Fibonacci Search Method for unconstrained optimization problems

This method makes use of the sequence of Fibonacci numbers $\{F_n\}$ for placing the experiment.

Fibonacci numbers:

The Fibonacci numbers are defined as

$$F_0 = F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2};\ n \geq 2$$

Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34,.....

Fibonacci Search Method:

Task is to max/min f(x) over a given interval [a, b].

Divide the initial interval [a, b] equally into F_n subintervals and hence the length of each subinterval is $\frac{(b-a)}{F_n}$

(Initial length) $L_0 = b - a$, $L_n = \frac{b-a}{F_n}$ $\Rightarrow$ $\boxed{\frac{L_n}{L_0} = \frac{1}{F_n}}$

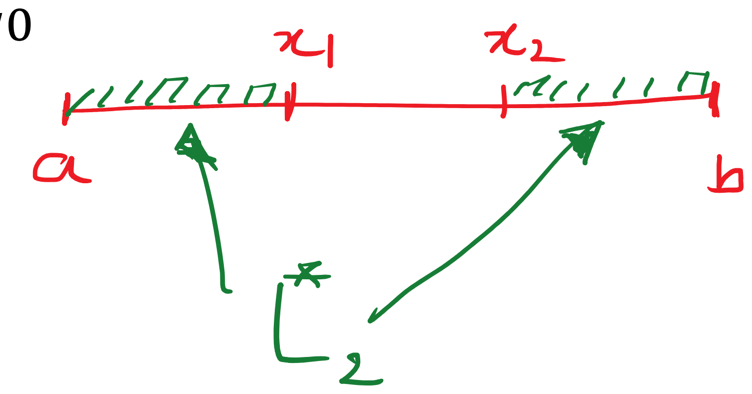
Fibonacci Method:

- Determine the number of iterations.
- Define $L_2^* = \frac{F_{n-2}}{F_n} L_0$; where $L_0 = b - a$ is the length of the initial interval of uncertainity.

  Choose the two initial points x_1 and x_2 be such that

  $$x_1 = a + L_2^* = a + \frac{F_{n-2}}{F_n} L_0 \text{ and } x_2 = b - L_2^* = b - \frac{F_{n-2}}{F_n} L_0$$

  Remark: $x_1 + x_2 = a + b$ or $b - x_2 = x_1 - a$



Formula

$$\boxed{x_2 = L + \left(\frac{F_{n-k}}{F_{n-k+1}}\right)(R - L)}$$

$\because x_1 + x_2 = L + R$

$\therefore \boxed{x_1 = L + R - x_2}$

- Compute f(x_1), f(x_2) and apply unimodal principle.
- The process is repeated again and again.

Unimodal:

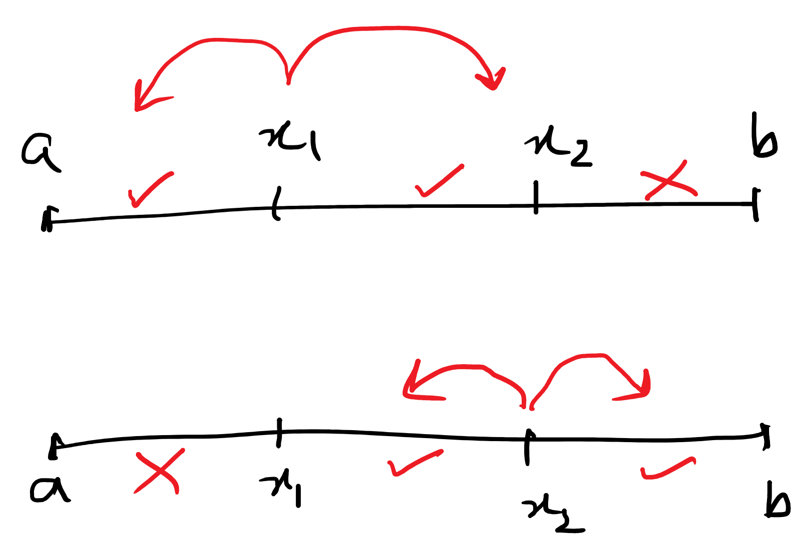
Let initial interval [a, b] be given and the number of iteration be n.

Set L= a; R = b; k = 1 (iteration number)

Compute x_1 and x_2 .

$If\ f(x_2) > f(x_1)$ then $L_{k+1} = L_k$ and $R_k = x_2$

$If\ f(x_2) \leq f(x_1)$ then $L_{k+1} = x_1$ and $R_{k+1} = R_k$



Pb 1 Min $f(x) = x(x-1.5)$ in $[0,1]$ within the interval of uncertainty 0.25 of the intial interval of uncertainty.

Sol:

$$L_n \leq 0.25 L_0$$

$$\Rightarrow \frac{L_n}{L_0} \leq \frac{1}{4} \quad \left(\because \frac{L_n}{L_0} = \frac{1}{F_n}\right)$$

$$\Rightarrow F_n \geq 4$$

1	1	2	3	$\boxed{5}$	8	13
F_0	F_1	F_2	F_3	$\boxed{F_4}$		

$\rightarrow n = 4$

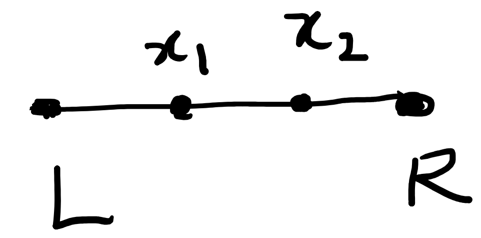
The smallest n satisfying this is 4.

$L + R - x_2$

$L + \left(\frac{F_{n-k}}{F_{n-k+1}}\right)(R-L)$

Since, problem is minimization, identify minimum occur at L or R points.

K	$\frac{F_{n-k}}{F_{n-k+1}}$	L	R	x_1	x_2	$f(x_1)$	$f(x_2)$	Preserve L or R	
1	3/5	0	1	0.4	0.6	−0.44	−0.54	R	
2	2/3	0.4	1	0.6	0.8	−0.54	−0.56	R	
3	1/2	0.6	1	0.8	0.8	−0.56	−0.56	L	(Choose any one)
4	1/1	0.6	0.8	—	—	—	—		



No need to calculate

Thus, $x_{min} \in [0.6, 0.8]$

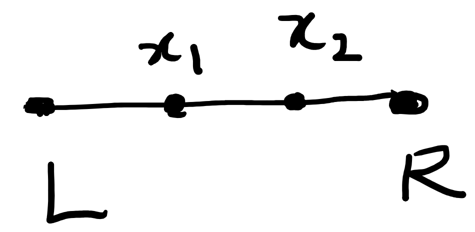
Hence, $x^* = \frac{0.6 + 0.8}{2} = 0.7$

$f(x^*) = -0.56$

Pb 2 Min $f(x) = x^2$ over $[-5, 15]$ using Fibonacci search method. Take $n=7$.

Sol :

Since, problem is minimization, identify minimum occur at L or R points.



$x_1 = L + R - x_2$ $\qquad$ $x_2 = L + \left(\frac{F_{n-k}}{F_{n-k+1}}\right)(R-L)$

K	$\frac{F_{n-k}}{F_{n-k+1}}$	L	R	x_1	x_2	$f(x_1)$	$f(x_2)$	Preserve L or R	
1	13/21	−5	15	2.6191	7.3809	6.8596	54.4776	L	
2	8/13	−5	7.3809	−0.2382	2.6191	0.0567	6.8596	L	
3	5/8	−5	2.6191	−2.1427	−0.2382	4.5911	0.0567	R	
4	3/5	−2.1427	2.6191	−0.2382	0.7146	0.0567	0.5106	L	
5	2/3	−2.1427	0.7146	−1.1899	−0.2382	1.4158	0.0567	R	
6	1/2	−1.1899	0.7146	−0.2382	−0.2377	0.0567	0.0565	R	
7	1/1	−0.2382	0.7146	−0.2377	0.7141	0.0565	0.5099		

No need to calculate

Thus, $x_{min} \in [-0.2382, 0.7146]$

Hence, $x^* = \dfrac{-0.2382 + 0.7146}{2} = 0.2382$

$f(x^*) = 0.05674$

Pb 3 $\quad$ Min $f(x) = x^2 - 2.6x + 2$; $x \in [-2, 3]$ using Fibonacci search technique. Perform 6 iterations.

Ans, $\quad x_{min} \in [1.0769, 1.4616]$; $\quad x^* = 1.26925$ and $f(x^*) = 0.31094$.

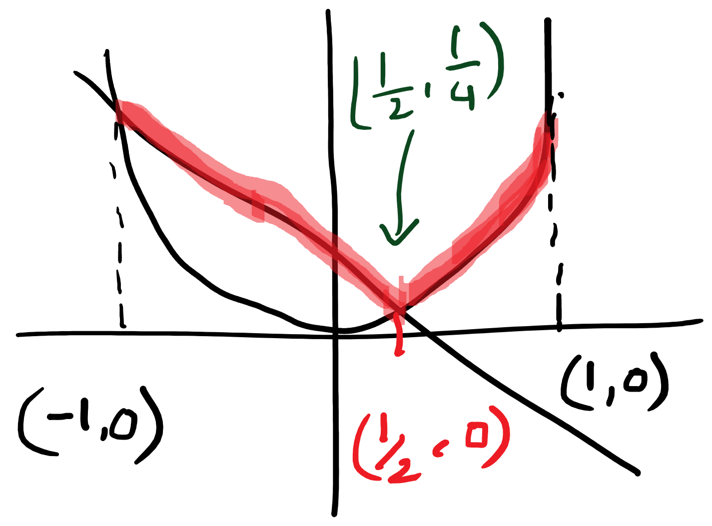
Pb 4 Perform 6 iterations of the Fibonacci search method to

$$\min f(x) = \text{Max}\left\{x^2, \frac{1-x}{2}\right\} \text{ over } [-1,1]$$ and hence identify the final interval I_6.

Sol

$$\text{Min } f(x) = \text{Max}\left\{x^2 ; \frac{1-x}{2}\right\}$$

$$= \begin{cases} \frac{1-x}{2} ; & \text{if } -1 \le x \le 1/2 \\ x^2 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

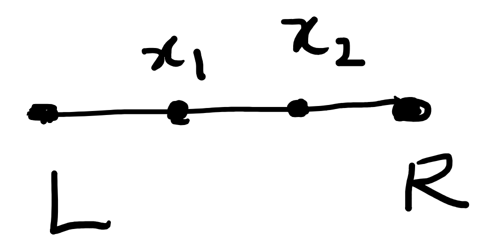


Since, problem is minimization, identify minimum occur at L or R points.

$L + R - x_2$

$L + \left(\frac{F_{n-k}}{F_{n-k+1}}\right)(R-L)$

K	$\frac{F_{n-k}}{F_{n-k+1}}$	L	R	x_1	x_2	$f(x_1)$	$f(x_2)$	Preserve L or R	
1	8/13	-1	1	-0.2307	0.2307	0.6153	0.3846	R	
2	5/8	-0.2307	1	0.2307	0.5386	0.3846	0.2901	R	
3	3/5								
4	2/3								
5	1/2								
6	1/1	0.3842	0.5877	—	—	—	—		



Thus, $x_{min} \in [0.3842, 0.5877]$

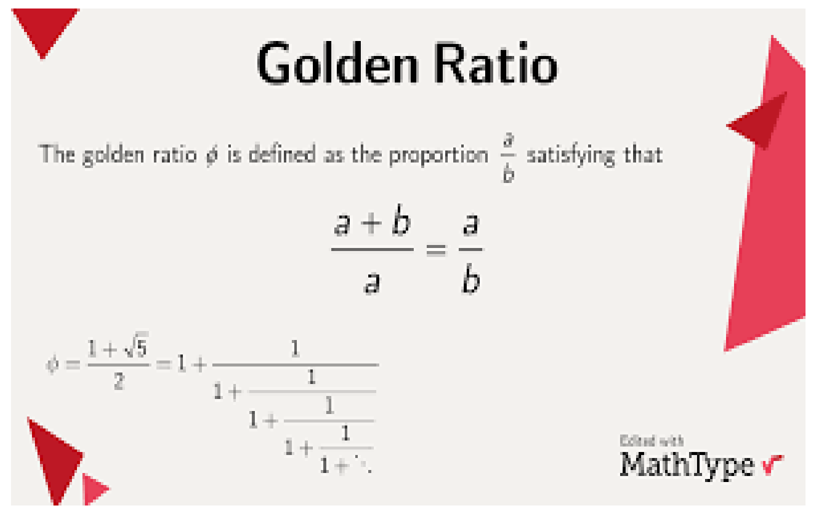
Hence, $x^* = \frac{0.3842 + 0.5872}{2} = 0.46095$

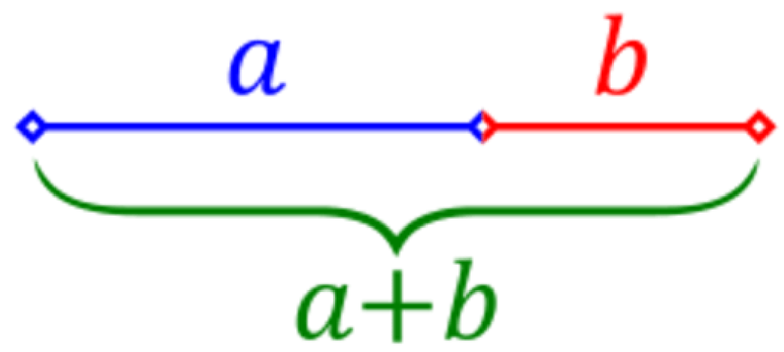
$f(x^*) = 0.26953$

Golden section search method: This method is same as the Fibonacci search method except that in the Fibonacci search method the total number of experiments to be conducted has to be specified before beginning the calculation, whereas this is not required in the golden section search method. In the Fibonacci search method, the location of the first two experiments is determined by the total number of experiments. In the golden section search method, we start with the assumption that we are going to conduct a large number of experiments.

- Golden Section Rule
- Difference between Golden Section and Fibonacci search method
- Golden ratio number ϕ = 1.618 (Approx.)
- Golden section rule algorithm and some examples.

$$\phi = \frac{1+\sqrt{5}}{2}$$





$a+b$ is to a as a is to b

$$\frac{a+b}{a} = \frac{a}{b} = \varphi = \frac{1+\sqrt{5}}{2} = 1.618$$

Golden Section Search:

$I_K = R - L$ length of the search interval at K^{th} iteration.

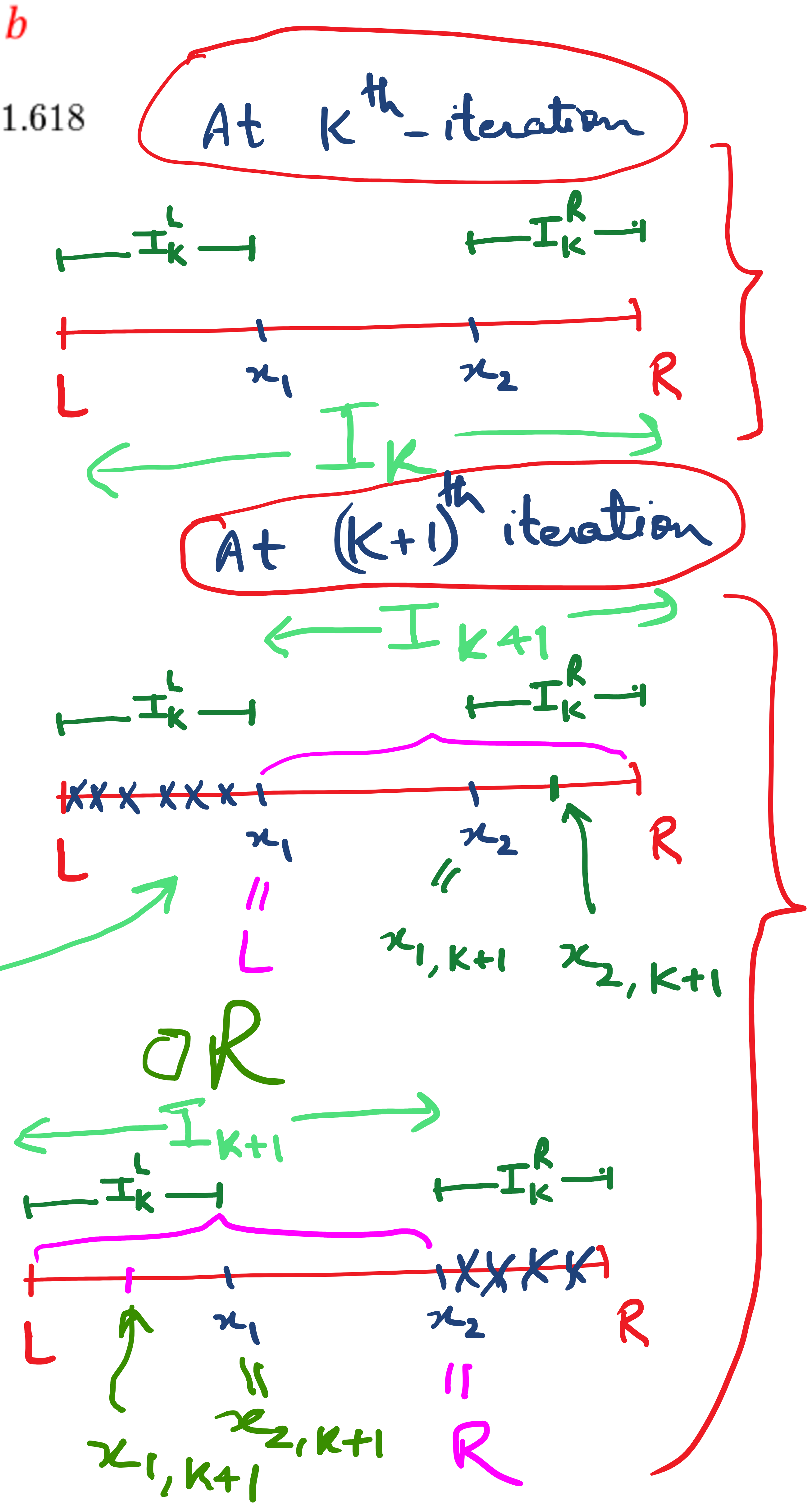
$I_K^L = x_{1,K} - L$ length of the left part of the search interval I_K

$I_K^R = R - x_{2,K}$ length of the right part of the search interval I_K

Suppose x_1 moves towards right

OR

Suppose x_2 moves towards left



Formula

$$\boxed{x_2 = L + 0.618(R-L)}$$

$$\because \quad x_1 + x_2 = L + R$$

$$\therefore \quad \boxed{x_1 = L + R - x_2}$$

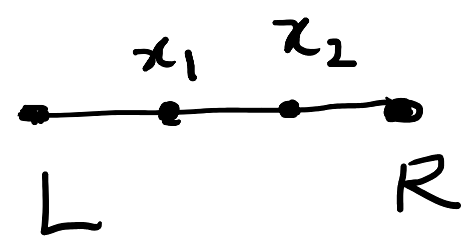
Pbl Min $f(x) = x(x-15)$ in $[0,1]$ by Golden Section rule with interval of uncertainty as 0.3.

Sol: Here, n is not given; but interval of uncertainty is given as 0.3 (say ϵ)

i.e; $R - L = \epsilon = 0.3$

Since, problem is minimization, identify minimum occur at L or R points.

$x_1 = L + R - x_2$, $\quad x_2 = L + 0.618(R-L)$



R-L < 0.3	K	L	R	x_1	x_2	$f(x_1)$	$f(x_2)$	Preserve L or R	
No	1	0	1	0.382	0.618	-0.42708	-0.54508	R	
No	2	0.382	1	0.618	0.764	-0.54508	-0.5623	R	
No	3	0.618	1	0.764	0.854	-0.5623	-0.55168	L	
Stop → Yes	4	0.618	0.854	0.708	0.764	-0.56074	-0.5623		

No need to calculate

$\because \quad I_4 = 0.854 - 0.618 = 0.236 < 0.3$ (stop)

Thus, $x_{min} \in [0.618, \ 0.854]$

Hence, $x^* = \dfrac{0.618 + 0.854}{2} = 0.736$

$f(x^*) = 0.19430$

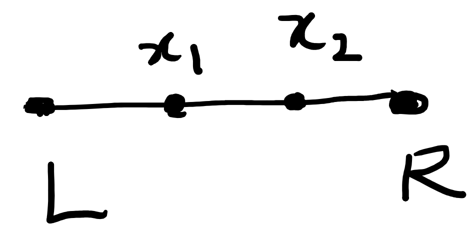
Pb 2 $\text{Min } f(x) = x^2$ over $[-5, 15]$ using Golden search method. Take $n=7$.

Sol:

Since, problem is minimization, identify minimum occur at L or R points.

$L + R - x_2$ $\qquad$ $L + 0.618\,(R-L)$

	K	L	R	x_1	x_2	$f(x_1)$	$f(x_2)$	Preserve L or R	
	1	−5	15	2.64	7.36	6.9696	54.17	L	
	2	−5	7.36	−0.28	2.64	0.0784	6.9696	L	
	3	−5	2.64	−2.08	−0.28	4.3264	0.0784	R	
	4	−2.08	2.64	−0.28	0.84	0.0784	0.7054	L	
	5	−2.08	0.84	−0.96	−0.28	0.9216	0.0784	R	
	6	−0.96	0.84	−0.28	0.16	0.0784	0.0256	R	
	7	−0.28	0.84						



Thus, $x_{min} \in [-0.28, 0.84]$

Hence, $x^* = \dfrac{-0.28 + 0.84}{2} = 0.06$

$f(x^*) = 0.0036$

Pb 3 $\text{Min } f(x) = x^2 - 2.6x + 2$; $x \in [-2, 3]$ using Golden section method. Perform 6 iterations.

Ans, $x_{min} \in [1.09, 1.54]$; $x^* = 1.23$ and $f(x^*) = 0.31490$.

Pb 4 Perform 6 iterations of the Golden Section method to minimize $f(x) = \text{Max}\left\{x^2, \dfrac{1-x}{2}\right\}$ over $[-1, 1]$ and hence identify the interval final interval I_6.

Ans:- $x_{min} \in [0.416, 0.596]$

Hence, $x^* = \dfrac{0.416 + 0.596}{2} = 0.472$, $\text{Min } f(x^*) = 0.2640$