MATH 6310 (FALL 2025) HOMEWORK SOLUTIONS

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UPDATED: WEDNESDAY 3 SEPTEMBER, 2025

1. Linear Algebra Problems

Problem (1)**.** If $A \in \mathbb{R}^{n\times n}$ is symmetric, invertible, and has spectral decomposition $A = V\Lambda V^{-1}$, give (with proof) a formula for A^{-1}?

Solution. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and invertible, with spectral decomposition

$$A = V\Lambda V^{-1},$$

where V is an orthogonal matrix ($V^{-1} = V^\top$) and Λ is diagonal.

Note: Λ contains the eigenvalues of A, which are all non-zero since A is invertible.

Proof.

$$Av = \Lambda v$$

If some $\lambda_i = 0$, then $Av_i = 0$ for some vector $v_i \neq 0$. This makes the $\mathcal{N}(A)$ non-trivial, contradicting the assumption that A is invertible. □

To find A^{-1}, we use the property that for invertible matrices:

$$(XYZ)^{-1} = Z^{-1}Y^{-1}X^{-1}$$

$$\begin{aligned}\Rightarrow (A)^{-1} &= (V\Lambda V^{-1})^{-1}\\ &= (V^{-1})^{-1}\Lambda^{-1}V^{-1}\\ &= V\Lambda^{-1}V^{-1}\end{aligned}$$

Thus, the inverse is given by

$$A^{-1} = V\Lambda^{-1}V^{-1}.$$

Problem (2)**.** (a) Prove that $\mathcal{N}(A^\top A) = \mathcal{N}(A)$

(b) Prove that $\mathcal{N}(A)$ is orthogonal to Row(A)

(c) How does $\mathcal{N}(AB)$ relate to $\mathcal{N}(A)$ and $\mathcal{N}(B)$? Prove any statements you claim.

(d) Use Sylvester's Inequality to prove that if $A \in \mathbb{R}^{m\times r}$ and $B \in \mathbb{R}^{r\times n}$ have rank r, then rank$(AB) = r$.

Solution. (a) *Proof.* Given $A \in \mathbb{R}^{m\times n}$, its **nullspace** is,

$$\mathcal{N}(A) := \{x \in \mathbb{R}^n : Ax = 0\}$$

If $x \in \mathcal{N}(A)$, then

$$A^\top(Ax) = A^\top 0 = 0$$

$$A^\top(Ax) = 0 \Leftrightarrow (A^\top A)x = 0$$

$$\Rightarrow x \in \mathcal{N}(A^\top A)$$

This also shows that $\mathcal{N}(A) \subseteq \mathcal{N}(A^\top A)$

If $x \in \mathcal{N}(A^\top A)$, then

$$A^\top Ax = 0$$

Multiplying both sides by $x^\top$

$$\begin{aligned} x^\top A^\top Ax &= 0 \\ \Rightarrow (Ax)^\top Ax &= 0 \\ \Rightarrow \|Ax\|_2^2 &= 0 \\ \Rightarrow Ax &= 0 \\ \Rightarrow x &\in \mathcal{N}(A) \end{aligned}$$

This also shows that $\mathcal{N}(A^\top A) \subseteq \mathcal{N}(A)$

Since both inclusions hold, we can conclude $\mathcal{N}(\boldsymbol{A}^\top \boldsymbol{A}) = \mathcal{N}(\boldsymbol{A})$. □

(b)

Claim. *For any $x \in \mathcal{N}(A)$ and $y \in \mathrm{Row}(A)$, $\mathcal{N}(A)$ is orthogonal to $\mathrm{Row}(A)$ if*

$$\langle x, y \rangle = 0$$

Proof. We know $\mathrm{Row}(A) = \mathrm{Col}(A^\top)$. Also, y can be written as a linear combination of the rows of A,

$$y = A^\top z : z \in \mathbb{R}^n$$

Computing the dot product $\langle x, y \rangle = 0$,. Hence,

$$\begin{aligned} \langle x, y \rangle &= y^\top x \\ &= (A^\top z)^\top x \\ &= z^\top Ax \end{aligned}$$

We know $\mathcal{N}(A) := \{x \in \mathbb{R}^n : Ax = 0\}$

$$\begin{aligned} &= z^\top 0 \\ &= 0 \end{aligned}$$

Thus, we prove $\langle x, y \rangle = 0$.

$\therefore \mathcal{N}(A)$ is orthogonal to $\mathrm{Row}(A)$ □

(c)

Claim (1)**.**

$$\mathcal{N}(B) \subseteq \mathcal{N}(AB)$$

Proof. We know $\mathcal{N}(B) := \{x \in \mathbb{R}^n : Bx = 0\}$. If $Bx = 0$, then

$$\begin{aligned} Bx &= 0 \\ \Rightarrow ABx &= 0 \\ \Rightarrow (AB)x &= 0 \\ \Rightarrow x &\in \mathcal{N}(AB). \end{aligned}$$

Thus, this proves the nullspace of B is a subspace of the nullspace of AB, i.e.

$$\mathcal{N}(B) \subseteq \mathcal{N}(AB) \tag{1}$$

□

Claim (2)**.** $\mathcal{N}(AB) = \mathcal{N}(B) \leftrightarrow \text{rank}(AB) = \text{rank}(B)$

Proof. By the Rank-nullity theorem, Let $Z \in \mathbb{R}^{m\times n}$. Then

$$\dim(\mathcal{N}(Z)) + \text{rank}(Z) = n$$

Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $AB \in \mathbb{R}^{m\times p}$ and $\text{rank}(AB) = \text{rank}(B)$. Then

$$\dim(\mathcal{N}(AB)) + \text{rank}(AB) = \dim(\mathcal{N}(B)) + \text{rank}(B)$$

$$\Rightarrow \dim(\mathcal{N}(AB)) = \dim(\mathcal{N}(B)) \tag{2}$$

From (1) and (2) we can conclude,

$$\mathcal{N}(AB) = \mathcal{N}(B)$$

□

(d) *Proof.* Given $A \in \mathbb{R}^{m\times r}$, $B \in \mathbb{R}^{r\times n}$ and $\text{rank}(A) = \text{rank}(B) = r$
We know, $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$. Then

$$\text{rank}(AB) \le \min\{r, r\}$$

$$\Rightarrow \text{rank}(AB) \le r \tag{1}$$

By **Sylvester's Inequality:** If $A \in \mathbb{R}^{m\times r}$, $B \in \mathbb{R}^{r\times n}$, then

$$\text{rank}(A) + \text{rank}(B) \le \text{rank}(AB) + r$$

$$\Rightarrow r + r \le \text{rank}(AB) + r$$

$$\Rightarrow \text{rank}(AB) \ge r \tag{2}$$

By (1) and (2) we prove,

$$\text{rank}(AB) = r$$

□

Problem (13)**.** Let $A \in \mathbb{R}^{m\times n}$ have SVD $A = U\Sigma V^\top$.

(a) If $A = U\Sigma V^\top$ is invertible, give a formula for A^{-1} in terms of U, Σ and V. Justify your answer.
(b) If $A = U\Sigma V^\top$, give a formula for $A^\top A$ and $AA^\top$. in terms of U, Σ, V. Justify your answer.

Solution. (a) If $A = U\Sigma V^\top$ then

$$A^{-1} = (U\Sigma V^\top)^{-1} = (V^\top)^{-1}\Sigma^{-1}U^{-1}$$

For orthogoal matrices the inverse is the same as transpose. Because

$$QQ^\top = I$$

$$\Rightarrow A^{-1} = V\Sigma^{-1}U^\top$$

Proof. Given $A = U\Sigma V^\top$, multiplying both sides by A^{-1}

$$A^{-1}A = (V\Sigma^{-1}U^\top)(U\Sigma V^\top)$$

$$A^{-1}A = V\Sigma^{-1}\Sigma V^\top \because UU^\top = I$$

$$A^{-1}A = VV^\top \because \Sigma\Sigma^{-1} = I$$

$$A^{-1}A = I \because VV^\top = I$$

□

(b) Given $A = U\Sigma V^\top$, multiplying by $A^\top$ to the left

$$A^\top A = (U\Sigma V^\top)^\top U\Sigma V^\top$$
$$A^\top A = (V^\top)^\top \Sigma^\top U^\top U\Sigma V^\top$$
$$A^\top A = V\Sigma^\top \Sigma V^\top$$
$$A^\top A = V\Sigma^2 V^\top$$

Similarly multiplying $A^\top$ to the right

$$AA^\top = U\Sigma V^\top (U\Sigma V^\top)^\top$$
$$A^\top A = U\Sigma V^\top (V^\top)^\top \Sigma^\top U^\top$$
$$A^\top A = U\Sigma^\top \Sigma U^\top$$
$$A^\top A = U\Sigma^2 U^\top$$

Problem (15)**.** Recall that the definition of the Frobenius norm is

$$\|A\|_{\mathrm{F}} := \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |A_{ij}|^2 \right)^{\frac{1}{2}}.$$

(a) Prove that the Frobenius norm is unitarily invariant (i.e., $\|QA\|_{\mathrm{F}} = \|AW\|_{\mathrm{F}} = \|A\|_{\mathrm{F}}$ for any orthogonal matrices Q, W).

(b) Prove that for any $A \in \mathbb{R}^{m\times n}$,

$$\|A\|_{\mathrm{F}} = \left(\sum_{i=1}^{\mathrm{rank}(A)} \sigma_i^2 \right)^{\frac{1}{2}}.$$

Solution. (a)

Claim (1)**.** $\|QAW\|_F = \|A\|_F \ \forall$ *orthogonal* Q, W

Proof. For the Frobenius norm, using $\|A\|_F^2 = \mathrm{trace}(A^\top A)$,

$$\|QAW\|_F^2 = \mathrm{trace}(W^\top A^\top Q^\top . QAW)$$

For $Q \in \mathbb{R}^{m\times n}$ it is orthogonal $\leftrightarrow Q^\top Q = I$

$$\|QAW\|_F^2 = \mathrm{trace}(W^\top A^\top AW)$$
$$\|QAW\|_F^2 = \mathrm{trace}(A^\top A) = \|A\|_F^2$$

$\square$

(b) *Proof.* Let SVD of A be $A = U\Sigma V^\top$, where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal and $\Sigma \in \mathbb{R}^{m\times n}$ {diagonal}

$$\begin{aligned}\|A\|_F^2 &= \mathrm{trace}(A^\top A)\\ &= \mathrm{trace}((U\Sigma V^\top)^\top U\Sigma V^\top)\\ &= \mathrm{trace}(V\Sigma^\top U^\top U\Sigma V^\top)\\ &= \mathrm{trace}(\Sigma^\top \Sigma)\end{aligned}$$

Let $r = \mathrm{rank}(A) \therefore$ the diagonal entries of $\Sigma = \{\sigma_1, \ldots, \sigma_r\}$

$$\Rightarrow \|A\|_F^2 = \sum_{i=1}^{r} \sigma_i^2$$

$$\Rightarrow \|A\|_F = (\sum_{i=1}^{\text{rank}(A)} \sigma_i^2)^{\frac{1}{2}}$$

□

Problem (18)**.** [CUR Decomposition] Let $A \in \mathbb{R}^{m\times n}$, and $I \subset [m]$, $J \subset [n]$ with $|I| = |J| = k = \text{rank}(A)$. If $C = A(:, J)$, $R = A(I, :)$ and $U = A(I, J)$, prove that if $\text{rank}(U) = k$, then $A = CU^{-1}R$. (*Hint:* First justify that one can write $A = CX$ for some unknown X. Next, write $R = P_I A$ where P_I is a row selection matrix, and consider what happens to $A = CX$ upon multiplication by P_I.)

Note: If you google this, you will find my published proof of it. Please try to do this from scratch using the hint above...

Solution. *Proof.* Since matrix C is subset of column vectors of A. Therefore, the $\text{Col}(C) \subseteq \text{Col}(A)$. We also know that $|I| = |J| = k = \text{rank}(A)$.

Assuming $\text{rank}(U) = k$, the column vectors of C become linearly independent i.e., $\text{rank}(C) = k \Rightarrow \text{Col}(C)$ forms a basis for $\text{Col}(A)$. Thus, we can write $A = CX$ for some unknown X. Now, let P_I be a row selection matrix that selects the rows indexed by I. Then we have:

$$R = P_I A = P_I CX$$

Since $P_I C$ selects the rows of C corresponding to I, we can write:

$$P_I C = U \Rightarrow R = UX$$

$$A = CX = CU^{-1}R$$

□