

Calculation for right-hand side

The right-hand side of the equation is described by the equation:

$$X_k := \langle \phi_{ek} | (V^* g) \rangle = \int_0^a \phi_{ek}(x) (V^* g)(x) dx \quad (0.0.1)$$

for $k \in \{1, \dots, n\}$ where:

$$\phi_{ek}(x) = \begin{cases} \frac{x - (k-1)l}{l} & x \in [(k-1)l, kl] \\ \frac{(k+1)l - x}{l} & x \in [kl, (k+1)l] \end{cases} \quad (0.0.2)$$

for $k \in \{1, \dots, n-1\}$ and for $k = n$:

$$\phi_{en}(x) = \frac{x - (n-1)l}{l} \quad x \in [(n-1)l, nl] \quad (0.0.3)$$

Furthermore we have:

$$(V^* g)(x) = p_0 c^2 \int_0^a G_{\frac{1}{2}}(x, y) e^{cy} dy \quad (0.0.4)$$

where the fractional derivative $G_{\frac{1}{2}}(x, y)$ is given by:

$$G_{\frac{1}{2}}(x, y) = \frac{1}{\pi} \ln \left[\frac{\tan \left(\frac{\pi(x+y)}{4a} \right)}{\tan \left(\frac{\pi|x-y|}{4a} \right)} \right] \quad (0.0.5)$$

Plugging all the results in 0.0.1 yields:

$$X_k = \frac{p_0 c^2}{1 + ca} \int_0^a \phi_{ek}(x) \int_0^a G_{\frac{1}{2}}(x, y) e^{cy} (y - a) dy dx \quad (0.0.6)$$

$$= \frac{p_0 c^2}{1 + ca} \int_0^a \int_0^a \phi_{ek}(x) G_{\frac{1}{2}}(x, y) e^{cy} (y - a) dy dx \quad (0.0.7)$$

We observe that X_k becomes singular for $x = y$. In order to eliminate the singularity, we will perform a Gauss-Jacobi integration scheme. Thus we introduce the term:

$$X_k = p_0 c^2 \int_0^a \int_0^a \phi_{ek}(x) f(x, y) |x - y|^{-\alpha} |x - y|^\alpha dy dx \quad (0.0.8)$$

with $-1 < \alpha < 0$ and:

$$f(x, y) = G_{\frac{1}{2}}(x, y) e^{cy} \quad (0.0.9)$$

$$g(x, y) = f(x, y) |x - y|^{-\alpha} \quad (0.0.10)$$

Splitting the integral at the critical point yields:

$$\begin{aligned} X_k &= \frac{p_0 c^2}{1 + ca} \int_0^a \left(\int_0^x \phi_{ek}(x) g(x, y) (x - y)^\alpha dy + \int_x^a \phi_{ek}(x) g(x, y) (y - x)^\alpha dy \right) dx \\ &= \frac{p_0 c^2}{1 + ca} \int_0^a \frac{x}{2} \int_{-1}^1 \phi_{ek}(x) g\left(x, \frac{x}{2}(1 + \xi)\right) \left(\frac{x}{2}\right)^\alpha (1 - \xi)^\alpha d\xi dx \\ &\quad + \frac{p_0 c^2}{1 + ca} \int_0^a \frac{a - x}{2} \int_{-1}^1 \phi_{ek}(x) g\left(x, \frac{a - x}{2}\xi + \frac{a + x}{2}\right) \left(\frac{a - x}{2}\right)^\alpha (1 + \xi)^\alpha d\xi dx \\ &= \frac{p_0 c^2}{1 + ca} \int_0^a \left(\frac{x}{2}\right)^{1+\alpha} \int_{-1}^1 \phi_{ek}(x) g\left(x, \frac{x}{2}(1 + \xi)\right) (1 - \xi)^\alpha d\xi dx \\ &\quad + \frac{p_0 c^2}{1 + ca} \int_0^a \left(\frac{a - x}{2}\right)^{1+\alpha} \int_{-1}^1 \phi_{ek}(x) g\left(x, \frac{a - x}{2}\xi + \frac{a + x}{2}\right) (1 + \xi)^\alpha d\xi dx \\ &= \frac{p_0 c^2}{1 + ca} \int_0^a \int_{-1}^1 \left(\frac{x}{2}\right)^{1+\alpha} \phi_{ek}(x) g\left(x, \frac{x}{2}(1 + \xi)\right) (1 - \xi)^\alpha d\xi dx \\ &\quad + \frac{p_0 c^2}{1 + ca} \int_0^a \int_{-1}^1 \left(\frac{a - x}{2}\right)^{1+\alpha} \phi_{ek}(x) g\left(x, \frac{a - x}{2}\xi + \frac{a + x}{2}\right) (1 + \xi)^\alpha d\xi dx \end{aligned}$$

$$\begin{aligned}
&= \frac{p_0 c^2}{1+ca} \frac{a}{2} \left[\int_{-1}^1 \int_{-1}^1 \left(\frac{a}{4}\right)^{1+\alpha} (1+\eta)^{1+\alpha} \phi_{ek} \left(\frac{a}{2}(1+\eta)\right) g \left(\frac{a}{2}(1+\eta), \frac{x}{2}(1+\xi)\right) (1-\xi)^\alpha d\xi d\eta \right. \\
&\quad \left. + \int_{-1}^1 \int_{-1}^1 \left(\frac{a}{4}\right)^{1+\alpha} (1-\eta)^{1+\alpha} \phi_{ek} \left(\frac{a}{2}(1+\eta)\right) g \left(\frac{a}{2}(1+\eta), \frac{a}{4}(1-\eta)\xi + \frac{a}{4}(3+\eta)\right) (1+\xi)^\alpha d\xi d\eta \right]
\end{aligned}$$

We substitute:

$$\tilde{g}_1(\xi, \eta) = \phi_{ek} \left(\frac{a}{2}(1+\eta)\right) g \left(\frac{a}{2}(1+\eta), \frac{x}{2}(1+\xi)\right) \quad (0.0.11)$$

$$\tilde{g}_2(\xi, \eta) = \phi_{ek} \left(\frac{a}{2}(1+\eta)\right) g \left(\frac{a}{2}(1+\eta), \frac{a}{4}(1-\eta)\xi + \frac{a}{4}(3+\eta)\right) \quad (0.0.12)$$

meaning:

$$\begin{aligned}
X_k &= 2 \frac{p_0 c^2}{1+ca} \left(\frac{a}{4}\right)^{2+\alpha} \int_{-1}^1 \int_{-1}^1 \tilde{g}_1(\xi, \eta) (1-\xi)^\alpha (1+\eta)^{1+\alpha} d\xi d\eta \\
&\quad + 2 \frac{p_0 c^2}{1+ca} \left(\frac{a}{4}\right)^{2+\alpha} \int_{-1}^1 \int_{-1}^1 \tilde{g}_2(\xi, \eta) (1+\xi)^\alpha (1-\eta)^{1+\alpha} d\xi d\eta
\end{aligned}$$

Performing the Gauss-Jacobi quadrature yields:

$$\begin{aligned}
X_k &\approx \frac{2p_0 c^2}{1+ca} \left(\frac{a}{4}\right)^{2+\alpha} \sum_i \sum_j \tilde{g}_1(\xi_i, \eta_j) w_i^{(\alpha,0)} w_j^{(0,1+\alpha)} \\
&\quad + \frac{2p_0 c^2}{1+ca} \left(\frac{a}{4}\right)^{2+\alpha} \sum_i \sum_j \tilde{g}_2(\xi_i, \eta_j) w_i^{(0,\alpha)} w_j^{(1+\alpha,0)}
\end{aligned} \quad (0.0.13)$$