Calculation for right-hand side

The right-hand size of the equation is described by the equation:

$$X_k := \langle \phi_{ek} | (V^* g) \rangle = \int_0^a \phi_{ek}(x)(V^* g)(x)\mathrm{d}x \tag{0.0.1}$$

for $k \in \{1, \dots, n\}$ where:

$$\phi_{ek}(x) = \begin{cases} \dfrac{x-(k-1)l}{l} & x \in [(k-1)l, kl] \\ \dfrac{(k+1)l-x}{l} & x \in [kl, (k+1)l] \end{cases} \tag{0.0.2}$$

for $k \in \{1, \dots, n-1\}$ and for $k = n$:

$$\phi_{en}(x) = \frac{x-(n-1)l}{l} \quad x \in [(n-1)l, nl] \tag{0.0.3}$$

Furthermore we have:

$$(V^* g)(x) = p_0 c^2 \int_0^a G_{\frac{1}{2}}(x,y) e^{cy} \mathrm{d}y \tag{0.0.4}$$

where the fractional derivative $G_{\frac{1}{2}}(x,y)$ is given by:

$$G_{\frac{1}{2}}(x,y) = \frac{1}{\pi} \ln \left[\frac{\tan\left(\dfrac{\pi(x+y)}{4a}\right)}{\tan\left(\dfrac{\pi|x-y|}{4a}\right)} \right] \tag{0.0.5}$$

Plugging all the results in 0.0.1 yields:

$$X_k = p_0 c^2 \int_0^a \phi_{ek}(x) \int_0^a G_{\frac{1}{2}}(x,y) e^{cy} \mathrm{d}y \mathrm{d}x \tag{0.0.6}$$

$$= p_0 c^2 \int_0^a \int_0^a \phi_{ek}(x) G_{\frac{1}{2}}(x,y) e^{cy} \mathrm{d}y \mathrm{d}x \tag{0.0.7}$$

We observe that X_k becomes singular for $x = y$. In order to eliminate the singularity, we will perform a Gauss-Jacobi integration scheme. Thus we introduce the term:

$$X_k = p_0 c^2 \int_0^a \int_0^a \phi_{ek}(x) f(x,y) |x-y|^{-\alpha} |x-y|^{\alpha} \mathrm{d}y \mathrm{d}x \tag{0.0.8}$$

with $-1 < \alpha < 0$ and:

$$f(x,y) = G_{\frac{1}{2}}(x,y) e^{cy} \tag{0.0.9}$$

$$g(x,y) = f(x,y)|x-y|^{-\alpha} \tag{0.0.10}$$

Spliting the integral at the critical point yields:

$$\begin{aligned}
X_k &= p_0 c^2 \int_0^a \left(\int_0^x \phi_{ek}(x) g(x,y)(x-y)^{\alpha} \mathrm{d}y + \int_x^a \phi_{ek}(x) g(x,y)(y-x)^{\alpha} \mathrm{d}y \right) \mathrm{d}x \\
&= p_0 c^2 \int_0^a \frac{x}{2} \int_{-1}^1 \phi_{ek}(x) g\left(x, \frac{x}{2}(1+\xi)\right) \left(\frac{x}{2}\right)^{\alpha} (1-\xi)^{\alpha} \mathrm{d}\xi \mathrm{d}x \\
&+ p_0 c^2 \int_0^a \frac{a-x}{2} \int_{-1}^1 \phi_{ek}(x) g\left(x, \frac{a-x}{2}\xi + \frac{a+x}{2}\right) \left(\frac{a-x}{2}\right)^{\alpha} (1+\xi)^{\alpha} \mathrm{d}\xi \mathrm{d}x \\
&= p_0 c^2 \int_0^a \left(\frac{x}{2}\right)^{1+\alpha} \int_{-1}^1 \phi_{ek}(x) g\left(x, \frac{x}{2}(1+\xi)\right) (1-\xi)^{\alpha} \mathrm{d}\xi \mathrm{d}x \\
&+ p_0 c^2 \int_0^a \left(\frac{a-x}{2}\right)^{1+\alpha} \int_{-1}^1 \phi_{ek}(x) g\left(x, \frac{a-x}{2}\xi + \frac{a+x}{2}\right) (1+\xi)^{\alpha} \mathrm{d}\xi \mathrm{d}x \\
&= p_0 c^2 \int_0^a \int_{-1}^1 \left(\frac{x}{2}\right)^{1+\alpha} \phi_{ek}(x) g\left(x, \frac{x}{2}(1+\xi)\right) (1-\xi)^{\alpha} \mathrm{d}\xi \mathrm{d}x \\
&+ p_0 c^2 \int_0^a \int_{-1}^1 \left(\frac{a-x}{2}\right)^{1+\alpha} \phi_{ek}(x) g\left(x, \frac{a-x}{2}\xi + \frac{a+x}{2}\right) (1+\xi)^{\alpha} \mathrm{d}\xi \mathrm{d}x
\end{aligned}$$

$$
= p_0 c^2 \frac{a}{2} \left[\int_{-1}^{1} \int_{-1}^{1} \left(\frac{a}{4}\right)^{1+\alpha} (1+\eta)^{1+\alpha} \phi_{ek}\left(\frac{a}{2}(1+\eta)\right) g\left(\frac{a}{2}(1+\eta), \frac{x}{2}(1+\xi)\right) (1-\xi)^{\alpha} \mathrm{d}\xi \mathrm{d}\eta \right.
$$

$$
\left. + \int_{-1}^{1} \int_{-1}^{1} \left(\frac{a}{4}\right)^{1+\alpha} (1-\eta)^{1+\alpha} \phi_{ek}\left(\frac{a}{2}(1+\eta)\right) g\left(\frac{a}{2}(1+\eta), \frac{a}{4}(1-\eta)\xi + \frac{a}{4}(3+\eta)\right) (1+\xi)^{\alpha} \mathrm{d}\xi \mathrm{d}\eta \right]
$$

We substitute:

$$
\tilde{g}_1(\xi,\eta) = \phi_{ek}\left(\frac{a}{2}(1+\eta)\right) g\left(\frac{a}{2}(1+\eta), \frac{x}{2}(1+\xi)\right) \tag{0.0.11}
$$

$$
\tilde{g}_2(\xi,\eta) = \phi_{ek}\left(\frac{a}{2}(1+\eta)\right) g\left(\frac{a}{2}(1+\eta), \frac{a}{4}(1-\eta)\xi + \frac{a}{4}(3+\eta)\right) \tag{0.0.12}
$$

meaning:

$$
\begin{aligned}
X_k = {} & 2p_0 c^2 \left(\frac{a}{4}\right)^{2+\alpha} \int_{-1}^{1} \int_{-1}^{1} \tilde{g}_1(\xi,\eta)(1-\xi)^{\alpha}(1+\eta)^{1+\alpha} \mathrm{d}\xi \mathrm{d}\eta \\
& + 2p_0 c^2 \left(\frac{a}{4}\right)^{2+\alpha} \int_{-1}^{1} \int_{-1}^{1} \tilde{g}_2(\xi,\eta)(1+\xi)^{\alpha}(1-\eta)^{1+\alpha} \mathrm{d}\xi \mathrm{d}\eta
\end{aligned}
$$

Performing the Gauss-Jacobi quadrature yields:

$$
\begin{aligned}
X_k \approx {} & 2p_0 c^2 \left(\frac{a}{4}\right)^{2+\alpha} \sum_i \sum_j \tilde{g}_1(\xi_i,\eta_j) w_i^{(\alpha,0)} w_j^{(0,1+\alpha)} \\
& + 2p_0 c^2 \left(\frac{a}{4}\right)^{2+\alpha} \sum_i \sum_j \tilde{g}_2(\xi_i,\eta_j) w_i^{(0,\alpha)} w_j^{(1+\alpha,0)}
\end{aligned} \tag{0.0.13}
$$