The left-hand side of the equation is given by:

$$M := \langle \phi_{ek} | [(A^*A)^{1/2} \phi_{em}] \rangle_{k,m=1,\dots,n} \tag{1}$$

where:

$$\phi_{ek}(x) = \begin{cases} \dfrac{x-(k-1)l}{l} & x \in [(k-1)l, kl] \\ \dfrac{(k+1)l - x}{l} & x \in [kl, (k+1)l] \end{cases} \tag{2}$$

for $k \in \{1, \dots, n-1\}$ and for $k = n$:

$$\phi_{en}(x) = \frac{x-(n-1)l}{l} \quad x \in [(n-1)l, nl] \quad x \in [(n-1)l, nl] \tag{3}$$

Meanwhile, it holds true that:

$$\begin{aligned} &[(A^*A)^{1/2}\phi_{em}](y) = \\ &\quad \frac{1}{\pi l} \left\{ \ln \left[\frac{\tan\left(\frac{\pi[ml+y]}{4a}\right) \tan\left(\frac{\pi|(m-1)l-y|}{4a}\right)}{\tan\left(\frac{\pi|ml-y|}{4a}\right) \tan\left(\frac{\pi[(m-1)l+y]}{4a}\right)} \right] \right. \\ &\qquad \left. - \ln \left[\frac{\tan\left(\frac{\pi[(m+1)l+y]}{4a}\right) \tan\left(\frac{\pi|ml-y|}{4a}\right)}{\tan\left(\frac{\pi|(m+1)l-y|}{4a}\right) \tan\left(\frac{\pi[ml+y]}{4a}\right)} \right] \right\} \end{aligned} \tag{4}$$

for every $m \in \{1, \dots, n-1\}$

$$[(A^*A)^{1/2}\phi_{en}](y) = \frac{1}{\pi l} \ln \left[\frac{\tan\left(\frac{\pi[nl+y]}{4a}\right) \tan\left(\frac{\pi|(n-1)l-y|}{4a}\right)}{\tan\left(\frac{\pi|nl-y|}{4a}\right) \tan\left(\frac{\pi[(n-1)l+y]}{4a}\right)} \right] \tag{5}$$

Unpacking the equation (1):

$$M_{km} = \int_0^a \phi_{ek}(x) [(A^*A)^{1/2}\phi_{em}](x) dx \tag{6}$$

We take into account that this integrand has a singularity at $x = kl$ with $k \in \{1, \dots, n-1\}$. We perform the following split. Also from now on we set $[(A^*A)^{1/2}\phi_{em}](x) = f_m(x)$:

$$M_{km} = \int_0^{kl} \phi_{ek}(x) f_m(x) dx + \int_{kl}^a \phi_{ek}(x) f_m(x) dx$$

We then introduce the term $|x - kl|^\alpha$, in order to get rid of the singularity:

$$M_{km}=\int_0^{kl}\phi_{ek}(x)f_m(x)|x-kl|^{-\alpha}|x-kl|^{\alpha}dx+\int_{kl}^{a}\phi_{ek}(x)f_m(x)|x-kl|^{-\alpha}|x-kl|^{\alpha}dx$$

Setting $f_m(x)|x-kl|^{-\alpha}=g_m(x)$, with $-1<\alpha<0$, we can write:

$$\begin{aligned}
M_{km} &= \int_0^{kl}\phi_{ek}(x)g_m(x)(kl-x)^{\alpha}dx+\int_{kl}^{a}\phi_{ek}(x)g_m(x)(x-kl)^{\alpha}dx\\
&= \frac{kl}{2}\int_{-1}^{1}\phi_{ek}\left(\frac{kl}{2}\xi+\frac{kl}{2}\right)g_m\left(\frac{kl}{2}\xi+\frac{kl}{2}\right)\left(\frac{kl}{2}\right)^{\alpha}(1-\xi)^{\alpha}d\xi+\\
&+\frac{a-kl}{2}\int_{-1}^{1}\phi_{ek}\left(\frac{a-kl}{2}\xi+\frac{a+kl}{2}\right)g_m\left(\frac{a-kl}{2}\xi+\frac{a+kl}{2}\right)\left(\frac{a-kl}{2}\right)^{\alpha}(1+\xi)^{\alpha}d\xi\\
&= \left(\frac{kl}{2}\right)^{1+\alpha}\int_{-1}^{1}\phi_{ek}\left(\frac{kl}{2}\xi+\frac{kl}{2}\right)g_m\left(\frac{kl}{2}\xi+\frac{kl}{2}\right)(1-\xi)^{\alpha}d\xi+\\
&+\left(\frac{a-kl}{2}\right)^{1+\alpha}\int_{-1}^{1}\phi_{ek}\left(\frac{a-kl}{2}\xi+\frac{a+kl}{2}\right)g_m\left(\frac{a-kl}{2}\xi+\frac{a+kl}{2}\right)(1+\xi)^{\alpha}d\xi
\end{aligned}$$

We substitute:

$$\tilde{g}_{1m}(\xi)=\phi_{ek}\left(\frac{kl}{2}\xi+\frac{kl}{2}\right)g_m\left(\frac{kl}{2}\xi+\frac{kl}{2}\right)$$

$$\tilde{g}_{2m}(\xi)=\phi_{ek}\left(\frac{a-kl}{2}\xi+\frac{a+kl}{2}\right)g_m\left(\frac{a-kl}{2}\xi+\frac{a+kl}{2}\right)$$

This yields:

$$M_{km}=\left(\frac{kl}{2}\right)^{1+\alpha}\int_{-1}^{1}\tilde{g}_{1m}(\xi)(1+\xi)^{\alpha}d\xi+\left(\frac{a-kl}{2}\right)^{1+\alpha}\int_{-1}^{1}\tilde{g}_{2m}(\xi)(1-\xi)^{\alpha}d\xi \quad (7)$$

Performing a Gauss-Jacobi integration scheme, we get:

$$M_{km}\approx\left(\frac{kl}{2}\right)^{1+\alpha}\sum_i\tilde{g}_{1m}(\xi_i)w_i^{(\alpha,0)}+\left(\frac{a-kl}{2}\right)^{1+\alpha}\sum_i\tilde{g}_{1m}(\xi_i)w_i^{(0,\alpha)} \qquad (8)$$

Special case k=n

For the case that $k = n$, we follow the same procedure as above:

$$M_{nn} \approx \left(\frac{a}{2}\right)^{1+\alpha} \sum_i \tilde{g}(\xi) w_i^{(\alpha,0)} \tag{9}$$

where:

$$\tilde{g}(\xi) = \phi_{en}\left(\frac{a}{2}\xi + \frac{a}{2}\right) f\left(\frac{a}{2}\xi + \frac{a}{2}\right)(a-x)^{-\alpha} \tag{10}$$