MATH 6310 (FALL 2025) HOMEWORK SOLUTIONS

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1. Homework 5 - Wednesday 29 October, 2025

Problem (9)**.** A matrix $A \in \mathbb{R}^{n\times n}$ is *strictly diagonally dominant (SDD)* if $|A_{ii}| > \sum_{j\neq i} |A_{ij}|$ for all $i \in [n]$. Prove that if A is SDD then A is invertible. (*Hint:* Show that 0 cannot be an eigenvalue of A by mimicking the proof of the Geršgorin circle theorem.)

Solution. A square matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if $\mathbf{0}$ is not an eigenvalue of A. We will prove that if A is strictly diagonally dominant (SDD), then $\mathbf{0}$ cannot be an eigenvalue, thereby showing that A must be invertible.

A matrix A is strictly diagonally dominant if for all $i \in [n] = \{1, 2, \dots, n\}$:

$$|A_{ii}| > \sum_{j\neq i} |A_{ij}|$$

Proof. Assume, that $\mathbf{0}$ is an eigenvalue of A.

If $\mathbf{0}$ is an eigenvalue, then there exists a non-zero eigenvector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top$ such that:

$$A\mathbf{x} = \mathbf{0x} = \mathbf{0}$$

Now, consider an index k such that $|x_k| = \max_i |x_i|$. Since $\mathbf{x}$ is a non-zero vector, we must have $|x_k| > 0$.

$$\sum_{j=1}^{n} A_{ij} x_j = 0$$

splitting at k^{th} index.

$$A_{kk}x_k + \sum_{j\neq k} A_{kj}x_j = 0$$
$$A_{kk}x_k = -\sum_{j\neq k} A_{kj}x_j$$

Taking the absolute value of both sides.

$$|A_{kk}x_k| = \left| -\sum_{j\neq k} A_{kj}x_j \right| = \left| \sum_{j\neq k} A_{kj}x_j \right|$$
$$|A_{kk}||x_k| = \left| \sum_{j\neq k} A_{kj}x_j \right|$$

By triangle inequality,

$$|A_{kk}||x_k| \le \sum_{j\neq k} |A_{kj}x_j|$$

$$|A_{kk}||x_k| \leq \sum_{j\neq k} |A_{kj}||x_j|$$

we know that $|x_j| \leq |x_k|$ for all $j \neq k$. Substituting this into the inequality:

$$|A_{kk}||x_k| \leq \sum_{j\neq k} |A_{kj}||x_k|$$

$$|A_{kk}||x_k| \leq |x_k| \sum_{j\neq k} |A_{kj}|$$

Since $\mathbf{x}$ is a non-zero vector, we established that $|x_k| > 0$. Therefore, we can divide both sides of the inequality by $|x_k|$:

$$|A_{kk}| \leq \sum_{j\neq k} |A_{kj}|$$

But this contradicts with our assumption that matrix A is SDD.

Thus, our initial assumption that $\mathbf{0}$ is an eigenvalue of A must be false and if matrix A is SDD it is invertible.

□

Problem (11)**.** Alternate Proof of the Spectral Decomposition; *Counts as 2 problems* The aim is to prove that $A = V\Lambda V^\top$ for symmetric $A \in \mathbb{R}^{n\times n}$.

(a) Let V be any subspace of $\mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ be symmetric. Prove that if $Ax \in V$ for every $x \in V$, then $Ay \in V^\perp$ for every $y \in V^\perp$.

(b) With the assumptions above, if V is any nontrivial subspace of $\mathbb{R}^n$ (i.e., $V \neq \{0\}$) and $Ax \in V$ for every $x \in V$, prove that V contains an eigenvector of A. (*Hint:* Start with an orthonormal basis of V, say $\{u_1, \dots, u_k\}$. Write $A = UR$ for some $R \in \mathbb{R}^{k\times k}$. Show that R is symmetric. Next, suppose λ is an eigenvalue of R with eigenvector v. Show that $w = \sum v_j u_j$ is an eigenvector of A with eigenvalue λ.)

(c) Prove the spectral theorem. (*Hint:* Start with λ_1, v_1 the leading eigenvalue/vector of A and let $V_1 = \text{span}(v_1)$. Use the above lemmas to find an orthogonal eigenvector $v_2 \perp v_1$. Iterate the argument to complete the proof.)

Solution. (a) To prove: $\langle Ay, x\rangle = 0$

Let $y \in V^\perp$, given $x \in V$ and $Ax \in V$.

Proof. Since $y \in V^\perp$,

$$\langle y, Ax\rangle = 0$$

$$\langle A^\top y, x\rangle = 0$$

$$\langle Ay, x\rangle = 0$$

Since $(A\mathbf{y})\cdot\mathbf{x} = 0$ holds for every $\mathbf{x} \in V$, it proves that $A\mathbf{y}$ is orthogonal to V i.e., $A\mathbf{y} \in V^\perp$. □

(b) To Prove: $w = \sum v_j u_j$ is an eigenvector of A with eigenvalue λ

Proof. Let $[u_1, \dots, u_k]$ be an orthonormal basis of the nontrivial subspace (V). Let $U = [u_1 \cdots u_k]$. Then $U^\top U = I_k$ and $\text{col}(U) = V$.

Because A maps V into V, each column Au_j lies in V and so can be written as a linear combination of the u_i. Hence there exists an $k \times k$ matrix R with

$$AU = UR.$$

Multiply on the left by $U^\top$. Using $U^\top U = I_k$ we get

$$R = U^\top AU.$$

Since A is symmetric, $A^\top = A$, so

$$R^\top = U^\top AU^\top = U^\top A^\top U = U^\top AU = R,$$

thus R is symmetric. Let $Rv = \lambda v$, where $v \neq 0$

For given,

$$w := Uv = \sum_{j=1}^{k} v_j u_j \in V.$$

Because $U^\top U = I_k$ we have $|w|^2 = v^\top U^\top Uv = v^\top v > 0$, so $w \neq 0$. Now

$$Aw = A(Uv) = (AU)v = (UR)v = U(Rv) = U(\lambda v) = \lambda(Uv) = \lambda w.$$

□

(c)

Theorem 1.1. *Let $A \in R^{n\times n}$, then $\exists$ orthonormal basis of eigenvectors of A.*

Proof. The whole space $V = \mathbb{R}^n$ is nontrivial and A-invariant, so by the lemma (b) there is a nonzero eigenvector v_1 of A with eigenvalue λ_1

We use the lemmas established in the previous problems:

The first problem showed that if A is symmetric and $A(\mathbf{x}) \in V$ for all $\mathbf{x} \in V$, then

$$A(\mathbf{y}) \in V^\perp \quad \forall \quad \mathbf{y} \in V^\perp.$$

Here, $V = V_1 = \text{span}(\mathbf{v}_1)$. The condition $A(\mathbf{x}) \in V_1$ is satisfied because for any $\mathbf{x} = c\mathbf{v}_1 \in V_1$, we have $A\mathbf{x} = A(c\mathbf{v}_1) = c(A\mathbf{v}_1) = c(\lambda_1\mathbf{v}_1) = (\lambda_1 c)\mathbf{v}_1$, which is clearly still in V_1.

Therefore, the previous lemma applies: A preserves the orthogonal complement $V_1^\perp$.

$$\mathbf{y} \in V_1^\perp \implies A\mathbf{y} \in V_1^\perp$$

We can choose an orthonormal basis for $V_1^\perp$,

$$\{\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$$

Let P be the $n \times (n-1)$ matrix with columns $\mathbf{u}_2, \ldots, \mathbf{u}_n$. We can define a matrix (similar to R in (b)):

$$A' = P^T AP$$

Since A is symmetric, A' must also be symmetric: $(A')^T = (P^T AP)^T = P^T A^T (P^T)^T = P^T AP = A'$. A' is an $(n-1)\times(n-1)$ symmetric matrix. By (b), A' has $n-1$ orthonormal eigenvectors, $\mathbf{w}_2, \ldots, \mathbf{w}_n$.

Each eigenvector $\mathbf{w}_i$ of A' corresponds to an eigenvector $\mathbf{v}_i \in V_1^\perp$ of A by the transformation:

$$\mathbf{v}_i = P\mathbf{w}_i$$

(This is exactly the construction from the previous problem, $\mathbf{w} = U\mathbf{v}$.)

These vectors $\mathbf{v}_2, \ldots, \mathbf{v}_n$ are orthonormal because the $\mathbf{w}_i$'s are orthonormal, and P has orthonormal columns ($P^T P = I_{n-1}$).

Finally, since $\mathbf{v}_2, \ldots, \mathbf{v}_n$ are all in $V_1^\perp$, they are all orthogonal to $\mathbf{v}_1$.

We have constructed a set of n vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ that are **Eigenvectors of** A and are **Orthogonal**.

□

Problem (12). Suppose $A \in \mathbb{R}^{m\times n}$ with SVD $A = U\Sigma V^\top$ and that $\operatorname{rank}(A) = k$. Write U, Σ, and $V^\top$ in block form as follows

$$U = \begin{bmatrix} U_k & U_{m-k} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix}, \quad V^\top = \begin{bmatrix} V_k^\top \\ \\ V_{n-k}^\top \end{bmatrix},$$

where U_k contains the first k left singular vectors, and U_{m-k} contains the remainder, and similarly for V_k, and $\Sigma_k = \operatorname{diag}(\sigma_1, \ldots, \sigma_k)$ is diagonal and contains all nonzero singular values of A.

(a) List the sizes of all of the submatrices involved in the expressions above (including the 0 blocks in Σ)

(b) Multiply $U\Sigma V^\top$ using block matrix multiplication from above, and prove that $U\Sigma V^\top = U_k\Sigma_k V_k^\top$ (hence $A = U_k\Sigma_k V_k^\top$).

Solution. (a) Sizes of the blocks

Given $A \in \mathbb{R}^{m\times n}$ and $\operatorname{rank}(A) = k$:

- U is $m \times m$ orthogonal.
  - U_k contains the first k columns of U, so U_k is $m \times k$.
  - U_{m-k} contains the remaining $m-k$ columns, so U_{m-k} is $m \times (m-k)$.
- Σ is $m \times n$. It is blocked as

$$\Sigma = \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix},$$

  where
  - Σ_k is the $k \times k$ diagonal matrix $\operatorname{diag}(\sigma_1, \ldots, \sigma_k)$ (the nonzero singular values),
  - the top-right zero block has size $k \times (n-k)$,
  - the bottom-left zero block has size $(m-k) \times k$,
  - the bottom-right zero block has size $(m-k) \times (n-k)$.
    (Together these blocks make an $(m \times n)$ matrix.)
- V is $n \times n$ orthogonal.
  - V_k (first (k) columns) is $n \times k$, so $V_k^\top$ is $k \times n$.
  - V_{n-k} is $n \times (n-k)$, so $V_{n-k}^\top$ is $(n-k) \times n$.
    Stacking these gives $V^\top$ which is $n \times n$.

(b) *Proof.* We start with the block product $U\Sigma V^\top$. First we multiply U and Σ. Using the block forms

$$U = \begin{bmatrix} U_k & U_{m-k} \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix},$$

we get

$$U\Sigma = \begin{bmatrix} U_k & U_{m-k} \end{bmatrix} \begin{bmatrix} \Sigma_k & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_k\Sigma_k & 0 \end{bmatrix}.$$

Here $U_k\Sigma_k$ is $m \times k$ and the zero block is $m \times (n-k)$.

Now multiply this with $V^\top = \begin{bmatrix} V_k^\top \\ V_{n-k}^\top \end{bmatrix}$:

$$U\Sigma V^\top = \begin{bmatrix} U_k\Sigma_k & 0 \end{bmatrix} \begin{bmatrix} V_k^\top \\ V_{n-k}^\top \end{bmatrix} = U_k\Sigma_k V_k^\top + 0 \cdot V_{n-k}^\top = U_k\Sigma_k V_k^\top.$$

$\square$

Problem (14). Let $A \in \mathbb{R}^{m\times n}$ have truncated SVD of order k given by $A_k = U_k\Sigma_k V_k^\top$.

(a) Give a formula for $A_k A_k^\top$ (make any simplifications you can)
(b) Give a formula for $A_k^\top A_k$ (make any simplifications you can)
(c) Interpret these formulas.

Solution. Let $A_k = U_k \Sigma_k V_k^\top$ where $U_k \in \mathbb{R}^{m\times k}$, $V_k \in \mathbb{R}^{n\times k}$ have orthonormal columns and $\Sigma_k = \mathrm{diag}(\sigma_1, \ldots, \sigma_k)$ with $\sigma_1 \geq \cdots \geq \sigma_k > 0$.

Lemma 1.2. $V_k^\top V_k = I_k$

Proof. We know $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices ($U^\top U = I_m$, $V^\top V = I_n$).

U_k and V_k are obtained by **keeping only the first k columns** of U and V corresponding to the nonzero singular values $\sigma_1, \ldots, \sigma_k$. Since the columns of U and V are orthonormal, any subset of them (like the first k) will also be orthonormal - i.e.

$$U_k^\top U_k = I_k \quad \text{and} \quad V_k^\top V_k = I_k.$$

□

(a) $\mathbf{A_k A_k^\top}$

$$\begin{aligned} A_k A_k^\top &= (U_k \Sigma_k V_k^\top)(U_k \Sigma_k V_k^\top)^\top \\ &= U_k \Sigma_k V_k^\top V_k \Sigma_k U_k^\top \\ &= U_k \Sigma_k^2 U_k^\top, \end{aligned}$$

since $V_k^\top V_k = I_k$.
Equivalently,

$$A_k A_k^\top = \sum_{i=1}^{k} \sigma_i^2, u_i u_i^\top.$$

(b) $\mathbf{A_k^\top A_k}$

$$\begin{aligned} A_k^\top A_k &= (U_k \Sigma_k V_k^\top)^\top (U_k \Sigma_k V_k^\top) \\ &= V_k \Sigma_k U_k^\top U_k \Sigma_k V_k^\top \\ &= V_k \Sigma_k^2 V_k^\top, \end{aligned}$$

since $U_k^\top U_k = I_k$.
Equivalently,

$$A_k^\top A_k = \sum_{i=1}^{k} \sigma_i^2, v_i v_i^\top.$$

(c) **Interpretation**

Both $A_k A_k^\top$ and $A_k^\top A_k$ are symmetric positive semidefinite matrices of rank k.

Proof. Take $A_k A_k^\top = U_k \Sigma_k^2 U_k^\top$.

$$(A_k A_k^\top)^\top = (U_k \Sigma_k^2 U_k^\top)^\top = U_k \Sigma_k^2 U_k^\top$$

Hence $A_k A_k^\top$ is symmetric.
Similarly, $(A_k^\top A_k)^\top = A_k^\top A_k$, so both are symmetric.
For any vector x,

$$x^\top (A_k A_k^\top) x = (A_k^\top x)^\top (A_k^\top x) = |A_k^\top x|^2 \geq 0.$$

Therefore, $A_k A_k^\top$ is positive semidefinite. □

Their eigenpairs are given directly by the truncated SVD: the columns u_i of U_k are eigenvectors of $A_k A_k^\top$ with eigenvalues σ_i^2; the columns v_i of V_k are eigenvectors of $A_k^\top A_k$ with the same eigenvalues σ_i^2s.

Proof. Multiplying $A_k A_k^\top$ by any left singular vector u_i:

$$A_k A_k^\top u_i ;=; U_k \Sigma_k^2 U_k^\top u_i.$$

Because the columns $u_1, \ldots, u_k$ are orthonormal, $U_k^\top u_i$ is the basis vector e_i. Thus

$$A_k A_k^\top u_i = U_k \Sigma_k^2 e_i = \sigma_i^2 u_i.$$

So each u_i (for $i \le k$) is an eigenvector with eigenvalue σ_i^2. □