MA6003D M.Tech NIT Calicut

Mathematical Methods For Power Engineering

Binary Operation:- A binary operation on a non-empty set A is a function from $A \times A$ to A.

i.e. $f : A \times A \longrightarrow A$

$$(a, b) \longmapsto f(a, b) \in A$$

Ex: $+ : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$$(a, b) \longmapsto a+b \in \mathbb{N}.$$

Algebraic Structure:- A set A together with one or more binary operation is called algebraic structure.

- Let A be any set and $*$ be the binary op^n on A then $(A, *)$ is called algebraic structure.
- If there are $+, \cdot, *$ are binary operations on A then $(A, +, \cdot, *)$ is also algebraic structure.

Ex: 1. $(\mathbb{N}, +, \cdot, *)$ is algebraic structure as addition '$+$', multiplication '$\cdot$' and $*$ are binary operation on $\mathbb{N}$. Where $*$ is given by $a*b = a^b$ which is binary.

2. $S = \{ f \mid f : A \xrightarrow{\text{function}} A,\ A \text{ is non empty set} \}$.

then composition of functions is a binary operation.

$$\circ : S \times S \longrightarrow S$$

$$(f, g) \mapsto f \circ g \in S$$

Thus, $(S, \circ)$ is an alg. structure.

1) <u>Closure Property</u>:-
Let $*$ be a binary opⁿ on G then we say that the set is <u>closed</u>.

$$\forall\, a, b \in G \Rightarrow a * b \in G$$

Note: Every alg. structure is closed with respect to all the binary operation in corporated in the structure.

2) <u>Associativity</u>:- Let $(G, *)$, $*$ - binary opⁿ on G.
then $*$ is called associative if $\forall\, a, b, c \in G$

$$a * (b * c) = (a * b) * c$$

3) <u>Identity element</u>:-
Let $*$ be an binary operation on G if $\exists$ an element e (say) in G s.t.

$$a * e = e * a = a \quad \forall\, a \in G$$

then e is called identity element in G w.r.t. $*$.

4) <u>Commutative</u>:- Let $*$ be an binary opⁿ on G.
$*$ is commutative if $\forall\, a, b \in G$

$$a * b = b * a$$

5) <u>Inverse element</u>: Let G with binary opⁿ $*$, $*$ - associative and G has an identity element e. We say an element $a \in G$ possesses inverse in G if $\exists$ $b \in G$ s.t.

$$a * b = e = b * a.$$

Module 1 (Linear Algebra)

- **Groupoid** :- A set $\phi \neq G$ together with only one binary op^n $(G, *)$ is called groupoid or quasi group.

Ex: $H_1 = \{2n \mid n \in I\}$, $(H_1, +)$ — Groupoid.

$H_2 = \{2n+1 \mid n \in I\}$, $(H_2, +)$ — not closed, thus not groupoid.

- **Semigroup** :- Let $(G, *)$ is a groupoid then $(G, *)$ is called semigroup if $*$ is associative on G.

- **Monoid** :- A semigroup $(G, *)$ is said to be monoid if $\exists$ an element $e \in G$ s.t.

$$a * e = e * a = a \quad \forall a \in G.$$

Ex:

Monoid:

$(P(\mathbb{N}), \cup)$, $e = \phi$

$(P(\mathbb{N}), \cap)$, $e = \mathbb{N}$

$(P(\mathbb{N}), \Delta)$, $e = \phi$ $\qquad A \Delta B = (A \cup B) \setminus (A \cap B)$

$(\mathbb{N}, *)$, $a * b = \text{lcm}(a, b)$ —— $e = 1$

→ $(\mathbb{N}, *)$, $a * b = \text{hcf}(a, b)$ —— There is no element $e \in \mathbb{N}$ s.t. $a * e = a \ \forall a \in \mathbb{N}$

Not monoid but semigroup $\because$ has no identity.

- **Group** :- A monoid in which every element has an inverse is called group.

- **Abelian Group** :- A group $(G, *)$ is called abelian group if $*$ is commutative on G. i.e.,

$$\forall a, b \in G, \quad a * b = b * a.$$

- Ring :- let R be an algebraic structure equiped with two binary operation denoted by '+' & '·' then $(R, +, \cdot)$ algebraic structure forms ring if following conditions are satisfies :

i) $(R, +)$ is an abelian gp.

ii) $(R, \cdot)$ is semigp.

iii) $\forall\ a, b, c \in R$

$$\left.\begin{aligned} a\cdot(b+c) &= a\cdot b + a\cdot c \\ (a+b)\cdot c &= a\cdot c + b\cdot c \end{aligned}\right\} \text{Distributive Property}$$

Ex: → $(\mathbb{C}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{Z}, +, \cdot)$, $(m\mathbb{Z}, +, \cdot)$

→ $M_{n\times m}(\mathbb{R})$, $M_n(\mathbb{C})$, $M_n(\mathbb{Z})$ under usual matrix addition and usual matrix multiplication form ring.

- Commutative Ring: If in a ring $(R, +, \cdot)$, $\cdot$ is commutative binary operation then ring is said to commutative ring.

- Ring with unity :- If in a ring $(R, +, \cdot)$, $\exists$ an element $a \in R$ such that

$$a\cdot b = b\cdot a = b \quad \forall\ b \in R$$

then the element a is called unity (multiplicative unity) of ring R. Unity is usually denoted by 1.

- Field :- A commutative ring with unity (C.R.U) in which every non-zero element possess multiplicative inverse then such ring is called field. Field is usually denoted by $(F, +, \cdot)$.

In linear algebra we are dealing with those variables which can be taken in linear form only.

$(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are field structures.

0 : Zero element (additive identity)

1 : Unity element (multiplicative identity)

$(\mathbb{Z}_p, \oplus_p, \odot_p)$ is finite field, p is prime.

Ex: $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$ ....

$\mathbb{Z}_p = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \ldots, \overline{p-1}\}$

Internal composition :- A composition $*$ is said to be an internal composition on a non-empty set A if for all $a, b \in A$,

$a * b \in A$ and it is unique.

$$\boxed{\begin{aligned} * : A \times A &\longrightarrow A \\ (a, b) &\longmapsto a * b \end{aligned}}$$

Ex: $+$ is internal compo. on $\{0\}$

$\times$ " " " " $\{1\}$

$+$ " " " " $\mathbb{R}$

$+$ " " " " $\mathbb{Q}$

$+$ " " " " $\mathbb{C}$

$\oplus_p$ " " " " $\mathbb{Z}_p$

$+$ " " " " $M_{m \times n}(\mathbb{R})$

$\cdot$ " " " " $M_{n \times n}(\mathbb{R})$

<u>External composition</u> :- A composition $\circ$ is said to be external composition in V over F if
$\forall\ a \in F$ & $\alpha \in V$

$a \circ \alpha \in V$ and it is a unique.

$$\boxed{\begin{aligned} \circ : F \times V &\longrightarrow V \\ (a, \alpha) &\longmapsto a \circ \alpha \in V \end{aligned}}$$

Pb 1) Is $\times$ (multiplication) an ext. comp. in $\mathbb{R}$ over $\mathbb{R}$ (T)

2) Is $\times$ (") an " " " $\mathbb{R}$ over $\mathbb{C}$ (F)

3) Is $\times$ (") an " " " $\mathbb{Q}$ over $\mathbb{R}$ (F)

True 4) Is $\times$ (") an " " in $\{0\}$ over $\mathbb{C}/\mathbb{R}/\mathbb{Q}$.

False 5) Is $\times$ (") " " " " $\{1\}$ over $\mathbb{C}/\mathbb{R}/\mathbb{Q}$.

False 6) Is $+$ (addition) " " " " " " "

False 7) Is $+/\times$ " " " " $\mathbb{R}\setminus\{0\}$ over $\mathbb{R}$.

False 8) Is $+/\times$ " " " " $\mathbb{R}\setminus\{1\}$ over $\mathbb{R}$.

<u>Vector Space (Linear Space)</u> :-

let $(F, +, \cdot)$ be a field whose elements will be known as scalar. Let V be a non-empty set whose elements will be knowns as vectors. Then $V(F)$ or V over F is a vector space, if

1) There is an internal composition '+' known as vector addition defined over V w.r.t which $(V, +)$ is an abelian gp.

$$\begin{aligned} + : V \times V &\longrightarrow V \\ (u, v) &\longmapsto u+v \end{aligned}$$

2) There is an external composition $\circ$ defined in V over F (known as scalar multiplication) s.t. $\forall\ \alpha \in F$ and $v \in V$.

$\Rightarrow \alpha \circ v \in V$

$$\boxed{\begin{aligned} \circ : F \times V &\longrightarrow V \\ (\alpha, v) &\longmapsto \alpha \circ v \end{aligned}}$$

3) $\forall\ \alpha, \beta \in F$ and $\forall\ v_1, v_2 \in V$ the following conditions must hold.

- (i) $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$

(ii) $(\alpha + \beta) v_1 = \alpha v_1 + \beta v_1$

(iii) $(\alpha\beta) v_1 = \alpha(\beta v_1)$

(iv) $1 \circ v_1 = v_1$, $1 \in F$

Examples of Vector spaces :-

1) If K is a field then it is a vector sp. over it's subfield F.

K(F) is Vector sp.

1) $(K, +)$ — is an abelian gp.

2) $\forall\ \alpha \in F$ and $v \in K$

$\Rightarrow \alpha \circ v = \alpha v \in K \quad (\because F \subseteq K)$

3) $\forall\ \alpha, \beta \in F$ & , $u, v \in K$

i) $\alpha(u + v) = \alpha u + \alpha v$

(ii) $(\alpha + \beta) u = \alpha u + \beta u$

iii) $(\alpha\beta) u = \alpha(\beta u)$

iv) $1 \cdot u = u \quad \forall\ u \in K$.

Subfields of $\mathbb{C}$.

$\mathbb{C}$: $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$ ......

Subfield of $\mathbb{Z}_p$

$\mathbb{Z}_p$: $\mathbb{Z}_p$. (only)

Examples 1. $\mathbb{C}(\mathbb{C})$, $\mathbb{C}(\mathbb{R})$, $\mathbb{C}(\mathbb{Q})$, $\mathbb{C}(\mathbb{Q}(\sqrt{2}))$, ...
$\mathbb{R}(\mathbb{R})$, $\mathbb{R}(\mathbb{Q})$, $\mathbb{Z}_p(\mathbb{Z}_p)$ etc. are Vector space.

2. Let $V = \mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{n\text{-times}}$

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}, \ \forall\, i = 1, 2, \dots, n\}$$

let $u = (x_1, x_2, \dots, x_n)$ and $v = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$

$+$ <u>Vector addition</u> $u + v = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

$\cdot$ <u>Scalar multiplication</u> $\alpha \cdot u = \alpha(x_1, x_2, \dots, x_n)$, $\alpha \in \mathbb{R}$

$$= (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

then $\mathbb{R}^n$ w.r.t. $+$ and $\cdot$ will form vector space over $\mathbb{R}$.

Properties: Let $(V(F), +, \cdot)$ — V. Space.

1. If α is a scalar and 0 is the zero vector then $\alpha \cdot 0 = 0$.

Pf: we can write

$$\underbrace{\alpha 0}_{\in V} = \alpha(0 + 0) \qquad (\because 0 + u = u, \ \forall\, u \in V)$$

$$= \underbrace{\alpha 0}_{\in V} + \underbrace{\alpha 0}_{\in V} \qquad (\because \alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2)$$

Adding $-(\alpha 0)$ on both side, we have

$\Rightarrow \quad O = \alpha O \quad$ ($\because -(\alpha O)$ is the additive inverse of αO in V.)

2. for $0 \in F$ and $u \in V$

$$0u = O \quad (\text{Here, } O \in V)$$

Pf: for any $u \in V$, we can write

$$0u = (0+0)u$$
$$= 0u + 0u$$

$\because$ $0u \in V$ and V has additive inverse of $0u$ i.e. $-(0u)$. So on adding both side.

$$\Rightarrow 0u + (-(0u)) = 0u + \underbrace{0u + (-(0u))}$$
$$\Rightarrow O = 0u + O$$
$$\Rightarrow O = 0u$$

3. Let for $\alpha \in F$ and $u \in V$ if $\alpha u = O$ then either $\alpha = 0$ or $u = O$.

Pf: Let for $\alpha \in F$ and $u \in V$, $\alpha u = O$

if $\alpha = 0$ then we have done

if $\alpha \neq 0$ then α^{-1} exist in F, so

$$\alpha^{-1}(\alpha u) = \alpha^{-1} O$$
$$\Rightarrow (\alpha^{-1}\alpha)u = O$$
$$\Rightarrow 1 \cdot u = O$$
$$\Rightarrow u = O$$

4. If $u \in V$ then $(-1)u = -u$.

Pf:- $O = 0u = (1-1)u = 1u + (-1)u = u + (-1)u$

$$\Rightarrow (-1)u = -u$$

Some more examples of ~~[illegible]~~ Vector space :-

1. The set of real valued continuous functions f on any closed interval $[a, b]$.

$$S = \{ f \mid f : [0,1] \xrightarrow{\text{cont.}} \mathbb{R} \}$$

let $f, g \in S$ and $\alpha \in \mathbb{R}$

Vector addition $(f+g)(x) = f(x) + g(x) \quad \forall\, x \in [0,1]$

Scalar Multiplication $(\alpha \cdot f)(x) = \alpha \cdot f(x)$

Then $(S, +, \cdot)$ will form vector space over $\mathbb{R}$.

2. The set of all polynomials P_n of degree less than or equal to n, i.e.,

$$S = \left\{ P_n \;\middle|\; P_n(x) = \sum_{i=0}^{n} a_i x^i ,\ a_i \in \mathbb{R} \right\}$$

let $P_n, Q_n \in S$ and $\alpha \in \mathbb{R}$.

Vector Addition

$$(P_n + Q_n)(x) = P_n(x) + Q_n(x)$$

$$\sum_{i=0}^{n} (a_i + b_i) x^i = \sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i$$

Scalar Multiplication

$$\alpha \cdot P_n(x) = \alpha \cdot \left(\sum_{i=0}^{n} a_i x^i \right)$$

$$= \sum_{i=0}^{n} (\alpha a_i) x^i$$

Here, $(S, +, \cdot)$ will form vector space over $\mathbb{R}$.

3. The set V of all $m \times n$ matrices.

$$V = \{ A \mid A = (a_{ij})_{m \times n},\ a_{ij} \in R \}$$

$(V, +, \cdot)$ will form vector space over $\mathbb{R}$.

Here $+$ stands for, ~~vector~~ usual matrix addition and $\cdot$ stands for, usual ~~matrix~~ scalar multiplication.

let $A, B \in V$ and $\alpha \in R$, wher $A = (a_{ij})$
$B = (b_{ij})$

$$A + B = (a_{ij})_{m \times n} + (b_{ij})_{m \times n}$$
$$= (a_{ij} + b_{ij})_{m \times n}$$

$$\alpha \cdot A = \alpha (a_{ij})_{m \times n} = (\alpha a_{ij})_{m \times n}$$

Here, 0 vector is the null ~~vector~~ matrix of order $m \times n$.

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Some examples which are not V. space:

1. The set V of all polynomials of degree n.

$$V = \{ P_n \mid P_n\text{-polynomials of degree } n \}$$

$$P_n = x^n + 1, \quad Q_n = -x^n \in V$$

$$P_n + Q_n = x^n + 1 + (-x^n)$$
$$= 1 \quad (\text{polynomial of degree zero})$$
$$\notin V$$

Hence, not closed. Thus not vector space.

2. The set V of all real-value functions of one variable x, defined and continuous on the closed interval $[a, b]$ such that the value of the function at b is some non-zero constant p.

$$V = \{ f \mid f : [a,b] \longrightarrow \mathbb{R} \text{ s.t. } f(b) = p, \; p \neq 0 \}$$

Let $f, g \in V$ then $f(b) = p$ and $g(b) = p$

$$(f+g)(b) = f(b) + g(b)$$
$$= p + p$$
$$\neq p$$

$\therefore f+g \notin V$

$\therefore (V, +, \cdot)$ not vector space.

3. let V be the set of all polynomials, with real coefficient, of degree n, where addition is defined by

$$a+b = ab \quad \forall\, a, b \in V$$

and under usual scalar multiplication.

Sol let $P_n, Q_n \in V$. Then

$$P_n + Q_n = P_n Q_n \notin V$$

which is a polynomial of degree $2n$, which is not in V.

Therefore, V does not form V. space.

4. $$V = \{ (x, y) \mid x, y \in \mathbb{R} \}$$

let $a = (x_1, y_1)$, $b = (x_2, y_2) \in V$

Define addition as

$$a+b = (x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$$

and scalar multiplication as

$$\alpha (x_1, y_1) = \left(\frac{\alpha x_1}{3}, \frac{\alpha y_1}{3} \right)$$

Show that V is not a vector space? which of the properties are not satisfied?

5. $V = \{(x, y) \mid x, y \in \mathbb{R}\}$

let $a = (x_1, y_1)$ and $b = (x_2, y_2) \in V$

Define addition as

$$a + b = (x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

and scalar multiplication as

$$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$$

show that V is not V. space. which of the properties are not satisfied.

Subspace :- Let S be any non-empty subset of a vector space $V(F)$, then S is called subspace of V if S itself forms a vector space w.r.t. same operations defined on V.

Remark: Let $S \subset V$, where V is any vector space, in order to prove that S is subspace of V, we have to prove the following:

1) if $u, v \in S \Rightarrow u+v \in S$
2) the additive identity i.e. the 0 vector is in S.
3) Every element of S has its additive inverse in S. i.e. for each $u \in S \Rightarrow -u \in S$.
4) for each $\alpha \in F$ and for each $u \in S$ $\Rightarrow \alpha u \in S$.

No need to check the remaining property like the commutativity, associativity of vector addition and so on. Since the remaining properties follow directly from the properties of the operations on V.

Thm 1. Let S be any non-empty subset of a vector space V, then S is subspace of V iff the following two conditions hold:

1) if $u, v \in S \Rightarrow u+v \in S$
2) $\alpha u \in S \quad \forall \alpha \in F$ and $\forall u \in S$.

Pf:- ($\Rightarrow$) Let S be the ~~st~~ subspace of V.

By definition, S itself is a vector space.

$\Rightarrow$ S is closed w.r.t. $+$ (vector addition)

$\forall\ u, v \in S \Rightarrow u+v \in S$.

and also $\forall\ \alpha \in F$, $u \in S \Rightarrow \alpha u \in S$.

($\Leftarrow$) Let ① and ② hold in S, where S is subset of V (vector sp.).

$\because$ $u+v \in S$ and $\alpha \cdot u \in S$ $\forall\ u, v \in S, \alpha \in F$

take $\alpha = 0 \Rightarrow 0 \cdot u = 0 \in S$ (which is identity)

$0 = u + (-u) \in S$

$\Rightarrow -u \in S$ (which is the inverse of u)

Other properties are invarient from V to S.

$\Rightarrow$ S is a vector space and hence S is a subspace of V.

Result Prove that conditions ① and ② of Thm 1. can be replaced by the single condition $\alpha u + \beta v \in S$ for all $u, v \in S$ and all scalars α and β.

Pf: ~~($\Leftrightarrow$)~~ ($\Leftarrow$) let $\alpha u + \beta v \in S$ where α, β are scalars, $u, v \in S$

Take $\alpha = 1 = \beta \Rightarrow 1 \cdot u + 1 \cdot v \in S$

$\Rightarrow u + v \in S$

and take $\beta = 0 \Rightarrow \alpha u + 0 \cdot v \in S$

$\Rightarrow \alpha u \in S$ "

($\Rightarrow$) let if $u, v \in S \Rightarrow u+v \in S$
and $\forall \alpha \in \mathbb{F} \Rightarrow \alpha u \in S$

Now as $\alpha u \in S$ $\quad$ α is any scaler and $u \in S$

$\Rightarrow \beta v \in S$ $\quad$ $\beta \in \mathbb{F}$, $v \in S$

$\Rightarrow \alpha u + \beta v \in S$ (by condition ①).

Ex 1. Let $W = \{(x_1, x_2, \ldots, x_n) \mid x_1 = 0\}$
Prove that W is a subspace of V_n, when $V_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$, a vector space of over $\mathbb{R}$.

Pf: let $u = (x_1, x_2, \ldots, x_n)$, $v = (y_1, y_2, \ldots, y_n) \in W$
then ~~such that~~ $x_1 = 0$ and $y_1 = 0$

1) $u + v = (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)$
$= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
$\in W$, since $x_1 + y_1 = 0 + 0 = 0$

2) $0 = (0, 0, 0 \ldots 0) \in W$

3) for $u \in W$, $-u = (-x_1, -x_2, -x_3, \ldots, -x_n)$
$\in W$, since $-x_1 = 0$

4) for $\alpha \in \mathbb{R}$ and $u \in W$
$\alpha u = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \in W$
since $\alpha x_1 = \alpha \cdot 0 = 0$

Hence W is the subspace of V.

Pb 1. Let V be the set of n-tuples $(x_1, x_2, \ldots, x_n)$ in R^n with usual addition and scalar multiplication

Then

i) $W = \{(x_1, x_2, \ldots, x_n) \in V \mid x_1 = 0\}$ is a subspace of V.

ii) $W = \{(x_1, x_2, \ldots, x_n) \in V \mid x_1 \geq 0\}$ is not a subspace of V. (why ?)

iii) $W = \{(x_1, x_2, \ldots, x_n) \in V \mid x_2 = x_1 + 1\}$ is not a subspace of V. (why ?)

Pb 2. Let V be the set of all real polynomials P of degree $\leq m$ with usual addition and scalar multiplication. Then

i) W consisting of all real polynomials of degree $\leq m$ with $P(0) = 0$ is a subspace of V. (Show)

ii) W consisting of all real polynomials of degree $\leq m$ with $P(0) = 1$ is not a subspace of V. (why ?)

iii) W consisting of all polynomials of degree $\leq m$ with real positive coefficient is not a ~~vector~~ subspace of V. ~~[illegible]~~ (why?)

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Pb 3. Let V be the set of all $n \times n$ real square matrices with usual matrix addition and scalar multiplication. Then which one is True/False. (Prove or provide counter examples).

~~i) W consisting~~

i) $W = \{ A \in V \mid A$ – symmetric matrices of order $n\}$ is a subspace of V.

ii) $W = \{ A \in V \mid A$ – skew symmetric matrices$\}$ is a subspace of V.

iii) $W = \{ A \in V \mid A$ – upper triangular matrices$\}$ is a subspace of V.

iv) $W = \{ A \in V \mid A$ – lower triangular " $\}$ is a subspace of V.

v) $W = \{ A \in V \mid A = (a_{ij}),\ a_{ij} \geq 0\ \forall\, i,j \}$ is a subspace of V.

Pb 4. Let V be the set of all $n \times n$ complex matrices with usual matrix addition and scalar multiplication.

i) W consisting of all Hermitian matrices of order n forms a vector space over $\mathbb{R}$. (True/False)

ii) W " " " " ......over $\mathbb{C}$ (T/F).

iii) W " " skew-Hermitian " ...... over $\mathbb{R}$ (T/F)

iv) W " " skew-Hermitian " ...... over $\mathbb{C}$ (T/F).

Linear combination of vectors:

Let V be any vector space and let $v_1, v_2, v_3, \dots, v_n$ be n vectors of V and $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ be any n scalars then the sum $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ is called as linear combination of vectors of $v_1, v_2, \dots, v_n$.

Eg: let $v_1, v_2, v_3 \in \mathbb{R}^3 = V$

$$1, 3, 4 \in \mathbb{R} = \mathbb{F}$$

then

$$\left.\begin{array}{l} v_1 + 3v_2 + 4v_3 \\ 3v_1 + v_2 + 4v_3 \\ 4v_1 + 3v_2 + v_3 \\ \vdots \\ v_1 + v_2 + v_3 \end{array}\right\} \text{These are all linear combinations of } v_1, v_2, v_3.$$

Span :- Let S be any non-empty subset of a vector space V, then the set of all finite linear combination of vectors of S is called span of S.
The span of S is denoted by $[S]$ or $\langle S \rangle$ or $L[S]$.

span of S is also called as linear span of S.

$$[S] = \left\{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \,\middle|\, \begin{array}{l} v_1, v_2, \dots, v_n \in S, \\ \alpha_1, \alpha_2, \dots, \alpha_n \in F \end{array} \right\}$$

Eg:1. Let $S \subset V_3 = \mathbb{R}^3$ such that

$$S = \{ (1,2,1), (0,-1,3), (2,4,5) \}$$

Find span of S ?

Sol: $[S] = \{ \alpha(1,2,1) + \beta(0,-1,3) + \gamma(2,4,5) \mid \alpha, \beta, \gamma \in F \}$

$= \{ (\alpha + 2\gamma, 2\alpha - \beta + 4\gamma, \alpha + 3\beta + 5\gamma) \mid \alpha, \beta, \gamma \in F \}$

Note: The order of $[S]$ is always infinite, where S is any subset of vector space.

Eg2. If $S = \{ (0,1,0), (1,0,0), (0,0,1) \} \subset V_3$ then find $[S]$?

$[S] = \{ \alpha(0,1,0) + \beta(1,0,0) + \gamma(0,0,1) \mid \alpha, \beta, \gamma \in F \}$

$= \{ (\beta, \alpha, \gamma) \mid \alpha, \beta, \gamma \in F \}$

$= \mathbb{R}^3$ if $F = \mathbb{R}$

Here, $\mathbb{R}^3 = \{ (x, y, z) \mid x, y, z \in \mathbb{R} \}$

Thm 2. let $S \neq \phi$ and $S \subset V$ (vector space) then span of S i.e., $[S]$ is always a subspace of vector space V.

Pf: let S be any non-empty subset of a vector space V, then

$$[S] = \left\{ \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \;\middle|\; \begin{array}{l} v_1, v_2, \dots, v_n \in S, \\ \alpha_1, \alpha_2, \dots, \alpha_n \in F \end{array} \right\}$$

Now we have to show that $[S]$ is a subspace of V.

let $u, v \in [S]$

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \quad , \quad \alpha_i \in F$$

$$v = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n \quad , \quad \beta_i \in F$$

$\forall \; i = 1, 2, \dots n$

Now, $u + v = (\alpha_1 v_1 + \dots + \alpha_n v_n) + (\beta_1 v_1 + \dots + \beta_n v_n)$

$$= (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + \dots + (\alpha_n + \beta_n) v_n$$

$$= \gamma_1 v_1 + \gamma_2 v_2 + \dots + \gamma_n v_n$$

$\in [S]$ where $\gamma_i \in F$, $\forall \; i = 1, 2 \dots n$

$\Rightarrow \quad u + v \in [S]$

Next $\alpha \cdot u = \alpha \cdot (\alpha_1 v_1 + \dots + \alpha_n v_n)$

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$= \alpha\alpha_1 v_1 + \alpha\alpha_2 v_2 + \dots + \alpha\alpha_n v_n$

$= \eta_1 v_1 + \eta_2 v_2 + \dots + \eta_n v_n$.

$\in [S]$

$\Rightarrow \alpha.u \in [S]$

Hence by Theorem 1, $[S]$ is a subspace of V.

Q Let $S = \{(1,2,1), (1,1,-1), (4,5,-2)\}$

Determine which of the following vectors are in $[S]$?

a) $(0,0,0)$ (b) $(1,1,0)$ (c) $(2,-1,-8)$

Solⁿ:

$[S] = \{\alpha(1,2,1) + \beta(1,1,-1) + \gamma(4,5,-2) \mid \alpha,\beta,\gamma \in F\}$

$= \{(\alpha+\beta+4\gamma, 2\alpha+\beta+5\gamma, \alpha-\beta-2\gamma) \mid \alpha,\beta,\gamma \in F\}$

a) $(0,0,0) \in [S]$ for $\alpha=\beta=\gamma=0$

b) Let $(1,1,-1) \in [S] \Leftrightarrow (1,1,0) = (\alpha+\beta+4\gamma, 2\alpha+\beta+5\gamma, \alpha-\beta-2\gamma)$

$\Leftrightarrow$
$\alpha+\beta+4\gamma = 1$ — ①
$2\alpha+\beta+5\gamma = 1$ — ②
$\alpha-\beta-2\gamma = 0$ — ③

Adding ① & ③, $2\alpha + 2\gamma = 1 \Rightarrow \boxed{\alpha+\gamma = \frac{1}{2}}$

& subtracting ① from ②, $\boxed{\alpha+\gamma=0}$ No solⁿ.

Hence $(1,1,-1) \notin [S]$

(c) Let $(2,-1,-8) \in [S]$

$\Leftrightarrow (2,-1,-8) = (\alpha+\beta+4\gamma,\ 2\alpha+\beta+5\gamma,\ \alpha-\beta-2\gamma)$

$$\Leftrightarrow \quad \alpha+\beta+4\gamma = 2 \quad \text{—— (1)}$$
$$2\alpha+\beta+5\gamma = -1 \quad \text{—— (2)}$$
$$\alpha-\beta-2\gamma = -8 \quad \text{—— (3)}$$

Using (1)+(3),

$2\alpha+2\gamma = -6$

$\boxed{\alpha+\gamma = -3}$

Using (2)−(1),

$\boxed{\alpha+\gamma = -3}$

Let $\gamma = k$ (scalar) then

$\alpha = -3-k$

from (1), $\beta = 2-(-3-k)-4k$

$= 5-3k$. ~~Hence~~

$(\alpha,\beta,\gamma) \longrightarrow (-3-k,\ 5-3k,\ k)$

Thus, $(2,-1,-8) \in [S]$. you have infinite choices

Ex:- for $k=0 \Rightarrow \alpha=-3,\ \beta=5,\ \gamma=0$

$(2,-1,-8) = -3(1,2,1)+5(1,1,-1)+0(4,5,-2)$

Ex1. Let V be the vector space of all 2×2 real matrices. Show that the sets

i) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

ii) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

Span V.

Ex2: Let V be the vector space of all polynomials of degree ≤ 3. Determine whether or not the set

i) $S = \{ t^3, t^2, t, 1 \}$

ii) $S = \{ t^3, t^2+t, t^3+t+1 \}$

Span V. ?

Sol^n: ii) Let $p(t) = \alpha t^3 + \beta t^2 + \gamma t + \delta$ be any polynomial (element) in V. We need to find (if exist) scalars $\alpha_1, \alpha_2, \alpha_3$ s.t.

$$\alpha t^3 + \beta t^2 + \gamma t + \delta = \alpha_1 t^3 + \alpha_2 (t^2+t) + \alpha_3 (t^3+t+1)$$
$$= (\alpha_1+\alpha_3) t^3 + \alpha_2 t^2 + (\alpha_2+\alpha_3) t + \alpha_3 .$$

$\Rightarrow$ on comparing coefficient of various powers of t, we get $\alpha_1 + \alpha_3 = \alpha$, $\alpha_2 = \beta$, $\alpha_2 + \alpha_3 = \gamma$, $\alpha_3 = \delta$

From the first three eq^n, we get

$$\alpha_1 = \alpha + \beta - \gamma$$
$$\alpha_2 = \beta$$
$$\alpha_3 = \gamma - \beta$$

substituting all of these in the last eq^n, we obtain

$$\gamma - \beta = \delta$$

which may not be true for all elements of V.

If we take $t^3 + 2t^2 + t + 3 \in V$

here $\gamma - \beta \neq \delta$

so, this polynomial can not be written in the linear combination of elements of S.

$\therefore$ S does not span the vector space V.

Thm 3. Let S be any non-empty subset of a vector space V, then $[S]$ is the smallest subspace of V containing S.

Pf: Let $S = \{v_1, v_2, \ldots, v_n\} \subset V$

then $v_i = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 1 \cdot v_i + 0 \cdot v_{i+1} + \cdots + 0 \cdot v_n$

$\Rightarrow v_i \in [S] \quad \forall \; i = 1, 2, \ldots, n$

$\Rightarrow S \subset [S]$

Now, we have to prove that $[S]$ is the <u>smallest</u> <u>subspace</u> of V containing S.

By Thm 2., $[S]$ is always a subspace of S.

Now, let T be any other subspace of V containing S i.e. $S \subset T$.

So, we have to show that $[S] \subset T$

Let $u \in [S]$

$\Rightarrow \quad u = \alpha_1 v_1 + \cdots + \alpha_n v_n$, $\alpha_i \in F$ and $v_1, v_2, \ldots, v_n \in S$

$\because \quad S \subset T$

$\Rightarrow \quad v_1, v_2, \ldots, v_n \in T$ and also T is subspa

$$\Rightarrow \left.\begin{array}{l} \alpha_1 v_1 \in T \\ \alpha_2 v_2 \in T \\ \vdots \\ \alpha_n v_n \in T \end{array}\right\}$$

$\Rightarrow \quad \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in T$

$\Rightarrow \quad u \in T$

$\Rightarrow \quad [S] \subset T$.

Hence $[S]$ is the smallest subspace containing S.

Important results:-

1. Let U and W are subspaces of a vector space V, then

   a) $U \cap W$ is a subspace of V.

   b) $U \cup W$ is a subspace of $V \iff U \subset W$ or $W \subset U$.

2. Let $U_1, U_2, \ldots, U_n$ be n-subspaces of a vector space V. Then

   $$U_1 \cap U_2 \cap \ldots \cap U_n = \bigcap_{i=1}^{n} U_i \text{ is also}$$

   a subspace of V.

3. If S is non-empty subset of V, then $[S]$ is the intersection of all subspaces of V containing S.

Linear dependence & independence of vectors

Let V be a vector space. A finite set $\{v_1, v_2, \ldots, v_n\}$ of the elements of V is said to be linearly dependent if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0.$$

If the above equation is satisfied only for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then the set of vectors is said to be linearly independent.

Pb: Check whether the following set of vectors are L.D or L.I over the field $\mathbb{R}$.

1) $\{(1,0), (0,1)\}$
2) $\{(3,2), (2,3)\}$
3) $\{(1,-1,0), (0,1,-1), (0,0,1)\}$
4) $\{(1,-1,0), (0,1,-1), (0,2,1), (1,0,3)\}$
5) $\{(1,0,1), (1,1,0), (1,-1,1), (1,2,-3)\}$
6) $\{(1,0,0), (0,1,0), (0,0,1)\}$

Res 4. Let V be any vector space then

① $v \in V$ is L.D $\Leftrightarrow$ $v = 0$

② $v_1, v_2 \in V$ are L.D $\Leftrightarrow$ v_1 and v_2 are colinear i.e., one is scalar multiple of other.

③ $v_1, v_2, v_3 \in V$ are L.D $\Leftrightarrow$ they are coplanar i.e., one of them is the linear combination of other two.

④ If S is any subset of vector space V such that $0 \in S$, then S is L.D.

Res 5 Let V be a vector space and S be any non-empty subset of V. Then

1) If S is L.I, then every subset of S is L.I.

2) If S is L.D, then every ~~sub~~ superset of S is L.D.

Basis & Dimensions

Basis: let V be any vector space, then $B \subset V$ is said to form Basis of V, if the following two conditions are satisfied.

1) B is L.I
2) B spans V or B generates V or $[B] = V$.

Ex1. Let $B_1 = \{(1,0),(0,1)\} \subset \mathbb{R}^2$. Is B_1 is a basis of $\mathbb{R}^2$.

1) B_1 is L.I: for $\alpha, \beta \in F = \mathbb{R}$

let $\alpha(1,0) + \beta(0,1) = (0,0)$

$\Rightarrow (\alpha, \beta) = (0,0)$

$\Rightarrow \alpha = 0 = \beta$. Hence B_1 is L.I

2) $[B_1] = \{\alpha(1,0) + \beta(0,1) \mid \alpha, \beta \in R\}$

$= \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}$

$= \mathbb{R}^2$

$\therefore B_1$ is a basis of $\mathbb{R}^2$. Also, B_1 is called standard Basis.

Ex 2. Is $B_2 = \{(-1,0),(2,1)\} \subset \mathbb{R}^2$ is a basis of $\mathbb{R}^2$.

1) B_2 is L.I: Let $\alpha, \beta \in \mathbb{R}$ s.t.

$\alpha(-1,0) + \beta(2,1) = (0,0)$

$\Rightarrow (-\alpha + 2\beta, \beta) = (0,0)$

$\beta = 0$, $\alpha = 0$. B_2 is L.I.

2) $[B_2] = \{\alpha(-1,0) + \beta(2,1) \mid \alpha, \beta \in \mathbb{R}\}$
$= \{(-\alpha + 2\beta, \beta) \mid \alpha, \beta \in \mathbb{R}\} = \mathbb{R}^2$.

Here $[B_2] = R^2$, Since every element of $\mathbb{R}^2$ can be written in the linear combination of $(-1,0)$ and $(2,1)$. Thus B_2 is a basis of $\mathbb{R}^2$.

Here, we observe that B_1 and B_2 are two basis of $\mathbb{R}^2$.

Note: Basis of a vector space is not unique.

Standard Basis of $V_n = \mathbb{R}^n$.

Let $e_1 = (1,0,0,\dots,0)$
$e_2 = (0,1,0,\dots,0)$
$\vdots$
$e_n = (0,0,0,\dots,0,1)$

Now, take a set $S = \{e_1, e_2, \dots, e_n\}$

As $S \subset V_n$

1) It can be easily varified that S is L.I.

2) $[S] = V_n$

Thus, S is called as standard basis of V_n.

Ex: 1) Standard basis of V_3 is $\{(1,0,0),(0,1,0),(0,0,1)\}$

2) " " " V_4 is $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$

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Standard basis of $M_{m\times n}(\mathbb{R})$

- $E_{11} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$, $E_{12} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$, ...

$$E_{ij} = \begin{cases} 1 & , \text{ at } i^{th}\text{-row and } j^{th}\text{-column} \\ 0 & , \text{ else.} \end{cases}$$

Coordinate Vector:- let $B = \{v_1, v_2, \ldots, v_n\}$ be a basis of vector space V, then $v \in V$ is a vector that can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

then $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is called as the coordinate vector of V w.r.t. basis B.

Pb Find coordinate vectors of $(1,4,3)$ w.r.t.

1) standard basis of V_3.
2) $B = \{(-1,2,0), (0,1,0), (1,-1,1)\}$ basis of V_3.

Sol(2) $(1,4,3) = \alpha(-1,2,0) + \beta(0,1,0) + \gamma(1,-1,1)$

$(1,4,3) = (-\alpha+\gamma,\ 2\alpha+\beta-\gamma,\ \gamma)$

$\Rightarrow \boxed{\gamma = 3}$, $-\alpha+\gamma = 1$ and $2\alpha+\beta-\gamma = 4$

$\rightarrow \boxed{\alpha = 2}$ $\qquad \boxed{\beta = 3}$

Thus, the $(2, 3, 3)$ is the coordinate vector.

Dimension :- Let V be a vector space, then the number of elements in the basis of V is known as dimension of V.

Note : Dimension of a vector space is unique.

Remark : A vector space may have infinite dim., finite dim. or zero dimension.

- A vector space with infinite dimension is called as infinite dimensional vector space.

  Eg: $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots\cdots \infty$

- A vector sp. with finite dim. is called as finite dimensional vector space.

  Eg: $V_1, V_2, V_3, \ldots, V_n$

  $V_1 \rightarrow$ one dim.; $V_2 \rightarrow$ two dim.; $V_n \rightarrow$ n-dim.

Example: 1) Dim. of $\mathbb{R}^n$ over $\mathbb{R} = n$

2) Dim of $\mathbb{C}^n$ over $\mathbb{C} = n$

3) Dim. of $\mathbb{C}^n$ over $\mathbb{R} = 2n$

4) Dim. of $M_{n\times n}(\mathbb{R})$ over $\mathbb{R} = n$

- $\{0\}$ is an example of zero dim. vector space of $\mathbb{R}$.

  Here, Basis $= \{\ \} = \phi$.

Thm 4. Let V be a vector space which is spanned by a finite set of vectors $\{v_1, v_2, \dots, v_n\}$. ~~Then~~ If $\{\omega_1, \omega_2, \dots, \omega_m\}$ is L.I set of vectors of V, then $m \le n$.

OR [The largest L.I subset of a vector space V is its basis itself.]

Cor 1. If V is a finite-dimensional vector space, then any two bases of V have the same (finite) number of elements.

Cor 2. Let V be a finite-dimensional vector space and let $n = \dim V$. Then

a) any subset of V which contains more than n vectors is linearly dependent.

b) no subset of V which contains less than n vectors can span V.

OR [Basis is the ~~~~ largest L.I. subset of V and the smallest ~~spanning~~ set which can span V.]

~~Addition of sets~~ **Additions of sets in a V. sp.**

let A and B be two subset of V sp. V. Then sum of A and B written as $A+B$, is the set of all vectors of the form $u+v$, $u \in A$, $v \in B$ i.e;

$$A+B = \{u+v \mid u \in A, v \in B\}$$

Ex:1. let $A = \{(1,2), (0,1)\}$ and $B = \{(1,1), (-1,2), (2,5)\}$. Then $A+B = ?$ and $A \cup B = ?$

sol^n: $A+B = \{(1,2)+(1,1), (1,2)+(-1,2), (1,2)+(2,5), (0,1)+(1,1), (0,1)+(-1,2), (0,1)+(2,5)\}$

$= \{(1,3), (0,4), (3,7), (1,2), (-1,3), (2,6)\}$

$A \cup B = \{(1,2), (0,1), (1,1), (-1,2), (2,5)\}$

Difference bet^n $A+B$ and $A \cup B$

let $A = \{(x,0) \mid x \in \mathbb{R}\}$ and $B = \{(0,y) \mid y \in \mathbb{R}\}$

$A+B = \{(x,0)+(0,y) \mid x,y \in \mathbb{R}\} = \{(x,y) \mid x,y \in \mathbb{R}\} = \mathbb{R}^2$.

$A \cup B = \{(x,0) \cup (0,y) \mid (x,0) \in A, (0,y) \in B\} = \{(x,0), (0,y) \mid x,y \in \mathbb{R}\}$

i.e; if A is x-axis and B is y-axis then $A+B$ is generating whole plane while $A \cup B$ is just x and y-axis.

Sum / Direct sum of two subspaces of a V.Sp.

Thm 2 If A and B are two subspaces of vector space V, then $A+B$ is subspace of V and also

$$A+B = [A \cup B] \text{ i.e. } span(A \cup B)$$

Pf: To prove $A+B$ subspace of V

$\because$ A, B subspace of V

$\Rightarrow$ $0 \in A$ and $0 \in B$

$\Rightarrow$ $0 = 0 + 0 \in A+B$

$\Rightarrow$ $A+B \neq \phi$

$\boxed{A+B = \{ a+b \mid a \in A, b \in B \}}$

let $a+b \in A+B$ and $c+d \in A+B$

also $\alpha, \beta \in F$.

then $\alpha(a+b) + \beta(c+d) = \underbrace{(\alpha a + \beta c)}_{\in A} + \underbrace{(\alpha b + \beta d)}_{\in B}$

$\in A+B$

$\Rightarrow$ $A+B$ is subspace of V.

To prove $A+B = [A \cup B]$

Since $A+B \subseteq [A \cup B]$ as every element of $A+B$ will be written as finite linear combination of elements of $A \cup B$.

Now, we will prove that $[A \cup B] \subseteq A+B$.

Let $v \in [A \cup B] = \left\{ \sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j \;\middle|\; a_i \in A,\ b_j \in B,\ \alpha_i, \beta_j \in F \right\}$

So,

$$v = \underbrace{\sum_{i=1}^{n} \alpha_i a_i}_{\in A} + \underbrace{\sum_{j=1}^{m} \beta_j b_j}_{\in B}$$

$$\in A+B$$

$$\Rightarrow [A \cup B] \subseteq A+B.$$

Hence, $A+B = [A \cup B]$ //

$[A \cup B] = \left\{ \sum_{i=1}^{n} \alpha_i v_i \;\middle|\; v_i \in A \cup B \right\}$

If $v \in [A \cup B]$

$v = \sum_{i=1}^{n} \alpha_i v_i$, $v_i \in A \cup B$

$= \underbrace{\sum \beta_i a_i}_{\in A} + \underbrace{\sum \gamma_i b_i}_{\in B}$,

$\in$ ~~A+B~~ $A+B$

Direct sum:-

We have seen that if A and B are subspaces of Vector space V then the sum A+B is also a subspace of V.

If in addition $A \cap B = \{0\}$, then the sum A+B is called direct sum and written as $A \oplus B$.

- The advantage of direct sum is that any element z of $A \oplus B$ will be uniquely represented as $\underbrace{z = a+b}$, $a \in A$, $b \in B$

This representation is unique for $z \in A \oplus B$.

Ex 1 let vector space $V = \mathbb{R}^3$

let U be xy-plane and W be yz plane.

then $U+W$ is ?

$U = \{(a,b,0) \mid a,b \in \mathbb{R}\}$

$W = \{(0,b,c) \mid b,c \in \mathbb{R}\}$

$U+W = \{(a,b,c) \mid a,b,c \in \mathbb{R}\}$

$= \mathbb{R}^3$

But $\mathbb{R}^3$ will not be direct sum of U & W.

Since $U \cap W = \{(0,b,0) \mid b \in \mathbb{R}\}$

$\neq \{(0,0,0)\}$

Ex 2 If $U = \{(a,b,0) \mid a,b \in \mathbb{R}\} = xy$-plane

and $W = \{(0,0,c) \mid c \in \mathbb{R}\} = z$-axis.

Then $\mathbb{R}^3 = U \oplus W$ as $U \cap W = \{(0,0,0)\}$

Thm 6 In an n-dimensional vector space, any set of n-linearly independent vectors is a basis.

Pf. Let V be an n-dim. Vector sp.

Then every basis of V contains n elements.

Let $B = \{v_1, v_2, \ldots, v_n\}$ be an L.I subset of V.

let $v \in V$, then the set

$$B' = \{v_1, v_2, \ldots, v_n, v\} \text{ is L.D}$$

$\Rightarrow$ One of vectors of B' can be written as the linear combination of other vectors.

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \quad , \alpha_i \in F$$

$\Rightarrow \quad v \in [B]$

$\Rightarrow \quad V \subseteq [B]$ ($\because v$ is arbitrary element of V)

But $[B] \subseteq V$ (always)

$\Rightarrow \quad [B] = V$

Hence, B is a basis.

Ex Prove that set $\{(1,1,1), (1,-1,1), (0,1,1)\}$ is a basis of V_3.

Pb1. Which of the following subsets form a basis for V_3?
In case, if S is not a basis for V_3, determine a basis for $[S]$?

1) $S_1 = \{(1,2,3), (3,1,0), (-2,1,3)\}$

2) $S_2 = \{(1,1,1), (1,2,3), (-1,0,1)\}$

Sol 1) Let $\alpha, \beta, \gamma \in \mathbb{R}$ s.t.

$$\alpha(1,2,3) + \beta(3,1,0) + \gamma(-2,1,3) = (0,0,0)$$

$$\left.\begin{aligned} \alpha + 3\beta - 2\gamma &= 0 \\ 2\alpha + \beta + \gamma &= 0 \\ 3\alpha + 3\gamma &= 0 \end{aligned}\right\} \qquad \begin{aligned} \beta &= \gamma \\ \alpha &= -\gamma \end{aligned}$$

Thus, S_1 is L.D. Hence, S_1 is not a basis of V_3.

Then the basis of $[S_1]$ is

$$B = \{(1,2,3), (3,1,0)\}.$$

Since $(-2,1,3) = \alpha(1,2,3) + \beta(3,1,0)$ for
$\alpha = 1$, $\beta = -1$

ii) let $\alpha, \beta, \gamma \in \mathbb{R}$ s.t.

$$\alpha(1,1,1) + \beta(1,2,3) + \gamma(-1,0,1) = (0,0,0)$$

$$\left.\begin{aligned} \alpha + \beta - \gamma &= 0 \\ \alpha + 2\beta &= 0 \\ \alpha + 3\beta + \gamma &= 0 \end{aligned}\right\} \qquad \begin{aligned} \alpha &= 2\gamma \\ \beta &= -\gamma \end{aligned}$$

Thus, S_2 is L.D. Hence S_2 is not a basis of V_3.

Then the basis of $[S_2]$ is

$$B = \{(1,1,1), (1,2,3)\}$$

Since, $(-1,0,1) = -2(1,1,1) + 1(1,2,3)$.

Pb 2. Extend the set $\{(3,-1,2)\}$ to two different basis of V_3.

Sol: Given $S = \{(3,-1,2)\}$

$$\therefore [S] = \{\alpha(3,-1,2) \mid \alpha \in \mathbb{R}\}$$

$$= \{(3\alpha, -\alpha, 2\alpha) \mid \alpha \in \mathbb{R}\}$$

$$\because (0,0,1) \notin [S]$$

Hence $S_1 = \{(3,-1,2), (0,0,1)\}$ is L.I.

Now, $[S_1] = \{\alpha(3,-1,2) + \beta(0,0,1) \mid \alpha, \beta \in \mathbb{R}\}$

$$= \{(3\alpha, -\alpha, 2\alpha + \beta) \mid \alpha, \beta \in \mathbb{R}\}$$

$(0,1,0) \notin [S_1]$

$(-1,0,0) \notin [S_1]$

Then

$S_2 = \{(3,-1,2), (0,0,1), (0,1,0)\}$ is L.I and hence forms a basis for V_3.

Smly, $S_3 = \{(3,-1,2), (0,0,1), (-1,0,0)\}$ is L.I and hence forms another basis for V_3.

(Extension-Theorem)

Thm 7: Let the set $\{v_1, v_2, \ldots, v_k\}$ be a L.I subset of an n-dimensional vector space V, then we can find vectors $v_{k+1}, \ldots, v_n$ in V such that the set $\{v_1, v_2, \ldots, v_k, \ldots, v_n\}$ is a basis for V.

Pf: Given $S = \{v_1, v_2, \ldots, v_k\}$ is L.I subset of V_n.

If $k = n$, then S is a basis of V_n (by Thm 6.)

If $k > n$, not possible ($\because$ S becomes L.D).

For $k < n$, $S = \{v_1, v_2, \ldots, v_k\}$ is L.I but $[S] \neq V$.

let $v_{k+1} \in V$ such that $v_{k+1} \notin [S]$

then the set $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$ is again L.I.

Take another $v_{k+2} \in V$ such that

$v_{k+2} \notin [v_1, v_2, \ldots, v_k, v_{k+1}]$

then set $\{v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}\}$ is L.I

proceeding Inductively, we get

set $\{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ as L.I subset of V_n. Hence forms a basis for V_n.

(Dimension Theorem)

Theorem 3.8: If U and W are the subspaces of finite dimensional vector space V, then

$$\boxed{\dim(U+W) = \dim U + \dim W - \dim(U \cap W)}$$

Pf: As we know if U and W are subspace of V then $U \cap W$ is subspace of both U and W.

let $\dim(U) = m$, $\dim W = n$ and $\dim(U \cap W) = r$.

Suppose $S = \{v_1, v_2, \ldots, v_r\}$ be the basis of $U \cap W$.

By extension theorem, S may be extended to the basis of U, W. ~~and~~

Let
$$B_U = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_{m-r}\}$$
and $B_W = \{v_1, v_2, \ldots, v_r, \omega_1, \omega_2, \ldots, \omega_{n-r}\}$

~~and $B = \{v_1, v_2,$~~

Let $B = \{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_{m-r}, \omega_1, \omega_2, \ldots, \omega_{n-r}\}$

Here B has exactly $(m+n-r)$ element. And hence theorem is proved if we show B is a basis of $U+W$.

To prove B is basis of $U+W$, we have to show

a) B is L.I in $U+W$

b) B span $U+W$ i.e. $[B] = U+W$.

<u>B - L.I :-</u>

Let $a_1 v_1 + \cdots + a_r v_r + b_1 u_1 + \cdots + b_{m-r} u_{m-r} + c_1 \omega_1 + \cdots + c_{n-r}\omega_{n-r} = 0$

where a_i, b_i, c_i are scalars.

$$\Rightarrow \sum_{i=1}^{r} a_i v_i + \sum_{j=1}^{m-r} b_j u_j + \sum_{k=1}^{n-r} c_k \omega_k = 0$$

$$\Rightarrow \underbrace{\sum a_i v_i + \sum b_j u_j}_{\in U} = \overbrace{-\sum c_k \omega_k}^{\in W} \quad \text{——} (1)$$
$$= v \text{ (say)}$$

Since LHS of ① is in $U \Rightarrow v \in U$
$\therefore$ RHS of ① is in $W \Rightarrow v \in W$

Combinedly $v \in U \cap W$

$\therefore \quad v = d_1 v_1 + d_2 v_2 + \cdots + d_r v_k$

$\Rightarrow \quad -\sum c_k w_k = \sum d_i v_i$

$\Rightarrow \quad \sum d_i v_i + \sum c_k w_k = 0$

As $\{v_i, w_k\}$ "B_W" are basis of W.

Therefore all scalars $d_i = c_k = 0$, $i = 1, \ldots r$, $k = 1, \ldots n-r$

Thus eqⁿ ① becomes

$$\sum a_i v_i + \sum b_j u_j = 0$$

As again $\{v_i, u_j\}$ "B_U" are L.I as basis of U.

$\Rightarrow \quad a_i = 0, \; b_j = 0 \quad \forall \; i = 1, \ldots, r$, $j = 1, 2, \ldots, m-r$.

Thus all $a_i, b_j, c_k = 0 \Rightarrow$ Elements of B are L.I.

<u>B-span $(U+W)$</u> let $z \in U+W$ then

$z = u + w, \quad u \in U, \; w \in W$

$= \left(\sum \alpha_i v_i + \sum \beta_j u_j\right) + \left(\sum \alpha_i' v_i + \sum \gamma_k w_k\right)$

$= \sum (\alpha_i + \alpha_i') v_i + \sum \beta_j u_j + \sum \gamma_k w_k$

$\Rightarrow \quad U+W \subseteq [B]$ and $[B] \subseteq U+W$ (always)

$\Rightarrow \quad [B] = U+W$.

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Pb1. What are the dim. of the following subspaces of $\mathbb{R}^3(\mathbb{R})$?

1) $A_1 = \{(x,y,z) \mid x-y=0,\ y+z=0\}$

2) $A_2 = \{(x,y,x) \mid x,y \in \mathbb{R}\}$

3) $A_3 = \{(x,y,z) \mid 7x+9y+4z=0\}$

4) $A_4 = \{(0,y,z) \mid y,z \in \mathbb{R}\}$

Sol 1) $x-y=0 \Rightarrow x=y$

$y+z=0 \Rightarrow z=-y$

so, $A_1 = \{(y,y,-y) \mid y \in \mathbb{R}\}$

Basis $= \{(1,1,-1)\}$

dim $= 1$

2) $A_2 = \{(x,y,x) \mid x,y \in \mathbb{R}\}$

Basis $= \{(1,0,1), (0,1,0)\}$

dim $= 2$

3) $A_3 = \{(x,y,z) \mid 7x+9y+4z=0\}$

$= \left\{\left(x, y, -\left(\frac{7x+9y}{4}\right)\right) \mid x,y \in \mathbb{R}\right\}$

Basis $= \{(1,0,-\frac{7}{4}), (0,1,-\frac{9}{4})\}$

dim $= 2$.

4)

Basis $= \{(0,1,0)\ (0,0,1)\}$

dim $= 2$

P62. Consider vector space $M_n(\mathbb{R})$. What are the dimension of its following subspaces?

1) $A_1 = \{ A \in M_n(\mathbb{R}) \mid A' = A \}$

2) $A_2 = \{ A \in M_n(\mathbb{R}) \mid A' = -A \}$

3) $A_3 = \{ A \in M_n(\mathbb{R}) \mid trace(A) = 0 \}$

where, $trace(A) = \sum_{i=1}^{n} a_{ii}$

4) $A_4 = \{ A \in M_n(\mathbb{R}) \mid \sum_{j=1}^{n} a_{ij} = 0 \quad \forall i \}$

5) $A_5 = \{ A \in M_n(\mathbb{R}) \mid \sum_{i=1}^{n} a_{ij} = 0 \quad \forall j \}$

6) $A_6 = \{ A \in M_n(\mathbb{R}) \mid \sum_{i=1}^{n} a_{ij} = 0 \ \forall j \text{ and } \sum_{j=1}^{n} a_{ij} = 0 \ \forall i \}$

7) $A_7 = \{ A \in M_n(\mathbb{R}) \mid A' = A \text{ and } tr(A) = 0 \}$

Ans

$\dim M_n(\mathbb{R}) = n^2$

1) $\dim(A_1) = \frac{n(n+1)}{2}$

2) $\dim A_2 = \frac{n(n-1)}{2}$

3) $\dim A_3 = n^2 - 1$

4) $\dim A_4 = n^2 - n$

5) $\dim A_5 = n^2 - n$

6) $\dim A_6 = (n-1)^2$

7) $\dim A_7 = \frac{n(n+1)}{2} - 1$

Pb3. Consider Vector space $M_n(\mathbb{R})$ and it's subspaces

$$W_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a+b=0 \;,\; c+d=0 \right\}$$

$$W_2 = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

then find

1) $\dim W_1$

2) $\dim W_2$

3) $\dim (W_1 \cap W_2)$

4) $\dim (W_1 + W_2)$

5) Give a basis of $(W_1 + W_2)$.

Sol^n:

$$\dim (M_2(\mathbb{R})) = 4$$

1) $\dim W_1 = 4 - 2 = 2$

2) $\dim W_2 = 1$

3) $$W_1 \cap W_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{array}{l} a+b=0 \;,\; c+d=0 \\ b=0,\; c=0 \text{ and } a=d \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$\dim (W_1 \cap W_2) = 0$.

(Here Basis of $(W_1 \cap W_2) = \phi$)

4) $\dim (W_1 + W_2) = \dim W_1 + \dim W_2 - \dim (W_1 \cap W_2)$

$= 2 + 1 - 0 = 3$

5) $$B_{W_1} = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} , \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

$$B_{W_2} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

So, $$B_{W_1+W_2} = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Pb 4 Consider a vector space $P_{10}(\mathbb{R})$ and it's subspaces.

$$P_{10}(\mathbb{R}) = \left\{ \sum_{i=0}^{10} a_i x^i = p(x) \;\middle|\; a_i \in \mathbb{R} \right\}$$

↳ (polynomial in x with degree upto 10.)

Find dim of the following

1) $W_1 = \{ P(x) \mid P(0) = 0 \}$

2) $W_2 = \{ P(x) \in P_{10}(\mathbb{R}) \mid P(1) = 0 \}$

3) $W_3 = \{ P(x) \in P_{10}(\mathbb{R}) \mid P(1) = 0,\ P(2) = 0,\ P(3) = 0 \}$

4) $W_4 = \{ P(x) \in P_{10}(\mathbb{R}) \mid \frac{d}{dx}(P(x)) = 0 \}$

Sol^n: $\dim P_{10}(\mathbb{R}) = 10 + 1 = 11$

Basis $= \{ 1, x, x^2, \dots, x^{10} \}$

1) If $P(0) = 0 \Rightarrow a_0 = 0$

So, $P(x) = a_1 x + a_2 x^2 + \dots + a_{10} x^{10}$

$\text{Basic}(W_1) = \{ x, x^2, \dots, x^{10} \}$

$\dim(W_1) = 10$

2) $\dim W_2 = 11 - 1 = 10$ ← No. of L.I restriction.

3) $\dim W_3 = 11 - 3 = 8$

4) If $\frac{d}{dx}(P(x)) = 0 \Rightarrow P(x) =$ constant poly. So, $\dim W_4 = 1$

Linear Transformations (L.T.)

Let U and V be any two vector spaces over the same field $\mathbb{F}$. Then the mapping $T: U(\mathbb{F}) \longrightarrow V(\mathbb{F})$ is said to be linear transformation if the following two conditions are satisfied.

1) $T(u+v) = T(u) + T(v) \quad \forall\, u, v \in U$

2) $T(\alpha u) = \alpha T(u) \quad \forall\, \alpha \in \mathbb{F},\ u \in U$.

Ex1. Show that the function $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x,y,z) = (x,y,0)$ is a L.T.

~~proof~~: let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{R}^3$

Then

1) $T(u+v) = T(u_1+v_1, u_2+v_2, u_3+v_3)$
$= (u_1+v_1, u_2+v_2, 0)$
$= (u_1, u_2, 0) + (v_1, v_2, 0)$
$= T(u) + T(v)$

2) $T(\alpha u) = T(\alpha u_1, \alpha u_2, \alpha u_3)$
$= (\alpha u_1, \alpha u_2, 0)$
$= \alpha(u_1, u_2, 0)$
$= \alpha T(u)$

Hence, T is L.T. //

Ex 2 Show that the function $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T(x,y,z) = (x+y, x+z+1)$ is not a L.T.

Sol:- Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathbb{R}^3$

$T(u+v) = T(u_1+v_1, u_2+v_2, u_3+v_3)$
$= (u_1+u_2+v_1+v_2, u_1+u_3+v_1+v_3+1)$

and

$T(u) + T(v) = T(u_1, u_2, u_3) + T(v_1, v_2, v_3)$
$= (u_1+u_2, u_1+u_3+1) + (v_1+v_2, v_1+v_3+1)$
$= (u_1+u_2+v_1+v_2, u_1+u_3+v_1+v_3+2)$

Thus,
$T(u)+T(v) \neq T(u+v)$. Hence not L.T.

~~Properties~~

Linear operator :- A linear transformation is called a linear operator if the vector space on both sides is same.

Identity Linear Transformation

Let $I: U \longrightarrow U$ be a L.T defined by
$I(u) = u \quad \forall\, u \in U$,
then such map is called Identity Linear Transf.

* Identity map is always bijective map.

<u>Zero Linear Transformation</u> :- We denote zero linear Transf^n by $O: U \to V$ and defined it as $O(u) = O_V \quad \forall\ u \in U$.

<u>Range of a Linear transformation</u> :-

Let $T: U \to V$ be a L.T., then the set:

$R(T) = \{ T(u) \mid u \in U \}$ is called range of T.

Pb Define $T: V_3 \to V_3$, defined by

$T(x_1, x_2, x_3) = (0, x_2, x_3)$. Find $R(T) = ?$

Sol $R(T) = \{ T(x_1, x_2, x_3) \mid (x_1, x_2, x_3) \in V_3 \}$

$= \{ (0, x_2, x_3) \mid (x_1, x_2, x_3) \in V_3 \}$

$=$ x_2x_3 - plane

Pb Which of the following maps are linear?

a) $T: V_2 \to V_3$ defined by $T(x,y) = (1, x, y)$

b) $T: V_1 \to V_3$ defined by $T(x) = (x, 2x, 3x)$

c) $T: V_1 \to V_3$ " " $T(x) = (x, x^2, x^3)$

d) $T: V_2 \to V_2$ " " $T(x,y) = (2x+3y, 3x-4y)$

<u>Properties</u> :-

1. Let $T: U \to V$ be a L.T, then

1) $T(O_U) = O_V$

2) $T(-u) = -T(u) \quad \forall\ u \in U$

3) $T(u_1 - u_2) = T(u_1) - T(u_2) \quad \forall\ u_1, u_2 \in U$

4) $T(\alpha_1 u_1 + \cdots + \alpha_n u_n) = \alpha_1 T(u_1) + \cdots + \alpha_n T(u_n)$

$\forall\ u_i \in U, \ \alpha_i \in F$.

Pf: Since $T: U \to V$ is a linear map, then

$T(u+v) = T(u) + T(v) \quad \forall\ u, v \in U$

and $T(\alpha u) = \alpha T(u) \quad \forall\ \alpha \in F, \ u \in U$

1) Choose $v = -u$

Then $T(u-u) = T(u) + T(-u)$

$\Rightarrow T(O_U) = T(u) - T(u) \quad (\because T(\alpha u) = \alpha T(u),\ \alpha = -1)$

$\Rightarrow T(O_U) = O_V$

2) Choose $\alpha = -1$ in $T(\alpha u) = \alpha T(u)$

$T(-1 \cdot u) = -1\, T(u)$

$T(-u) = -T(u)$

3) H/A

4) H/A

Thm 1. (Statement) A Linear transformation T is completely determined by the values on the elements of a basis.

Precisely, if $B = \{u_1, u_2, \dots, u_n\}$ is a basis for U and $v_1, v_2, \dots, v_n$ be n-vectors (not necessarily distinct) in V, there exists a unique linear transformation $T: U \to V$ such that

$$T(u_i) = v_i \quad \text{for } i = 1, 2, \dots n.$$

Case 1. To find T when image of a basis set is given:-

Pb1. let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a Linear transformⁿ such that $T(1,2) = (3,0)$ and $T(2,1) = (1,2)$. Find T=?

sol let $u = (a_1, a_2) \in \mathbb{R}^2$ be any vector.

Since, $\{u_1 = (1,2),\ u_2 = (2,1)\}$ is a basis of $\mathbb{R}^2$ as this set is L.I (by Thm 6).

Then any vector $u \in \mathbb{R}^2$ can be written as

$$u = (a_1, a_2) = \alpha u_1 + \beta u_2, \quad \alpha, \beta \in R \qquad \text{— (1)}$$

$$(a_1, a_2) = (\alpha + 2\beta,\ 2\alpha + \beta)$$

$$x = \alpha + 2\beta, \quad y = 2\alpha + \beta$$

$$\alpha = \frac{2a_2 - a_1}{3}, \quad \beta = \frac{2a_1 - a_2}{3}$$

from (1),

$$u = (a_1, a_2) = \left(\frac{2a_2 - a_1}{3}\right)(1,2) + \left(\frac{2a_1 - a_2}{3}\right)(2,1)$$

Now, operating T on both sides,

$$T(u) = \left(\frac{2a_2 - a_1}{3}\right) \cdot T(1,2) + \left(\frac{2a_1 - a_2}{3}\right) T(2,1)$$

$$= \left(\frac{2a_2 - a_1}{3}\right)(3,0) + \left(\frac{2a_1 - a_2}{3}\right)(1,2)$$

$$T(a_1, a_2) = \left(\frac{5a_2 - a_1}{3},\ \frac{4a_1 - 2a_2}{3}\right) \quad \text{Ans.}$$

Pb2. let $T: V_2 \to V_2$ be a L.T., such that $T(2,1) = (2,1)$ and $T(1,2) = (4,2)$

Solⁿ Since $\{(2,1), (1,2)\}$ is L.I and hence is a basis of V_2, then any $(x,y) \in V_2$ can be written as

$$(x,y) = \alpha(2,1) + \beta(1,2) \qquad \text{— (1)}$$

$$= (2\alpha + \beta,\ \alpha + 2\beta)$$

$$\alpha = \frac{2x - y}{3}, \quad \beta = \frac{2y - x}{3}$$

from ①

$$(x,y) = \frac{2x-y}{3}(2,1) + \frac{2y-x}{3}(1,2)$$

Operating Linear transformation T on both sides,

$$T(x,y) = \frac{2x-y}{3}T(2,1) + \frac{2y-x}{3}T(1,2)$$

$$= \frac{2x-y}{3}(2,1) + \frac{2y-x}{3}(4,2)$$

$$= (2y, y)$$

Pb 3 Let $u_1 = (1,1,-1)$, $u_2 = (4,1,1)$, $u_3 = (1,-1,2)$ be a basis of $\mathbb{R}^3$. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a Linear transformation such that

$T(u_1) = (1,0)$, $T(u_2) = (0,1)$ and $T(u_3) = (1,1)$.

Find T?

Sol^n Since $\{u_1, u_2, u_3\}$ is a basis of $\mathbb{R}^3$, so any element $u = (x,y,z) \in \mathbb{R}^3$ can be written as

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \quad \alpha_i \in \mathbb{R} \quad \text{—— ①}$$

from ①, we have

$$\left.\begin{aligned} \alpha_1 + 4\alpha_2 + \alpha_3 &= x \\ \alpha_1 + \alpha_2 - \alpha_3 &= y \\ -\alpha_1 + \alpha_2 + 2\alpha_3 &= z \end{aligned}\right\} \text{—— ②}$$

Solving ②, we get

$$\left.\begin{aligned} \alpha_1 &= 3x - 7y - 5z \\ \alpha_2 &= -x - 3y + 2z \\ \alpha_3 &= 2x - 5y - 3z \end{aligned}\right\} \text{—— ③}$$

Now, by using ③ in ① and applying T on both sides, we have

$$T(u) = T(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3)$$

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \alpha_3 T(u_3)$$

$$= (3x-7y-5z)(1,0) + (-x-3y+2z)(0,1) + (2x-5y-3z)(1,1)$$

$$T(x,y,z) = (5x-12y-8z,\ x-2y-z) \quad \text{Ans.}$$

Algorithm:- To find T when image of a basis set is given:-

Given: let $T: U \to V$ be a Linear transf^n A basis $\{u_1, u_2, \dots, u_n\}$ of U is given and $T(u_i) = v_i$, $1 \le i \le n$ are given.

Working step:

S1 - Take any arbitrary vector $u \in U$ and let

$$u = \sum_{i=1}^{n} x_i u_i \quad , \quad x_i \in F \qquad \text{①}$$

S2 - Solving ①, find the values of x_i.

S3 - Substitute the values of x_i's in ① itself and then apply T on both sides to obtain the relation

$$T(u) = \sum_{i=1}^{n} x_i \, T(u_i)$$

S4 - Now use the given values of $T(u_i)$'s to get $T(u) = \sum_{i=1}^{n} x_i v_i$ and thus T is found.

2. To find T when image of a subset of a basis is given.

Given:- let $T: U \to V$ be a L·T and $\dim U = n$. A linearly independent set $\{u_1, u_2, \dots, u_m\}$ where $m < n$ is given and $T(u_i) = v_i$, $1 \le i \le m$.

Working steps

S1 - Find extend the set $\{u_1, u_2, \dots, u_m\}$ to a basis of U and let the basis obtained be

$$\{u_1, u_2, \dots, u_m, u_{m+1}, \dots, u_n\}$$

S2. There is no condition on $T(u_{m+1}), \dots, T(u_n)$. So, for our simplicity, we assume that $T(u_{m+1}) = 0_V = T(u_{m+2}) = \dots = T(u_n)$

S3. Now find T by using the method discussed in Case I.

Pb1 Find a Linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,1,0) = (1,0)$ and $T(1,-1,0) = (1,1)$ and then verify your result.

Sol:- We know that to determine a linear transformation T. We need a basis of domain and the images of all vectors of basis under T.

Here $T(1,1,0) = (1,0)$ and $T(1,-1,0) = (1,1)$.

But $S = \{(1,1,0), (1,-1,0)\}$ $\xrightarrow{L.I}$ does not form a basis of R^3. Let us extend this to a basis.

$$[S] = \{\alpha(1,1,0) + \beta(1,-1,0) \mid \alpha, \beta \in \mathbb{R}\}$$

$$= \{(\alpha+\beta, \alpha-\beta, 0) \mid \alpha, \beta \in \mathbb{R}\}$$

$\therefore$ $(0,0,1) \notin [S]$

Thus $B = \{(1,1,0), (1,-1,0), (0,0,1)\}$ is a L.I set and since no. of elements in B is equal to $\dim \mathbb{R}^3_{\mathbb{R}} = 3$.

Thus B will form a basis of $\mathbb{R}^3$.

Now, we need image of $(0,0,1)$.

For our simplicity, we take

$$T(0,0,1) = (0,0).$$

Now, for $u = (x,y,z) \in \mathbb{R}^3$, we have

$$u = (x,y,z) = a(1,1,0) + b(1,-1,0) + c(0,0,1) \quad \text{①}$$

$$\left.\begin{aligned} a+b &= x \\ a-b &= y \\ c &= z \end{aligned}\right\} \quad a = \frac{x+y}{2}, \quad b = \frac{x-y}{2}, \quad c = z$$

$\therefore$ Apply T on both sides of ①, we have

$$T(u) = aT(1,1,0) + bT(1,-1,0) + cT(0,0,1)$$

$$T(x,y,z) = \frac{x+y}{2}(1,0) + \frac{x-y}{2}(1,1) + c(0,0)$$

$$= \left(x, \frac{x-y}{2}\right)$$

<u>Varification</u> $T(x,y,z) = \left(x, \frac{x-y}{2}\right)$

$T(1,1,0) = (1,0)$ and

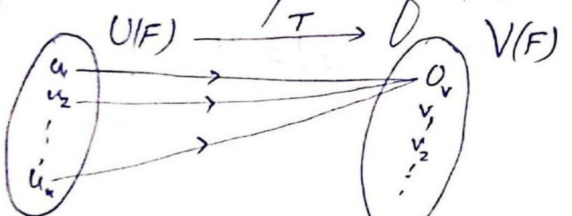
$T(1,-1,0) = (1,1)$.

Null space or Kernel of a linear transformation

Let U and V be any two vector space over the field F.
Let $T: U \longrightarrow V$ be a linear transformation.
Then the set denoted by $N(T)$ or $\ker T$ or $K(T)$ and defined as

$$\ker T = \{ u_i \in U \mid T(u) = O_v \}$$ is called

Kernel of T or null space of T.



Pb! Find the range and kernel of the following L.T.

1) $T: V_3 \longrightarrow V_3$ defined by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$

$R(T) = \{ T(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R} \}$

$= \{ (x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R} \}$

$= [(1,0,0), (0,1,0)] = x_1x_2$ plane.

Now, $\ker T = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid T(x_1, x_2, x_3) = (0,0,0) \}$

$= \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2, 0) = (0,0,0) \}$ $\rightarrow x_1 = 0,\ x_2 = 0$

$= \{ (0, 0, x_3) \mid x_3 \in \mathbb{R} \}$

$= [(0,0,1)] = x_3$ axis.

2) $T: V_3 \rightarrow V_2$ defined by $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$

$R(T) = \{ (x_1 - x_2, x_1 + x_3) \mid x_1, x_2, x_3 \in \mathbb{R} \}$

$= \{ x_1(1,1) + x_2(-1,0) + x_3(0,1) \mid x_1, x_2, x_3 \in \mathbb{R} \}$

$= [(1,1), (-1,0), (0,1)] \longrightarrow$ Linearly Dependent

$= [(1,1), (-1,0)]$ $\quad \because (1,1) + (-1,0) = (0,1)$

$\rightarrow$ Linearly independent.

for $\ker T$, we put $T(x_1, x_2, x_3) = (0,0)$

$(x_1 - x_2, x_1 + x_3) = (0,0)$

$$\left.\begin{aligned} x_1 - x_2 &= 0 \\ x_1 + x_3 &= 0 \end{aligned}\right\} \quad x_1 = x_2 \ \& \ x_3 = -x_1$$

$\therefore \ker T = \{ (x_1, x_1, -x_1) \mid x_1 \in \mathbb{R} \}$

$= \{ x_1(1,1,-1) \mid x_1 \in \mathbb{R} \}$

$= [(1,1,-1)]$

3) $T: V_2 \rightarrow V_2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1)$

4) $T: V_3 \rightarrow V_3$ " " $T(x_1, x_2, x_3) = (x_1, x_3, x_2)$

Ans 3) $R(T) = [(1,1), (1,0)]$ and $\ker T = \{(0,0)\}$
$= V_2$

4) $R(T) = [(1,0,0), (0,1,0), (0,0,1)] = V_3$

and

$\ker T = \{(0,0,0)\}$.

Thm 2 Let $T: U \to V$ be a linear transf., then

1) $R(T)$ is a subspace of V.

2) $N(T)$ or $\text{Ker}\, T$ is a subspace of V.

Pf:- 1) Since $T: U \to V$ is a L. transf^n then

$$T(0) = 0_V \in R(T)$$

Thus $R(T) \neq \phi$ and $R(T) \subset V$

Let $v_1, v_2 \in R(T)$ then $\exists$ $u_1, u_2 \in U$

s.t. $T(u_1) = v_1$ and $T(u_2) = v_2$

$$\Rightarrow v_1 + v_2 = T(u_1) + T(u_2)$$
$$= T(u_1 + u_2) \quad (\because T \text{ is linear})$$
$$= T(u_3)$$

$$\Rightarrow v_1 + v_2 \in R(T)$$

Let $v \in R(T)$ and for any scalar $\alpha \in F$

$$\alpha v = \alpha T(u) \quad \text{for } u \in U$$
$$= T(\alpha u) \quad (\because T(\alpha u) = \alpha T(u))$$
$$\in R(T)$$

Thus $R(T)$ is subspace of V.

Note: Dimension of $R(T)$ is called rank of T and it's denoted by $r(T)$ or $\rho(T)$.

2) $\because 0 \in \text{Ker}\, T \Rightarrow \text{Ker}\, T \neq \phi$ and moreover $\text{Ker}\, T \subset U$.

Let $u_1, u_2 \in \text{Ker}\, T$ then $T(u_1) = 0 = T(u_2)$

then $T(u_1 + u_2) = T(u_1) + T(u_2)$ ($\because T$ is linear)

$$= 0_V + 0_V$$
$$= 0_V$$

$$\Rightarrow u_1 + u_2 \in \text{Ker}\, T$$

Let $u \in \text{ker}\, T$ and $\forall$ $\alpha \in \mathbb{F}$

$$T(\alpha u) = \alpha T(u) \quad (\because T \text{ is linear})$$
$$= \alpha \cdot 0_V \quad (\because u \in \ker T \Rightarrow T(u) = 0_V)$$
$$= 0_V$$

$$\Rightarrow \alpha u \in \text{Ker}\, T$$

$\therefore$ $\text{Ker}\, T$ is subspace of U.

Note: Dimension of $\text{Ker}\, T$ is called nullity of T and it's denoted by $\eta(T)$.

3) T is one-one $\Leftrightarrow$ $\text{Ker}\, T$ is the zero subspace of U. $\Leftrightarrow \eta(T) = 0$

4) If $[u_1, u_2, \ldots, u_n] = U$, then

$$R(T) = [T(u_1), T(u_2), \ldots, T(u_n)].$$

5). If U is finite dimensional space, then

$$\dim R(T) \leq \dim U.$$

6) T is onto $\Leftrightarrow$ $R(T) = V \Leftrightarrow$ $\dim R(T)$ is equal to $\dim V$. (in finite dim-sp.)

Rank Nullity Theorem :- (statement)

Let $T: U \to V$ be a linear transformation and U is a finite dimensional vector space, then

$$\dim U = \text{rank of } T + \text{nullity of } T$$

i.e. $\boxed{\dim U = r(T) + \eta(T)}$

i.e $\boxed{\text{Rank} + \text{Nullity} = \text{Dimension of domain space}}$

Pb Determine the range and kernel of following linear transformation. Find the rank and nullity of T and also verify Rank-Nullity Theorem. Check T is one to one, onto, both or neither?

1) $T: V_2 \to V_2$ defined by $T(x_1, x_2) = (x_1 - x_2, x_1)$

Solⁿ Let $R(T)$ be range of T, then

$$R(T) = \{ T(x_1, x_2) \mid x_1, x_2 \in \mathbb{R} \}$$
$$= \{ (x_1 - x_2, x_1) \mid x_1, x_2 \in \mathbb{R} \}$$
$$= \{ x_1(1,1) + x_2(-1,0) \mid x_1, x_2 \in \mathbb{R} \}$$
$$= [(1,1), (-1,0)]$$

$\because \{(1,1), (-1,0)\}$ is linearly independent, hence a basis of $R(T)$.

$\Rightarrow \dim R(T) = 2$

$\Rightarrow \text{rank}(T) = 2 = r(T)$

$\because r(T) = 2 = \dim V_2$ (codomain) $\therefore$ T is onto.

Further let $N(T)$ is null space then for kernel, we put

$$T(x_1, x_2) = (0,0)$$
$$\Rightarrow (x_1 - x_2, x_1) = (0,0)$$
$$\Rightarrow x_1 = 0 = x_2$$

$\therefore N(T) = \{(0,0)\}$

$\Rightarrow \dim N(T) = 0 = \text{Nullity of } T$

$\Rightarrow \eta(T) = 0$ (Thus, T is one-one map.)

$\because$ ~~rank(T)+~~ $\dim V_2 = 2$ (domain)

and $r(T) + \eta(T) = 2 + 0 = 2$

Thus, $\boxed{\dim V_2 = r(T) + \eta(T)}$ varified.

~~2) $T: V_3 \to V_2$ defined by $T(x_1, x_2, x_3) =$~~

~~$T(x_1, x_2, x_3) = \left(\frac{1}{2}x_1 + x_2 + x_3,\ x_1 - \frac{1}{3}x_2\right)$~~

2) $T: V_3 \to V_3$ defined by $T(x_1, x_2, x_3) = (x_3, x_2 + x_1, 0)$

Solⁿ let $R(T)$ be range of T, then

$$R(T) = \{ T(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R} \}$$
$$= \{ (x_3, x_2 + x_1, 0) \mid x_1, x_2, x_3 \in \mathbb{R} \}$$
$$= \{ x_1(0,1,0) + x_2(0,1,0) + x_3(1,0,0) \mid x_i \in \mathbb{R} \}$$
$$= [(0,1,0), (1,0,0)]$$

$\because \{(0,1,0),(1,0,0)\}$ is L.I and hence a basis of $R(T)$.

$\therefore \dim R(T) = 2 = \gamma(T)$ [Here, dim Codomain $= \dim V_3 = 3$, $\gamma(T) \neq 3$. Thus not onto map]

Let $N(T)$ denotes the nullspace of T, then for nullspace, we put

$$T(x_1, x_2, x_3) = (0,0,0)$$

$$\Rightarrow (x_3, x_2 + x_1, 0) = (0,0,0)$$

$$\Rightarrow x_3 = 0, \quad x_2 = -x_1$$

$$\therefore N(T) = \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0,0,0)\}$$

$$= \{(x_1, -x_1, 0) \mid x_1 \in \mathbb{R}\}$$

$$= [(1,-1,0)]$$

[Here, $\eta(T) \neq 0$ $\therefore$ T is not one-one]

$\therefore \dim N(T) = 1 = \eta(T)$

Varification

$\because \dim V_3 = 3$,

and $\gamma(T) + \eta(T) = 2 + 1 = 3$ $\Big\}$ Rank-Nullity Theorem hold.

3) $T: M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ defined by

$$T\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & w \end{pmatrix}, \quad x, y, z, w \in \mathbb{R}$$

Sol^n $R(T) = \left\{ T\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, y, z, w \in \mathbb{R} \right\}$

$$= \left\{ \begin{pmatrix} x & y \\ 0 & w \end{pmatrix} \mid x, y, w \in \mathbb{R} \right\}$$

$$= \left\{ x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + w\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid x, y, w \in \mathbb{R} \right\}$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$$

$\because \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is L.I set and hence will be a basis of $R(T)$.

Thus, $\dim R(T) = 3 = \gamma(T)$ [$\because \gamma(T) \neq 4$ T is not onto.]

Nullity $N(T) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid T\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

$$= \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid \begin{pmatrix} x & y \\ 0 & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} \mid z \in \mathbb{R} \right\}$$

$x = 0$
$y = 0$
$w = 0$

$$= \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right]$$

[$\because \eta(T) \neq 0$ $\therefore$ T is not one-one]

$\dim N(T) = 1 = \eta(T)$

Varification

$\because \dim M_2(\mathbb{R}) = 4$

and $\gamma(T) + \eta(T) = 3 + 1 = 4$ $\Big\}$ Rank-Nullity Theorem hold.

4) $T: M_3(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ defined by

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} - a_{12} + a_{13} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$$

Sol: $R(T) = \left\{ T\begin{pmatrix} a_{ij} \end{pmatrix} \in M_2(\mathbb{R}) \;\; a_{ij} \in \mathbb{R} \;\; \forall\; 1 \le i,j \le 3 \right\}$

$= \left\{ \begin{pmatrix} a_{11} - a_{12} + a_{13} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} \;\Big|\; a_{ij} \in \mathbb{R} \right\}$

$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$

These two are dependent vector. So write only one.

$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]$

$\because \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is L.I set.

Thus form a basis of $R(T)$.

$\dim R(T) = 4 = \gamma(T)$. $\left[\text{Here, } \gamma(T) = 4 = \dim M_2(\mathbb{R}) \text{ Thus, } T \text{ is onto.} \right]$

$N(T) = \left\{ \begin{pmatrix} a_{ij} \end{pmatrix} \in M_3(\mathbb{R}) \;\Big|\; a_{ij} \in \mathbb{R} \;\&\; T(a_{ij}) = 0 \right\}$

Let $T(a_{ij}) = 0$

$\begin{pmatrix} a_{11} - a_{12} + a_{13} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow a_{13} = 0,\; a_{31} = 0,\; a_{33} = 0,\; a_{11} = a_{12}$

Thus, $N(T) = \left\{ \begin{pmatrix} a_{11} & a_{11} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix} \;\Big|\; a_{ij} \in \mathbb{R} \right\}$

$= \left[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right]$

These all are L.I vectors, so form a basis for $N(T)$.

$\text{Dim } N(T) = 5 = \eta(T)$ $\left[\because \eta(T) \neq 0 \Rightarrow T \text{ is not one-one} \right]$

Verification

$\dim M_3(\mathbb{R}) = 9$

and $\gamma(T) + \eta(T) = 4 + 5 = 9$

$\Big\}$ R.N. Theorem Verified

5) $T : P_3(x) \longrightarrow P_3(x)$ defined by

$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + 2a_2 x + 3a_3 x^2 \quad a_i \in \mathbb{R}$

or

$T(p(x)) = \frac{d}{dx}(p(x))$

Solⁿ

$R(T) = \left\{ T(p(x)) \;\big|\; p(x) \in P_3(x) \right\}$

$= \left\{ a_1 + 2a_2 x + 3a_3 x^2 \;\big|\; a_1, a_2, a_3 \in \mathbb{R} \right\}$

$= \left\{ a_1 \cdot 1 + a_2 \cdot 2x + a_3 \cdot 3x^2 \;\big|\; a_1, a_2, a_3 \in \mathbb{R} \right\}$

$= [1, 2x, 3x^2]$

$\because \{1, 2x, 3x^2\}$ is a L.I set and hence form a basis for $R(T)$. $\text{Dim } R(T) = 3 = \gamma(T)$ $\left[\because \gamma(T) \neq 4 \;\therefore T \text{ is not onto} \right]$

To find Null space, we put

$$T(p(x)) = 0 \quad , \quad p(x) = \sum_{i=0}^{3} a_i x^i$$

$$\Rightarrow \frac{d}{dx}(p(x)) = 0$$

$$\Rightarrow a_1 + 2a_2 x + 3a_3 x^2 = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$$

$$\Rightarrow a_1 = 0, \ 2a_2 = 0, \ 3a_3 = 0$$

$$\Rightarrow a_1 = 0, \ a_2 = 0, \ a_3 = 0$$

Thus, $N(T) = \left\{ p(x) = \sum_{i=0}^{3} a_i x^i \ \middle| \ T(p(x)) = 0 \right\}$

$$= \{ a_0 \mid a_0 \in \mathbb{R} \}$$

$$= [1] \qquad \left[\because \eta(T) \neq 0 \ \therefore T \text{ is not one-one} \right]$$

$$\text{Dim } N(T) = 1 = \eta(T)$$

So, $r(T) + \eta(T) = 3 + 1 = 4 = \dim P_3(x)$

6) $T : P_5(x) \longrightarrow P_3(x)$ defined by

$$T\left(\sum_{i=0}^{5} a_i x^i\right) = a_0 \quad , \quad a_i \in \mathbb{R}$$

Sol H/A

Matrix representation of a Linear Transformation

1. **Ordered basis:-** A basis of a vector space is called an ordered basis if its vectors are written in a certain specific order.

Eg 1.
Let $B = \{v_1, v_2, v_3\}$ is a basis of a vector space V. Let us change the order of its vectors and let $B' = \{v_2, v_1, v_3\}$, then B and B′ are same basis but different ordered basis.

Ex 2. Let $V = M_2(\mathbb{R})$

$$B = \left\{ \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix} \right\}$$

and $$B' = \left\{ \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix} \right\}$$

Here, B and B′ are same basis of $M_2(\mathbb{R})$ but different ordered basis.

2. **Matrix of a L.T relative to ordered basis**

Let $B = \{u_1, u_2, \dots, u_n\}$ and $B' = \{v_1, v_2, \dots, v_m\}$ be ordered basis for the finite dimensional vector space U and V respectively, and let $T : U \longrightarrow V$ be a Linear transformation.

Since, $T(u_i) \in V$ and B′ is a basis of V.
So, each $T(u_i)$ can be expressed uniquely as a linear combination of the vectors of B′.

Let
$$T(u_1) = \alpha_{11} v_1 + \cdots + \alpha_{1m} v_m$$
$$T(u_2) = \alpha_{21} v_1 + \cdots + \alpha_{2m} v_m$$
$$\vdots$$
$$T(u_n) = \alpha_{n1} v_1 + \cdots + \alpha_{nm} v_m, \quad \alpha_{ij} \in F$$

The coefficient matrix of the above system is a $n \times m$ matrix given by

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nm} \end{bmatrix}_{n \times m}$$

The transpose of this matrix is a $m \times n$ matrix called the matrix of T with respect to ordered basis of B and B', denoted by $[T:B,B']$ and is given by

$$[T:B,B'] = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & & \alpha_{n2} \\ \vdots & \vdots & & \vdots \\ \alpha_{1m} & \alpha_{2m} & & \alpha_{nm} \end{bmatrix}$$

Remarks

a) It should be noted that the order of matrix of a L.T, $T: U \longrightarrow V$ is $m \times n$ i.e. $\dim V \times \dim U$.

b) Whenever basis B and B' are not mentioned in any problem, they are understood to taken as standard basis.

c) If $U = V$ and B is the basis used on both sides, then the matrix of T is denoted by $[T:B]$ instead of $[T:B,B]$

Pb 1. Find the matrix $[T:B,B']$ of the linear transformation $T: R^2 \longrightarrow R^2$ defined by

$$T(x,y) = (x+y,\ x-y)$$

in the following cases

a) $B = \{(1,0),(0,1)\}$ and $B' = \{(1,1),(1,-1)\}$

b) $B = \{(0,1),(1,0)\}$ and $B' = \{(1,0),(0,1)\}$

c) $B = \{(0,1),(1,0)\}$ and $B' = \{(0,1),(1,0)\}$

d) $B = \{(1,1),(1,-1)\}$ and $B' = \{(1,1),(1,-1)\}$

Sol a) Given $T: R^2 \longrightarrow R^2$ defined by

$$T(x,y) = (x+y,\ x-y)$$

$$T(1,0) = (1,1) = 1\cdot(1,1) + 0\cdot(1,-1)$$

and $T(0,1) = (1,-1) = 0\cdot(1,1) + 1\cdot(1,-1)$

then $[T,B,B'] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

b)

$$T(0,1) = (1,-1) = 1\cdot(1,0) - 1\cdot(0,1)$$

and $T(1,0) = (1,1) = 1\cdot(1,0) + 1\cdot(0,1)$

then $[T,B,B'] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}_{2\times 2}$

c) $T(0,1) = (1,-1) = -1\cdot(0,1) + 1\,(1,0)$

and $T(1,0) = (1,1) = 1\cdot(0,1) + 1\cdot(1,0)$

$$[T:B,B] = [T:B] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2}$$

d) $T(1,1) = (2,0) = 1\cdot(1,1) + 1\cdot(1,-1)$

and $T(1,-1) = (0,2) = 1\cdot(1,1) - 1\cdot(1,-1)$

$$[T:B,B] = [T:B] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{2\times 2}$$

Pb 2. $T: R^3 \longrightarrow R^2$ defined by

$T(x,y,z) = (x+y, y+z)$ in the following cases:-

a) $B = \{(1,1,1),(1,0,0),(1,1,0)\}$ and $B' = \{(1,2),(0,1)\}$

b) $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ and $B' = \{(1,0),(0,1)\}$

Sol (a) Given $T: R^3 \longrightarrow R^2$ is a L.T defined by

$$T(x,y,z) = (x+y, y+z)$$

let $(a,b) \in \mathbb{R}^2$ $\exists\ \alpha,\beta \in \mathbb{R}$ such that

$$(a,b) = \alpha(1,2) + \beta(0,1)$$

on comparison, $\left.\begin{aligned} \alpha &= a \\ 2\alpha+\beta &= b \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha &= a \\ \beta &= b-2a \end{aligned}$

so, $(a,b) = a\cdot(1,2) + (b-2a)\cdot(0,1)$ —— (1)

Now,

$$\left.\begin{aligned} T(1,1,1) &= (2,2) = 2\cdot(1,2) + (-2)(0,1) \\ T(1,0,0) &= (1,0) = 1\cdot(1,2) + (-2)(0,1) \\ T(1,1,0) &= (2,1) = 2\cdot(1,2) + (-3)\cdot(0,1) \end{aligned}\right\} \text{by using (1)}$$

$$[T:B,B'] = \begin{bmatrix} 2 & 1 & 2 \\ -2 & -2 & -3 \end{bmatrix}_{2\times 3}$$

b) let $(a,b) \in \mathbb{R}^2$, then there exists $\alpha,\beta \in \mathbb{R}$ such that

$$(a,b) = \alpha(1,0) + \beta(0,1)$$
$$= (\alpha,\beta)$$

$\alpha = a$ and $\beta = b$

so, $(a,b) = a\cdot(1,0) + b\cdot(0,1)$ —— (1)

$T(1,0,0) = (1,0) = 1\cdot(1,0) + 0\cdot(0,1)$

$T(0,1,0) = (1,1) = 1\cdot(1,0) + 1\cdot(0,1)$

$T(0,0,1) = (0,1) = 0\cdot(1,0) + 1\cdot(0,1)$

$$[T:B,B'] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$$

Pb 3 Find the matrix $[T:B,B']$ for the linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$T(x,y) = (x,-y)$ where

a) $B = \{(1,0),(0,1)\}$ and $B' = \{(1,1),(1,-1)\}$

b) $B = \{(1,1),(1,0)\}$ and $B' = \{(2,3),(4,5)\}$

Ans (a) $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$, (b) $\begin{bmatrix} -9/2 & -5/2 \\ 5/2 & 3/2 \end{bmatrix}$

• Pb 4 Find the matrix representing the linear transformation $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ defined by

$$T(x,y,z) = (x+y+z,\ 2x+z,\ 2y-z,\ 6y)$$

relative to the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$.

Ans: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & 0 \end{bmatrix}_{4\times 3}$

To find T when matrix is given

Matrices as linear Mappings:-

Let $A_{m\times n}$ be a matrix of order $m\times n$, then A determines a mapping $F_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $F_A(u) = Au$ (where the vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ are written as columns).

F_A is linear:-

$$\begin{aligned} F_A(u+v) &= A(u+v) \\ &= Au + Av \\ &= F_A(u) + F_A(v) \end{aligned}$$

and
$$\begin{aligned} F_A(\alpha u) &= A(\alpha u) \\ &= \alpha (Au) \\ &= \alpha F_A(u). \end{aligned}$$

Eg: Let $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 7 \end{bmatrix}$ be a matrix of order 2×3.

then A determines a map (say) $F_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$.

Let $u = (x\ y\ z)^T \in \mathbb{R}^3$, then

$$F_A(u) = F_A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+5y+4z \\ 3x+y+7z \end{bmatrix}_{2\times 1}$$

i.e. $F_A(x,y,z) = (2x+5y+4z,\ 3x+y+7z)$

~~Pb. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation~~

Pb1. Find the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ whose matrix $[T: B, B']$ is $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$,

where, $B = \{(1,1,1), (1,2,3), (1,0,0)\}$

and $B' = \{(1,1), (1,-1)\}$.

Sol^n: Given, $[T: B, B'] = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$.

So, we can write

$$T(1,1,1) = 1\cdot(1,1) + 3\cdot(1,-1) = (4,-2)$$

Similarly, $T(1,2,3) = -1(1,1) + 1\cdot(1,-1) = (0,-2)$

$$T(1,0,0) = 2(1,0) + 0\cdot(1,-1) = (2,2)$$

Let $(x,y,z) \in \mathbb{R}^3$ then $\exists\ \alpha, \beta, \gamma \in \mathbb{R}$ s.t.

$$(x,y,z) = \alpha(1,1,1) + \beta(1,2,3) + \gamma(1,0,0) \quad \text{--- (1)}$$
$$= (\alpha+\beta+\gamma,\ \alpha+2\beta,\ \alpha+3\beta)$$

$$\left.\begin{aligned} \alpha &= 3y - 2z \\ \beta &= z - y \\ \gamma &= x - 2y + z \end{aligned}\right\} \text{--- (2)}$$

Operating T on both sides of eq^n (1)

$$T(x,y,z) = \alpha T(1,1,1) + \beta T(1,2,3) + \gamma T(1,0,0)$$
$$= (3y-2z)(4,-2) + (z-y)(0,-2) + (x-2y+z)(2,2)$$
$$= (2x + 8y - 6z,\ 2x - 8y + 4z)$$

Ans.

Pb2. Find the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ whose matrix $[T: B, B']$ is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2}$ where,

a) $B = \{(1,0), (0,1)\}$ and $B' = \{(1,0,0), (0,1,0), (0,0,1)\}$

b) $B = \{(1,1), (1,-1)\}$ and $B' = \{(1,0,0), (0,1,0), (0,0,1)\}$

Row-Echelon form of Matrix :-

Matrix 'A' is said to be in row-echelon form if (where leading non-zero element of row of A is the first non-zero element in the row).

1) all zero rows, if any, are at the bottom of the matrix.

2) each leading non-zero entry in a row is to the right of the leading non-zero entry in the preceding row.

(Similar results for column-echelon form).

Ex: of row-echelon form

$$\begin{bmatrix} 0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 3 & 4 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} 4 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Yes Yes No

Elementary Row-operation

Let A be the matrix with rows $R_1, R_2, \ldots, R_n$. Then following operations are called elementary row operations.

1) Row interchange – $R_i \leftrightarrow R_j$
2) Row scaling – $R_i \longmapsto kR_i$, $(k \neq 0)$
3) Row addition – $R_i \mapsto R_i + kR_j$

Row-Canonical form :-

A matrix 'A' is said to be in row-Canonical form if it is an echelon form matrix and if it satisfy given additional two properties :-

1) each pivot (leading non-zero entry) is equal to 1.
2) each pivot is the only non-zero entry in its column.

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Canonical form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ → Only in row-echelon form not in Canonical form.

Pb 1 Consider matrix $A = \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix}$

a) Reduce it to echelon form.

b) Reduce it to canonical form.

Sol a)

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 6 & 7 \end{bmatrix} \quad \begin{matrix} R_2 \mapsto R_2 - 2R_1 \\ R_3 \mapsto R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \quad R_3 \longmapsto R_3 - \tfrac{3}{2}R_2$$

$\to$ Echelon form.

b) To obtain canonical form always proceed from pivot element from bottom side. First make it one and then make all entries above it to zero. Similarly proceed for other pivot going up from bottom side.

$$\begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 7 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{Canonical form}$$

Pb 2. Reduce $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{bmatrix}$ to

a) Echelon form b) Canonical form.

Ans (a) $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Rank of Matrix :-

Def^n: Rank of a matrix is the no. of linearly independent rows or linearly independent columns.

i.e. Convert the given matrix into echelon form and then rank will be number of L.I rows or columns.

→ Rank of matrix 'A' is equal to the number of pivots in an echelon form of matrix A.

→ Rank is the order of largest non-vanishing minor (determinant of square sub-matrices).

Eg: 1)
$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{No. of L.I rows} = 1$$

only one pivot

$$\rho(A_1) = 1$$

2)
$$A_2 = \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix}$$

Sol^n
$$A_2 \sim \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \rightarrow \text{Row-Echelon form}$$

$\rho(A_2) = 3$ = No. of L.I rows.
= No. of pivots.

3)
$$A_3 = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 3 & 4 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row-Echelon form

$\rho(A_3) = 4$
= No. of L.I rows.

Pb4) Check whether the following vectors are L.D or L.I ?

a) $\{(1,0,1)\ (1,1,0)\ ,(1,-1,1)\}$

b) $\{(1,1,0)\ (1,-1,1)\ .\ (1,2,-3)\}$

Sol^n

a)
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Row E.F

$\rho(A) = 3$ = No. of L.I vectors

Hence, (a) is L.I.

System of linear equation (SOLE)

1. **System of non-homogeneous linear equations:**

Consider a system of m-linear non-homo. eqn in n-unknowns $x_1, x_2, \dots, x_n$. Let $a_{ij} \in \mathbb{R}$ or $\mathbb{C}$, $b_i \in \mathbb{R}/\mathbb{C}$.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}\right\} \longrightarrow (1)$$

This system can be written as a single matrix equation $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \neq 0$$

$[A:B]$ – Augmented matrix.

Thm Consider a system of linear equation $AX = B$, $B \neq 0$, then it has either
1) a unique solⁿ
2) no solⁿ or
3) an infinite number of solutions.

- SOLE
  - Inconsistent
    - No solⁿ
  - Consistent
    - Unique solⁿ
    - Infinite number of solⁿ

Working Rule

- $AX = B$
  - $\rho(A:B) \neq \rho(A)$ — $\text{rank}(A:B) \neq \text{rank}(A)$
  - $\rho(A:B) = \rho(A)$
    - $\rho(A:B) = \rho(A)$ = Number of Unknowns
    - $\rho(A:B) = \rho(A)$ < Number of Unknowns

SOLE in one variable:

Thm:- Consider the linear eqⁿ $ax = b$
1) If $a \neq 0$, then $x = b/a$ is a unique solⁿ
2) If $a = 0$ but $b \neq 0$, then system has no solⁿ.
3) If $a = 0 = b$, then every ~~scalar~~ scalar K is a solⁿ of the system.

SOLE in two variable

Consider a system of ~~two~~ linear eq^n in two variable.

$$\left.\begin{aligned} A_1x + B_1y &= C_1 \\ A_2x + B_2y &= C_2 \end{aligned}\right\}$$

Atleast $A_i \neq 0$ & $B_j \neq 0$ for some i & j.

Unique sol^n :- $\boxed{\dfrac{A_1}{A_2} \neq \dfrac{B_1}{B_2}}$

No sol^n :- $\boxed{\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} \neq \dfrac{C_1}{C_2}}$

Infinite no. of sol^n :- $\boxed{\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2}}$

P61 Show that the equations

$$x+y+z = 6$$
$$x+2y+3z = 14$$
$$x+4y+7z = 30$$

are consistent and solve them.

sol^n :- The given system of linear eq^n can be written as

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix} = B$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 20 \end{array}\right] \quad \begin{matrix} R_2 \longmapsto R_2 - R_1 \\ R_3 \longmapsto R_3 - R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right] \quad R_3 \longmapsto R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ which is the row-echelon form.

Since, $\rho(A:B) = 2 = \rho(A) < 3$.

Therefore, the given systems of linear eq^n are consistent and have infinite sol^n.

We see that the given system of eq^n is equivalent to the matrix eq^n:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

OR

$$x+y+z = 6$$
$$y+2z = 8$$

$\Rightarrow$ Let $z = K$ (free variable)

then $y = 8-2K$ and $x = K-2$,

where $K \in \mathbb{R}$.

Pb2. Investigate for what value of λ, μ the system of eqn :

$$x+y+z=6,\quad x+2y+3z=10,\quad x+2y+\lambda z=\mu$$

have
1) no soln 2) unique soln 3) infinite soln.

Soln:

$$[A:B]=\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \quad \begin{array}{l} R_2 \mapsto R_2 - R_1 \\ R_3 \mapsto R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \quad R_3 \mapsto R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] \hookrightarrow \text{Echelon form}$$

1) No soln: If $\rho(A:B) \neq \rho(A)$
which is only possible if $\lambda = 3$ and $\mu \neq 10$

2) Unique soln: If $\rho(A:B) = \rho(A) = 3$
which is only possible if $\lambda \neq 3$ & $\mu \in \mathbb{R}$.

3) Infinite soln: If $\rho(A:B) = \rho(A) < 3$
which is only possible if $\lambda = 3$ & $\mu = 10$.

Homogeneous System of linear eqn

$$\boxed{AX=0}$$

→ $X=0$ is always a soln (trivial soln)

→ If X_1 and X_2 are two soln, then $k_1X_1 + k_2X_2$ is also soln, where $k_i \in \mathbb{F}$.
But, this result is not true for non-homogn system of linear eqn.

→ If A is a square matrix then $AX=0$ has trivial soln only if $|A| \neq 0$.

Pb1 Find all the soln of the following system of equation :

$$3x + 4y - z - 6w = 0$$
$$2x + 3y + 2z - 3w = 0$$
$$2x + y - 14z - 9w = 0$$
$$x + 3y - 13z + 3w = 0.$$

Soln: $AX = 0$

$$A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & -13 & 3 \end{bmatrix} \quad R_4 \longleftrightarrow R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix} \quad \begin{array}{l} R_2 \mapsto R_2 - 2R_1 \\ R_3 \mapsto R_3 - 2R_1 \\ R_4 \mapsto R_4 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix} \quad \begin{array}{l} R_2 \mapsto -\frac{1}{3}R_2 \\ R_3 \mapsto -\frac{1}{5}R_3 \\ R_4 \mapsto -\frac{1}{5}R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \end{bmatrix} \quad \begin{array}{l} R_3 \mapsto R_3 - R_2 \\ R_4 \mapsto R_4 - R_2 \end{array}$$

$$\begin{array}{ccccc} & x & y & z & \omega \end{array}$$
$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 2 < 4$$

$\therefore$ z and ω are free variables.

Let $z = k_1$ and $\omega = k_2$, then general sol^n

$x = 11k_1 + 6k_2$, $y = -8k_1 - 3k_2$

$z = k_1$ and $\omega = k_2$. Ans.

Eigen Values and Eigen Vectors :-

1. Eigen values and eigen vectors of a linear operator

Let T be a linear operator on a vector space $V(\mathbb{F})$. If there exist a non-zero vector $v \in V$ such that $T(v) = \lambda v$ for some $\lambda \in \mathbb{F}$, then v is called an eigen vector of T corresponding to λ and λ is called an eigen value of T corresponding to v.

2. Eigen values and eigen vectors of a matrix :-

Let A be a square matrix of order n over a field $\mathbb{F}$. If there exists a non-zero column vector $X \in \mathbb{F}^n$ such that $AX = \lambda X$ for some $\lambda \in \mathbb{F}$, then X is called an eigen vector of A corresponding to λ and λ is called an eigen value of A corresponding to X.

~~Results~~
Results:

1. Let A be an $n \times n$ matrix over a field F. Then $\lambda \in F$ is an eigen value of A iff $A - \lambda I$ is singular i.e. $|A - \lambda I| = 0$.

2. Characteristic polynomial :-
Let A be ~~a~~ $n \times n$ matrix over a field F.
The function $f(\lambda) = |A - \lambda I|$ ← determinant
is a polynomial in λ of degree n.
This polynomial is called characteristic polynomial of A.

3. Characteristic equation (ch. eq^n)
Characteristic equation of a matrix A is given by $|A - \lambda I| = 0$.

4. Roots of ch. eq^n of a matrix are the eigen values of that matrix.

5. Let T be a linear operator on a finite dim. vector space $V(F)$. If A is the matrix of T with respect to any ordered basis B; ~~then~~
a) then a scalar $\lambda \in F$ is an eigen value of T iff λ is an eigen value of A.
b) then a vector $v \in V$ is an eigen vector of T corresponding to the eigen value λ iff its coordinate vector X relative to the basis B is an eigen vector ~~value~~ of A corresponding to its eigen value λ.

Pbl. Find the eigenvalues and the corresponding eigenvectors of the following matrices.

1) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ 2) $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ 3) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

4) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ 5) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ 7) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Sol 1. ① The ch. eq^n of A is given by

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$$

$$\Rightarrow \lambda = -2, 5$$

$\underline{\underline{\lambda = -2}}$

$$(A+2I)X = 0$$

$$\Rightarrow \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 3x + 4y = 0 \Rightarrow x = -\frac{4}{3}y$$

Hence, eigenvector X is given by

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{4}{3}y \\ y \end{pmatrix} = y\begin{pmatrix} -4/3 \\ 1 \end{pmatrix}$$

$$= \frac{y}{3}\begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

Since an eigenvector is unique upto a constant multiple, we can take the eigen vector as $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

Corresponding to the eigenvalue $\underline{\underline{\lambda = 5}}$,

$$(A-5I)X = \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-4x + 4y = 0 \text{ and } 3x - 3y = 0$$

$$\Rightarrow \boxed{x = y}$$

Therefore, the eigenvector is given by

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ or simply } \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

②. The ch. eq^n of A is given by

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda = 1, 3.$$

$$(A-\lambda I)X = 0$$

$\underline{\underline{\lambda = 1}}$:-

$$\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x + y = 0$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix} = x\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\underline{\underline{\lambda = 3}}$

$$\begin{bmatrix} -2 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2x = 0 \Rightarrow \boxed{x = 0}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} = y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvector corr. to 1 and 3.

Pb 2. Find the eigen values and the corresponding eigen vectors of the following matrices:

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Pb 3 Find the eigen values and the corresponding eigen vectors of the following matrices

a) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ b) $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Ans 2. a) $\lambda = 1, 1, 1$, $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

b) $\lambda = 1, 1, 1$, $X_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

c) $\lambda = 1, 1, 1$, $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Ans 3. a) $\lambda = 1, 2, 3$, $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

b) $\lambda = 2, 2, 3$, $X = [-5, -2, 5]^T$, $[1, 1, -2]^T$

c) $\lambda = 1, 1, 4$, $X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Properties:

1. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigen values of $n \times n$ matrix then corresponding eigen vectors $X_1, X_2, \ldots, X_n$ form a linearly independent set.

2. Eigenvector X of a matrix A is not unique.

3. If two or more eigen values are equal, it may or may not possible to get linearly independent eigen vectors corresponding to equal roots.

4. Eigen values of a lower triangular, upper triangular and diagonal matrices are just the diagonal elements.

5. All eigen values of a non-singular matrix are non-zero.

6. Atleast one eigen value of a singular matrix is 0.

7. If λ is an eigen value of A then λ^k is an eigen value of A^k, where k is a positive integer.

Pb Consider $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$.

Find the eigenvalues of $3A^3 + 5A^2 - 6A + 2I$.

Solⁿ $A \to$ upper triangular matrix.

Thus, eigen values $\lambda = 1, 3, -2$

$\therefore$ eigen values of $A^3 = 1, 27, -8$

eigen " of $A^2 = 1, 9, 4$

" " " $A = 1, 3, -2$

" " " $I = 1, 1, 1$

Now, eigen values of $3A^3 + 5A^2 - 6A + 2I$

first eigen value $= 3\cdot(1) + 5(1) - 6(1) + 2(1) = 4$

Second " " $= 3\cdot(27) + 5\cdot(9) - 6(3) + 2(1) = 110$

Third " " $= 3(-8) + 5(4) - 6(-2) + 2(1) = 10$

Cayley - Hamilton Theorem (statement)

Every square matrix A satisfies its own characteristic equation i.e.,

if $|A - \lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n) = 0$ be the characteristic eqⁿ, then matrix A satisfy it

i.e. $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0$.

Pb Varify Cayley-Hamilton theorem for the matrices.

1) $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ 2) $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

3) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Sol (1) The Char. eqⁿ of A is given by

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}$$

$$A^2 - 4A + 3I = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Pb1 Show that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = X_1$ and $\begin{bmatrix} i \\ -1-i \\ 2 \end{bmatrix} = X_2$ are eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix}$ but their sum is not an eigen vector.

Pf

$$A X_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot X_1$$

Here, $\lambda_1 = 1$ for $AX_1 = \lambda_1 X_1$

and

$$A X_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} i \\ -1-i \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1-i \\ 2i \end{bmatrix} = i \begin{bmatrix} i \\ -1-i \\ 2 \end{bmatrix} = i \cdot X_2$$

Here, $\lambda_2 = i$ for $AX_2 = \lambda_2 X_2$

Thus, X_1 and X_2 are eigen vectors of A.

Now,

$$X_1 + X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} i \\ -1-i \\ 2 \end{bmatrix} = \begin{bmatrix} 1+i \\ -1-i \\ 2 \end{bmatrix} = X$$

$$AX = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1+i \\ -1-i \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1-i \\ 2i \end{bmatrix} \neq \lambda \begin{bmatrix} 1+i \\ -1-i \\ 2 \end{bmatrix} \text{ for any } \lambda \in F.$$

Thus, $AX \neq \lambda X$ for any $\lambda \in F$.

Hence X is not an eigen vector of A.

Pb2. Find all the eigen values ~~of A~~ and the corresponding eigen vectors of the following linear operator?

1.) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x,y) = (x+2y, 3x+2y)$

2.) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x,y) = (x+y, x-y)$

3.) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x,y,z) = (x+y, y+z, z+x)$

4.) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x,y,z) = (x+y, x-z, 0)$

Sol (1) First of all, let us find matrix representation of T relative to standard basis, $B = \{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

$$T(1,0) = (1,3) = 1 \cdot (1,0) + 3(0,1)$$

$$T(0,1) = (2,2) = 2 \cdot (1,0) + 2(0,1)$$

$$A = [T : B; B] = [T : B] = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

Characteristic eqn of A: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$$

$$\Rightarrow \lambda = -1, 4$$

(say) $\rightarrow$ $\lambda_1 = -1$ and $\lambda_2 = 4$.

Thus -1 and 4 are ~~a~~ eigen value of T.
Let us find eigen ~~s~~ vector corresponding to these eigen values.

If X is the eigen vector corresponding to eigen value λ, then we have

$$AX = \lambda X$$
$$\Rightarrow (A - \lambda I)X = 0$$
$$= \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$$

$\lambda_1 = -1$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x + 2y = 0 \ \& \ 3x + 3y = 0$$

$$\Rightarrow \boxed{x = -y}$$

Taking $y = -1$, we get $x = 1$
and so, $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda_1 = -1$.

$\lambda_2 = 4$ using (1), we have

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x = 2y$$

Taking $y = 3$, we get $x = 2$
and so, $X_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigen vector corresponding to $\lambda_2 = 4$.