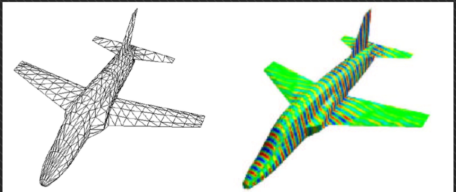

2D Finite Element Method

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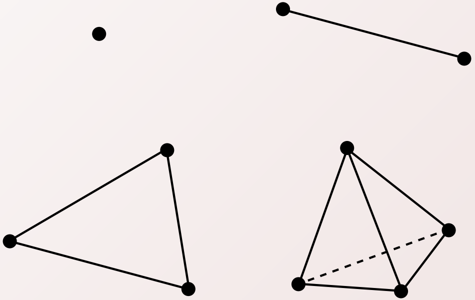
Outline

Outline

ND Finite Element Method

- Same steps as in 1D:
  1. Mesh
     - 2D - triangles, rectangles, hexagons ...
     - 3D - tetrahedrons, bricks ...
  2. Approximation.
  3. Compact form.
  4. Solve SLE.

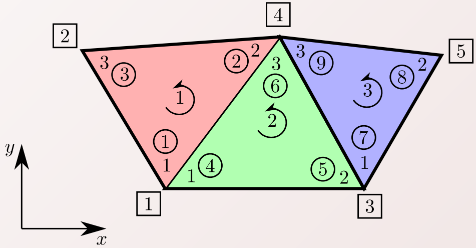


2D Mesh

- Set of elements (triangles) $\mathcal{E}$ and nodes $\mathcal{N}$.
- Element set (matrix) - pointers to nodes:

$$\mathcal{E} = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 3 & 4 \\ 3 & 5 & 4 \end{bmatrix}$$

- Distmesh - free MATLAB mesh generator.



Outline

2D Problem

2D Poisson equation

PDE:

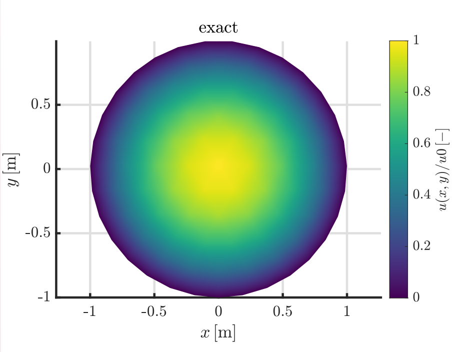
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4, \;\; \Omega : x^2 + y^2 \leq 1.$$

Boundary conditions:

$$u\left(\{x, y\} \in \delta\Omega\right) = 0\,\mathrm{V}.$$

Analytical solution:

$$u\left(x, y\right) = 1 - x^2 - y^2.$$



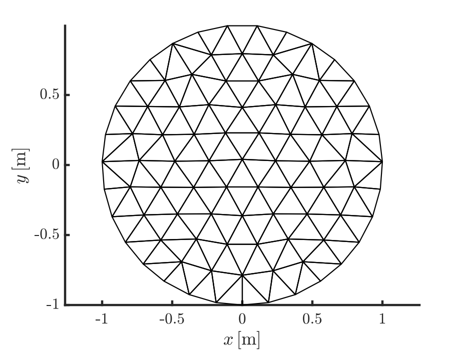
2D Mesh

- Equilateral triangles are the best!
- Counterclockwise order of nodes!
- Area of individual triangles:

$$\Delta^{(e)} = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2}\left(b_1 c_2 - b_2 c_1\right)$$

(see next slides for meaning of b, and c triangle coeffs.)



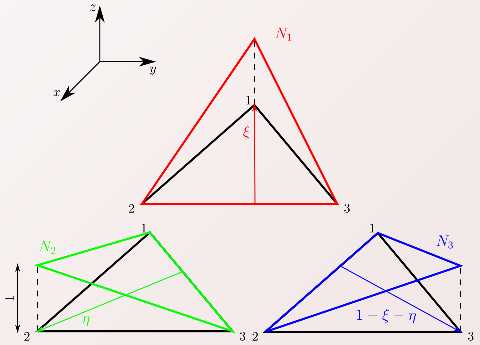
2D Approximation

- First order basis functions:

$$N_j(x, y) = \frac{1}{2\Delta}(a_j + b_j x + c_j y)$$

- Relation to simplex coordinates?

$$\xi + \eta + \zeta = 1,$$
$$\zeta = 1 - \xi - \eta.$$



2D Approximation

- First order basis functions:

$$N_j(x,y) = \frac{1}{2\Delta}\left(a_j + b_j x + c_j y\right)$$

- Triangle coefficients:

$$\begin{aligned} a_1 &= x_2 y_3 - y_2 x_3, & b_1 &= y_2 - y_3, & c_1 &= x_3 - x_2, \\ a_2 &= x_3 y_1 - y_3 x_1, & b_2 &= y_3 - y_1, & c_2 &= x_1 - x_3, \\ a_3 &= x_1 y_2 - y_1 x_2. & b_3 &= y_1 - y_2. & c_3 &= x_2 - x_1. \end{aligned}$$

- Basis function derivatives:

$$\frac{\partial N_j}{\partial x} = \frac{b_j}{2\Delta}. \qquad \frac{\partial N_j}{\partial y} = \frac{c_j}{2\Delta}.$$

Outline

Rayleigh–Ritz Formulation

- Minimization of functional $\mathcal{F}$:

$$\mathcal{F} = \sum_{e=1}^{M} \mathcal{F}^{(e)} \left(N^{(e)} \right)$$

Functional

Higher–order function $\simeq$ function that takes functions as arguments (or return them).

- Variational analysis:

$$I\left(y\right) = \int_{a}^{b} \mathcal{F}\left(x, y, \partial y / \partial x\right) dx$$

- Variation δy - infinitesimal change in y for a fixed value of the independent variable x
- Variation δy of y vanishes for points where y is known (prescribed).

Variational principle of common PDEs

- Minimization of $\mathcal{F}$ - no variation:

$$\delta I = 0$$

- For expansion (basis) functions N_i:

$$\frac{\partial I}{\partial u_1} = 0, \ \frac{\partial I}{\partial u_2} = 0, \ \dots, \ \frac{\partial I}{\partial u_N} = 0,$$

Name of eq.	PDE	$I(\Phi)$
Inhom. wave eq.	$\nabla^2\Phi + k^2\Phi = g$	$1/2 \int_\Omega \left[\lvert\nabla\Phi\rvert^2 - k^2\Phi^2 + 2g\Phi \right] d\Omega$
Hom. wave eq.	$\nabla^2\Phi + k^2\Phi = 0$	$1/2 \int_\Omega \left[\lvert\nabla\Phi\rvert^2 - k^2\Phi^2 \right] d\Omega$
TD hom. wave eq.	$\nabla^2\Phi - 1/k^2\Phi_{tt} = 0$	$1/2 \int_{\Delta t} \int_\Omega \left[\lvert\nabla\Phi\rvert^2 - 1/k^2{\Phi_t}^2 \right] d\Omega\, dt$
Diffusion eq.	$\nabla^2\Phi - k^2\Phi_t = 0$	$1/2 \int_{\Delta t} \int_\Omega \left[\lvert\nabla\Phi\rvert^2 - k^2\Phi\Phi_t \right] d\Omega\, dt$
Poisson's eq.	$\nabla^2\Phi = g$	$1/2 \int_\Omega \left[\lvert\nabla\Phi\rvert^2 + 2g\Phi \right] d\Omega$
Laplace's eq.	$\nabla^2\Phi = 0$	$1/2 \int_\Omega \left[\lvert\nabla\Phi\rvert^2 \right] d\Omega$

Rayleigh–Ritz Formulation

- Back to our 2D Poisson's eq.:

$$\mathcal{F}^{(e)} = \frac{1}{2} \iint\limits_{\Omega^{(e)}} \left[\left(\frac{\partial u^{(e)}}{\partial x} \right)^2 + \left(\frac{\partial u^{(e)}}{\partial y} \right)^2 \right] d\Omega - \iint\limits_{\Omega^{(e)}} f^{(e)} u^{(e)} \, d\Omega$$

- Minimization of functional over element e:

$$\frac{\partial \mathcal{F}^{(e)}}{\partial {N_i}^{(e)}} = \sum_{j=1}^{3} {u_j}^{(e)} \iint\limits_{\Omega^{(e)}} \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \, d\Omega - \iint\limits_{\Omega^{(e)}} f N_i \, d\Omega$$

- Matrix form:

$$\boldsymbol{K} \boldsymbol{u} = \boldsymbol{g}$$

Element Matrices

- Matrix $\boldsymbol{K}$:

$$K_{i,j} = \iint\limits_{\Omega^{(e)}} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\, dy\, dx$$

- Using barycentric coords.:

$$K_{i,j} = 2\Delta \int_0^1 \int_0^{1-\xi_i} \frac{\partial \xi_i}{\partial x}\frac{\partial \xi_j}{\partial x} + \frac{\partial \xi_i}{\partial y}\frac{\partial \xi_j}{\partial y}\, d\xi_j\, d\xi_i = \frac{b_i b_j + c_i c_j}{4\Delta}$$

Element Matrices

- Right–hand side g:

$$g_j = \iint\limits_{\Omega^{(e)}} f N_j \, dy \, dx$$

- Using barycentric coords.:

$$g_j = 2\Delta \int\limits_0^1 \int\limits_0^{1-\xi_i} f \xi_j \, d\xi_j \, d\xi_i = \frac{\Delta f}{3}$$

Matrix Assembly

- Put matrices $\boldsymbol{K}^{(e)}$ on diagonal:

$$\underset{3M\times 3M}{\boldsymbol{K}} = \begin{bmatrix} \boldsymbol{K}^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{K}^{(2)} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{K}^{(M)} \end{bmatrix}$$

$$\underset{3M\times 1}{\boldsymbol{g}} = \begin{bmatrix} \boldsymbol{g}^{(1)} \\ \boldsymbol{g}^{(2)} \\ \dots \\ \boldsymbol{g}^{(M)} \end{bmatrix}$$

- Compact form:

$$\boldsymbol{K}_\mathrm{c} = \boldsymbol{C}^\mathrm{T}\boldsymbol{K}\boldsymbol{C},$$
$$\boldsymbol{g}_\mathrm{c} = \boldsymbol{C}^\mathrm{T}\boldsymbol{g}.$$

- Apply boundary condition:

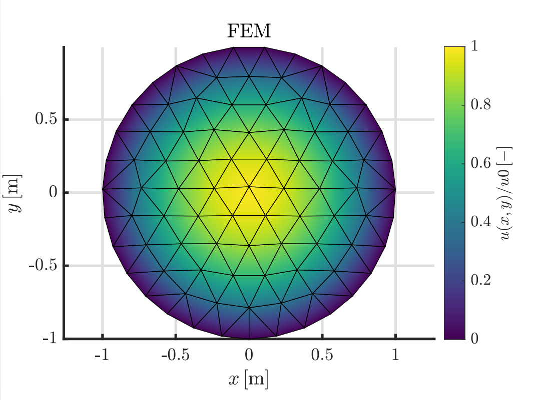
$$u\left(\{x,y\} \in \delta\Omega\right) = 0$$

  - delete rows and columns corresponding to global nodes on boundary $\delta\Omega$ from $\boldsymbol{K}_\mathrm{c}$ and $\boldsymbol{g}_\mathrm{c}$

Solution

- Solve the reduced system:

$$\boldsymbol{K}_{\mathrm{r}}\boldsymbol{u}_{\mathrm{r}} = \boldsymbol{g}_{\mathrm{r}}$$

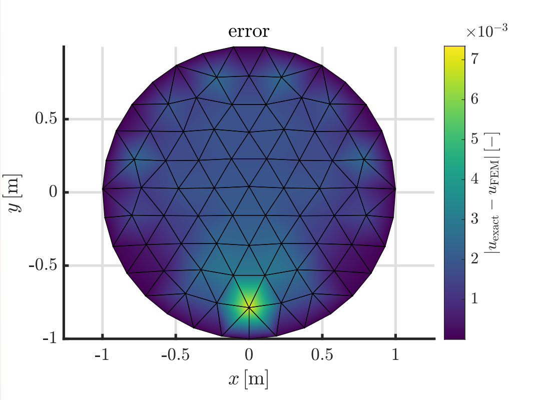


Solution

- Solve the reduced system:

$$\boldsymbol{K}_{\mathrm{r}}\boldsymbol{u}_{\mathrm{r}} = \boldsymbol{g}_{\mathrm{r}}$$

- Analytical vs. FEM error (three orders of magnitude lower).



Outline

General Boundary Conditions

- Homogeneous BC: 0 for boundary point $\boldsymbol{r} \in \delta\Omega$.
- Three types:
  1. Dirichlet: $\Phi(\boldsymbol{r}) = 0$.
  2. Neumann: $\dfrac{\partial \Phi(\boldsymbol{r})}{\partial n} = 0$.
  3. Mixed: $\dfrac{\partial \Phi(\boldsymbol{r})}{\partial n} + h(\boldsymbol{r})\,\Phi(\boldsymbol{r}) = 0$.

- Inomogeneous BC: $q(\boldsymbol{r})$ for boundary point $\boldsymbol{r} \in \delta\Omega$.
- Three types:
  1. Dirichlet: $\Phi(\boldsymbol{r}) = q(\boldsymbol{r})$.
  2. Neumann: $\dfrac{\partial \Phi(\boldsymbol{r})}{\partial n} = q(\boldsymbol{r})$.
  3. Mixed: $\dfrac{\partial \Phi(\boldsymbol{r})}{\partial n} + h(\boldsymbol{r})\,\Phi(\boldsymbol{r}) = q(\boldsymbol{r})$.
- $\boldsymbol{n}$ - unit vector perpendicular to boundary pointing outwards.

Boundary Conditions Example

Solve for u:

ODE:

$$\frac{\partial^2 u}{\partial x^2} = f, \; 0 \leq x \leq L \; (L = 1\,\mathrm{m}).$$

Neumann B.C.:

$$\frac{\partial u\,(x=0)}{\partial x} = a,$$

Dirichlet B.C.:

$$u\,(x=L) = b.$$

- FDM ystem $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{g}$:

$$\boldsymbol{A} = \frac{1}{\Delta x^2}\begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \dots \\ u_n \end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix} \frac{a}{\Delta x} \\ f_2 \\ f_3 \\ \dots \\ b \end{bmatrix}$$

Boundary Conditions

- Neumann B.C.:
  - Incorporate to the system matrices.
  - Used for excitation of the system.
- Dirichlet B.C.:
  - Reduce the size of the system by N pre–defined values.
  - Significant reduction for 2D and 3D.

Thank you for your attention!

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