Análise de Algoritmos - T1

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17/09/2018

Multiplicação de N bits

O problema

$$
\begin{array}{r}
1100 \\
\times\, 1101 \\
\hline
1100 \\
0000 \\
1100 \\
1100 \\
\hline
10011100
\end{array}
\qquad
\begin{array}{r}
12 \\
\times\, 13 \\
\hline
36 \\
12 \\
\hline
156
\end{array}
$$

12	1100
× 13	× 1101
36	1100
12	0000
156	1100
	1100
	10011100
(a)	(b)

Figure 5.8 The elementary-school algorithm for multiplying two integers, in (a) decimal and (b) binary representation.

Divide-and-conquer

$$m = \lceil n/2 \rceil$$

$$a = \lfloor x/2^m \rfloor \quad b = x \bmod 2^m$$

$$c = \lfloor y/2^m \rfloor \quad d = y \bmod 2^m$$

← use bit shifting to compute 4 terms

$$x\,y \;=\; (2^m a + b)(2^m c + d) \;=\; 2^{2m} ac \;+\; 2^m (bc + ad) \;+\; bd$$

①   **②**   **③**   **④**

Ex. $x = 1\,0\,0\,0\,1\,1\,0\,1$ $y = 1\,1\,1\,0\,0\,0\,0\,1$

$\underbrace{}_{a}$ $\underbrace{}_{b}$ $\underbrace{}_{c}$ $\underbrace{}_{d}$

MULTIPLY(x, y, n)

IF $(n = 1)$

    RETURN $x \times y$.

ELSE

    $m \leftarrow \lceil n / 2 \rceil$.

    $a \leftarrow \lfloor x / 2^m \rfloor$; $b \leftarrow x \bmod 2^m$. $\longleftarrow$ $\Theta(n)$

    $c \leftarrow \lfloor y / 2^m \rfloor$; $d \leftarrow y \bmod 2^m$.

    $e \leftarrow$ MULTIPLY(a, c, m).

    $f \leftarrow$ MULTIPLY(b, d, m). $\longleftarrow$ $4\, T(\lceil n / 2 \rceil)$

    $g \leftarrow$ MULTIPLY(b, c, m).

    $h \leftarrow$ MULTIPLY(a, d, m).

    RETURN $2^{2m}\, e + 2^m\, (g + h) + f$. $\longleftarrow$ $\Theta(n)$

Divide-and-conquer: Análise

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 4T(\lceil n/2 \rceil) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Karatsuba

$$m = \lceil n / 2 \rceil$$

$$a = \lfloor x / 2^m \rfloor \quad b = x \bmod 2^m$$

$$c = \lfloor y / 2^m \rfloor \quad d = y \bmod 2^m$$

middle term

$$x\,y \;=\; (2^m\,a + b)\,(2^m\,c + d) \;=\; 2^{2m}\,ac + 2^m\,(bc + ad) \;+\; bd$$

$$=\; 2^{2m}\,ac + 2^m\,(ac + bd \;-\; (a-b)(c-d)) + bd$$

❶   **①** **③**   **❷**   **❸**

$$x = \underbrace{1\,0\,0\,0}_{a}\,\underbrace{1\,1\,0\,1}_{b}$$

$$y = \underbrace{1\,1\,1\,0}_{c}\,\underbrace{0\,0\,0\,1}_{d}$$

KARATSUBA-MULTIPLY(x, y, n)

IF ($n = 1$)

    RETURN $x \times y$.

ELSE

    $m \leftarrow \lceil n / 2 \rceil$.

    $a \leftarrow \lfloor x / 2^m \rfloor$; $b \leftarrow x \bmod 2^m$. $\longleftarrow$ $\Theta(n)$

    $c \leftarrow \lfloor y / 2^m \rfloor$; $d \leftarrow y \bmod 2^m$.

    $e \leftarrow$ KARATSUBA-MULTIPLY(a, c, m).

    $f \leftarrow$ KARATSUBA-MULTIPLY(b, d, m). $\longleftarrow$ $3\,T(\lceil n / 2 \rceil)$

    $g \leftarrow$ KARATSUBA-MULTIPLY($|a - b|, |c - d|, m$).

    Flip sign of g if needed.

    RETURN $2^{2m}\, e + 2^m\, (e + f - g) + f$. $\longleftarrow$ $\Theta(n)$

Karatsuba: Análise

$$T(n) \;=\; \begin{cases} \Theta(1) & \text{if } n = 1 \\[2em] 3T(\lceil n/2 \rceil) \;+\; \Theta(n) & \text{if } n > 1 \end{cases}$$

$$\implies \quad T(n) \;=\; \Theta(n^{\log_2 3}) \;=\; O(n^{1.585})$$

Fonte

Algorithm Design - Jon Kleinberg, Eva Tardos - Copyright © 2005
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