A general NLPP with equality constraints is

$$\text{Max/Min } Z = f(X)$$
$$\text{s.t. } g(X) = 0$$
$$X \geq 0 \quad ; \quad X = (x_1, x_2, \dots, x_n)$$

Assume that $f(X)$ and $g(X)$ are differentiable w.r.t X.

Langrange Multiplier Method. To find the necessary condition for the maxima (minima) value of Z, a new function is formed by introducing some multipliers λ_i known as *Langrange multipliers as*

$$L = L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$; where, λ_i are constant and unrestricted in sign.

The function L is called as Langrangian function with Langrange multipliers λ_i.

Ex: opt $f(x) = 2x_1^2 + x_2^2 + 10x_1$

s.t.
$$x_1 + x_2 = 20$$
$$2x_1 - x_2 = 6$$
$$x_1, x_2 \geq 0$$

Sol
$$g_1(x): \; x_1 + x_2 - 20 = 0$$
$$g_2(x): \; 2x_1 - x_2 - 6 = 0$$
$$L = L(x, \lambda) = 2x_1^2 + x_2^2 + 10x_1 + \lambda_1(x_1 + x_2 - 20) + \lambda_2(2x_1 - x_2 - 6)$$

$$L = L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \quad ; \quad x = (x_1, x_2, \dots, x_n)$$

$$\boxed{\begin{array}{l}\text{Max/Min } Z = f(X) \\ \text{s.t. } g(X) = 0 \\ \quad X \geq 0\end{array}} \xrightarrow[\text{Multiplier.}]{\text{Using Langrangian}} \boxed{\begin{array}{l}\text{Max/Min } L(X,\lambda) = f(X) + \lambda g(X) \\ \text{s.t. } X \geq 0\end{array}}$$

Constraint opt. problem — Unconstraint opt^n problem

Necessary Condition (stationary point)

$$\frac{\partial L}{\partial x_j} = 0 \text{ and } \frac{\partial L}{\partial \lambda_i} = 0 \Rightarrow \begin{cases} x^* = ? & \forall\, j = 1, 2, \dots, n \\ \lambda^* = ? & \forall\, i = 1, 2, \dots, m \end{cases}$$

Sufficient Condition

1st Method :- The necessary conditions become sufficient conditions for a maximum (minimum), if
- the objective function $f(X)$ is concave (convex) and
- the constraints are equality sign i.e., $g(X) = 0$.

2nd Method

Bordered Hessian Matrix :-

$$H^B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix}_{(m+n)\times(m+n)} ; \quad O = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{m\times n} ; \quad P = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} ; \quad Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

$\hookrightarrow$ Null matrix

Note: The necessary condition become sufficient conditions for a maximum (minimum) if objective function is concave (convex) and constraints are of equality type.

Algorithm :- Consider (x^*, λ^*) as stationary point for the function $L(x, \lambda)$.
Let H^B be the corresponding Bordered Hessian Matrix.
Then x^* is a :

i) Maximum point ; if starting with principal minor of order $(2m+1)$, compute the last $(n-m)$ principal minors of H^B form an alternating sign pattern and $\det(H_{m+n})$ has sign $(-1)^n$.

ii) Minimum point ; if starting with principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H^B have the sign of $(-1)^m$.

Constraint Optimization problem with one equality constraint:

Pb1 Use the Langrange Multiplier method to solve NLPP

Sol^n: opt $f(x) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

s.t. $x_1 + x_2 + x_3 = 20$

$x_1, x_2, x_3 \geq 0$

$L(x,\lambda) = f(x) + \lambda g(x) = (2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100) + \lambda(x_1 + x_2 + x_3 - 20)$

Necessary Cond^n for stationary point:

$\frac{\partial L}{\partial x_1} = 4x_1 + 10 + \lambda = 0 \longrightarrow x_1 = -\frac{(10+\lambda)}{4}$

$\frac{\partial L}{\partial x_2} = 2x_2 + 8 + \lambda = 0 \longrightarrow x_2 = -\frac{(8+\lambda)}{2}$

$\frac{\partial L}{\partial x_3} = 6x_3 + 6 + \lambda = 0 \longrightarrow x_3 = -\frac{(6+\lambda)}{6}$

put all these in eq^n (*);

$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 20 = 0 \longrightarrow$ (*)

we have $-\left(\frac{10+\lambda}{4}\right) - \left(\frac{8+\lambda}{2}\right) - \left(\frac{6+\lambda}{2}\right) - 20 = 0$

$\Rightarrow \lambda = -30$

$\Rightarrow X^* = (5, 11, 4) \longrightarrow$ Stationary point.

Method 2nd

$H^B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix}$; $O \rightarrow$ Null matrix of order $m \times m$ ($m \rightarrow$ No. of constraint.)

bordered Hessian matrix

$P = \left[\frac{\partial g_1}{\partial x_1} \quad \frac{\partial g_1}{\partial x_2} \quad \frac{\partial g_1}{\partial x_3}\right]_{1\times 3} = [1 \quad 1 \quad 1]_{1\times 3}$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$$

(In this case, $n = 3$)

$$H^B = \left[\begin{array}{c|ccc} 0 & 1 & 1 & 1 \\ \hline 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{array}\right]$$

$n \rightarrow$ no. of variables (x_1, x_2, x_3) ; $n = 3$, $m = 1$; $2m+1 = 2\cdot 1 + 1 = 3$

$m \rightarrow$ no. of constraints ;

$D_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$; $D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{vmatrix} = -44$

Both D_3 and D_4 have sign of $(-1)^m$;

x^* is a point of Minimum;

$f(x^*) = f(5, 11, 4) = 281$ (optimal value)

Method 1st

Hessian Metrix for f(X)

$H = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$D_1 = 4 > 0$, $D_2 > 0$, $D_3 > 0$

or all eigen values are +ve.

Thus, $f(x)$ is +ve definite. $\Rightarrow$ Convex function

Constraint Optimization problem with two equality constraints:

Pb 2. Solve NLPP

opt. $Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

s.t. $x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20$

Sol: $L = f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x)$

$= (4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2) + \lambda_1(x_1 + x_2 + x_3 - 15) + \lambda_2(2x_1 - x_2 + 2x_3 - 20)$

$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 + \lambda_1 + 2\lambda_2 = 0$

$\frac{\partial L}{\partial x_2} = 4x_2 - 4x_1 + \lambda_1 - \lambda_2 = 0$

$\frac{\partial L}{\partial x_3} = 2x_3 + \lambda_1 + 2\lambda_3 = 0$

$\frac{\partial L}{\partial \lambda_1} = x_1 + x_2 + x_3 - 15 = 0$

$\frac{\partial L}{\partial \lambda_2} = 2x_1 - x_2 + 2x_3 - 20 = 0$

stationary point.

$X^* = (x_1, x_2, x_3) = \left(\frac{33}{9}, \frac{10}{3}, 8\right)$

$\lambda^* = (\lambda_1, \lambda_2) = \left(\frac{40}{9}, \frac{52}{9}\right)$

Method 2:-

$$H^B = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ \hline 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{array}\right]$$

$n = 3$; $m = 2$

$\therefore 2m+1 = 5$;

$D_5 = |H^B| = 90 > 0$

Sign $= (-1)^m = (-1)^2 = +1$

$\therefore X^*$ is a minimum point.

$f(x^*) \approx 91.11$ (opt. value).

Method 1:-

Hessian Matrix for $f(x)$ $H = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$D_1 = 8 > 0$

$D_2 = 32 - 16 > 0$

$D_3 > 0$

$\longrightarrow$ All minors are positive and hence it is positive definite.

Thus, it is convex function.

Hence, the stationary point is a minimum point.

Pb 3 Solve the nonlinear programming problem:

$$\text{Max. } Z = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$$

s.t. $x_1 + x_2 + x_3 = 11$

$x_1, x_2, x_3 \geq 0$

Hint: Stationary point is $(x^*; \lambda^*) = (6, 2, 3; 0)$

2nd Method:-

$$H^B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix} \rightarrow D_3 = -8 < 0 \;;\; D_4 = -48 < 0$$

Thus, $(6, 2, 3)$ is a point of minima.

1st Method

$$H = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$ All eigen values are +ve. Hence, it is positive definite. Thus, it is convex function. Hence, the stationary point is a minimum point.

Pb 4 Solve the non-linear programming problem

$$Z = 3e^{2x_1+1} + 2e^{x_2+5}$$

s.t. $x_1 + x_2 = 7$

and $x_1, x_2 \geq 0$

Sol: Here, it's not given whether to maximize or minimize the objective function. So, first check the convexity or concavity of the function. If the given objective function is of convex (concave) type then the objectivity of the function is to minimize (maximize).

$$f(x) = 3e^{2x_1+1} + 2e^{x_2+5}$$

$$\frac{\partial f(x)}{\partial x_1} = 6e^{2x_1+1}, \quad \frac{\partial f}{\partial x_2} = 2e^{x_2+5}$$

$$H = \begin{bmatrix} 12e^{2x_1+1} & 0 \\ 0 & 2e^{x_2+5} \end{bmatrix}$$

The principal minors are

$$D_1 = 12e^{2x_1+1} > 0 \quad \forall\; x_1, x_2 \geq 0$$

$$D_2 = 24e^{2x_1+x_2+6} > 0 \quad \forall\; x_1, x_2 \geq 0$$

All minors are positive, so it is positive definite.

Thus, it is convex function, so the stationary point is a point of local minima. Now our objective function can be consider as of minimization type.

We need only what is the stationary point.

$$L = \left(3e^{2x_1+1} + 2e^{x_2+5}\right) - \lambda(x_1 + x_2 - 7)$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 6e^{2x_1+1} - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2e^{x_2+5} - \lambda = 0$$

$$\boxed{\lambda = 6e^{2x_1+1} = 2e^{x_2+5}}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow \boxed{x_1 + x_2 - 7 = 0}$$

$$x_1 = \frac{1}{3}\left(11 - \log_e 3\right)$$

$$x_2 = \frac{1}{3}\left(11 + \log_e 3\right)$$

Pb 5 Solve the following NLPP

$$\text{Min } Z = x_1^2 + x_2^2 + x_3^2$$

s.t. $4x_1 + x_2^2 + 2x_3 = 14$

and $x_1, x_2, x_3 \geq 0$

Sol

$$L = x_1^2 + x_2^2 + x_3^2 + \lambda\left(4x_1 + x_2^2 + 2x_3 - 14\right)$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 + 4\lambda = 0 \Rightarrow x_1 = -2\lambda \quad \text{--- (1)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 + 2x_2\lambda = 0 \Rightarrow x_2(1+\lambda) = 0 \quad \text{--- (2)}$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 + 2\lambda = 0 \Rightarrow x_3 = -\lambda \quad \text{--- (3)}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 4x_1 + x_2^2 + 2x_3 - 14 = 0 \quad \text{--- (4)}$$

from (2), we have

$x_2 = 0$ or $\lambda = -1$

$$(x_1, x_2, x_3; \lambda) = \left(\frac{14}{5}, 0, \frac{14}{10}; \frac{-14}{10}\right) \qquad (x_1, x_2, x_3; \lambda) = (2, 2, 1; -1)$$

$$H^B = \begin{bmatrix} 0 & 4 & 2x_2 & 2 \\ 4 & 2 & 0 & 0 \\ 2x_2 & 0 & 2(1+\lambda) & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}_{4\times 4}$$

$m = 1$ (# of const.)
$n = 3$ (# of variables)
$n - m = 2$ and
$2m + 1 = 3$

At $(2, 2, 1; -1)$ $D_3 = -16 < 0$; $D_4 = -64 < 0$

The given pt. is minimum & Min $Z = 9$.

At $\left(\frac{14}{5}, 0, \frac{14}{10}; \frac{-14}{10}\right)$: $D_3 = \frac{64}{5} > 0$; $D_4 = -80 < 0$

This pt. is max as the sign of D_3 & D_4 are alternative and sign of $D_4 = (-1)^n = (-1)^3 = -ve$.

Hence, the optimal sol^n is $Z_{min} = 9$ at $(2, 2, 1; -1)$

Pb 6. A positive quantity b is to be divided into n parts in such a way that the product of n parts is to be maximum. Use Langrage's multiplier technique to obtain the optimal sub-division.

Sol: Let b be divided into n parts $x_1, x_2, \ldots, x_n$ so that we have to

$$\text{Max } Z = x_1 x_2 \cdots x_n$$

$$\text{s.t.} \quad x_1 + x_2 + \cdots + x_n = b$$

$$\text{and} \quad x_1, x_2, \ldots, x_n \geq 0$$

$$L = (x_1 x_2 \cdots x_n) - \lambda(x_1 + x_2 + \cdots + x_n - b)$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow [x_2 x_3 \cdots x_n = \lambda] \times x_1 \quad \text{--- (1)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow [x_1 x_3 \cdots x_n = \lambda] \times x_2 \quad \text{--- (2)}$$

$$\vdots$$

$$\frac{\partial L}{\partial x_n} = 0 \Rightarrow [x_1 x_2 \cdots x_{n-1} = \lambda] \times x_n \quad \text{--- (n)}$$

and $\frac{\partial L}{\partial \lambda} = 0$

$$\Rightarrow \boxed{x_1 + x_2 + \cdots + x_n = b}$$

$$n(x_1 x_2 \cdots x_n) = \lambda(x_1 + x_2 + \cdots + x_n)$$

$$\lambda = \frac{n(x_1 x_2 \cdots x_n)}{b}$$

Hence, from (1), (2), ... (n), we get

$$x_1 = \frac{b}{n}, \quad x_2 = \frac{b}{n}, \quad \ldots, \quad x_n = \frac{b}{n}$$

$$\boxed{\text{Max } Z = \left(\frac{b}{n}\right)^n}$$

Constrained Optimization problem with inequality constraint

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Let $f, g_i : R^n \longrightarrow R$ for $i = 1, 2, \ldots, m$ be continuously diff. function

Consider $\left\{ \begin{array}{ll} \text{Max/Min} & f(x) \\ \text{s.t.} & g_i(x) \leq 0 \quad \text{for } i = 1, 2, \ldots, m \end{array} \right\} \longrightarrow (*)$

Method - 1

$$\left\{ \begin{array}{ll} \text{Max/Min} & f(x) \\ \text{s.t.} & g_i(x) + s_i^2 = 0 \quad , \; i = 1, 2, \ldots m \end{array} \right.$$

slacks variable.

$$\left\{ \begin{array}{ll} \text{Max/Min} & f(x) \\ \text{s.t.} & h(x) = 0 \; ; \text{ where } \; h(x) = g_i(x) + s_i^2 \end{array} \right.$$

Apply Langrangian Method.

Method 2 Karush - Kuhn - Tucker Condition (KKT - condition)

(Necessary & Sufficient Conditions)

We can solve the constrained problem with inequality constraint using KKT - condition under certain circumstances.

Maximize $f(x)$

s.t. $g_i(x) \leq 0$

$$L(\lambda, x) = f(x) - \sum_i \lambda_i g_i(x)$$

Necessary Condition For critical points

1) $\dfrac{\partial f}{\partial x_j} - \sum_i \lambda_i \dfrac{\partial g_i}{\partial x_j} = 0 \quad \forall j = 1, 2, \ldots, n$

2) $\lambda_i g_i(x) = 0 \quad , \forall i = 1, \ldots, m$

3) $g_i(x) \leq 0 \quad \forall i$

4) $\lambda_i \geq 0 \quad \forall i$

Minimization $f(x)$

s.t. $g_i(x) \leq 0$

$$L(\lambda, x) = f(x) - \sum_i \lambda_i g_i(x)$$

Necessary Condition For critical points

1) $\dfrac{\partial f}{\partial x_j} - \sum_i \lambda_i \dfrac{\partial g_i}{\partial x_j} = 0 \quad \forall j = 1, 2, \ldots, n$

2) $\lambda_i g_i(x) = 0 \quad , \forall i = 1, \ldots, m$

3) $g_i(x) \leq 0 \quad \forall i$

4) $\lambda_i \leq 0 \quad \forall i$

Pbl

Write the KKT condition of the NLPP

1) $\text{Max } Z = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

s.t. $x_1 + x_2 \le 10$

$x_2 \le 8$

$x_1, x_2 \ge 0$

Sol

$$L(X,\lambda) = f(X) - \lambda_1 g_1(X) - \lambda_2 g_2(X)$$

$$= \left(12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2\right) - \lambda_1\left(x_1 + x_2 - 10\right) - \lambda_2\left(x_2 - 8\right)$$

The necessary conditions are

$\frac{\partial L}{\partial x} = 0$: $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$

$\lambda_i g_i = 0$: $\lambda_1 g_1 = 0$, $\lambda_2 g_2 = 0$

$g_i \le 0$ → $g_1(x) \le 0$, $g_2(x) \le 0$

$\lambda_i \ge 0$ → $\lambda_1 \ge 0$, $\lambda_2 \ge 0$

$X \ge 0$ → $\boxed{x_1, x_2 \ge 0}$

Given feasible region

	Problem Types	Necessary Condition For critical point	Sufficient Conditions for KKT	Conclusion for optimal point
i)	Max $f(x)$ s.t. $g_i(x) \le 0$	$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial f(x)}{\partial x} - \sum \lambda_i \frac{\partial g_i(x)}{\partial(x)} = 0$; $\lambda g_i = 0$; $g_i \le 0$; $\lambda_i \ge 0$; $L = f(x) - \sum \lambda_i g_i(x)$	f is concave and all g_i are convex.	Global maximum
ii)	Min $f(x)$ s.t. $g_i(x) \le 0$	$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial f(x)}{\partial x} - \sum \lambda_i \frac{\partial g_i(x)}{\partial(x)} = 0$; $\lambda g_i = 0$; $g_i \le 0$; $\lambda_i \le 0$; $L = f(x) - \sum \lambda_i g_i(x)$	f is convex and all g_i are convex.	Global minimum
iii)	Max $f(x)$ s.t. $g_i(x) \ge 0$	$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial f(x)}{\partial x} + \sum \lambda_i \frac{\partial g_i(x)}{\partial(x)} = 0$; $\lambda g_i = 0$; $g_i \ge 0$; $\lambda_i \ge 0$	f is concave and all g_i are concave	Global maximum
iv)	Min $f(x)$ s.t. $g_i(x) \ge 0$	$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial f(x)}{\partial x} + \sum \lambda_i \frac{\partial g_i(x)}{\partial(x)} = 0$; $\lambda g_i = 0$; $g_i \ge 0$; $\lambda_i \le 0$	f is convex and all g_i are concave	Global minimum

Remark:

- A function which is both convex and concave, then it has to be linear function.
- Domain of the convex function is convex set.
- Every U-shaped function is convex function. **Ex:** $f(x) = x^2 \quad \forall\ x \in \mathbb{R}$
- A function may neither be convex nor concave. **Ex:** $f(x) = x^3 \quad \forall\ x \in \mathbb{R}$
- **Convex function** is always to **minimize the problem.**
- **Concave function** is always to **maximize the problem.**

Pb 1 Solve the NLPP

$$\text{Max } Z = 36x_1 - 4x_1^2 + 16x_2 - 2x_2^2$$

$$\text{s.t.} \quad 2x_1 + x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

Sol For the KKT conditions to be necessary and sufficient for Z to a maximum, f(X) should be concave and for g(X) ≤ 0 type, g(X) must be convex.

for $f(x)$

$$\frac{\partial f}{\partial x_1} = 36 - 8x_1$$

$$\frac{\partial f}{\partial x_2} = 16 - 4x_2$$

$$H = \begin{bmatrix} -8 & 0 \\ 0 & -4 \end{bmatrix} \longrightarrow D_1 < 0, \quad D_2 > 0$$

H — Negative definite matrix

$\Rightarrow f(x)$ concave function.

for $g(X)$: linear (always convex, why?)

Every linear function is convex.

$$H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \text{+ve semi-definite}$$

$\Rightarrow g(X)$ is convex.

Hence the KKT conditions are the sufficient conditions for the maximum

Define the Lagrangian function as

$$L = f(x) - \lambda\, g(x)$$

$$= (36x_1 - 4x_1^2 - 16x_2 - 2x_2^2) - \lambda(2x_1 + x_2 - 10)$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 36 - 8x_1 - 2\lambda = 0 \quad \text{— (1)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 16 - 4x_2 - \lambda = 0 \quad \text{— (2)}$$

$$\lambda g = 0 \Rightarrow \lambda(2x_1 + x_2 - 10) = 0 \quad \text{— (3)}$$

$$\lambda \geq 0 \Rightarrow \lambda \geq 0 \quad \text{— (4)}$$

$$g \leq 0 \Rightarrow 2x_1 + x_2 \leq 10 \quad \text{— (5)}$$

$$X \geq 0 \Rightarrow x_1, x_2 \geq 0 \quad \text{— (6)}$$

Case 1 when $\lambda = 0$;

from (1) & (2), we have

$$x_1 = 4.5 \text{ and } x_2 = 4$$

which does not satisfy (5) and hence this case is discard.

Case 2 :- when $\lambda \neq 0$;

from (3), we get

$$2x_1 + x_2 = 10$$

from (1) & (2),

$$x_1 = \frac{36 - 2\lambda}{8} \; ; \quad x_2 = \frac{16 - \lambda}{4}$$

$$\therefore \; 2\left(\frac{36 - 2\lambda}{8}\right) + \left(\frac{16 - \lambda}{4}\right) = 10$$

$$\Rightarrow \lambda = 4$$

and hence $x_1 = 3.5$; $x_2 = 3$

Thus the stationary point is

$$(x_1, x_2 ; \lambda) = (3.5, 3 ; 4)$$

and the optimal value is

$$Z = 107 \quad \text{Ans}$$

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Pb 2 Solve the following NLPP

$$\text{Min } Z = -\log x_1 - \log x_2$$

s.t. $x_1 + x_2 \leq 2$

$x_1, x_2 \geq 0$

Pb 3 Solve the following NLPP

$$\text{Max. } Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$$

s.t. $3x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

Pb4 Solve the NLPP

$$\text{Max } f(x) = 4x_1 + 6x_2 - x_1^2 - x_2^2 - x_3^2$$

s.t. $x_1 + x_2 \le 2$

$2x_1 + 3x_2 \le 12$

$x_1, x_2 \ge 0$

For the KKT conditions to be necessary and sufficient for Z to a maximum, f(X) should be concave and for g(X) ≤ 0 type, g(X) must be convex.

Pb5 Solve the NLPP

$$\text{Min } Z = (x_1-2)^2 + (x_2-1)^2$$

s.t. $x_1^2 - x_2 \le 0$

$x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

For the KKT conditions to be necessary and sufficient for Z to a minimum, f(X) should be convex and for g(X) ≤ 0 type, g(X) must be convex.

for $f(X) = (x_1-2)^2 + (x_2-1)^2$ $\qquad \dfrac{\partial f}{\partial x_1} = 2(x_1-2)$, $\quad \dfrac{\partial f}{\partial x_2} = 2(x_2-1)$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \text{+ve def.}$$

$\Rightarrow f(x)$ is convex.

for $g_1(X) = x_1^2 - x_2$

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \text{+ve definite}$$

$\Rightarrow g_1(x)$ is convex.

for $g_2(X) = x_1 + x_2 - 2$ Linear hence convex.

Thus, KKT conditions will be necessary & sufficient for minimum.

$$L = (x_1-2)^2 + (x_2-1)^2 - \lambda_1(x_1^2 - x_2) - \lambda_2(x_1 + x_2 - 2)$$

$$\boxed{\frac{\partial L}{\partial x} = 0\ ; \quad \lambda_i g_i = 0\ ; \quad \lambda_i \le 0\ ; \quad g_i \le 0\ ; \quad x \ge 0}$$

$2(x_1-2) - 2\lambda_1 x_1 - \lambda_2 = 0$ — (1)

$2(x_2-1) + \lambda_1 - \lambda_2 = 0$ — (2)

$\lambda_1(x_1^2 - x_2) = 0$ — (3)

$\lambda_2(x_1 + x_2 - 2) = 0$ — (4)

$\lambda_1, \lambda_2 \le 0$ — (5)

$x_1^2 - x_2 \le 0$ — (6)

$x_1 + x_2 \le 2$ — (7)

$x_1, x_2 \ge 0$ — (8)

Case 1:- when $\lambda_1 = 0$; $\lambda_2 = 0$

from (1) & (2), we have

$x_1 = 2$, $x_2 = 1$

but this not satisfy (7) $\qquad$ X

Case 2:- When $\lambda_1 \ne 0$; $\lambda_2 \ne 0$

from (3) & (4), we have

$$\left.\begin{aligned} x_1^2 - x_2 &= 0 \\ x_1 + x_2 - 2 &= 0 \end{aligned}\right\} \Rightarrow x_1 = 1, -2 \quad (\because x_1 \ge 0)$$

Thus, $x_2 = 1$

$\Rightarrow$ from (1) & (2); $\underbrace{\lambda_1 = \lambda_2 = -2/3}_{\text{also satisfy (5)}}$

Thus, stationary pt. $\left(1, 1; \frac{-2}{3}, \frac{-2}{3}\right)$
$(x_1, x_2; \lambda_1, \lambda_2)$

Thus, Min Z = 1 Ans

Case 3 when $\lambda_1 = 0$ & $\lambda_2 \ne 0$

Case 4 when $\lambda_1 \ne 0$ & $\lambda_2 = 0$

both of these cases fail (Check - why)