Nested Circles

Ting Gong, Nick VanderLaan, and Nikhil Shankar

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Abstract

Consider any sequence of nested circles. If the radii of the circles decreases sufficiently quickly, the sequence will converge to some point. This paper will explore how imposing certain conditions on the sequence of circles produces different sets of limit points.

1 Introduction

We will define our sequence of nested circles (C_n) in terms of $(r_n) \subset \mathbb{R}$ and $(x_n) \subset \mathbb{C}$. The radius of each C_k is $r_k = 1/2^k$, and the centerpoint is $x_k = x_{k-1} + \lambda_k r_k$ with $\lambda_k \in \{+1, -1, +i, -i\}$. Let $x_0 = 0$, implying C_0 is the unit circle, and now we may describe (C_n) simply by specifying (λ_n).

1.1 Notation

We will use the following definitions and notation throughout the paper

Definition 1.1. Let $z = a + bi \in \mathbb{C}$. We define the **real part** of z as $\Re(z) = a$ and the **imaginary part** as $\Im(z) = b$.

Definition 1.2. The $\mathcal{L}^1$ or **taxicab norm** is a metric on $\mathbb{C}$ given by

$$z \mapsto |\,\Re(z)| + |\,\Im(z)|$$

Definition 1.3. The $\mathcal{L}^2$ or **Euclidean norm** is a metric on $\mathbb{C}$ given by

$$z \mapsto \sqrt{\Re(z)^2 + \Im(z)^2}$$

Definition 1.4. The **unit diamond** is the set $\{z \in \mathbb{C} : \mathcal{L}^1(0, z) \leq 1\}$.

Definition 1.5. The **unit circle** is the set $C_0 = \{z \in \mathbb{C} : \mathcal{L}^2(0, z) = 1\}$. The **unit disk** is the set $D = \{z \in \mathbb{C} : \mathcal{L}^2(0, z) \leq 1\}$.

Since a sequence (C_n) is fully determined by the corresponding sequence (λ_n), we will introduce what we believe is a convenient way to describe (λ_n) in the complex plane. For the remainder of this paper a given (λ_n) will be identified with a sequence of cardinal directions drawn from $\{N, S, E, W\}$. For example, $(i, i, -i, 1\ldots) \cong (N, N, S, E\ldots)$ where N and S represent i and $-i$, and E and W represent 1 and -1. We would informally describe the above example as moving N twice, then S, and then E. Also informally, the N circle refers to the C_k obtained by moving N on the kth step of the sequence.

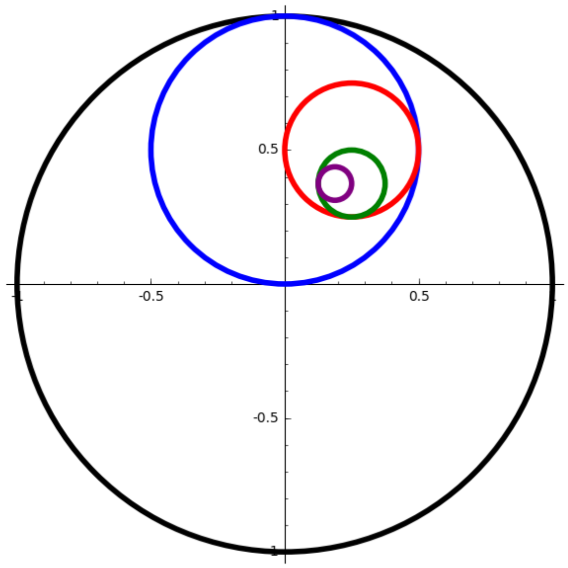

Figure 1: C_0 with the sequence N, E, S, W

We will now explore the set of limit points of such sequences of circles.

2 Preliminary Results

Lemma 2.1. *$\forall z \in D$, the unit disk, s.t. $\mathcal{L}^1(0, z) > 1$ there is no sequence (C_n) which converges to z.*

Proof. Fix such a z, and pick an arbitrary (C_n). Consider the sequence of centerpoints and radii associated with it: (x_n) and (r_n). Note that $\mathcal{L}^1(0, z) > \sum_{i=1}^{k} |r_i| = 1 \geq \mathcal{L}^1(0, x_k)\ \forall k \in \mathbb{N}$. But then any z s.t. $\mathcal{L}^1(0, z) > 1$ cannot be the limit of (C_n). $\square$

Theorem 2.1. *There exists a sequence of circles that converges to $z \in D$ if and only if z is in the unit diamond.*

Proof. The above lemma provides the forward direction, we will now show the reverse direction is true.

Remark. Each step in the sequence is simply a scaled and translated version of the original problem (i.e. it makes little geometric difference whether we start at C_0 or C_{43}). This means if we prove a fact for the first step, we can often inductively argue that it is true for the entire sequence (C_n).

Fix z in the unit diamond and note it satisfies $\mathcal{L}^1(0,z) = \mathcal{L}^1(x_0,z) \leq 1 = r_0$. We would like to show that we can choose a C_1 such that $\mathcal{L}^1(x_1,z) \leq r_1 = \frac{1}{2}$. If this is possible, then we can redefine x_1 as the origin and use a scaled argument to choose a C_2 within C_1, and continue in this way.

Well, let's split up the unit diamond by considering each possible x_1 and then the sets $\{z \in \mathbb{C} \,|\, \mathcal{L}^1(x_1,z) \leq r_1\}$. We will refer to these sets as the sub-diamonds. It is clear z must be in at least one of the sub-diamonds.

Remark. It is possible that more than one choice still exist for C_1 if $\Re(z) = \pm\,\Im(z)$, in that scenario either choice is valid, and we will return to this idea in the following section.

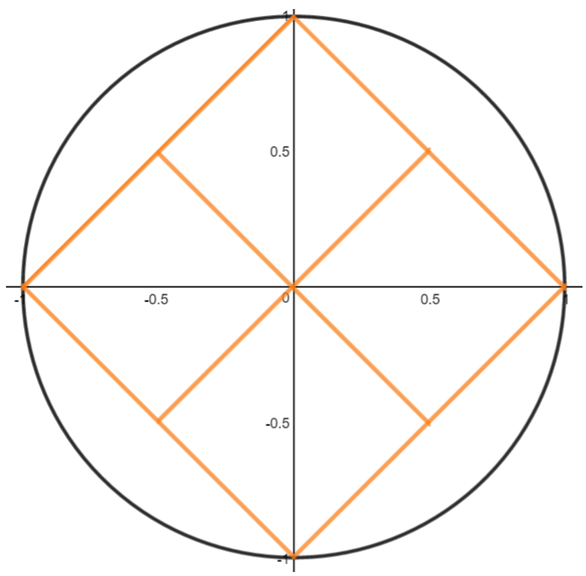

Figure 2: z must be in one of the sub-diamonds

Simply choose a sub-diamond that contains z, and note that x_1 is the "center" of this sub-diamond, and is also the center of a possible C_1. At this point we've reached an analogous position to C_0, wherein $\mathcal{L}^1(x_1,z) \leq r_1 = \frac{1}{2}$. We can now repeat this process arbitrarily many times with $\mathcal{L}^1(x_k,z) \leq r_k$, but then certainly $x_k \to z$. $\square$

Remark. We may also consider the possible sub-diamonds of any kth step, which visually appear as a criss-crossed net that grows finer on every iteration. This idea will be further discussed in the following section.

3 Uniqueness

When are points z in the unit diamond the limit of a unique sequence of circles? And how many distinct sequences for other points?

We can see that not every point has a unique sequence of circles converging to it: 0 is the limit of the sequences $(N, S, S, \ldots)$ and also $(S, N, N, \ldots)$. However, there are also points that do have unique sequences, namely 1 is uniquely the limit of $(E, E, E, \ldots)$.

Definition 3.1. If we consider an arbitrary possible centerpoint x_k for $k \in \mathbb{N}$, a **sub-diamond** is said to be $\{z \in \mathbb{C} \,|\, \mathcal{L}^1(x_k, z) = r_k\}$.

Lemma 3.1. *Suppose z a point in the interior of the unit diamond.*

1. *If z is the corner of some sub-diamond, z has at least 4 distinct sequences converging to it.*
2. *If z is on the boundary of some sub-diamond (excluding the corners), z has at least 2 distinct sequences converging to it.*
3. *If z is never on the boundary of any sub-diamond for any kth step, z has a unique sequence converging to it.*

Proof.

1. If z an interior point, and on the corner of a sub-diamond, it must be on the corner of four equally sized sub-diamonds. But by definition these 4 sub-diamonds correspond to 4 circles which intersect at z. Since we may reach any of these circles in a finite number of moves, it suffices to simply move towards z for all remaining moves. Since the 4 sequences have at least one term different from one another, they are distinct.

2. If z an interior point and on the boundary of a sub-diamond, it must be on the boundary of at least two sub-diamonds. But then similar to the above proof, we may reach the two corresponding circles that contain z in finitely many moves. From that point onwards we can converge to z, in at least two distinct ways.

3. If z an interior point and never on the boundary of any sub-diamond, then on each kth move there is a unique choice of sub-diamond which contains z. Hence there is a unique C_k we are forced to pick to converge to z, making the sequence unique.

□

Lemma 3.2. *Suppose z a point on the boundary of the unit diamond excluding the corners.*

1. *If z is also a corner of a some sub-diamond, z has 2 distinct sequences converging to it.*
2. *If z is never a corner of any sub-diamond for any kth step, z has a unique sequence converging to it.*

Proof.

1. This proof is extremely similar to proof (1) above, but the corner is only on the intersection of 2 possible circles instead of 4 possible circles.

2. This proof is similar to proof (3) above. □

Definition 3.2. A number is said to be ***dyadic*** if it can be written as $\frac{l}{2^k}$ for some $l \in \mathbb{Z}, k \in \mathbb{N}$.

Theorem 3.1. *Given $z = a + bi \in D$, the number of sequences converging to z is as following:*

1. *for all z such that $\mathcal{L}^1(0, z) < 1$, i.e, z is in the interior of the unit diamond.*
   - *(a) if both $a+b = d_1$ and $a-b = d_2$ hold for some d_1, d_2 dyadic numbers, then there are at least 4 distinct sequences converging to z.*
   - *(b) if either $a+b = d_1$ or $a-b = d_2$ holds for some d_1, d_2 dyadic numbers, then there are at least 2 distinct sequences converging to z.*
   - *(c) if neither $a+b = d_1$ nor $a-b = d_2$ holds for all d_1, d_2 dyadic numbers, then there is a unique sequence converging to z.*
2. *for all z such that $\mathcal{L}^1(0, z) = 1$, i.e, z is on the boundary of the unit diamond.*
   - *(d) if $0 < |a| < 1$ and $|a| = d_1$ for some d_1 dyadic number, then there are at least two distinct sequences converging to z.*
   - *(e) if either $|a| = 1$ or $|b| = 1$, then there is a unique sequences converging to z.*
   - *(f) if neither $a = d_1$ nor $b = d_2$ holds for all d_1, d_2 dyadic numbers, then there is a unique sequence converging to z.*

Proof.

(a) Note that if the point z satisfies both equations, that means now the point z in the graph is the intersection of the two lines $a+b-d_1 = 0$ and $a-b-d_2 = 0$. Thus, it is a corner of a diamond. Thus, by the Lemma 3.1.1, we can conclude that it has at least 4 distinct sequences.

(b) Note that if the point z satisfies only one of the two equations, then the point can only be on either line $a + b - d_1 = 0$ or line $a - b - d_2 = 0$ but can never be on both, i.e, the intersection of the two lines. Thus, by the Lemma 3.1.2, we can conclude that it has at least 2 distinct sequences.

(c) Note that if the point z satisfies neither one of the two equations, then the point can neither be on the line $a+b-d_1 = 0$ nor the line $a-b-d_2 = 0$. Thus, by the Lemma 3.1.3, we can conclude that it has a unique sequence.

(d) Note that if $0 < |a| < 1$ and $|a| = d_1$ for some d_1 dyadic number, then since $|a| + |b| = 1$, $0 < |b| < 1$ and $|b|$ is also dyadic. Also, now both $a + b$ and $a - b$ are some dyadic numbers. Then by the proof (a) and Lemma 3.2.1, we can conclude that there are at least two distinct sequences converging to z.

(e) Let (C_n) be a sequence of circles converging to 1. Since each C_i is closed, we have $1 \in C_i \; \forall i$. Thus we know that C_1 must correspond to an initial direction of E. From there, we proceed inductively. Suppose the k-th direction in our sequence is E. The $k+1$-th direction must again be E, else $C_{k+1} \not\ni 1$, therefore C_n is uniquely determined by the sequence $(E, E, E, \ldots)$. The proofs for $-1, i, -i$ are identical, with unique sequences $(W, W, W, \ldots)$ converging to -1; $(N, N, N, \ldots)$ converging to i; and $(S, S, S, \ldots)$ converging to $-i$.

(f) This can be easily proved by (c) and Lemma 3.2.2.

$\square$