The left-hand side of the equation is given by:

$$M := \langle \phi_{ek} | [(A^*A)^{1/2} \phi_{em}] \rangle_{k,m=1,\dots,n} \tag{1}$$

where:

$$\phi_{ek}(x) = \begin{cases} \dfrac{x-(k-1)l}{l} & x \in [(k-1)l, kl] \\ \dfrac{(k+1)l-x}{l} & x \in [kl, (k+1)l] \end{cases} \tag{2}$$

for $k \in \{1, \dots, n-1\}$ and for $k = n$:

$$\phi_{en}(x) = \frac{x-(n-1)l}{l} \quad x \in [(n-1)l, nl] \quad x \in [(n-1)l, nl] \tag{3}$$

Meanwhile, it holds true that:

$$[(A^*A)^{1/2}\phi_{em}](y) =$$

$$\frac{1}{\pi l} \left\{ \ln \left[\frac{\tan\left(\dfrac{\pi[ml+y]}{4a}\right) \tan\left(\dfrac{\pi|(m-1)l-y|}{4a}\right)}{\tan\left(\dfrac{\pi|ml-y|}{4a}\right) \tan\left(\dfrac{\pi[(m-1)l+y]}{4a}\right)} \right] \right.$$

$$\left. - \ln \left[\frac{\tan\left(\dfrac{\pi[(m+1)l+y]}{4a}\right) \tan\left(\dfrac{\pi|ml-y|}{4a}\right)}{\tan\left(\dfrac{\pi|(m+1)l-y|}{4a}\right) \tan\left(\dfrac{\pi[ml+y]}{4a}\right)} \right] \right\} \tag{4}$$

for every $m \in \{1, \dots, n-1\}$

$$[(A^*A)^{1/2}\phi_{en}](y) = \frac{1}{\pi l} \ln \left[\frac{\tan\left(\dfrac{\pi[nl+y]}{4a}\right) \tan\left(\dfrac{\pi|(n-1)l-y|}{4a}\right)}{\tan\left(\dfrac{\pi|nl-y|}{4a}\right) \tan\left(\dfrac{\pi[(n-1)l+y]}{4a}\right)} \right] \tag{5}$$

Unpacking the equation (1):

$$M_{km} = \int_0^a \phi_{ek}(x)[(A^*A)^{1/2}\phi_{em}](x)dx \tag{6}$$

We take into account that this integrand has a singularity at $x = kl$ with $k \in \{1, \dots, n-1\}$. We perform the following split. Also from now on we set $[(A^*A)^{1/2}\phi_{em}](x) = f_m(x)$:

$$M_{km} = \int_0^{kl} \phi_{ek}(x)f_m(x)dx + \int_{kl}^a \phi_{ek}(x)f_m(x)dx$$

We then introduce the term $|x - kl|^\alpha$, in order to get rid of the singularity:

1

$$M_{km} = \int_0^{kl} \phi_{ek}(x)f_m(x)|x-kl|^{-\alpha}|x-kl|^{\alpha}dx + \int_{kl}^{a} \phi_{ek}(x)f_m(x)|x-kl|^{-\alpha}|x-kl|^{\alpha}dx$$

Setting $f_m(x)|x-kl|^{-\alpha} = g_m(x)$, with $-1 < \alpha < 0$, we can write:

$$M_{km} = \int_0^{kl} \phi_{ek}(x)g_m(x)(kl-x)^{\alpha}dx + \int_{kl}^{a} \phi_{ek}(x)g_m(x)(x-kl)^{\alpha}dx$$

$$= \frac{kl}{2}\int_{-1}^{1} \phi_{ek}\left(\frac{kl}{2}\xi + \frac{kl}{2}\right)g_m\left(\frac{kl}{2}\xi + \frac{kl}{2}\right)\left(\frac{kl}{2}\right)^{\alpha}(1-\xi)^{\alpha}d\xi +$$

$$+ \frac{a-kl}{2}\int_{-1}^{1} \phi_{ek}\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right)g_m\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right)\left(\frac{a-kl}{2}\right)^{\alpha}(1+\xi)^{\alpha}d\xi$$

$$= \left(\frac{kl}{2}\right)^{1+\alpha}\int_{-1}^{1} \phi_{ek}\left(\frac{kl}{2}\xi + \frac{kl}{2}\right)g_m\left(\frac{kl}{2}\xi + \frac{kl}{2}\right)(1-\xi)^{\alpha}d\xi +$$

$$+ \left(\frac{a-kl}{2}\right)^{1+\alpha}\int_{-1}^{1} \phi_{ek}\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right)g_m\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right)(1+\xi)^{\alpha}d\xi$$

We substitute:

$$\tilde{g}_{1m}(\xi) = \phi_{ek}\left(\frac{kl}{2}\xi + \frac{kl}{2}\right)g_m\left(\frac{kl}{2}\xi + \frac{kl}{2}\right) \tag{7}$$

$$\tilde{g}_{2m}(\xi) = \phi_{ek}\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right)g_m\left(\frac{a-kl}{2}\xi + \frac{a+kl}{2}\right) \tag{8}$$

This yields:

$$M_{km} = \left(\frac{kl}{2}\right)^{1+\alpha}\int_{-1}^{1} \tilde{g}_{1m}(\xi)(1+\xi)^{\alpha}d\xi + \left(\frac{a-kl}{2}\right)^{1+\alpha}\int_{-1}^{1} \tilde{g}_{2m}(\xi)(1-\xi)^{\alpha}d\xi \tag{9}$$

Performing a Gauss-Jacobi integration scheme, we get:

$$M_{km} \approx \left(\frac{kl}{2}\right)^{1+\alpha}\sum_i \tilde{g}_{1m}(\xi_i)w_i^{(\alpha,0)} + \left(\frac{a-kl}{2}\right)^{1+\alpha}\sum_i \tilde{g}_{1m}(\xi_i)w_i^{(0,\alpha)} \tag{10}$$

Special case k=n

For the case that $k = n$, we follow the same procedure as above:

$$M_{nn} \approx \left(\frac{a}{2}\right)^{1+\alpha} \sum_i \tilde{g}(\xi) w_i^{(\alpha,0)} \tag{11}$$

where:

$$\tilde{g}(\xi) = \phi_{en}\left(\frac{a}{2}\xi + \frac{a}{2}\right) f\left(\frac{a}{2}\xi + \frac{a}{2}\right)(1 - \xi)^{-\alpha} \tag{12}$$