Double Pendulum System

The constants we are using are gravitational constant g, length of two strings l_1, l_2, angle of two strings from vertical θ_1, θ_2, and two masses m_1, m_2. The general assumptions is negligible friction and conservation of mechanic energy.

We deduce the control function of the spring using Euler-Lagrange Equation. The kinetic term of Lagrangian can be deduced with coordinate x and y in which case

$$
\begin{gathered}
x_1 = l_1 \sin\theta_1 \\
y_1 = -l_1 \cos\theta_1 \\
x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \\
y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2
\end{gathered}
$$

The kinetic energy is

$$
\begin{gathered}
T = \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2) \\
= \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2 l_2^2\dot{\theta_2}^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)
\end{gathered}
$$

Or, another way to view it is take $\vec{v_1} = l_1\dot{\vec{\theta_1}}$, $\vec{v_2} = l_2\dot{\vec{\theta_2}}$, therefore the kinetic energy is written as

$$
\frac{1}{2}m_1(\vec{v_1}\cdot\vec{v_1}) + \frac{1}{2}m_2(\vec{v_1} + \vec{v_2})\cdot(\vec{v_1} + \vec{v_2})
$$

which gives the same result.

The potential energy term is simply

$$
V = -(m_1 + m_2)gl\cos\theta_1 - m_2 gl\cos\theta_2
$$

The Lagrangian is

$$L = T - V$$

The system has a degree of freedom of two, with generalized coordinate of θ_1 and θ_2, therefore we apply the Lagrange Equation

$$\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= \frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial L}{\partial \dot{\theta}_1} \\
\frac{\partial L}{\partial \theta_2} &= \frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial L}{\partial \dot{\theta}_2}
\end{aligned}$$

Collect the terms, there is

$$\begin{gathered}
(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 = -m_2 l_2 \dot{\theta}_2^{\,2} \sin(\theta_1 - \theta_2) \\
-(m_1 + m_2) g \sin\theta_1 \\
l_1 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 + l_2 \ddot{\theta}_2 = l_1 \dot{\theta}_1^{\,2} \sin(\theta_1 - \theta_2) - g \sin\theta_2
\end{gathered}$$

Therefore we have a binary linear system. The system can be solved with scipy.linalg library in the form of matrix:

$$\begin{bmatrix} (m_1 + m_2) l_1 & m_2 l_2 \cos(\theta_1 - \theta_2) \\ l_1 \cos(\theta_1 - \theta_2) & l_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -m_2 l_2 \dot{\theta}_2^{\,2} \sin(\theta_1 - \theta_2) - (m_1 + m_2) g \sin\theta_1 \\ l_1 \dot{\theta}_1^{\,2} \sin(\theta_1 - \theta_2) - g \sin\theta_2 \end{bmatrix}$$