Spectral theory of signal simulation and AI

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Special thanks to Vladimir Syuzev and Elena Smirnova

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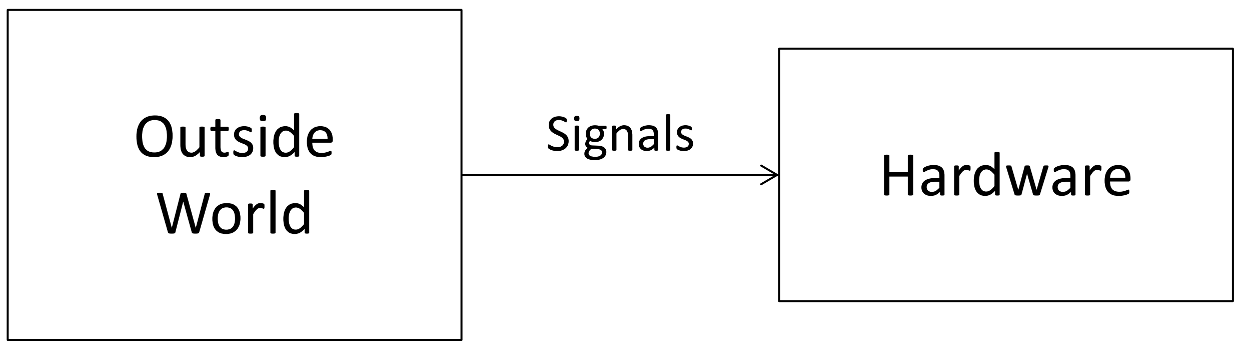
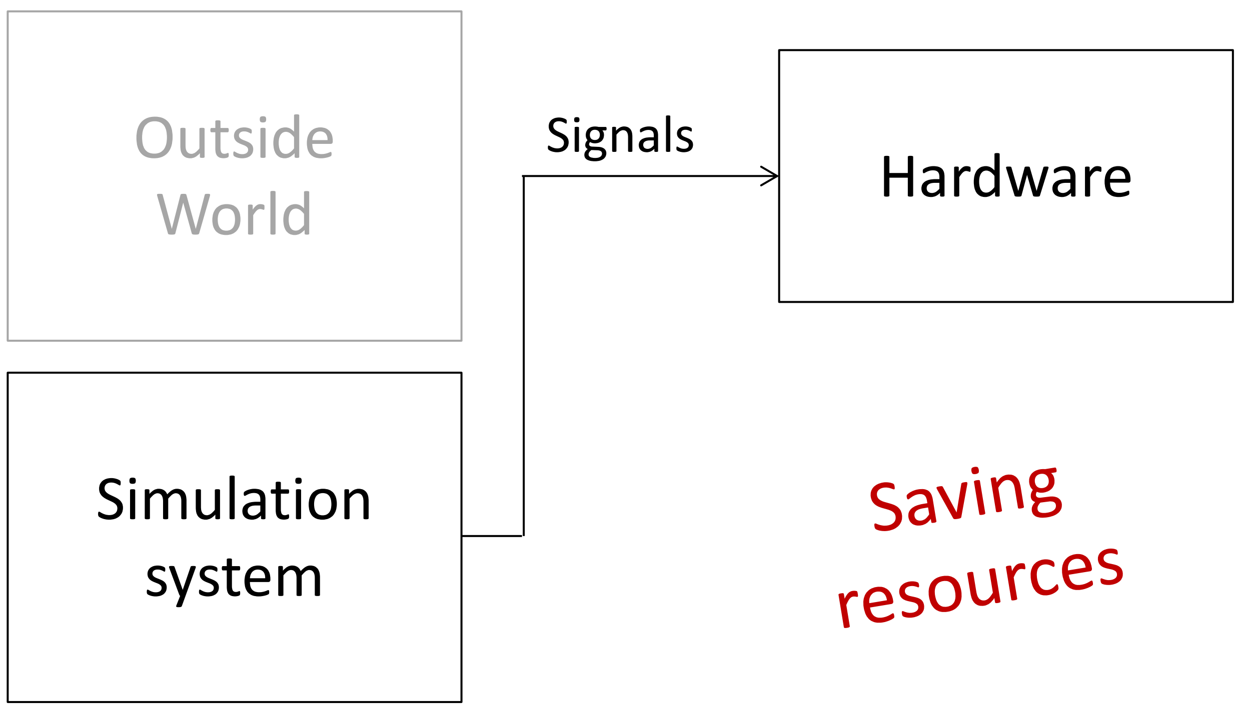
Signal simulation

Hardware-in-the-loop



Signal simulation

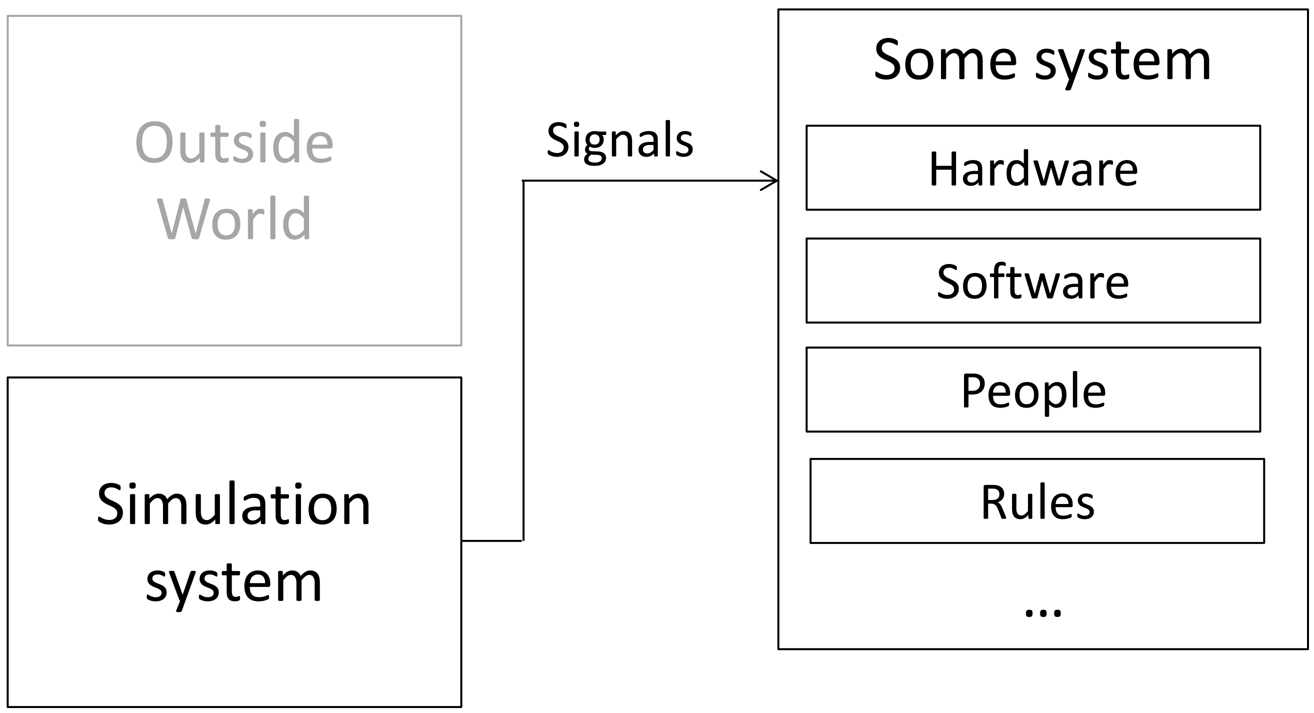
Hardware-in-the-loop



Saving resources

Signal simulation

Hardware-in-the-loop

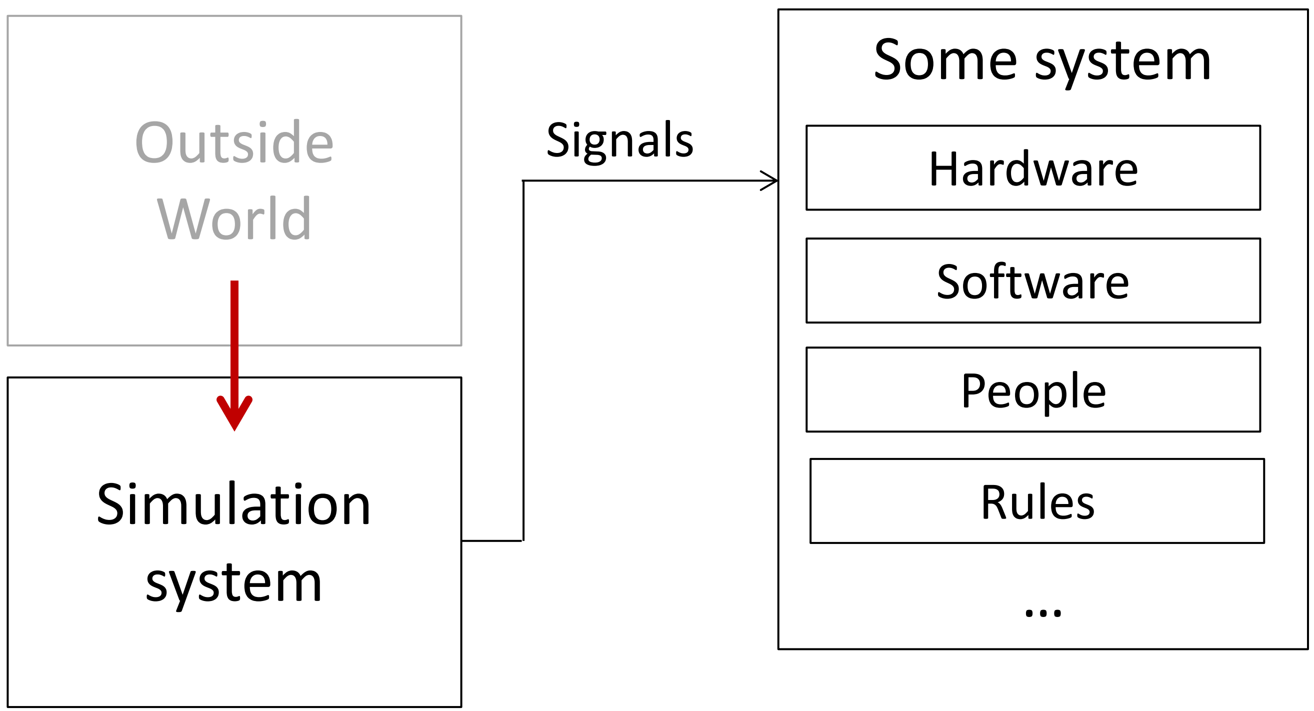


Example: forest fire prevention system

- Does hardware work correctly?
- Is software OK?
- Is staff properly trained?
- Are the rules adequate?

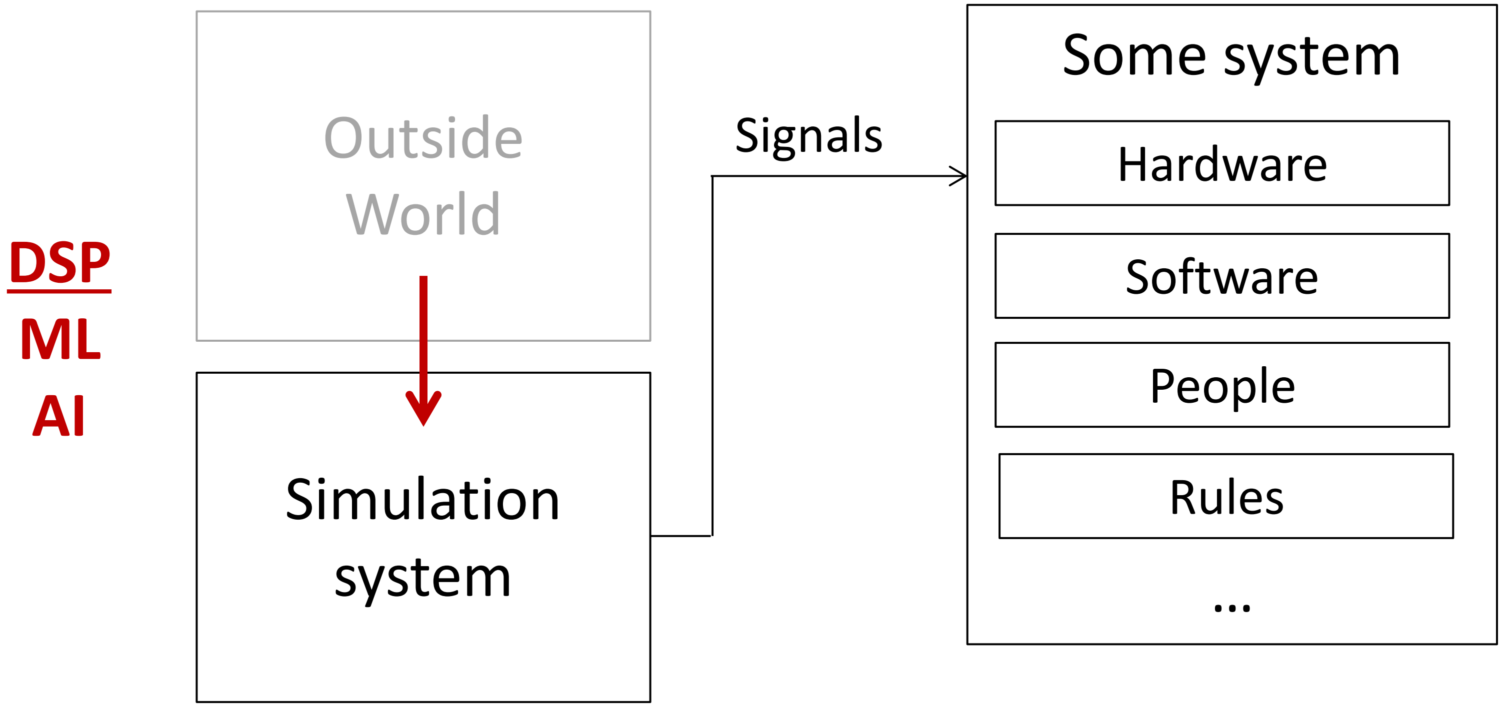
Signal simulation

Hardware-in-the-loop



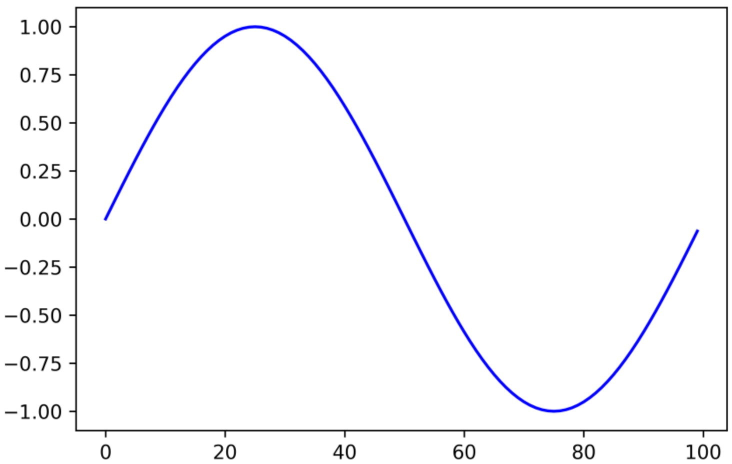
Signal simulation

Hardware-in-the-loop



1D signal

$$x(\mathrm{t}) = \sin(t) + \sin(8t)\,;\; T = 2\pi, N = 100.$$



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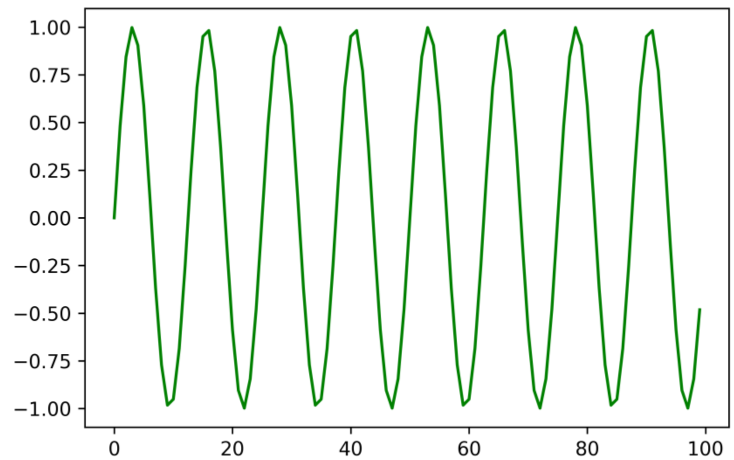

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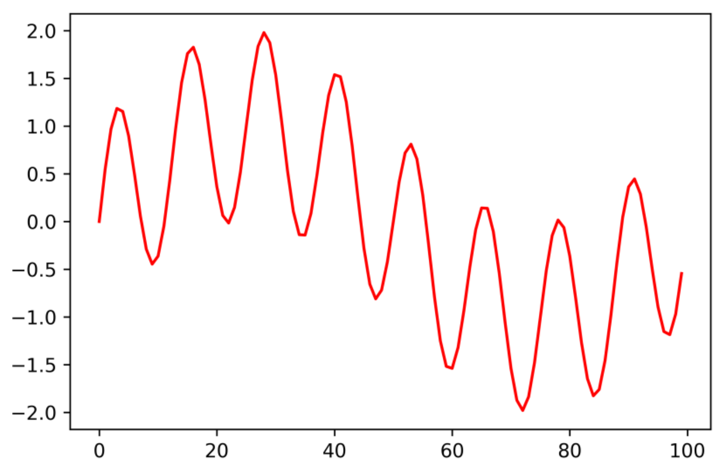


Time and frequency

$$f = \frac{n}{t} = \frac{1}{T}.$$

f – frequency – “the number of occurrences of a repeating event per unit of time”;

n – repeats of something within T;

T – period.

Fourier Transform

Fourier – from time domain to frequency domain:

$$\mathcal{F}\big(x(\mathrm{t})\big) = X(f) = \int_{-\infty}^{+\infty} x(\mathrm{t})e^{-2\pi jf\mathrm{t}}dt,$$

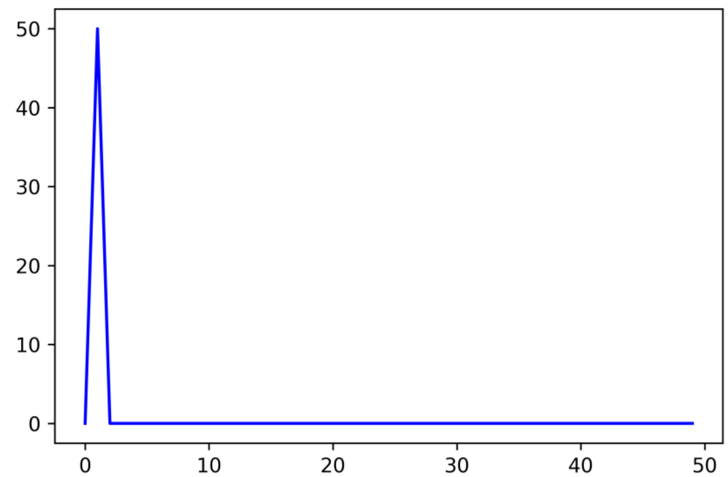
Inverse Fourier – from frequency to time:

$$\mathcal{F}^{-1}\big(X(f)\big) = x(t) = \int_{-\infty}^{+\infty} X(f)e^{2\pi jf\mathrm{t}}df\,.$$

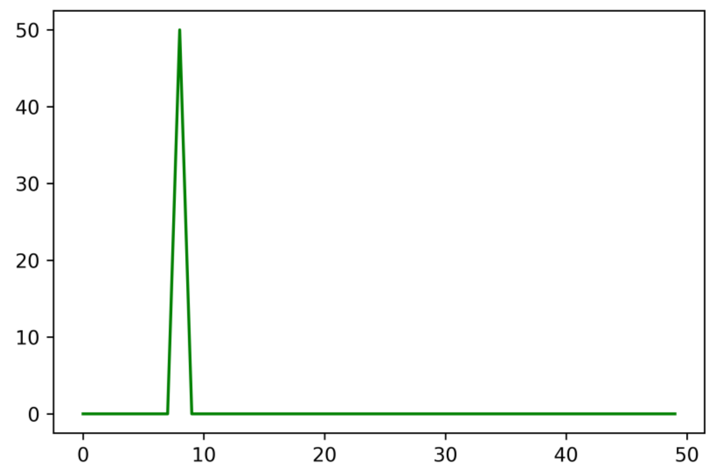
$$\omega = 2\pi f.$$

Fourier Transform

$$\mathcal{F}\big(sin(\mathrm{t})\big) =$$

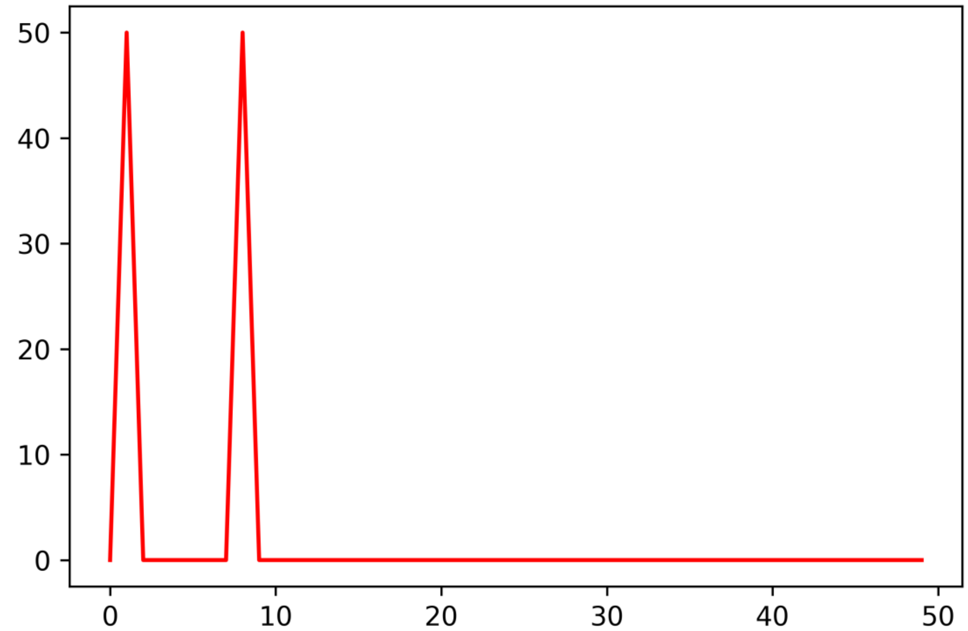


$$\mathcal{F}\big(sin(8\mathrm{t})\big) =$$



Fourier Transform

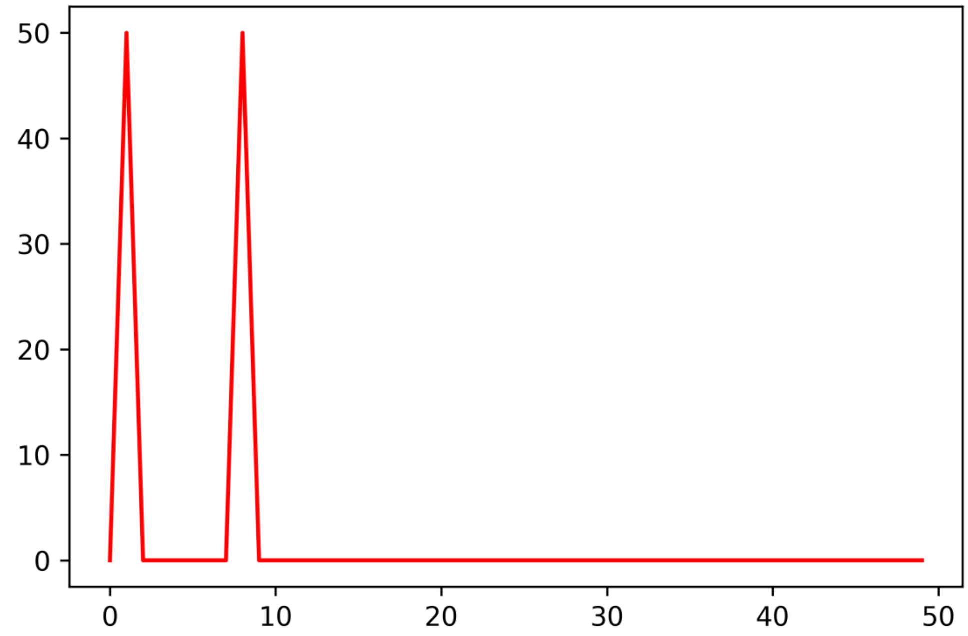
$$\mathcal{F}\big(x(\mathrm{t})\big) =$$



$\mathcal{F}\big(x(\mathrm{t})\big)$ shows how "strong" or "loud" different frequencies are within a signal.

Fourier Transform

$$\mathcal{F}\big(x(\mathrm{t})\big) =$$

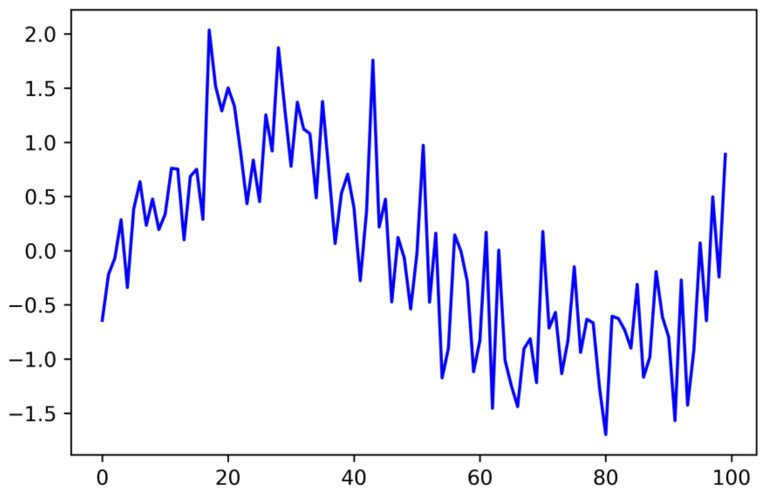


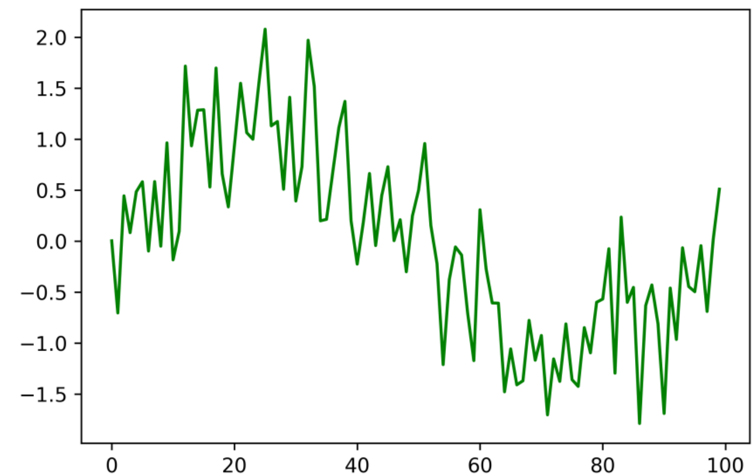
$\mathcal{F}\big(x(\mathrm{t})\big)$=$\mathcal{F}\big(sin(\mathrm{t})\big) + \mathcal{F}\big(sin(8\mathrm{t})\big)$ - we can filter $\mathcal{F}\big(x(\mathrm{t})\big)$ and use $\mathcal{F}^{-1}\big(X'(f)\big)$ to reproduce the new signal

Random signal

is a signal that changes when experiment is repeated (the world is noisy).

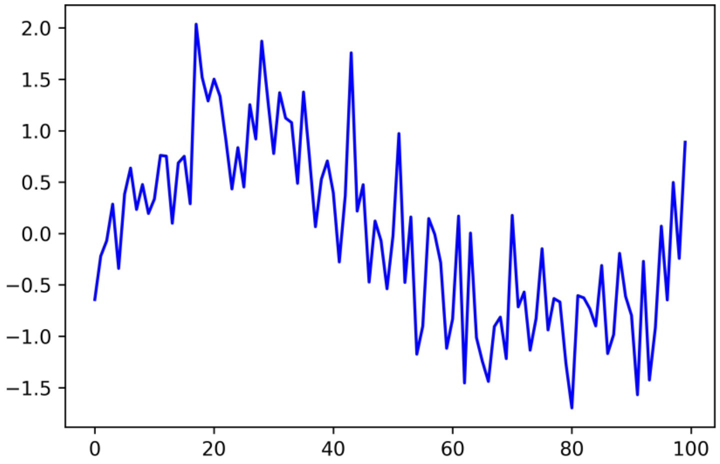
Example: $sin(\mathrm{t})$ + 0.5*noise

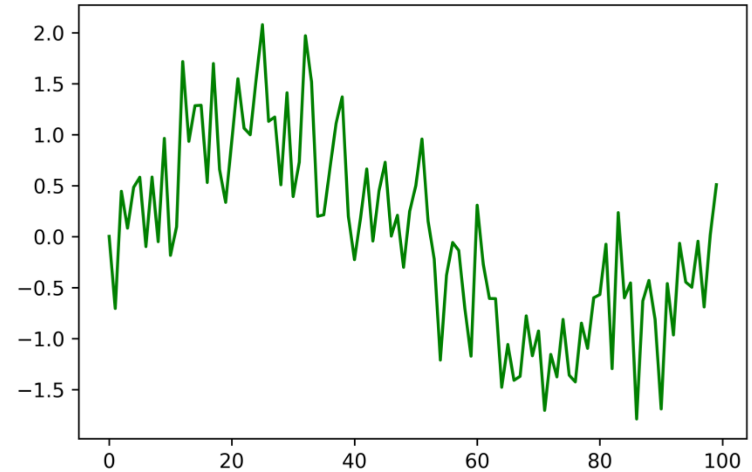




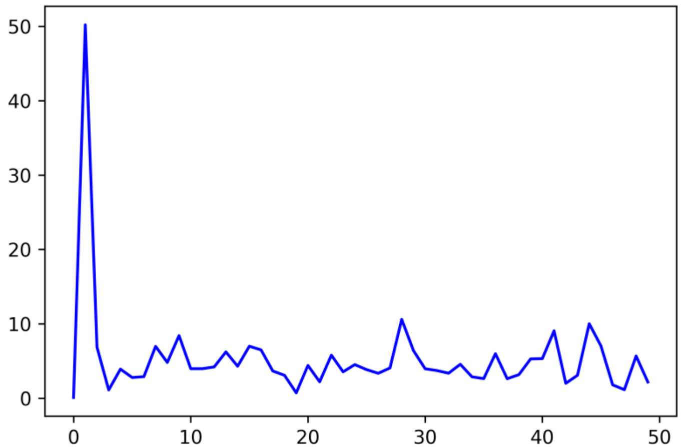
Random signal

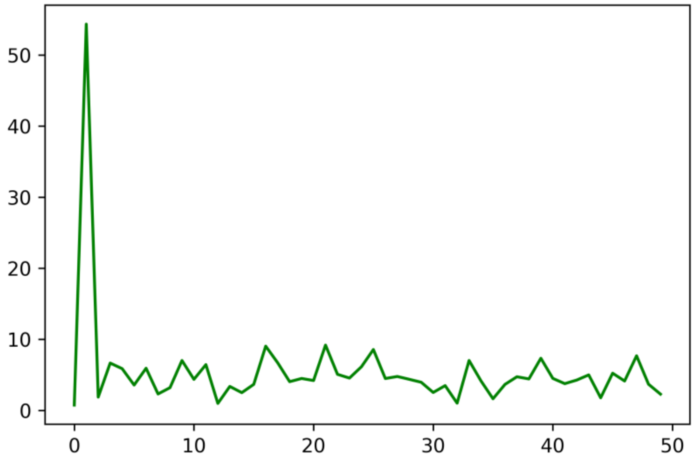
$sin(\mathrm{t})$ + 0.5*noise





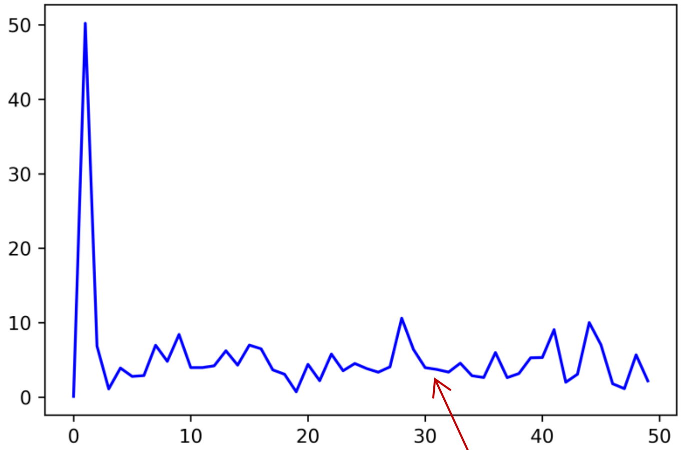
Fourier Transforms

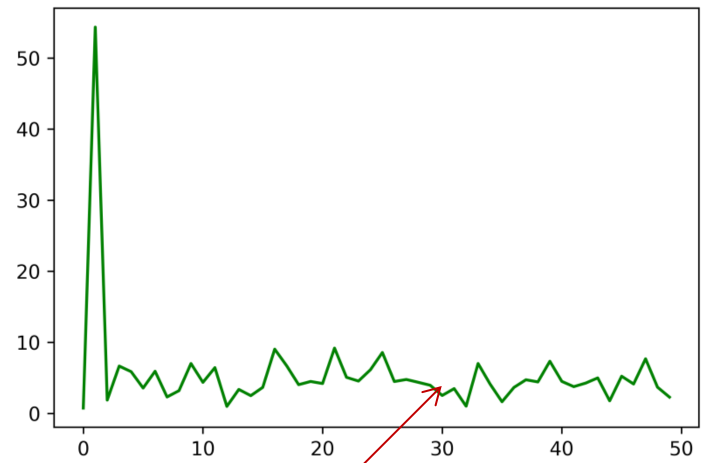




Random signal

Fourier Transforms





How often do I get a spike in that frequency?

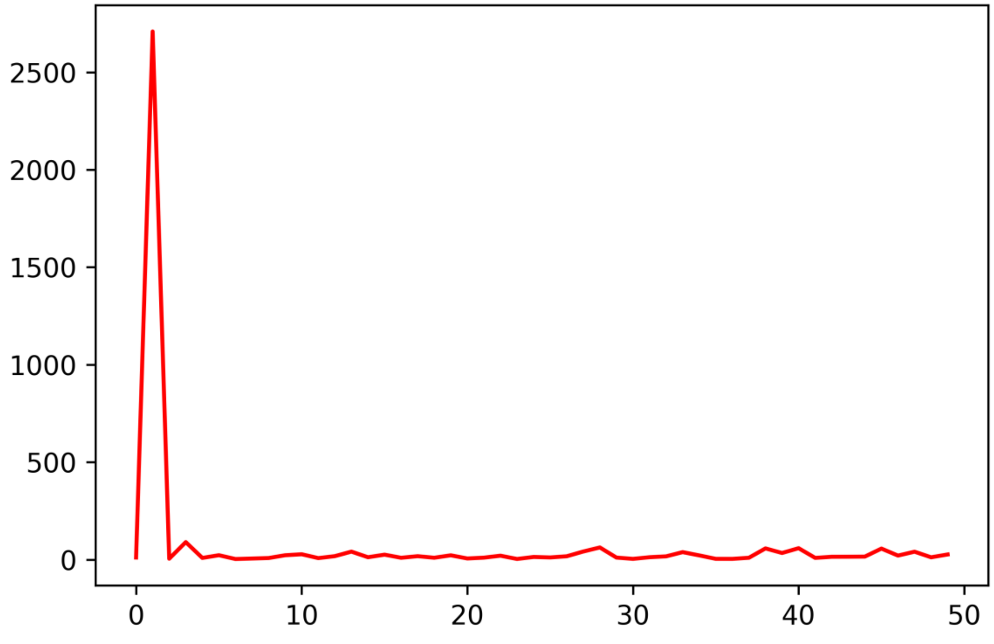
What are the chances? How tall is it usually?

How much energy/power is in that frequency?

Spectral density

Energy spectral density (if we had K experiments):

$$S_E(f) = \frac{1}{K}\sum_{i=1}^{K}|X_i(f)|^2 = \frac{1}{K}\sum_{i=1}^{K}|\mathcal{F}(x_i(t))|^2$$



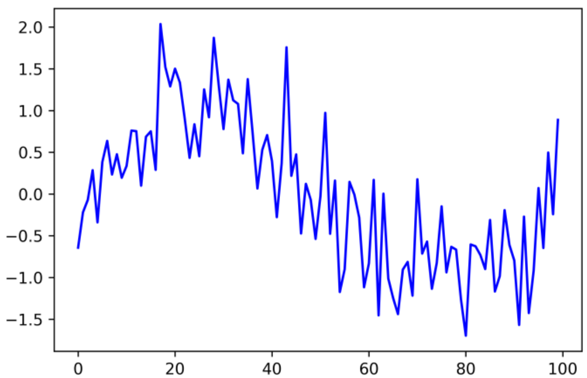
Simulation

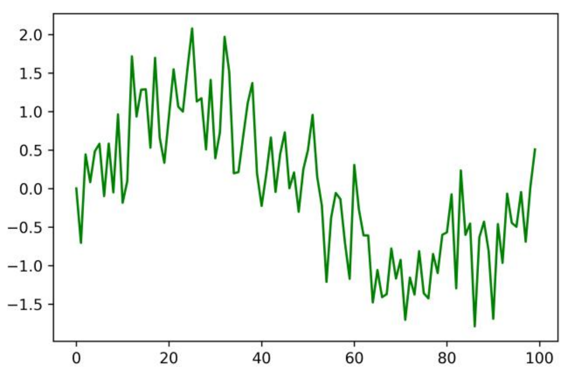
Spectral density (energy or power) can be used to simulate random signals. One way is by using shaping filters:

- Generate white noise
- Get its Fourier transform
- Multiply this transform with the ESD (filter the spectrum of the noise through the ESD)
- Get an inverse transform of the result

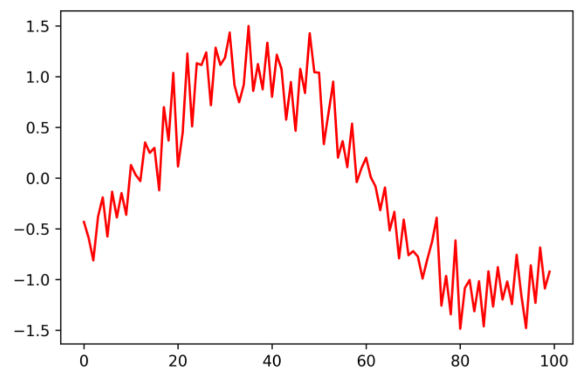
Shaping filters simulation

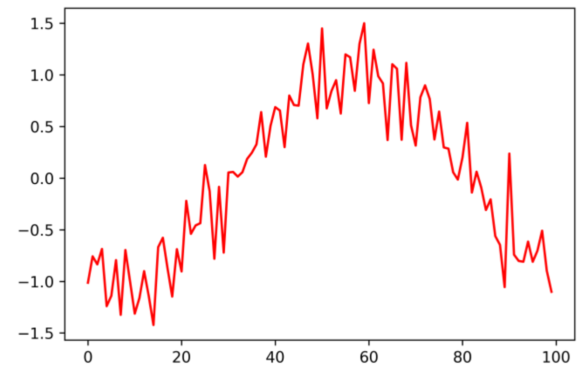
Real

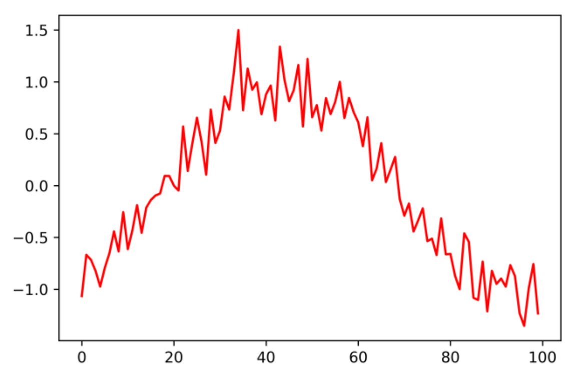


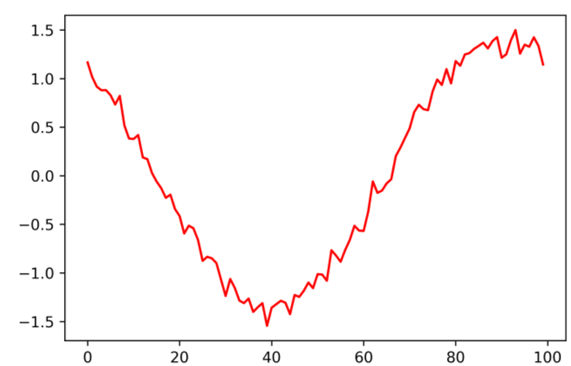


Simulated from ESD









Spectral simulation

We generate Fourier coefficients:

$$X_F(k) = \sqrt{\frac{S_E\left(\frac{2\pi}{T}k_1\right)}{T^2(1+\lambda_k^2)}} = \frac{1}{T}\sqrt{\frac{S_E\left(\frac{2\pi}{T}k_1\right)}{1+\lambda_k^2}},$$

$$k = 0,1,\ldots,M \geq N.$$

Then those coefficients provide the simulated signal:

$$x(i) = \sum_{k=0}^{\frac{N}{2}} X_F(k)\exp\left[j2\pi\left(\frac{ki}{N}\right)\right], i \in [0,N).$$

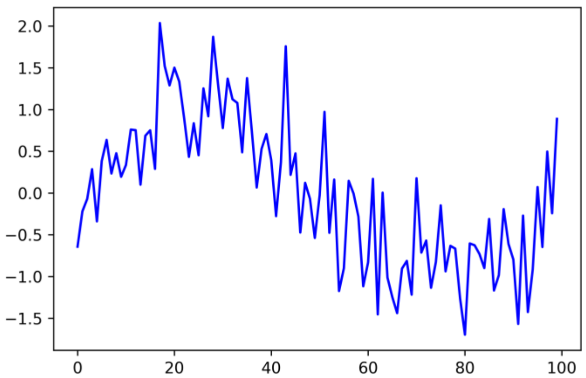
This is simulation in Fourier basis, you can use others: Hartley, etc.

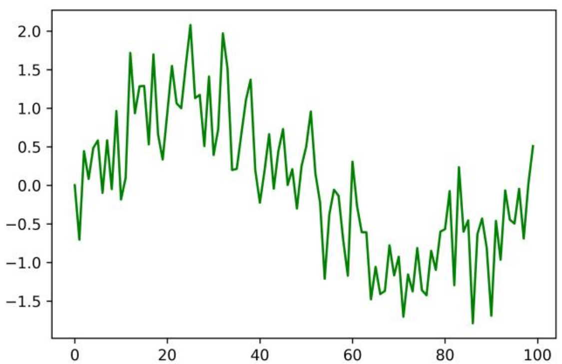
I might have cheated a bit...

```
XF = np.sqrt(ESD)
XF /= XF.max() # just normalized sqrt(ESD)
# turning noise into random signs
n3 = np.random.normal(0,1,50)
n3 /= abs(n3)
import cmath
z = np.empty([100], dtype=complex)
for i in range(0, 100):
    z[i] = 0
    for k in range(0, 50):
        # adding random signs to get random signals
        z[i] += XF[k] * n3[k] * cmath.exp(1j * k * i *math.pi/100)
```

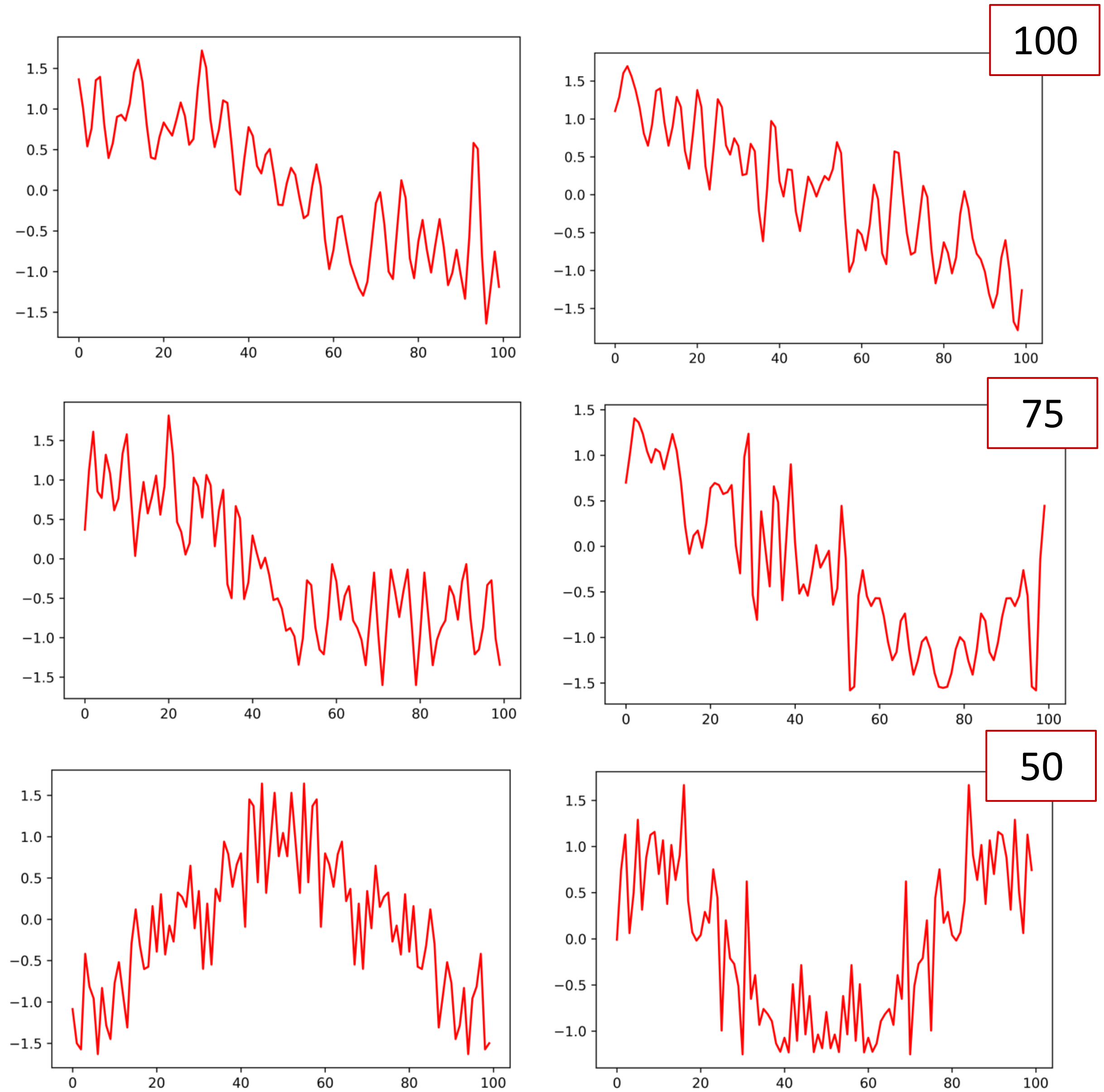
Spectral simulation

Real





Simulated from ESD



2D signals

What kinds of data?

- 2D table data
- Spatial data
- Images
- Graphs

What changes?

- 1D space + time or 2D space (t1, t2 instead of t)
- Spatial frequencies

Computers “see” with help of image processing. Graphs may represent semantic webs. DSP gets close to AI.

What changes:

Fourier transform:

$$\mathcal{F}\big(x(t_1,t_2)\big) = X(f_1,f_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t_1,t_2)e^{-2\pi j(f_1 t_1 + f_2 t_2)}dt_1 dt_2\,.$$

Energy spectral density:

$$\mathrm{S_E}(f_1,f_2) = \frac{1}{\mathrm{N}}\sum_{\mathrm{i}=1}^{\mathrm{N}}|\mathrm{X_i}(f_1,f_2)|^2 = \frac{1}{\mathrm{N}}\sum_{\mathrm{i}=1}^{\mathrm{N}}|\mathcal{F}(x_{\mathrm{i}}(t_1,t_2))|^2\,.$$

What happens then:

We generate Fourier coefficients:

$$X_F(k_1, k_2) = \sqrt{\frac{S_E\left(\frac{2\pi}{T_1}k_1, \frac{2\pi}{T_1}k_2\right)}{T_1^2 T_2^2\left(1+\lambda_{k_1,k_2}^2\right)}} = \frac{1}{T_1 T_2}\sqrt{\frac{S_E\left(\frac{2\pi}{T_1}k_1, \frac{2\pi}{T_1}k_2\right)}{1+\lambda_{k_1,k_2}^2}},$$

$$k_1, k_2 = 0,1, \dots,$$

Then those coefficients provide the simulated signal:

$$x(i_1, i_2) = \sum_{k_1=0}^{\frac{N_1}{2}} \sum_{k_2=0}^{\frac{N_2}{2}} X_F(k_1, k_2) \exp\left[j2\pi\left(\frac{k_1 i_1}{T_1} + \frac{k_2 i_2}{T_2}\right)\right], i_1 \in [0, N_1), i_2 \in [0, N_2).$$

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$$X_F(k_1, k_2) = \sqrt{\frac{S_E\left(\frac{2\pi}{T_1}k_1, \frac{2\pi}{T_1}k_2\right)}{T_1^2 T_2^2\left(1 + \lambda_{k_1,k_2}^2\right)}} = \frac{1}{T_1 T_2}\sqrt{\frac{S_E\left(\frac{2\pi}{T_1}k_1, \frac{2\pi}{T_1}k_2\right)}{1 + \lambda_{k_1,k_2}^2}},$$

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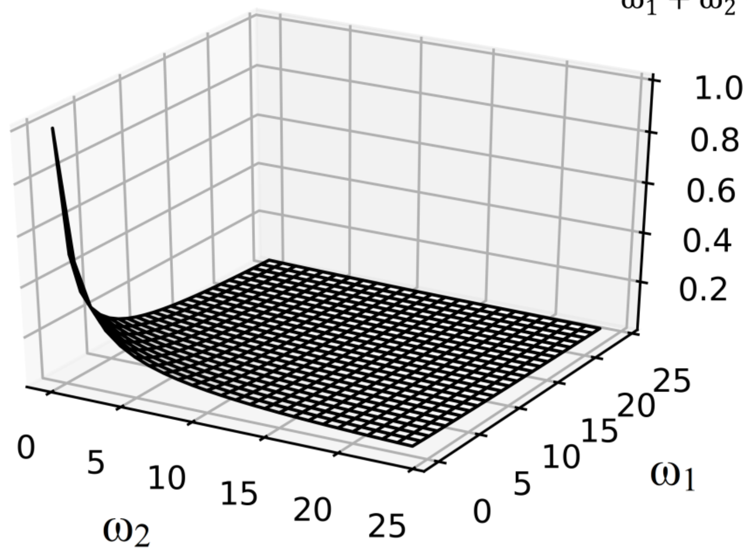
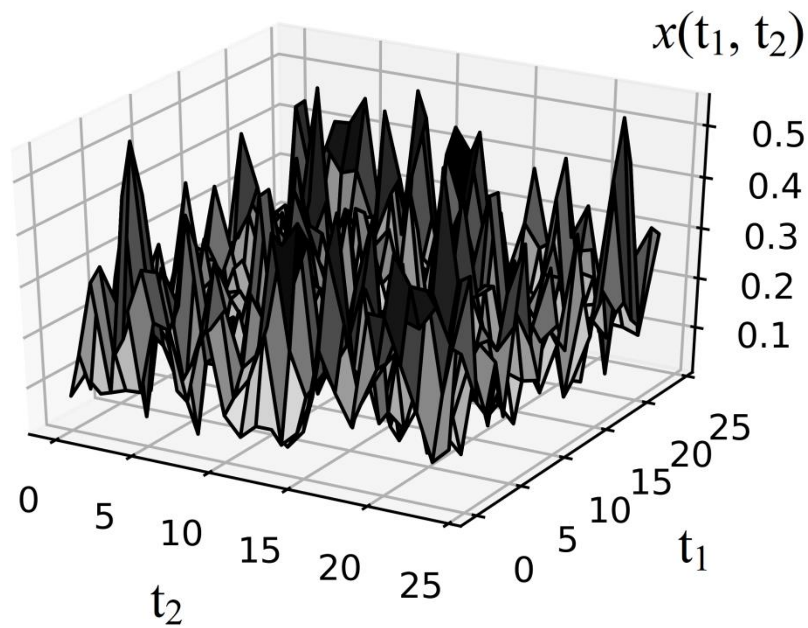
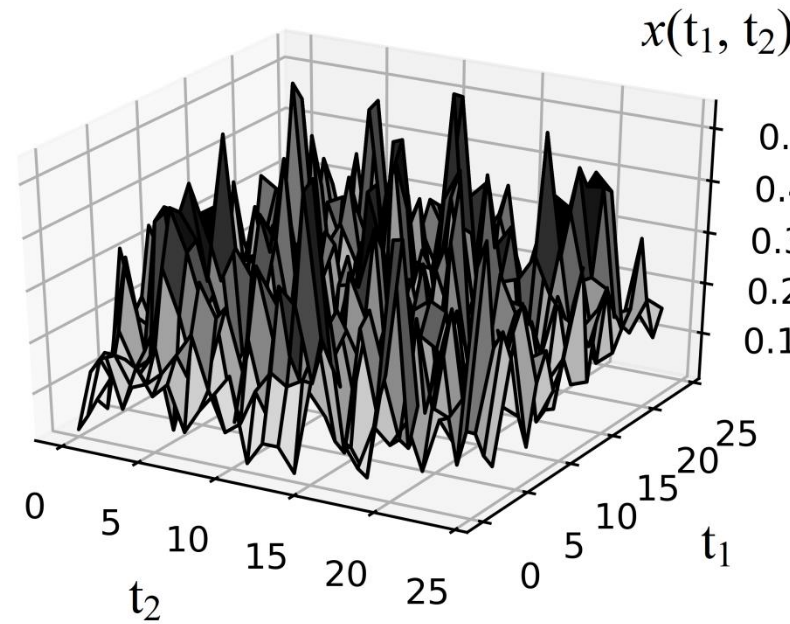
ESD

$$S_E(\omega_1, \omega_2) = \frac{1}{\omega_1 + \omega_2 + 1}$$



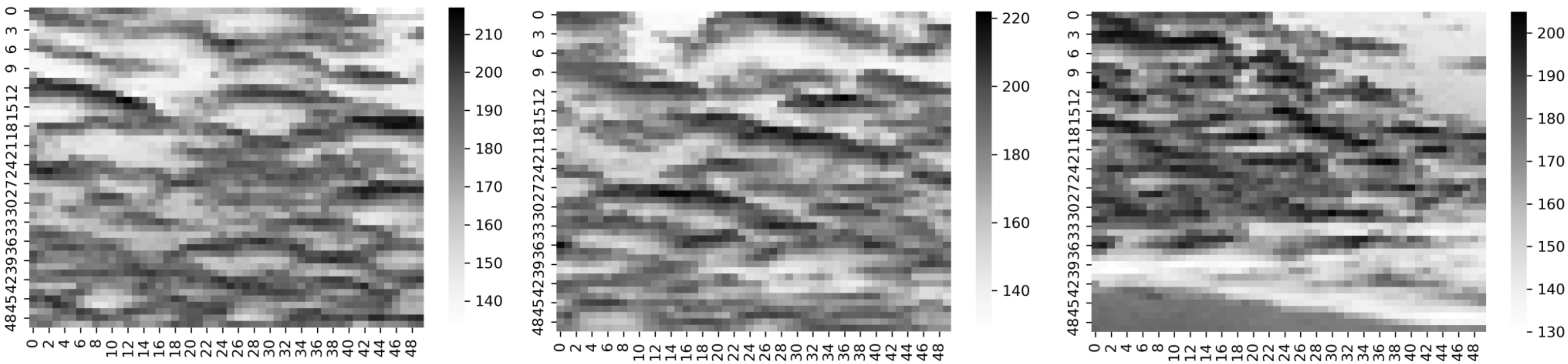
Random Signals in Fourier basis



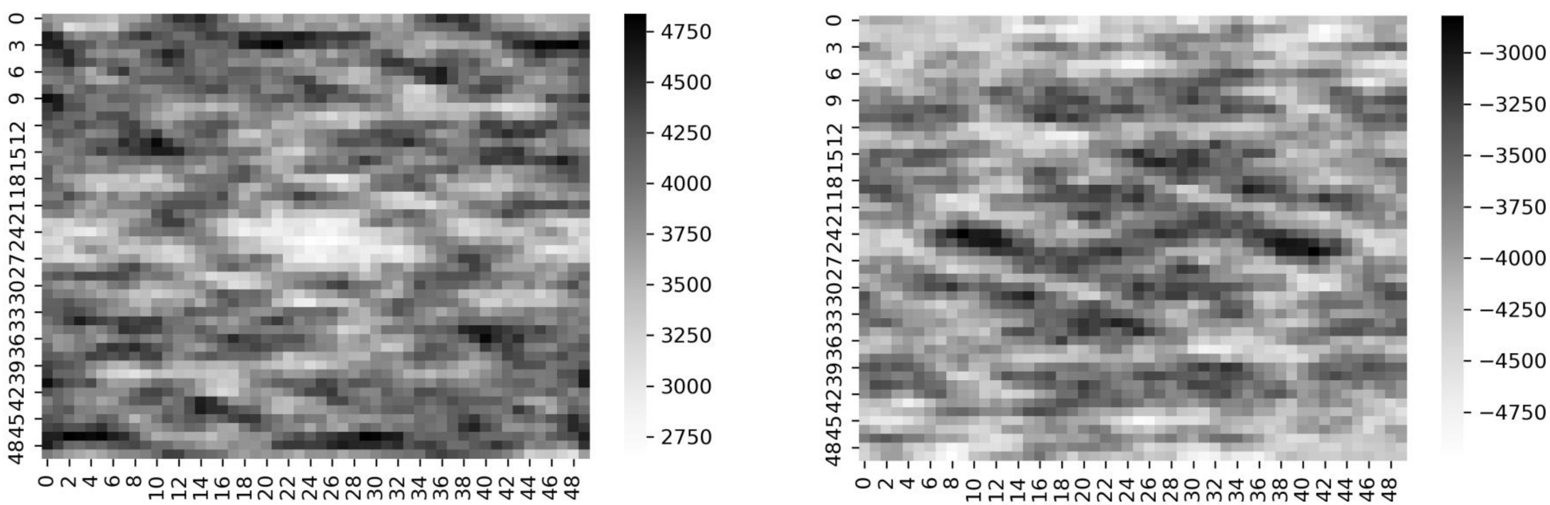


Clouds

Real

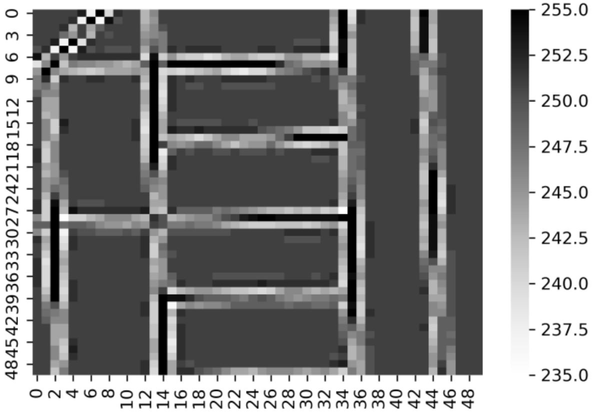


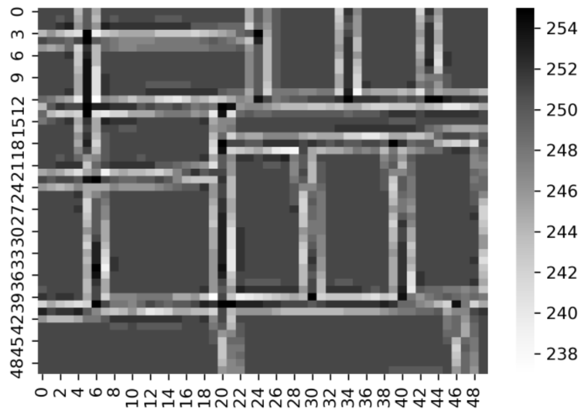
Fourier simulated

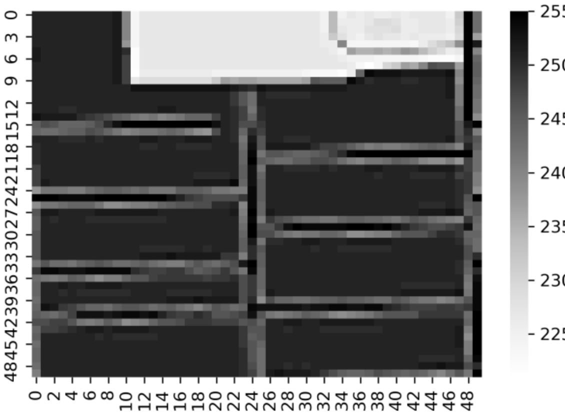


Streets

Real







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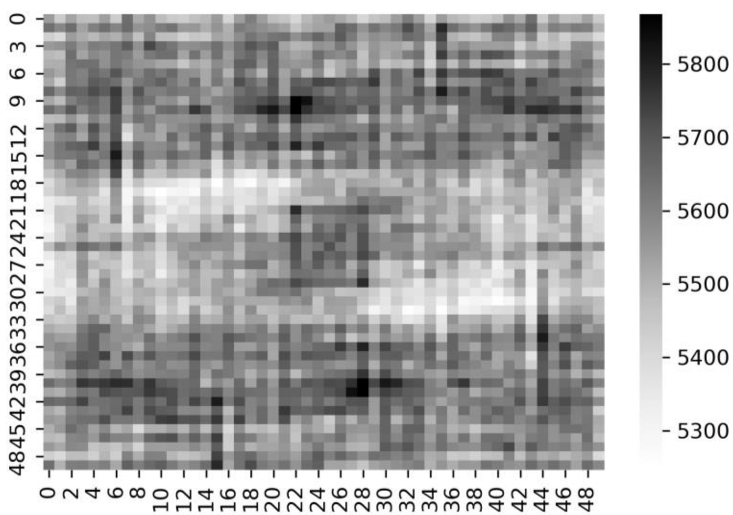

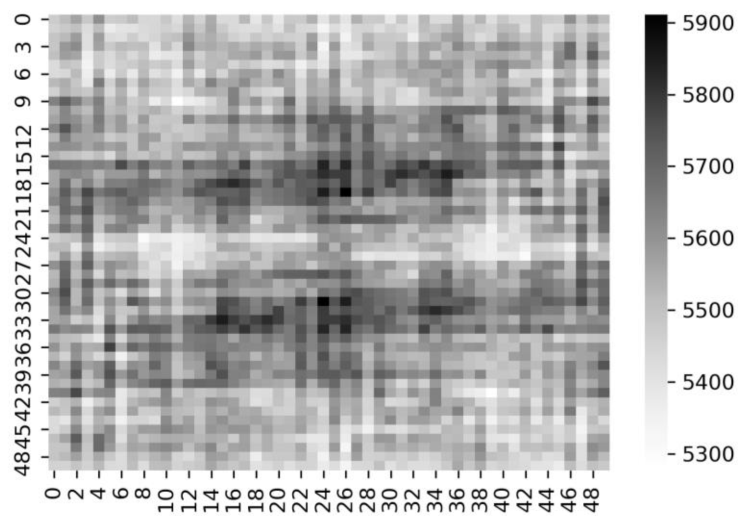

Thanks!