6 Solution of the Barometric Equation via Finite Elements

6.1 Approach 1

6.1.1 Transformation of the Barometric Equation into an Operator Equation

Let $a > 0$, $I := (0, a)$, $q \in C(I, \mathbb{C})$, $p \in C^1(I, \mathbb{C})$ be such that

$$p'(h) + cp(h) = q(h) \ , \tag{6.1.1}$$

for every $h \in I$ and

$$\lim_{h \to 0} p(h) = p_0 \ . \tag{6.1.2}$$

Further, let $P \in C^1(I, \mathbb{C})$ be such that

$$\lim_{x \to 0} P(x) = p_0 \ .$$

We define a new "unknown" function $f \in C^1(I, \mathbb{C})$ by

$$f := e^{c.\mathrm{id}_I}(p - P) \ .$$

Then

$$\lim_{h \to 0} f(h) = 0$$

and

$$\begin{aligned}
f'(h) &= e^{ch}[p'(h) - P'(h)] + cf(h) \\
&= e^{ch}[p'(h) - P'(h)] + ce^{ch}[p(h) - P(h)] \\
&= e^{ch}[p'(h) + cp(h)] - e^{ch}[P'(h) + cP(h)] \\
&= e^{ch}q(h) - e^{ch}[P'(h) + cP(h)] \\
&= e^{ch}[q(h) - P'(h) - cP(h)] \\
&= \{e^{c.\mathrm{id}_I}[q - P'(h) - cP]\}(h) \ ,
\end{aligned}$$

for every $h > 0$. Hence we arrive at the transformation of the system of equations, (6.1.1) and (6.1.2), into an operator equation

$$Af = g \tag{6.1.3}$$

where

$$A := \overline{D_{I,1}} \ , \ g := e^{c.\mathrm{id}_I}[q - P'(h) - cP] \ ,$$

and $D_{I,1}$ denotes the derivative operator in $L^2_{\mathbb{C}}(I)$, which has the unique solution

$$f = A^{-1}g \ . \tag{6.1.4}$$

In particular, if $q = 0$, P is constant of value p_0, then

$$g(x) = -cp_0e^{cx} \ ,$$

for every $c \in I$.

6.1.2 Transformation of the Operator Equation into a Suitable Form for the Application of Finite Elements

In the following, we use that the *polar decomposition of* A, which is uniquely associated with A, i.e.,

$$A = V(A^*A)^{1/2} \ ,$$

where $V \in L(X, X)$ is a partial isometry with initial space $\overline{\mathrm{Ran}A^*}$ and end space $\overline{\mathrm{Ran}A}$. Since A is bijective U, we have

$$\overline{\mathrm{Ran}A^*} = \overline{\mathrm{Ran}A}$$

and V *is a unitary transformation*, i.e.,

$$V^*V = \mathrm{id}_X \ .$$

We note that (6.1.3), since,

$$(A^*A)^{1/2}f = V^*V(A^*A)^{1/2}f = V^*Af = V^*g \ ,$$

is equivalent to the equation

$$(A^*A)^{1/2}f = V^*g \ . \tag{6.1.5}$$

The operator in X, $(A^*A)^{1/2}$ is densely-defined, linear and positive self-adjoint and bijective, therefore (6.1.5) has the unique solution

$$f = [(A^*A)^{1/2}]^{-1}V^*g \ .$$

Now (6.1.5) can be solved by the "usual finite element methods." For this purpose, let $n \in \mathbb{N}^*$, $\varphi_1, \dots, \varphi_n$ be linearly independent elements of $D((A^*A)^{1/2})$. Further, we denote by $P_n \in L(X, X)$ the orthogonal projection onto $\mathcal{L}(\{\varphi_1, \dots, \varphi_n\})$. Then

$$\begin{aligned}\langle \varphi_k | V^* g \rangle &= \langle \varphi_k | (A^*A)^{1/2} f \rangle = \langle (A^*A)^{1/2} \varphi_k | f \rangle \\ &= \langle (A^*A)^{1/2} \varphi_k | P_n f \rangle + \langle (A^*A)^{1/2} \varphi_k | (1 - P_n) f \rangle \ ,\end{aligned}$$

for every $k \in \{1, \dots, n\}$. In the following, we are going to solve the corresponding "approximate" system

$$\langle \varphi_k | V^* g \rangle = \langle (A^*A)^{1/2} \varphi_k | P_n f \rangle \ , \tag{6.1.6}$$

where $k \in \{1, \dots, n\}$. The vector $P_n f$ is an element of $\mathcal{L}(\{\varphi_1, \dots, \varphi_n\})$. Hence, there are uniquely determined $\alpha_1, \dots, \alpha_n$ such that

$$P_n f = \sum_{l=1}^{n} \alpha_l \varphi_l \ .$$

As a consequence,

$$\begin{aligned}\langle \varphi_k | V^* g \rangle &= \langle (A^*A)^{1/2} \varphi_k | P_n f \rangle = \sum_{l=1}^{n} \alpha_l \, \langle (A^*A)^{1/2} \varphi_k | \varphi_l \rangle \\ &= \sum_{l=1}^{n} \langle \varphi_k | (A^*A)^{1/2} \varphi_l \rangle \, \alpha_l \ ,\end{aligned}$$

for every $k \in \{1, \dots, n\}$. Hence, we arrive at the system of linear equations

$$\sum_{l=1}^{n} M_{kl} \alpha_l = \langle \varphi_k | V^* g \rangle \ ,$$

$k = 1, \dots, n$, *where the real $n \times n$ matrix*

$$M := \Big(\langle \varphi_k | (A^*A)^{1/2} \varphi_l \rangle \Big)_{k,l=1,\dots,n}$$

is symmetric and positive definite. As a consequence,

$$\alpha_l = \sum_{k=1}^{n} (M^{-1})_{lk} \, \langle \varphi_k | V^* g \rangle \ ,$$

for every $l \in \{1, \dots, n\}$.

6.1.3 Calculation of the Polar Decomposition of A

Since

$$A = \overline{D_{\mathrm{I,l}}} \ ,$$

where $I := (0, a)$, is bijective, we need to calculate the corresponding operators

$$(A^*A)^{1/2} \ , \ (A^*A)^{1/4} \text{ and } V^* = (A^*A)^{1/2}A^{-1} \ .$$

First, we note that

$$A^*A = \overline{D_{\mathrm{I,l}}}^{\,*} \, \overline{D_{\mathrm{I,l}}} = \overline{D_{\mathrm{I,l,ad}}} \ \overline{D_{\mathrm{I,l}}} \ .$$

Hence if $f \in D_0$, where

$$D_0 := \{f \in C^2(\bar{I}, \mathbb{C}) : \lim_{x \to 0} f(x) = 0 \ , \ \lim_{x \to a} f'(x) = 0\} \subset D(A^*A) \ ,$$

then

$$A^*Af = \overline{D_{\mathrm{I,l,ad}}} f' = -f'' \ .$$

Hence, A^*A is a densely-defined, linear and self-adjoint extension of the densely-defined, linear, symmetric and essentially self-adjoint operator A_0, defined by

$$A_0 : D_0 \to X \ ,$$

and

$$A_0 f := -f'' \ ,$$

for every $f \in D_0$. Since A^*A is, in particular, closed, it follows that

$$A^*A \supset \bar{A}_0$$

and hence that

$$A^*A = \bar{A}_0 \ .$$

Let $\lambda > 0$. In the following, we are going to calculate $(A^*A + \lambda)^{-1}$. According to the theory of regular Sturm-Liouville operators,

$$(A^*A + \lambda)^{-1}$$

is given by

$$[(A^*A + \lambda)^{-1} f](x)$$

$$= -u_2(x) \int_0^x u_1(y) f(y)\, dy - u_1(x) \int_x^a u_2(y) f(y)\, dy$$
$$= \int_0^a K_\lambda(x, y) f(y)\, dy \quad ,$$

where $K_\lambda; I^2 \to \mathbb{R}$ is defined by

$$K_\lambda(x, y) := \begin{cases} -u_2(x)u_1(y) & \text{for } 0 < y < x \\ -u_1(x)u_2(y) & \text{for } x < y < a \end{cases} ,$$

for $(x, y) \in I^2$, for every $f \in X$ and $x \in I$, where $u_1, u_2 \in C^2(I, \mathbb{C})$ satisfy

$$-u_1''(x) + \lambda u_1(x) = -u_2''(x) + \lambda u_2(x) = 0 ,$$

for every $x \in I$ and

$$\lim_{x \to 0} u_1(x) = \lim_{x \to a} u_2'(x) = 0$$

as well as

$$[W(u_1, u_2)](x) = u_1(x)u_2'(x) - u_1'(x)u_2(x) = 1 ,$$

for some $x \in I$. In the following, we use for this purpose $u_1 : I \to \mathbb{C}$ and $u_2 : I \to \mathbb{C}$, defined by

$$u_1(x) := \frac{\sinh(\sqrt{\lambda}\, x)}{\sqrt{\lambda}} \ , \ u_2(x) := -\frac{\cosh(\sqrt{\lambda}\, (x - a))}{\cosh(a\sqrt{\lambda}\,)} ,$$

for every $x \in I$. Hence,

$$K_\lambda(x, y) = \begin{cases} \frac{\cosh(\sqrt{\lambda}\,(x-a)) \sinh(\sqrt{\lambda}\, y)}{\sqrt{\lambda} \cosh(a\sqrt{\lambda}\,)} & \text{for } 0 < y < x \\ \frac{\sinh(\sqrt{\lambda}\, x) \cosh(\sqrt{\lambda}\,(y-a))}{\sqrt{\lambda} \cosh(a\sqrt{\lambda}\,)} & \text{for } x < y < a \end{cases} ,$$

for $(x, y) \in I^2$. $0 < y < x, \alpha \in (0, 1)$:

$$\int_0^\infty \lambda^{\alpha-1} \frac{\cosh(\sqrt{\lambda}\,(x - a)) \sinh(\sqrt{\lambda}\, y)}{\sqrt{\lambda} \cosh(a\sqrt{\lambda}\,)}\, d\lambda$$
$$= 2a^{1-2\alpha} \int_0^\infty u^{2(\alpha-1)} \frac{\cosh(\,[1 - (x/a)]\, u\,) \sinh((y/a)u)}{\cosh(u)}\, du$$

$$= 2a^{1-2\alpha} \int_0^\infty u^{2(\alpha-1)} \, \frac{\sinh(\,[1-(x/a)+(y/a)]\,u\,)}{\cosh(u)} \, du$$
$$- 2a^{1-2\alpha} \int_0^\infty u^{2(\alpha-1)} \, \frac{\sinh(\,[1-(x/a)-(y/a)]\,u\,)}{\cosh(u)} \, du \ .$$

Hence it follows, with the help of 6.16 on page 63 of [25], that

$$\int_0^\infty \lambda^{\alpha-1} \, \frac{\cosh(\sqrt{\lambda}\,(x-a))\sinh(\sqrt{\lambda}\,y)}{\sqrt{\lambda}\cosh(a\sqrt{\lambda}\,)} \, d\lambda$$
$$= 2(4a)^{1-2\alpha}\,\Gamma(2\alpha-1)$$
$$\cdot \left[\zeta\left(2\alpha-1, \frac{x-y}{4a}\right) - \zeta\left(2\alpha-1, \frac{2a-(x-y)}{4a}\right) \right.$$
$$+\zeta\left(2\alpha-1, \frac{4a-(x-y)}{4a}\right) - \zeta\left(2\alpha-1, \frac{2a+(x-y)}{4a}\right)$$
$$-\zeta\left(2\alpha-1, \frac{x+y}{4a}\right) + \zeta\left(2\alpha-1, \frac{2a-(x+y)}{4a}\right)$$
$$\left. -\zeta\left(2\alpha-1, \frac{4a-(x+y)}{4a}\right) + \zeta\left(2\alpha-1, \frac{2a+(x+y)}{4a}\right) \right] \ ,$$

where ζ denotes Hurwitz's zeta function. We note that for $\alpha = 1/2$, the formula is ill-defined and must be interpreted in terms of analytic continuation. For all values of $\alpha \in (0, 1)$, there is available a series representation for $\zeta(2\alpha-1, \cdot)$ given by [16].

In the following, we are going to calculate $(A^*A)^{1/2}$ and V^*, using another approach via Proposition 3.2.2 and the theory of Sturm-Liouville operators. For this purpose, we need the eigenvalues and eigenfunctions of A^*A. If $\lambda > 0$, the solutions $u \in C^2(I, \mathbb{C})$ of

$$-u''(x) - \lambda u(x) = 0 \ ,$$

for every $x \in I$ are given by

$$u(x) = \alpha \sin(\sqrt{\lambda}\,x) + \beta \cos(\sqrt{\lambda}\,x) \ ,$$

for every $x \in I$, where $\alpha, \beta \in \mathbb{C}$. The boundary condition

$$\lim_{x \to 0} u(x) = 0 \ ,$$

gives

$$u(x) = \alpha \sin(\sqrt{\lambda}\,x) \ ,$$

for every $x \in I$, where $\alpha \in \mathbb{C}$, and the boundary condition

$$\lim_{x \to a} u'(x) = 0$$

gives a non-trivial u iff

$$\cos(\sqrt{\lambda}\, a) = 0 \ ,$$

i.e., iff

$$\lambda = \lambda_k := \frac{\pi^2}{a^2} \left(k + \frac{1}{2} \right)^2 \ ,$$

for some $k \in \mathbb{N}$. For such $k \in \mathbb{N}$, it follows that

$$\int_0^a \sin^2(\sqrt{\lambda}\, x)\, dx = \int_0^a \frac{1}{2} \left[1 - \cos(2\sqrt{\lambda}\, x) \right] dx$$
$$= \frac{a}{2} - \frac{1}{2} \int_0^a \cos(2\sqrt{\lambda}\, x)\, dx = \frac{a}{2} - \frac{1}{2} \frac{\sin(2\sqrt{\lambda}\, x)}{2\sqrt{\lambda}} \bigg|_0^a = \frac{a}{2}$$

and hence that, if $e_k \in C^{\infty}(I, \mathbb{R})$ is defined by

$$e_k(x) := \sqrt{\frac{2}{a}} \sin\left(\left(k + \frac{1}{2} \right) \frac{\pi x}{a} \right) \ ,$$

for every $x \in I$, where $k \in \mathbb{N}$, then

$$e_0 \, , \, e_1 \, , \, e_2 \, , \, \dots$$

is a Hilbert basis of X. For

$$f \in \left\{ g \in C^2(\bar{I}, \mathbb{C}) : \lim_{x \to 0} g(x) = 0 \wedge \lim_{x \to a} g'(a) = 0 \right\} \ ,$$

it follows that

$$A^*Af = A^*A \sum_{k=0}^{\infty} \langle e_k | f \rangle \, e_k = \sum_{k=0}^{\infty} \langle e_k | f \rangle \, A^*Ae_k = \sum_{k=0}^{\infty} \lambda_k \, \langle e_k | f \rangle \, e_k$$

and hence that

$$\langle e_k | -f'' \rangle = \langle e_k | A^*Af \rangle = \lambda_k \, \langle e_k | f \rangle \ , \ \langle e_k | f \rangle = \frac{1}{\lambda_k} \, \langle e_k | -f'' \rangle \ ,$$

for every $k \in \mathbb{N}$. Hence it follows from Proposition 3.2.2 that

$$
\begin{aligned}
(A^*A^{1/2})f &= (A^*A)^{1/2} \sum_{k=0}^{\infty} \langle e_k | f \rangle \, e_k = \sum_{k=0}^{\infty} \langle e_k | f \rangle \, (A^*A)^{1/2} e_k \\
&= \sum_{k=0}^{\infty} \langle e_k | f \rangle \sqrt{\lambda_k} \, e_k = \sum_{k=0}^{\infty} \frac{1}{\lambda_k} \langle e_k | - f'' \rangle \sqrt{\lambda_k} \, e_k \\
&= \sum_{k=0}^{\infty} \frac{1}{\sqrt{\lambda_k}} \langle e_k | - f'' \rangle \, e_k \ ,
\end{aligned}
$$

For almost all $x \in I$,

$$
\begin{aligned}
[(A^*A^{1/2})f](x) &= \sum_{k=0}^{\infty} \frac{1}{\sqrt{\lambda_k}} \langle e_k | - f'' \rangle \, e_k(x) \\
&= \langle \sum_{k=0}^{\infty} \frac{1}{\sqrt{\lambda_k}} \, e_k(x) e_k | - f'' \rangle \\
&= \int_0^a \left[\sum_{k=0}^{\infty} \frac{1}{\sqrt{\lambda_k}} \, e_k(x) e_k(y) \right] (-f''(y)) \, dy \ ,
\end{aligned}
$$

where we used that

$$
\left(\sum_{k=0}^{n} \frac{1}{\sqrt{\lambda_k}} \, e_k(x) e_k \right)_{n \in \mathbb{N}}
$$

is convergent in X, as a consequence of the estimations

$$
\left| \frac{1}{\sqrt{\lambda_k}} \, e_k(x) \right|^2 = \frac{1}{\lambda_k} |e_k(x)|^2 \leqslant \frac{2}{a} \cdot \frac{a^2}{\pi^2} \frac{1}{\left(k + \frac{1}{2}\right)^2} = \frac{8a}{\pi^2} \, \frac{1}{(2k+1)^2} \ ,
$$

$k \in \mathbb{N}^*$, and as consequence of the existence of

$$
\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \ .
$$

With the help of Corollary 5.3.5, consistent with Formula 19 on page 1052 of [8], it follows for $x, y \in I$ satisfying $x \neq y$ that

$$
\sum_{k=0}^{\infty} \frac{1}{\sqrt{\lambda_k}} \, e_k(x) e_k(y)
$$

$$
\begin{aligned}
&= \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{k+\frac{1}{2}} \sin\left(\left(k+\frac{1}{2}\right)\frac{\pi x}{a}\right) \sin\left(\left(k+\frac{1}{2}\right)\frac{\pi y}{a}\right) \\
&= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{1}{k+\frac{1}{2}} \left[\cos\left(\left(k+\frac{1}{2}\right)\frac{\pi(x-y)}{a}\right) - \cos\left(\left(k+\frac{1}{2}\right)\frac{\pi(x+y)}{a}\right)\right] \\
&= \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \left[\cos\left((2k+1)\frac{\pi(x-y)}{2a}\right) - \cos\left((2k+1)\frac{\pi(x+y)}{2a}\right)\right] \\
&= \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \left[\cos\left((2k+1)\frac{\pi|x-y|}{2a}\right) - \cos\left((2k+1)\frac{\pi(x+y)}{2a}\right)\right] \\
&= \frac{1}{\pi} \left\{\ln\left[\cot\left(\frac{\pi|x-y|}{4a}\right)\right] - \ln\left[\cot\left(\frac{\pi(x+y)}{4a}\right)\right]\right\} \\
&= \frac{1}{\pi} \ln\left[\frac{\cot\left(\frac{\pi|x-y|}{4a}\right)}{\cot\left(\frac{\pi(x+y)}{4a}\right)}\right] .
\end{aligned}
$$

As a consequence,

$$
[(A^*A)^{1/2} f](x) = -\int_0^a G_{\frac{1}{2}}(x,y) f''(y)\, dy ,
$$

for every $x \in I$, where

$$
G_{\frac{1}{2}}(x,y) := \frac{1}{\pi} \ln\left[\frac{\cot\left(\frac{\pi|x-y|}{4a}\right)}{\cot\left(\frac{\pi(x+y)}{4a}\right)}\right] = \frac{1}{\pi} \ln\left[\frac{\tan\left(\frac{\pi(x+y)}{4a}\right)}{\tan\left(\frac{\pi|x-y|}{4a}\right)}\right] ,
$$

for almost all $(x,y) \in I^2$. Further, from the latter, we conclude that

$$
(V^* f)(x) = -\int_0^a G_{\frac{1}{2}}(x,y) f'(y)\, dy ,
$$

for every $x \in I$.

Let $n \in \mathbb{N} \setminus \{0,1\}$, $l := \frac{a}{n}$ and $c_0 := 0$, $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. We define $\varphi_c : \bar{I} \to \mathbb{R}$ by

$$
\varphi_c(x) := c_m - m(c_{m+1} - c_m) + \frac{c_{m+1} - c_m}{l}\, x ,
$$

for every $x \in [ml, (m+1)l]$ and $m \in \{0, \ldots, n-1\}$. We note that φ_c is well-defined, since

$$\begin{aligned}
& c_{m-1} - (m-1)(c_{m-1+1} - c_{m-1}) + \frac{c_{m-1+1} - c_{m-1}}{l}\, ml \\
& = c_{m-1} - (m-1)(c_m - c_{m-1}) + \frac{c_m - c_{m-1}}{l}\, ml \\
& = c_{m-1} + (c_m - c_{m-1}) = c_m \\
& = c_m - m(c_{m+1} - c_m) + \frac{c_{m+1} - c_m}{l}\, ml \ ,
\end{aligned}$$

for every $m \in \{1, \ldots, n-1\}$. Hence, φ_c is piecewise linear, continuous and satisfies

$$\varphi_c(ml) = c_m \ ,$$

for every $m \in \{0, \ldots, n\}$. We note that

$$\varphi_c(x) = \begin{cases} c_0\varphi_1(x) + c_1\varphi_2(x) & \text{for } x \in [0, l] \\ c_1\varphi_1(x-l) + c_2\varphi_2(x-l) & \text{for } x \in [l, 2l] \\ \quad\cdot & \\ \quad\cdot & \\ \quad\cdot & \\ c_{n-1}\varphi_1(x-(n-1)l) + c_n\varphi_2(x-(n-1)l) & \text{for } x \in [(n-1)l, nl] \end{cases} ,$$

for every $x \in I$, where

$$\varphi_1(x) := 1 - \frac{x}{l} \ , \ \varphi_2(x) := \frac{x}{l} \ ,$$

for every $x \in [0, l]$. Further, for $k \in \mathbb{N}$,

$$\begin{aligned}
\langle e_k | \varphi_c \rangle & = \sqrt{\frac{2}{a}} \int_0^a \sin(\sqrt{\lambda_k}x)\varphi_c(x)\, dx \\
& = \sqrt{\frac{2}{a}} \sum_{m=0}^{n-1} \int_{ml}^{(m+1)l} \sin(\sqrt{\lambda_k}x) \left[c_m - m(c_{m+1} - c_m) + \frac{c_{m+1} - c_m}{l}\, x \right] dx \\
& = \sqrt{\frac{2}{a}} \sum_{m=0}^{n-1} [c_m - m(c_{m+1} - c_m)] \left[-\frac{\cos(\sqrt{\lambda_k}x)}{\sqrt{\lambda_k}} \right]\Bigg|_{ml}^{(m+1)l}
\end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{\frac{2}{a}} \sum_{m=0}^{n-1} \frac{c_{m+1}-c_m}{l} \int_{ml}^{(m+1)l} x \sin(\sqrt{\lambda_k} x)\, dx \\
&= \sqrt{\frac{2}{a}} \sum_{m=0}^{n-1} [c_m - m(c_{m+1}-c_m)] \left[-\frac{\cos(\sqrt{\lambda_k} x)}{\sqrt{\lambda_k}} \right]\Bigg|_{ml}^{(m+1)l} \\
&+ \sqrt{\frac{2}{a}} \sum_{m=0}^{n-1} \frac{c_{m+1}-c_m}{l} \left[\frac{\sin(\sqrt{\lambda_k} x)}{\lambda_k} - \frac{x \cos(\sqrt{\lambda_k} x)}{\sqrt{\lambda_k}} \right]\Bigg|_{ml}^{(m+1)l} \\
&= \sqrt{\frac{2}{a}} \frac{1}{\lambda_k} \sum_{m=0}^{n-1} \frac{c_{m+1}-c_m}{l} [\sin((m+1)l\sqrt{\lambda_k}) - \sin(ml\sqrt{\lambda_k})] \\
&- \sqrt{\frac{2}{a}} \frac{1}{\sqrt{\lambda_k}} \sum_{m=0}^{n-1} [c_m - m(c_{m+1}-c_m)] \\
&\qquad [\cos((m+1)l\sqrt{\lambda_k}) - \cos(ml\sqrt{\lambda_k})] \\
&- \sqrt{\frac{2}{a}} \frac{1}{\sqrt{\lambda_k}} \sum_{m=0}^{n-1} (c_{m+1}-c_m) \\
&\qquad [(m+1)\cos((m+1)l\sqrt{\lambda_k}) - m\cos(ml\sqrt{\lambda_k})] \ .
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\sum_{m=0}^{n-1} [c_m - m(c_{m+1}-c_m) + (m+1)(c_{m+1}-c_m)] \cos((m+1)l\sqrt{\lambda_k}) \\
&= \sum_{m=0}^{n-1} (c_m + c_{m+1} - c_m) \cos((m+1)l\sqrt{\lambda_k}) \\
&= \sum_{m=0}^{n-1} c_{m+1} \cos((m+1)l\sqrt{\lambda_k}) = \sum_{m=1}^{n} c_m \cos(ml\sqrt{\lambda_k}) \ , \\
&- \sum_{m=0}^{n-1} [c_m - m(c_{m+1}-c_m) + m(c_{m+1}-c_m)] \cos(ml\sqrt{\lambda_k}) \\
&= -\sum_{m=0}^{n-1} c_m \cos(ml\sqrt{\lambda_k}) \ ,
\end{aligned}
$$

$$
\sum_{m=1}^{n} c_m \cos(ml\sqrt{\lambda_k}\,) - \sum_{m=0}^{n-1} c_m \cos(ml\sqrt{\lambda_k}\,) = c_n \cos(a\sqrt{\lambda_k}\,) - c_0
$$
$$
= -c_0 = 0
$$

Hence,

$$
\langle e_k|\varphi_c\rangle = \frac{1}{l}\sqrt{\frac{2}{a}}\,\frac{1}{\lambda_k}\,\sum_{m=0}^{n-1}(c_{m+1}-c_m)\,[\,\sin((m+1)l\sqrt{\lambda_k}\,) - \sin(ml\sqrt{\lambda_k}\,)]\ ,
$$

and

$$
\begin{aligned}
&\sqrt{\lambda_k}\,\langle e_k|\varphi_c\rangle \\
&= \frac{1}{l}\sqrt{\frac{2}{a}}\sum_{m=0}^{n-1}(c_{m+1}-c_m)\left[\,\frac{\sin((m+1)l\sqrt{\lambda_k}\,)}{\sqrt{\lambda_k}} - \frac{\sin(ml\sqrt{\lambda_k}\,)}{\sqrt{\lambda_k}}\right] \\
&= \frac{2^{3/2}\sqrt{a}}{\pi l(2k+1)} \\
&\quad \cdot \sum_{m=0}^{n-1}(c_{m+1}-c_m)\left[\,\sin\left(\frac{(m+1)l\pi}{2a}(2k+1)\right) - \sin\left(\frac{ml\pi}{2a}(2k+1)\right)\right]\ ,
\end{aligned}
$$

for every $k \in \mathbb{N}$. As a consequence,

$$
\left(|\sqrt{\lambda_k}\,\langle e_k|\varphi_c\rangle\,|^2\right)_{k\in\mathbb{N}}
$$

is summable, and therefore

$$
\left(\sqrt{\lambda_k}\,\langle e_k|\varphi_c\rangle\, e_k\right)_{k\in\mathbb{N}}
$$

is summable. The latter implies that

$$
\varphi_c \in D((A^*A)^{1/2})\ .
$$

and that

$$
(A^*A)^{1/2}\varphi_c = \sum_{k=0}^{\infty}\sqrt{\lambda_k}\,\langle e_k|\varphi_c\rangle\, e_k\ .
$$

Further, for $y \in I$

$$\sqrt{\lambda_k}\,\langle e_k|\varphi_c\rangle\, e_k(y) = \frac{4}{\pi l}\sum_{m=0}^{n-1}(c_{m+1}-c_m)$$
$$\cdot\left[\frac{1}{2k+1}\sin\left((2k+1)\frac{(m+1)l\pi}{2a}\right)\sin\left((2k+1)\frac{\pi y}{2a}\right)\right.$$
$$\left.-\frac{1}{2k+1}\sin\left((2k+1)\frac{ml\pi}{2a}\right)\sin\left((2k+1)\frac{\pi y}{2a}\right)\right],$$

for every $k \in \mathbb{N}$. We know that for $x, y \in I$ satisfying $x \neq y$

$$\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left((2k+1)\frac{\pi x}{2a}\right)\sin\left((2k+1)\frac{\pi y}{2a}\right)$$
$$=\frac{1}{4}\ln\left[\frac{\cot\left(\frac{\pi|x-y|}{4a}\right)}{\cot\left(\frac{\pi(x+y)}{4a}\right)}\right].$$

Hence

$$[(A^*A)^{1/2}\varphi_c](y) = \sum_{k=0}^{\infty}\sqrt{\lambda_k}\,\langle e_k|\varphi_c\rangle\, e_k(y)$$
$$=\frac{1}{\pi l}\sum_{m=0}^{n-1}(c_{m+1}-c_m)\left[\ln\left[\frac{\cot\left(\frac{\pi|(m+1)l-y|}{4a}\right)}{\cot\left(\frac{\pi[(m+1)l+y]}{4a}\right)}\right]-\ln\left[\frac{\cot\left(\frac{\pi|ml-y|}{4a}\right)}{\cot\left(\frac{\pi[ml+y]}{4a}\right)}\right]\right]$$
$$=\frac{1}{\pi l}\sum_{m=0}^{n-1}(c_{m+1}-c_m)\ln\left[\frac{\cot\left(\frac{\pi|(m+1)l-y|}{4a}\right)}{\cot\left(\frac{\pi[(m+1)l+y]}{4a}\right)}\frac{\cot\left(\frac{\pi[ml+y]}{4a}\right)}{\cot\left(\frac{\pi|ml-y|}{4a}\right)}\right]$$
$$=\frac{1}{\pi l}\sum_{m=0}^{n-1}(c_{m+1}-c_m)\ln\left[\frac{\tan\left(\frac{\pi[(m+1)l+y]}{4a}\right)}{\tan\left(\frac{\pi|(m+1)l-y|}{4a}\right)}\frac{\tan\left(\frac{\pi|ml-y|}{4a}\right)}{\tan\left(\frac{\pi[ml+y]}{4a}\right)}\right],$$

for almost all $y \in I$. As a consequence,

$$[(A^*A)^{1/2}\varphi_c](x)$$

$$= \frac{1}{\pi l} \sum_{m=0}^{n-1} (c_{m+1} - c_m) \ln \left[\frac{\tan\left(\frac{\pi[(m+1)l+x]}{4a}\right)}{\tan\left(\frac{\pi|(m+1)l-x|}{4a}\right)} \frac{\tan\left(\frac{\pi|ml-x|}{4a}\right)}{\tan\left(\frac{\pi[ml+x]}{4a}\right)} \right],$$

for almost all $x \in I$.

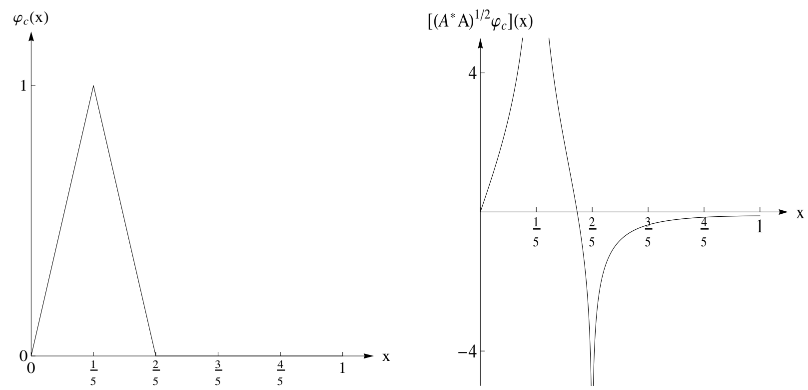

Figure 9: Graphs of φ_c and $(A^*A)^{1/2}\varphi_c$ for $a = 1, n = 5, l = 1/5, c_0 = 0, c = e_1$.

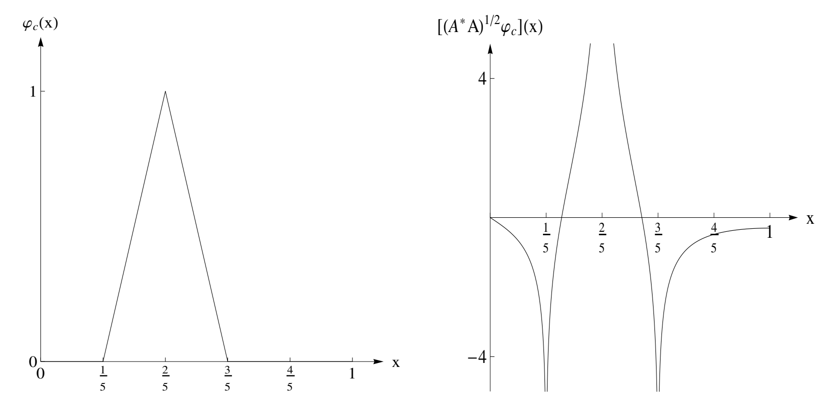

Figure 10: Graphs of φ_c and $(A^*A)^{1/2}\varphi_c$ for $a = 1, n = 5, l = 1/5, c_0 = 0, c = e_2$.

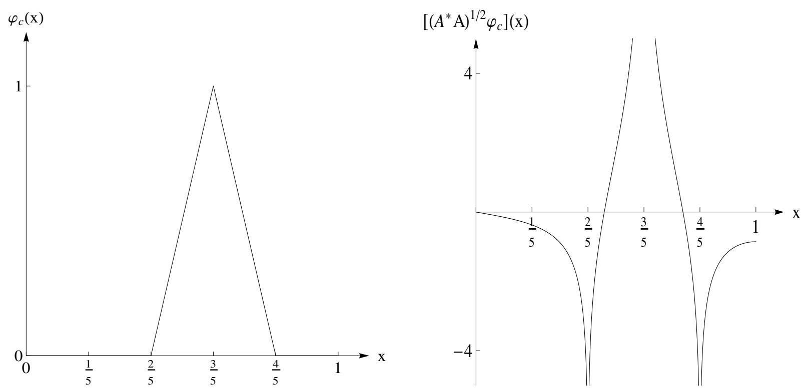

Figure 11: Graphs of φ_c and $(A^*A)^{1/2}\varphi_c$ for $a = 1, n = 5, l = 1/5, c_0 = 0, c = e_3$.

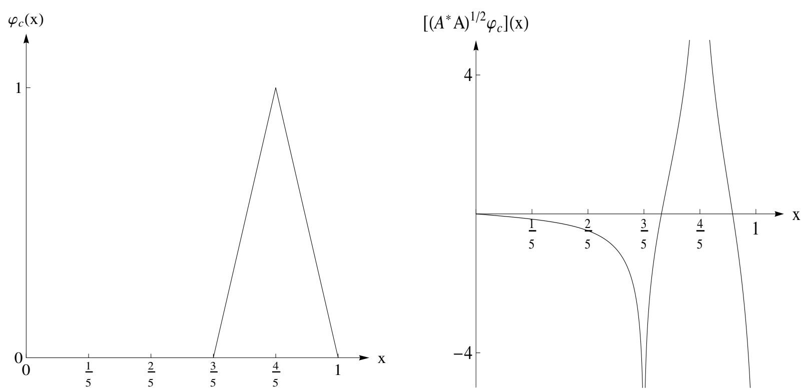

Figure 12: Graphs of φ_c and $(A^*A)^{1/2}\varphi_c$ for $a = 1, n = 5, l = 1/5, c_0 = 0, c = e_4$.

We note that for every $j \in \{1, \dots, n\}$, the corresponding φ_{e_j}, where $e_1, \dots, e_n$ are the canonical basis vectors of $\mathbb{R}^n$, vanishes outside the interval

$$((j-1)l, (j+1)l) \ .$$

Also $\varphi_{e_1}, \dots, \varphi_{e_n}$ are linearly independent, since if $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ are such that

$$\sum_{k=1}^{n} \alpha_k \varphi_{e_k} = 0 \ ,$$

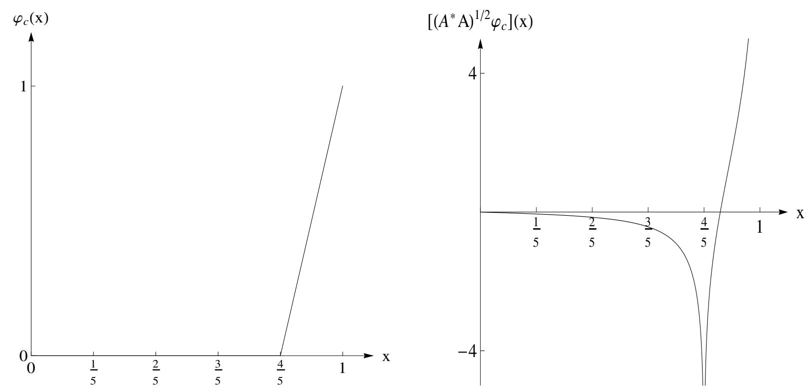

Figure 13: Graphs of φ_c and $(A^*A)^{1/2}\varphi_c$ for $a = 1, n = 5, l = 1/5, c_0 = 0, c = e_5$.

then

$$0 = \sum_{k=1}^{n} \alpha_k \varphi_{e_k}(jl) = \alpha_j \ ,$$

for every $j \in \{1, \ldots, n\}$. Hence, following the approach in Section 6.1.2, we arrive at the system of linear equations

$$\sum_{m=1}^{n} M_{km} \alpha_m = \langle \varphi_{e_k} | V^* g \rangle \ ,$$

$k = 1, \ldots, n$, where

$$\varphi_{e_k}(x) = \begin{cases} \frac{x-(k-1)l}{l} & \text{for } x \in [(k-1)l, kl] \\ \frac{(k+1)l-x}{l} & \text{for } x \in [kl, (k+1)l] \end{cases} ,$$

and zero otherwise, for every $k \in \{1, \ldots, n-1\}$,

$$\varphi_{e_n}(x) = \begin{cases} \frac{x-(n-1)l}{l} & \text{for } x \in [(n-1)l, nl] \end{cases} ,$$

and zero otherwise,

$$(V^*g)(x) = -\int_0^a G_{\frac{1}{2}}(x,y) g'(y)\, dy = p_0 c^2 \int_0^a G_{\frac{1}{2}}(x,y) e^{cy}\, dy \ ,$$

for every $x \in I$, where

$$G_{\frac{1}{2}}(x, y) = \frac{1}{\pi} \ln \left[\frac{\tan\left(\frac{\pi(x+y)}{4a}\right)}{\tan\left(\frac{\pi|x-y|}{4a}\right)} \right] ,$$

for almost all $(x, y) \in I^2$,

$$g(x) = -c p_0 e^{cx} \ , \ c = \frac{Mg}{RT} \ ,$$

for every $x \in I$,

$$M := \Big(\langle \varphi_{e_k} | (A^*A)^{1/2} \varphi_{e_m} \rangle \Big)_{k,m=1,\ldots,n}$$

is a symmetric and positive definite $n \times n$-matrix, where

$$\begin{aligned}
&[(A^*A)^{1/2}\varphi_{e_m}](y) \\
&= \frac{1}{\pi l} \left\{ \ln \left[\frac{\tan\left(\frac{\pi[ml+y]}{4a}\right)}{\tan\left(\frac{\pi|ml-y|}{4a}\right)} \frac{\tan\left(\frac{\pi|(m-1)l-y|}{4a}\right)}{\tan\left(\frac{\pi[(m-1)l+y]}{4a}\right)} \right] \right. \\
&\qquad \left. - \ln \left[\frac{\tan\left(\frac{\pi[(m+1)l+y]}{4a}\right)}{\tan\left(\frac{\pi|(m+1)l-y|}{4a}\right)} \frac{\tan\left(\frac{\pi|ml-y|}{4a}\right)}{\tan\left(\frac{\pi[ml+y]}{4a}\right)} \right] \right\} ,
\end{aligned}$$

for almost all $y \in I$, for every $m \in \{1, \ldots, n-1\}$,

$$[(A^*A)^{1/2}\varphi_{e_n}](y) = \frac{1}{\pi l} \ln \left[\frac{\tan\left(\frac{\pi[nl+y]}{4a}\right)}{\tan\left(\frac{\pi|nl-y|}{4a}\right)} \frac{\tan\left(\frac{\pi|(n-1)l-y|}{4a}\right)}{\tan\left(\frac{\pi[(n-1)l+y]}{4a}\right)} \right] ,$$

for almost all $y \in I$. As a consequence,

$$\alpha_l = \sum_{k=1}^{n} (M^{-1})_{lk} \, \langle \varphi_{e_k} | V^* g \rangle \ ,$$

for every $l \in \{1, \ldots, n\}$,