The left-hand side of the equation is given by:

$$M := \langle \phi_{ek} | [(A^*A)^{1/2}\, \phi_{em}] \rangle_{k,m=1,\dots,n} \tag{0.0.1}$$

where:

$$\phi_{ek}(x) = \begin{cases} \dfrac{x-(k-1)l}{l} & x \in [(k-1)l, kl] \\ \dfrac{(k+1)l-x}{l} & x \in [kl, (k+1)l] \end{cases} \tag{0.0.2}$$

for $k \in \{1, \dots, n-1\}$ and for $k = n$:

$$\phi_{en}(x) = \frac{x-(n-1)l}{l} \quad x \in [(n-1)l, nl] \quad x \in [(n-1)l, nl] \tag{0.0.3}$$

Meanwhile, it holds true that:

$$\begin{aligned} &[(A^*A)^{1/2}\phi_{em}](y) = \\ &\quad \frac{1}{\pi l}\left\{ \ln\left[\frac{\tan\left(\dfrac{\pi[ml+y]}{4a}\right)\tan\left(\dfrac{\pi|(m-1)l-y|}{4a}\right)}{\tan\left(\dfrac{\pi|ml-y|}{4a}\right)\tan\left(\dfrac{\pi[(m-1)l+y]}{4a}\right)}\right]\right. \\ &\qquad \left. -\ln\left[\frac{\tan\left(\dfrac{\pi[(m+1)l+y]}{4a}\right)\tan\left(\dfrac{\pi|ml-y|}{4a}\right)}{\tan\left(\dfrac{\pi|(m+1)l-y|}{4a}\right)\tan\left(\dfrac{\pi[ml+y]}{4a}\right)}\right]\right\} \end{aligned} \tag{0.0.4}$$

for every $m \in \{1, \dots, n-1\}$

$$[(A^*A)^{1/2}\phi_{en}](y) = \frac{1}{\pi l}\ln\left[\frac{\tan\left(\dfrac{\pi[nl+y]}{4a}\right)\tan\left(\dfrac{\pi|(n-1)l-y|}{4a}\right)}{\tan\left(\dfrac{\pi|nl-y|}{4a}\right)\tan\left(\dfrac{\pi[(n-1)l+y]}{4a}\right)}\right] \tag{0.0.5}$$

Unpacking the equation (0.0.1):

$$M_{km} = \int_0^a \phi_{ek}(x)[(A^*A)^{1/2}\phi_{em}](x)dx \tag{0.0.6}$$

We take into account that this integrand has a singularity at $x = kl$ with $k \in \{1, \ldots, n-1\}$. We perform the following split. Also from now on we set $[(A^*A)^{1/2}\phi_{em}](x) = f_m(x)$:

$$M_{km} = \int\limits_0^{kl} \phi_{ek}(x) f_m(x) dx + \int\limits_{kl}^{a} \phi_{ek}(x) f_m(x) dx$$

We then introduce the term $|x - kl|^\alpha$, with $-1 < \alpha < 0$, in order to get rid of the singularity:

$$M_{km} = \int\limits_0^{kl} \phi_{ek}(x) f_m(x) |x-kl|^{-\alpha} |x-kl|^{\alpha} dx + \int\limits_{kl}^{a} \phi_{ek}(x) f_m(x) |x-kl|^{-\alpha} |x-kl|^{\alpha} dx$$

Setting $f_m(x)|x - kl|^{-\alpha} = g_m(x)$ we can write:

$$\begin{aligned}
M_{km} &= \int\limits_0^{kl} \phi_{ek}(x) g_m(x) (kl-x)^\alpha dx + \int\limits_{kl}^{a} \phi_{ek}(x) g_m(x) (x-kl)^\alpha dx \\
&= \frac{kl}{2} \int\limits_{-1}^{1} \phi_{ek}\left(\frac{kl}{2}\xi + \frac{kl}{2}\right) g_m\left(\frac{kl}{2}\xi + \frac{kl}{2}\right) \left(\frac{kl}{2}\right)^\alpha (1-\xi)^\alpha d\xi + \\
&+ \frac{a-kl}{2} \int\limits_{-1}^{1} \phi_{ek}\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right) g_m\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right) \left(\frac{a-kl}{2}\right)^\alpha (1+\xi)^\alpha d\xi \\
&= \left(\frac{kl}{2}\right)^{1+\alpha} \int\limits_{-1}^{1} \phi_{ek}\left(\frac{kl}{2}\xi + \frac{kl}{2}\right) g_m\left(\frac{kl}{2}\xi + \frac{kl}{2}\right) (1-\xi)^\alpha d\xi + \\
&+ \left(\frac{a-kl}{2}\right)^{1+\alpha} \int\limits_{-1}^{1} \phi_{ek}\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right) g_m\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right) (1+\xi)^\alpha d\xi
\end{aligned}$$

We substitute:

$$\tilde{g}_{1m}(\xi) = \phi_{ek}\left(\frac{kl}{2}\xi + \frac{kl}{2}\right) g_m\left(\frac{kl}{2}\xi + \frac{kl}{2}\right) \tag{0.0.7}$$

$$\tilde{g}_{2m}(\xi) = \phi_{ek}\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right) g_m\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right) \tag{0.0.8}$$

This yields:

$$M_{km} = \left(\frac{kl}{2}\right)^{1+\alpha} \int_{-1}^{1} \tilde{g}_{1m}(\xi)(1+\xi)^{\alpha} d\xi + \left(\frac{a-kl}{2}\right)^{1+\alpha} \int_{-1}^{1} \tilde{g}_{2m}(\xi)(1-\xi)^{\alpha} d\xi \tag{0.0.9}$$

Performing a Gauss-Jacobi integration scheme, we get:

$$M_{km} \approx \left(\frac{kl}{2}\right)^{1+\alpha} \sum_i \tilde{g}_{1m}(\xi_i) w_i^{(\alpha,0)} + \left(\frac{a-kl}{2}\right)^{1+\alpha} \sum_i \tilde{g}_{1m}(\xi_i) w_i^{(0,\alpha)} \tag{0.0.10}$$

where $w_i^{(\alpha,0)}$ means that we use the Gauss-Jacobi integration nodes and weights for $\alpha = -0.5\,, \beta = 0$ and so on.

Special case k=n

For the case that $k = n$, we follow the same procedure as above:

$$M_{nn} = \int_0^a \phi_{en}(x) f_n(x)(a-x)^{\alpha}(a-x)^{-\alpha} dx \tag{0.0.11}$$

We set $f_n(x)(a-x)^{-\alpha} = g_n(x)$, so:

$$\begin{aligned} M_{nn} &= \int_0^a \phi_{en}(x) g_n(x)(a-x)^{\alpha} dx \\ &= \frac{a}{2}\int_{-1}^{1} \phi_{en}\left(\frac{a}{2}\xi + \frac{a}{2}\right) g_n\left(\frac{a}{2}\xi + \frac{a}{2}\right)\left(\frac{a}{2}\right)^{\alpha}(1-\xi)^{\alpha} d\xi \end{aligned}$$

$$= \left(\frac{a}{2}\right)^{1+\alpha} \int\limits_{-1}^{1} \phi_{en}\left(\frac{a}{2}\xi + \frac{a}{2}\right) g_n\left(\frac{a}{2}\xi + \frac{a}{2}\right)(1-\xi)^{\alpha} d\xi$$

If we substitute:

$$\tilde{g}_n(\xi) = \phi_{en}\left(\frac{a}{2}\xi + \frac{a}{2}\right) g_n\left(\frac{a}{2}\xi + \frac{a}{2}\right) \tag{0.0.12}$$

we can approximate the integral above as:

$$M_{nn} \approx \left(\frac{a}{2}\right)^{1+\alpha} \sum_i \tilde{g}(\xi) w_i^{(\alpha,0)} \tag{0.0.13}$$