqrt{3} \times 6.6} = 874.77\ \text{A}$$

$$1.0\ \text{p.u. reactance at 10 MV A} = \frac{3\,810}{874.77} = 4.355\ \Omega$$

$$\text{Reactance of generator in ohms} = 4.355 \times 0.1 = 0.435\,5\ \Omega$$

$$\text{Reactance of cables in parallel (0.022 4 p.u)} = 4.355 \times 0.224 = 0.097\,56\ \Omega$$

$$\text{Total reactance} = 0.435\,5 + 0.097\,56 = 0.533\ \Omega$$

$$\text{Short-circuit current} = \frac{3810}{0.533} = 7147\ \text{A}$$

$$\text{Short-circuit MV A} = \sqrt{3} \times 6.6 \times 7\,147 \times 10^{-3} = 81.7\ \text{MV A}$$

A1.3 DIRECT CURRENT MACHINES

When only d.c. machines are being investigated it is not usually necessary to use the per-unit notation. In multi-machine systems, however, the parameters of d.c. machines can also be written in per-unit form, referred to a selected kV A base. This may be, for example, the rating of an induction motor, driving a d.c. generator.

A1.4 ALTERNATOR PARAMETERS

The application of the per-unit system to an alternator is straightforward, except where the field values are concerned. The main complication here lies in the concept of an acceptable turns ratio. The flux set up by balanced three-phase current in the armature winding has a value which is constant in magnitude and rotates at constant angular velocity. As shown in section 2.12 this is 3/2 times the maximum value of the flux set up by the alternating current in each phase. Using this factor, Rankin [Rankin, A. W., 'Per-Unit Impedances of Synchronous Machines, Parts I and II', *TAIEE*, **64**, 569 and 839 (1945)] derived expressions for the per-unit resistance and reactance of the field winding referred to the armature, as follows

$$\text{Resistance } (R_f)_{\text{p.u.}} = \frac{3}{2}\frac{R_f}{Z_{\text{base}}}\left(\frac{i_f}{\frac{3}{2}i_a}\right)^2 \tag{A1.2}$$

$$\text{Reactance } (X_f)_{\text{p.u.}} = \frac{3}{2}\frac{X_f}{Z_{\text{base}}}\left(\frac{i_f}{\frac{3}{2}i_a}\right)^2 \tag{A1.3}$$

Here the current i_a is the rated armature current and i_f is the field current which will set up rated phase voltage in the stator winding on open-circuit. Typical values for the parameters of a large 3-phase turbo-alternator are given below,

Rating 200 MV A, generated voltage 16.5 kV.

$X_d = 1.6$ p.u.	$R_a = 0.002$ p.u.
$X_q = 1.6$ p.u.	$R_f = 0.001$ p.u.
$X_f = 1.6$ p.u.	$R_{kd} = 0.009$ p.u.
$X_{md} = 1.45$ p.u.	$R_{kq} = 0.009$ p.u.
$X_{mq} = 1.45$ p.u.	Inertia constant $H = 3.0$ MW s/MV A
$X_{kd} = 1.51$ p.u.	
$X_{kq} = 1.51$ p.u.	

A1.5 THE INERTIA CONSTANT

If dynamical machine studies involve large changes in speed the general equation is

$$J\dot{\omega}_m = \text{torque input} - \text{torque output}$$

or

$$J\dot{\omega}_m = \text{torque differential} \tag{A1.4}$$

where ω_m is the mechanical angular velocity of the rotor in radian/second.

The concept of power differential is complicated by the fact that this also involves the angular velocity which is now a variable, the dynamical power equation being

$$\omega_m J \omega_m = \text{power input} - \text{power output} \tag{A1.5}$$

However, in many synchronous machine studies, mechanical disturbances cause variations in speed about the synchronous value, the machines remaining in synchronism at constant mean speed. Here the rotor excursions will be in terms of variations in the synchronous load angle δ and the dynamical equation can be written

$$J p^2 \delta = \text{torque input} - \text{torque output} \tag{A1.6}$$

where $\delta = \theta_1 - \theta_2$, the synchronous angular position of the rotor structure with respect to the rotating flux wave on the stator. The power equation is then

$$\omega_m J p^2 \delta = \text{power input} - \text{power output} \tag{A1.7}$$

or

$$M p^2 \delta = \Delta P \tag{A1.8}$$

In power system stability and control studies, the constant M is often given in terms of another constant H, defined below. The advantage is that H does not vary much over quite a wide range of sizes, for alternators of the same class, namely turbo- or hydro-salient-pole. The relationships amongst the constants are as follows.

Let J be the polar moment of inertia of the rotating mass, in kg m^2.

Then the stored energy

$$W = \tfrac{1}{2} J \omega_m^2$$

If $W = H\,G$ where G is the machine rating, then H is the stored energy in the rotating mass per unit of V A (or MV A)

Thus

$$H = \frac{\frac{1}{2} J \omega_m^2}{G} \text{ p.u. of the machine rated V A} \tag{A1.9}$$

and

$$M = \omega_m J = \frac{2W}{\omega_m} = \frac{2HG}{\omega_m} \tag{A1.10}$$

or

$$M = \frac{2H}{\omega_m} \text{ p.u. machine rated V A} \tag{A1.11}$$

In a.c. machines with n pairs of poles, the steady-state and transient electrical equations in the general machine theory still apply when they are written in terms of the electrical angular velocity $\omega_e = 2\pi f$ electrical rad/s. However, since the inertial power and torque effects are associated with mechanical angular velocity, care is required in relating these to the generated electrical power and torque.

Consider, for example, a machine wound for n pairs of poles, with the rotor rotating at angular velocity ω_m rad/s and undergoing hunting excursions $\Delta\theta_m$. We define the hunting frequency as a function of synchronous frequency by

$$h = \frac{\omega_{\text{rotor hunting}}}{\omega_{\text{rotor angular vel.}}} = \frac{\omega_{\text{elect hunting osc.}}}{\omega_{\text{supply frequency}}} = \frac{2\pi f_{\text{hunting}}}{2\pi f_{\text{supply}}}$$

In the case of a machine with n pole pairs,

$$\omega_e = n\omega_m$$

$$n\,\omega_{\text{rotor hunting}} = \omega_{\text{elect. hunting osc.}}$$

and

$$n\Delta\theta_{\text{mech. excursion}} = \Delta\theta_{\text{elect. excursion}}$$

The steady hunting equation can be written

$$\Delta T = -h^2\omega_m^2 J\Delta\theta_m + (T_s + jhT_{de})\Delta\theta_m \tag{A1.12}$$

The power, in watts, at angular velocity ω_m will be

$$\begin{aligned}\Delta P = \omega_m \Delta T &= -h^2\omega_m^3 J\Delta\theta_m + (P_s + jhP_{de})\Delta\theta_m \\ &= -h^2\omega_m^2 M\Delta\theta_m + (P_s + jhP_{de})\Delta\theta_m \\ &= -h^2 M'\Delta\theta_m + (P_s + jhP_{de})\Delta\theta_m \end{aligned} \tag{A1.13}$$

where

$$M = \omega_m J$$

and

$$M(\text{p.u.}) = (\omega_m J)/G$$

The hunting inertia constant is

$$M' = \omega_m^2 M$$

and

$$M'(\text{p.u.}) = \omega_m^2 M(\text{p.u.}) \tag{A1.14}$$

The hunting equations may be expressed throughout in terms of the electrical supply frequency and the frequency of electrical hunting oscillation at the machine terminals. In terms of the electrical angular velocity ω_e and displacement $\Delta\theta_e$ in electrical radians, we write

$$\Delta T = -h^2\frac{\omega_e^2}{n} J\frac{\Delta\theta_e}{n} + (T_s + jhT_{de})\frac{\Delta\theta_e}{n} \tag{A1.15}$$

To express the hunting torque in synchronous watts, we multiply by ω_e giving,

$$\begin{aligned}\Delta P_{\text{synch. W}} = \omega_e \Delta T &= -\frac{h^2\omega_e^3}{n^2} J\Delta\theta_e + \omega_e(T_s + jhT_{de})\frac{\Delta\theta_e}{n} \\ &= -\frac{h^2\omega_e^3}{n^2} J\Delta\theta_e + (P'_s + jhP'_{de})\frac{\Delta\theta_e}{n}\end{aligned} \tag{A1.16}$$

where P'_s and P'_{de} are now in synchronous watts.
The hunting power at angular velocity ω_m is given by equation (A1.13), from which we see, when comparing with equation (A1.16), that

$$\Delta P = [\Delta P_{\text{synch. W}}]\frac{1}{n} = [\Delta P_{\text{synch. W}}]\frac{\omega_m}{\omega_e} \tag{A1.17}$$

When the rotor undergoes periodic or transient displacement about the mean constant synchronous speed, the computation of torque and power is simplified if we use per-unit values. We can then write the following relationships for a hunting alternator on infinite busbars.

Putting $p = \mathrm{j}h\omega_{\mathrm{m}}$

$$(-h^2\omega_{\mathrm{m}}^2 J + \mathrm{j}hT_{\mathrm{de}} + T_{\mathrm{s}}) = 0$$

Thus, equating real parts

$$h^2\omega_{\mathrm{m}}^2 J = T_{\mathrm{s}} \quad \text{(N m/mech. rad)} \tag{A1.18}$$

and

$$h^2\omega_{\mathrm{m}}^3 J = P_{\mathrm{s}} \quad \text{(W/mech. rad)}$$

Since

$$M\text{(p.u.)} = \omega_{\mathrm{m}} J/G$$

we see that

$$h^2\omega_{\mathrm{m}}^2 M\text{(p.u.)} = P_{\mathrm{s}} \quad [\text{(p.u.)/mech. rad}]$$

Now rated torque $= \dfrac{\text{rated power}}{\omega_{\mathrm{m}}} = \dfrac{G}{\omega_{\mathrm{m}}}$

and

$$\frac{h^2\omega_{\mathrm{m}}^2 J}{\text{rated torque}} = T_{\mathrm{s}} \quad [\text{(p.u.)/mech. rad}]$$

or

$$\frac{h^2\omega_{\mathrm{m}}^2 J}{G/\omega_{\mathrm{m}}} = T_{\mathrm{s}} \quad [\text{(p.u.)/mech. rad}]$$

or

$$\frac{h^2\omega_{\mathrm{m}}^3 J}{G} = T_{\mathrm{s}} \quad [\text{(p.u.)/mech. rad}]$$

or

$$\begin{aligned} h^2\omega_{\mathrm{m}}^2 M\text{(p.u.)} &= T_{\mathrm{s}} \quad [\text{(p.u.)/mech. rad}] \\ &= P_{\mathrm{s}} \quad [\text{(p.u.)/mech. rad}] \end{aligned}$$

or

$$h^2 M'\text{(p.u.)} = T_{\mathrm{s}} \quad [\text{(p.u.)/mech. rad}] \tag{A1.19}$$

$$= P_{\mathrm{s}} \quad [\text{(p.u.)/mech. rad}] \tag{A1.20}$$

where again

$$M' = \omega_{\mathrm{m}}^2 M$$

Thus we can now write, in terms of per-unit torque and power

$$\Delta T = [-M'h^2 + T_{\mathrm{s}} + \mathrm{j}hT_{\mathrm{de}}]\Delta\theta \tag{A1.21}$$

and

$$\Delta P = [-M'h^2 + P_s + jhP_{de}]\Delta\theta \tag{A1.22}$$

In this form of the hunting equations, the power and torque have equal per-unit values. Reactance and inductance have equal values per-unit. In equations (A1.21) and (A1.22), we still have

$$M' = \omega_m^2 M = \omega_m^3 J/G$$

where ω_m is in mechanical radians per second. The displacement angle $\Delta\theta$ is also expressed in per-unit terms, as a fraction of the electrical or mechanical radian

$$\Delta\theta\,(\text{p.u.}) = \Delta\theta_m/\text{mech. rad} = \Delta\theta_e/\text{elect. rad}$$

since

$$\Delta\theta\,(\text{p.u.}) = \frac{\Delta\theta_e}{\text{elect. rad}} = \frac{n\Delta\theta_m}{n(\text{mech. rad})} = \frac{\Delta\theta_m}{\text{mech. rad}}$$

where n is the number of pole pairs.

Since torque T and power $P\,(=\omega T)$ are equal in the per-unit system, there is an implication that the synchronous speed may be taken as unity. However, as equations (A1.12) to (A1.22) show, care is required in the use of angular velocity in the inertia constants.

Appendix 2 Measurement of machine parameters

There has sometimes been some confusion in generalised machine theory as to the choice of direct and quadrature axes in multipolar machines—especially in the cases of d.c. machines and salient-pole alternators. The usual method is to select equivalent direct and quadrature axes at the terminals of a machine, measure all of the parameters also from the terminals and express these along the equivalent d and q axes.

A2.1 THE INDUCTION MOTOR

The parameters of the induction motor can be measured by standard balanced three-phase tests. In most of the machine analysis given in the text, the inductances used are the full values of the self and mutual inductance of the windings. Typically, the relationship between the self-inductance of coil (a) and its leakage inductance with respect to coil (b) is $L_a = l_a + L_{ab}$ where l_a is the leakage inductances due to flux set up in coil (a) which does not link with coil (b) and L_{ab} is the mutual inductance due to flux set up in coil (a) which links both coils. Two points must be made here.

(1) When electromagnetically coupled coils are energised, leakage flux is a quantity which exists between pairs of coils. The concept of components of leakage flux allocated in some proportion to each coil is simply a convenience, as the reader will find if he tries to devise an experiment which will 'correctly' divide the leakage flux between two or more coils. When total leakage inductance is measured by a short-circuit test on a transformer, it

makes virtually no difference to calculated results whether this is associated with one coil or another or divided equally amongst two or more coils involved in the test—as long as all quantities are referred to the same coil by the squares of the respective turns ratios.

(2) The full self-inductances of the windings of a transformer or induction motor can be accurately measured under normal conditions of flux density, if reasonable care is taken, by standard open-circuit tests described in most text-books on machines. To obtain sufficiently accurate values of the mutual inductances is more difficult. It is advisable to use values of leakage inductance which have been accurately measured by short-circuit tests, along with the measured values of self-inductance. The total leakage inductance can safely be divided equally amongst the *referred* windings.

Winding resistance can be measured using a light-current d.c. supply. To allow for skin-effect at 50 Hz a multiplier of about 1.4 is usually necessary. Resistance values must be adjusted for operating temperature rise. The resistance of carbon brushes is, of course, nonlinear and should be measured at the operating current density.

When the transformations given by Stanley (reference 7 Chapter 5) are used, the measured parameters of the 3-phase induction motor per phase, give directly the d and q axis quantities.

A2.2 THE SYNCHRONOUS MACHINE

A wealth of literature has been published on the measurement of three-phase synchronous machine parameters. The synchronous reactance can be measured by direct tests at low power-factor. The familiar Potier test will yield values of the armature reaction effect and the leakage reactance. The reactance in the quadrature axis between the poles can be obtained by standard slip tests or flux reversal tests if the machine is not too large. The inductance of the d.c. field coils can be obtained by direct measurement of the electrical exponential decay time-constant. Alternatively, the self and mutual inductances of most of the windings (including perhaps amortisseur windings, mentioned in Appendix 3) can be measured directly by flux-meter techniques—as described, for example in reference 1, Chapter 2. Equivalent direct and quadrature axis quantities can then be obtained by resolving the coupled phase quantities along these directions.

The classical three-phase transient short-circuit test will give directly the subtransient and transient reactances and time constants in the direct axis. The parameters which can be obtained by straightforward measurement (and which are usually quoted by manufacturers) are

X_d = direct axis synchronous reactance

X'_d = direct axis transient reactance

X''_d = direct axis subtransient reactance

X_q = quadrature axis synchronous reactance ($= X'_q$)

X''_q = quadrature axis subtransient reactance

together with

T'_d = transient short-circuit time constant

T''_d = subtransient time constant

T_{do} = open-circuit field time constant

We can perhaps appreciate the significance of these quantities by examining the equivalent circuits for the alternator in the direct and quadrature axes, shown in Figures A2.1 and A2.2. Here the values seen from the armature terminals are the subtransient impedances, with reactances X''_d and X''_q respectively. In Figure A2.1

$$x_a = X_d - X_{md} \tag{A2.1}$$

where x_a is the armature leakage reactance, X_d is the direct axis synchronous reactance, and X_{md} is the direct axis field/armature mutual reactance.

Also,

$$x_f = X_f - X_{md} \tag{A2.2}$$

where x_f is the field-winding leakage reactance.

Similarly in the amortisseur-winding circuit,

$$x_{kd} = X_{kd} - X_{md} \tag{A2.3}$$

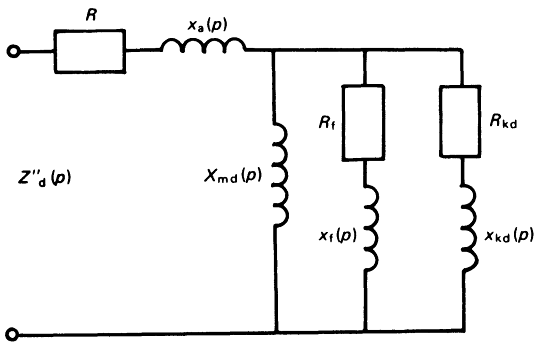

Figure A2.1

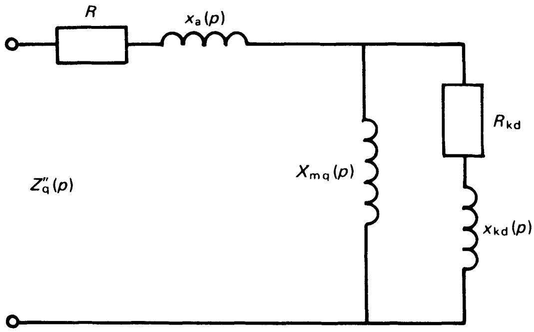

Figure A2.2

where x_{kd} is the amortisseur-winding leakage reactance and X_{kd} is the amortisseur-winding self reactance.

Similar expressions can be written for the q axis circuit shown in Figure A2.2.

When the subtransient part of the normal three-phase short-circuit current has died away, the equivalent transient circuits are as shown in Figures A2.3 and A2.4. When the transient components have decayed, the circuits of Figures A2.4

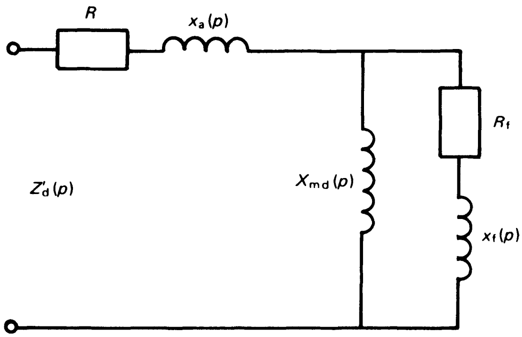

Figure A2.3

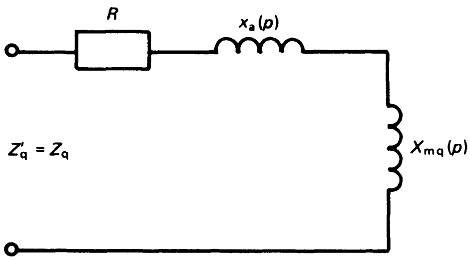

Figure A2.4

and A2.5 give the synchronous impedances, with reactances X_d and X_q. It will be noticed that due to the absence of a field winding in the q axis, $X'_q = X_q$.

If we now neglect the resistance of the windings, we can write

$$X'_d = x_a + \frac{X_{md}x_f}{X_{md} + x_f} \tag{A2.4}$$

$$X''_d = x_a + \frac{X_{md}x_f x_{kd}}{X_{md}x_f + X_{md}x_{kd} + x_f x_{kd}} \tag{A2.5}$$

$$X''_q = x_a + \frac{X_{mq}x_{kq}}{X_{mq} + x_{kq}} \tag{A2.6}$$

The subtransient and transient armature time constants can be written by inspection of the circuits of Figures A2.6 and A2.7, with the armature input terminals short-circuited as shown, giving

$$T''_d = \frac{1}{\omega R_{kd}}\left\{x_{kd} + \frac{X_{md}x_a x_f}{X_{md}x_a + X_{md}x_f + x_a x_f}\right\} \tag{A2.7}$$

$$T'_d = \frac{1}{\omega R_f}\left\{x_f + \frac{X_{md}x_a}{X_{md} + x_a}\right\} \tag{A2.8}$$

$$T''_q = \frac{1}{\omega R_{kq}}\left\{x_{kq} + \frac{X_{mq}x_a}{X_{mq} + x_a}\right\} \tag{A2.9}$$

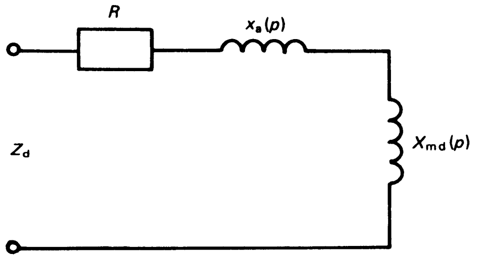

Figure A2.5

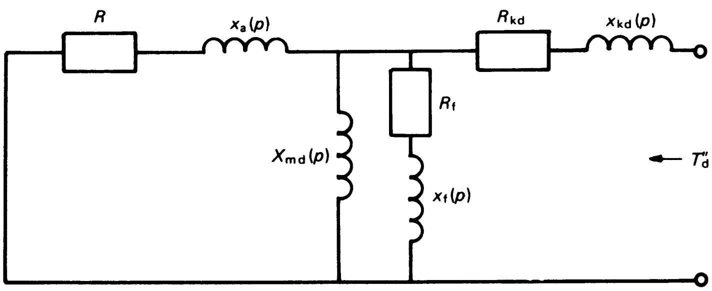

Figure A2.6

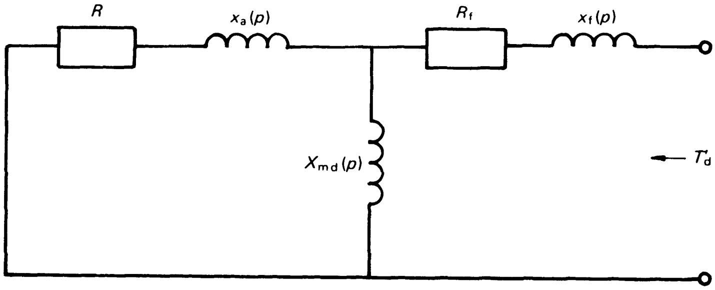

Figure A2.7

The d axis open-circuit time constant of the field circuit is

$$T_{\mathrm{do}} = \frac{X_{\mathrm{f}}}{\omega R_{\mathrm{f}}} \qquad \text{(A2.10)}$$

One further step is necessary in obtaining the required parameters for the general analysis of synchronous machines. This is the accurate determination of the resistance and reactance of the amortisseur winding shown in the equivalent circuits. An iterative computer programme can be written to yield these from the circuit, using the other known values. This is discussed briefly in Appendix 3.

A2.3 DIRECT CURRENT MACHINES

The measurement of the parameters of d.c. machines presents no difficulties where laboratory machines are concerned. Obviously, with very large machines heavy-current equipment and massive cabling will be necessary in order to test the machines under the working magnetic saturation conditions and current loading.

The self and mutual inductances of the windings can be measured by transient decay techniques or by flux-meter methods—or the equivalent of the latter, the inductance bridge described by Jones (reference 1 Chapter 2). The *rotational* self and mutual inductances L' and M' (equation 5.7) can be measured directly when the machine is energised and running at normal speed.

A2.4 THE POLAR MOMENT OF INERTIA AND THE ROTATIONAL FRICTION COEFFICIENT

These two important parameters are classed together here, since measurement of one of them can be used in the determination of the other. The most reliable method of determining the polar moment of inertia of the rotating mass is direct measurement, by removing the rotor and suspending it as a bifilar pendulum.

The method is described in text-books on elementary mechanics. When the polar moment of inertia is known, a speed retardation test will give the rotational coefficient of friction R_F since under these conditions

$$J\dot{\omega} + R_F\omega = 0$$

If a disc of known moment of inertia can be fixed to the shaft of the machine, then there is no need to remove the rotor, since simultaneous equations can be written which will yield both J and R_F.

On the other hand R_F can be measured by a modification of the Swinburne test and then used in the retardation test to give J.

A2.5 MEASUREMENT OF SYNCHRONOUS LOAD ANGLE δ

The measurement of speed of a machine is straightforward. The synchronous load angle of an alternator or synchronous motor can be clearly observed stroboscopically but accurate measurement of this angle and its derivatives during transient or hunting excursions of the rotor is quite difficult. Clearly this must be carried out by a speed comparator, with a synchronous speed signal as reference. This usually means a small structural appendage on the shaft which will give a pulse signal at a particular point during rotation, together with a fast accurate timing device. Several reliable electronic devices are now available which will give the load angle δ together with the rate of change $p\delta$ and the angular acceleration $p^2\delta$ (see, for example, reference 2 Chapter 7).

The methods described in this appendix give very good approximate values of the parameters of sufficient accuracy for most purposes. More rigorous techniques will be found in Canay, I. M., 'Causes of Descrepancies on Calculation of Rotor Quantities and Exact Equivalent Diagrams of the Synchronous Machine', *Trans. I.E.E.E.*, **PAS 88,** 1114–1120 (1969); Shackshaft, G., 'New Approach to the Determination of Synchronous Machine Parameters from Tests', *Proc. I.E.E.*, **121,** 1385 (1974); and Shackshaft, G., and Poray, A. T., 'Implementation of a New Approach to Determination of Synchronous Machine Parameters from Tests', *Proc. I.E.E.*, **124,** No. 12, Dec. (1977).

Appendix 3
Estimation of alternator parameters using the equivalent circuits for direct and quadrature axes

Standard tests give the machine subtransient and transient reactances and their time constants, shown on the equivalent circuits. These values can be used to give all of the parameters required in the generalised equations, for dynamical analysis.

For example, using equations (A2.4)–(A2.8), equation (A2.4) can be written

$$X'_d = \frac{X_f X_d - X_{md}^2}{X_f}$$

since

$$X_d = x_a + X_{md}$$

and

$$X_f = x_f + X_{md}$$

Knowing X'_d we can calculate X_{md} and thus x_f and x_a can be found. These latter values can be checked using equation (A2.8). We can then calculate x_{kd} and R_{kd} using equations (A2.5) and (A2.4). When x_a is known, X_{mq} is known from X_q. Hence we can find x_{kq} and $X_{kd} = x_{kd} + X_{md}$ and so on.

Iterative computer programmes can be designed to check all of the parameters in this manner using both test and estimated values of reactances and time constants and to converge to a self-consistent set.

In some hunting analyses flux penetration in the iron is significant. In these cases, frequency-dependent operational inductances must be used [see Adkins, B. and Sen, S. K., 'The Application of the Frequency-response Method to Electrical Machines', *Proc. I.E.E.*, **103,** Part C, 378 (1956) and Walshaw, M. H. and Lynn, J. W., 'A Hunting Analysis of a Permanent Magnet Alternator and a Synchronous Motor', *Proc. I.E.E.*, **108,** Part C, 516 (1961)].

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