Spring-Pendulum system

The constants we are using are gravitational constant g, hook's constant of the spring k, original length of spring l_0, temporary length of spring l, angle of spring from vertical θ, and mass of the pendulum m. The general assumptions are negligible friction and light spring.

We deduce the control function of the spring using Euler-Lagrange Equation. The kinetic term of Lagrangian can be deduced with x and y, in which case

$$x = l \sin \theta$$
$$y = -l \cos \theta$$

taking derivative against time on both sides gives

$$\dot{x} = \dot{l} \sin \theta + l\dot{\theta} \cos \theta$$
$$\dot{y} = -\dot{l} \cos \theta + \dot{l}\theta \sin \theta$$

Kinetic energy is given by

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

Substitute $\dot{x}$ and $\dot{y}$ in yields

$$T = \frac{1}{2} m(\dot{l}^2 + l^2\dot{\theta}^2)$$

While another more intuitional approach is decompose the movement to radical and tangential, which naturally gives the same result.

The potential energy is

$$V = \frac{1}{2}k(l - l_0)^2 - mgl\cos\theta$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{l}^2 + l^2\dot{\theta}^2) - \frac{1}{2}k(l - l_0)^2 + mgl\cos\theta$$

The degree of freedom is two, l and θ. Apply the Lagrange Equation and collect the terms gives

$$\ddot{l} = l\dot{\theta}^2 + g\cos\theta - \frac{k}{m}(l - l_0)$$
$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{2\dot{l}}{l}\dot{\theta}$$

Which is the control equations.