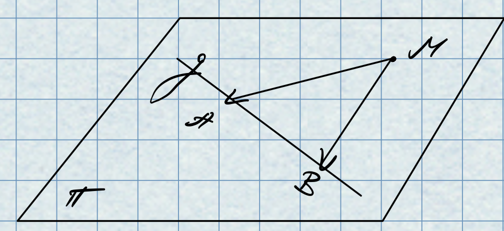
1 Да се намери равнината π, която минава през т. M и правата g, ако:

а) $M(2,1,-1)$ и $g: \begin{cases} x-z=0 \\ y+2=0 \end{cases}$



б) $M(1,0,-1)$ и $g: \begin{cases} x=1+\lambda \\ y=1+\lambda \\ z=-2\lambda \end{cases}$

а) $z=0 \Rightarrow x=0,\ y=-2$ | $z=1 \Rightarrow x=1,\ y=-2$

① $\Rightarrow$ т. $A(0,-2,0) \in g \in \pi$ | $\Rightarrow$ т. $B(1,-2,1) \in g \in \pi$

② $\overrightarrow{MA}(-2,-3,1)$ и $\overrightarrow{MB}(-1,-3,2)$

③ $\pi: \begin{vmatrix} x-2 & -2 & -1 \\ y-1 & -3 & -3 \\ z+1 & 1 & 2 \end{vmatrix} = 0$; $\pi: -6(x-2)-6(z+1)-(y-1)-3(z+1)+4(y-1)+3(x-2)=0$

$\Rightarrow \pi: -3x+3y+6=0 \ /:(-3)$; $\underline{x-y-2=0}$

б) ① $\lambda=0$ (ние го избираме) $\Rightarrow x=y=1$ и $z=0$

$\Rightarrow$ т. $A(1,1,0) \in g \in \pi$

$\lambda=1 \Rightarrow x=y=2,\ z=-2 \Rightarrow$ т. $B(2,2,-2) \in g \in \pi$

② $\overrightarrow{MA}(0,1,1)$ и $\overrightarrow{MB}(1,2,-1)$

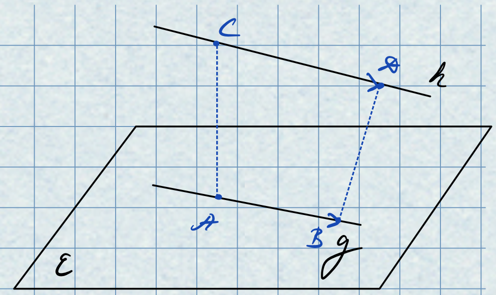
③ $\pi: \begin{vmatrix} x-1 & 0 & 1 \\ y-0 & 1 & 2 \\ z+1 & 1 & -1 \end{vmatrix} = 0$; $\pi: -(x-1)+y-(z+1)-2(x-1)=0$

$\rightarrow \pi: -3x+y-z+2=0 \ /.(-1)$

$\Rightarrow \pi: 3x-y+z-2=0$

2 Да се намери равнината ε, която съдържа правата g и е успоредна на правата $\underline{h}$, ако:

а) $g: \begin{cases} x = 2+\lambda \\ y = 2-\lambda \\ z = \lambda \end{cases}$ и $h: \begin{cases} x = 1+\mu \\ y = 0 \\ z = \mu \end{cases}$



б) $g: \begin{cases} x+y+2z-1=0 \\ y+3z-1=0 \end{cases}$ и $h: \begin{cases} x = 1+\lambda \\ y = -1+\lambda \\ z = 1+\lambda \end{cases}$

а) $\lambda = 0 \Rightarrow x=2,\ y=2,\ z=0$ | $\lambda = 1 \Rightarrow x=3,\ y=z=1$

① $\Rightarrow$ т. $A(2,2,0) \in g \in \varepsilon$ | $\Rightarrow$ т. $B(3,1,1) \in g \in \varepsilon$

② $\overrightarrow{AB}(1,-1,1) \in \varepsilon$

③ $\mu = 0 \Rightarrow x=1,\ y=z=0$ | $\mu = 1 \Rightarrow x=2,\ y=0,\ z=0$

$\Rightarrow$ т. $C(1,0,0) \in h$ | $\Rightarrow$ т. $D(2,0,1) \in h$

④ $\overrightarrow{CD}(1,0,1) \parallel h \parallel \varepsilon \Rightarrow \overrightarrow{CD} \parallel \varepsilon$

$$\Rightarrow \varepsilon: \begin{vmatrix} x-3 & 1 & 1 \\ y-1 & -1 & 0 \\ z-1 & 1 & 1 \end{vmatrix} = 0;$$

$\varepsilon: -(x-3)+y-1+z-1-(y-1)=0$

$\Rightarrow \varepsilon: -x+z+2=0 \quad /.(-1)$

$\varepsilon: x-z-2=0$

б) ① $z=0 \Rightarrow \left| \begin{array}{l} x+y-1=0 \\ y-1=0 \end{array} \right\}$ $x=0,\ y=1,\ z=0$ $\Rightarrow$ т. $A(0,1,0) \in g \in \varepsilon$

$t=1 \Rightarrow \left| \begin{array}{l} x+y-1=0 \\ y+2=0 \end{array} \right\}$ $x=1,\ y=-2,\ z=1$ $\Rightarrow$ т. $B(1,-2,1) \in g \in \varepsilon$

② $\overrightarrow{AB}(1,-3,1) \in g \in \varepsilon$

③ Векторът $\vec{v}(1,1,1)$ и $h: \begin{cases} x = 1+\lambda \\ y = -1+\lambda \\ z = 1+\lambda \end{cases}$ $\quad \vec{v} \parallel h \parallel \varepsilon \Rightarrow \vec{v} \parallel \varepsilon$

$$\Rightarrow \varepsilon: \begin{vmatrix} x & 1 & 1 \\ y-1 & -3 & 1 \\ z & 1 & 1 \end{vmatrix} = 0 \; ; \quad \begin{aligned} &\varepsilon: -3x+z+y-1+3z-(y-1)-x=0 \\ &\varepsilon: -4x+4z=0 \quad |:(-4) \\ &\varepsilon: x-z=0 \end{aligned}$$

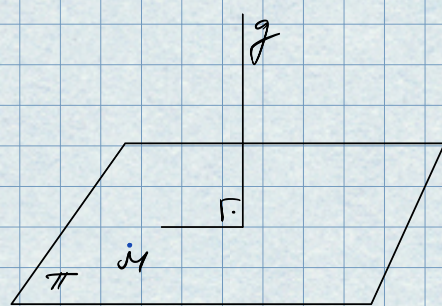
Нека $\pi: Ax+By+Cz+D=0$. Векторът $\vec{N_\pi}(A,B,C)$ се нарича „нормален вектор" на равнината π и $\vec{N_\pi} \perp \pi$

Ако векторът $\vec{v}(p,q,r)$ е перпендикулярен на π, то

$\pi: px+qy+rz+D=0, \; D=const$

$\boxed{3}$ Да се намери равнината π, която минава през т. M и е перпендикулярна на правата g, ако:

а) $M(0,-1,2)$ и $g: \begin{cases} x=\lambda-1 \\ y=-\lambda \\ z=2\lambda \end{cases}$



б) $M(2,2,-1)$ и $g: \begin{cases} x-2y-z+3=0 \\ 2x-y-z+3=0 \end{cases}$

а) $\vec{v}(1,-1,2) \parallel g \Rightarrow \vec{v} \perp \pi$ ② $M \in \pi: 1.0+(-1)(-1)+2.2+D=0$

① $\Rightarrow \pi: 1x+(-1)y+2z+D=0$ $\rightarrow D=-5 \Rightarrow \pi: x-y+2z-5=0$

б) ① $y=0$ (ние го взимаме) $\Rightarrow \left| \begin{aligned} x-z+3=0 \\ 2x-z+3=0 \end{aligned} \right\} \; x=0, \; z=3 \Rightarrow$ т. $A(0,0,3) \in g$

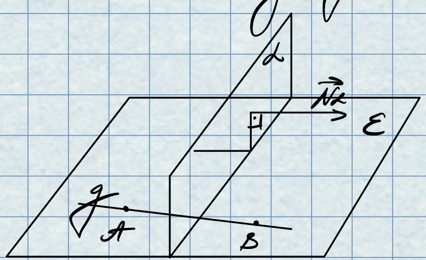
② $y=1 \Rightarrow \left| \begin{aligned} x-z+1=0 \\ 2x-z+2=0 \end{aligned} \right\} \; x=-1, \; z=0 \Rightarrow$ т. $B(-1,1,0) \in g$

③ $\vec{AB}(-1,1,-3) \parallel g, \; g \perp \pi \Rightarrow \vec{AB} \perp \pi$

④ $\pi: -x+y-3z+D=0, \quad M\in\pi$

$(-1)2+1.2-3(-1)+D=0, \quad D=-3 \Rightarrow \pi: x-y-3z+3=0$

4| Да се намери равнината ε, която съдържа правата $g: \begin{cases} -13\lambda = x \\ -3-5\lambda = y \\ 2+13\lambda = z \end{cases}$ и е перпендикулярна на равнината $\alpha: x-3y-z+4=0$



① $\vec{N_\alpha}(1,-3,-1)\perp\alpha$, но $\alpha\perp\varepsilon \Rightarrow \vec{N_\alpha}\parallel\varepsilon$

$\lambda=0: x=0, y=-3, z=2 \Rightarrow$ т. $A(0,-3,2)\in g\in\varepsilon$

$\lambda=1: x=-13, y=-8, z=15 \Rightarrow$ т. $B(-13,-8,15)\in g, \in\varepsilon$

② $\vec{BA}(13,5,-13)\in\varepsilon$, т.е. $\vec{BA}\parallel\varepsilon$

$\Rightarrow \varepsilon: \begin{vmatrix} x & 1 & 13 \\ y+3 & -3 & 5 \\ z-2 & -1 & -13 \end{vmatrix}=0$, $\quad\varepsilon: 39x+5(z-2)-13(y+3)+39(z-2)+13(y+3)+5x=0$

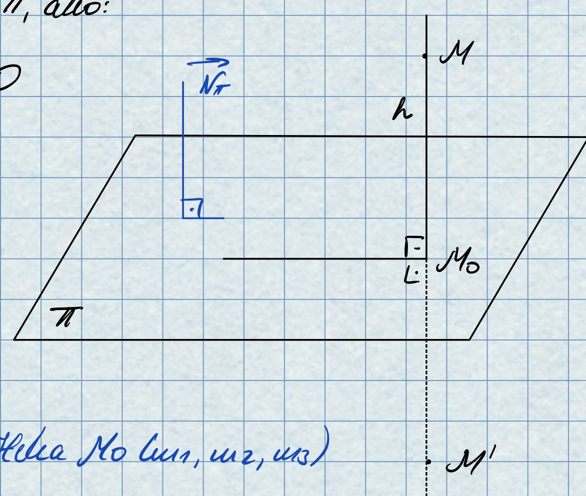
$\varepsilon: 44x+0y+44z-88=0 \quad /:44$

$\varepsilon: x+z-2=0$

5| Да се намерят права h през т. M, която е перпендикулярна на равнината π, както и ортогоналната проекция M_0 на M/π, както и симетричната т. M' относно π, ако:

а) $M(0,-3,2)$ и $\pi: x-3y-z+4=0$

б) $M(2,2,2)$ и $\pi: x+y+z-3=0$



а) $\vec{N_\pi}(1,-3,-1)\perp\pi$, но $h\perp\pi$

① $\rightarrow \vec{N_\pi}\parallel h$

② $h: \begin{cases} x = 0+1.\lambda = \lambda \\ y = -3-3\lambda = -3-3\lambda \\ z = 2-1\lambda = 2-\lambda \end{cases}$

③ Нека $M_0(m_1, m_2, m_3)$

$M_0\in h \Rightarrow$

$$M_0: \left| \begin{array}{l} m_1 = \lambda \\ m_2 = -3-3\lambda \\ m_3 = 2-\lambda \\ m_1 - 3m_2 - m_3 + 4 = 0 \end{array} \right. \quad M_0 \in \pi$$

$$\lambda - 3(-3-3\lambda) - (2-\lambda) + 4 = 0$$

$$\Rightarrow \lambda = -1$$

⑤ $m_1 = -1,\ m_2 = 0,\ m_3 = 3;$

$\Rightarrow M_0(-1,0,3)$

⑥ Нека $M'(m_1', m_2', m_3')$ и M_0 е среда MM' $\Rightarrow$

$$\left| \begin{array}{l} -1 = \frac{0+m_1'}{2} \\ 0 = \frac{-3+m_2'}{2} \\ 3 = \frac{2+m_3'}{2} \end{array} \right\} M'(-2,3,4)$$

б) $\vec{N_\pi}(1,1,1) \perp \pi$, но $h \perp \pi \Rightarrow \vec{N_\pi} \parallel h$

① $\Rightarrow h: \left| \begin{array}{l} x = 2+\lambda \\ y = 2+\lambda \\ z = 2+\lambda \end{array} \right.$

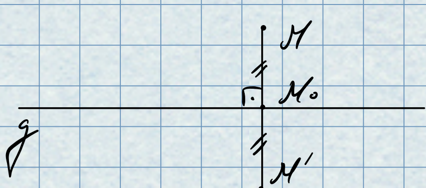
② Нека $M_0(m_1, m_2, m_3)$ и $M_0 \in h$

$$\Rightarrow \left| \begin{array}{l} m_1 = 2+\lambda \\ m_2 = 2+\lambda \\ m_3 = 2+\lambda \\ m_1 + m_2 + m_3 + 3 = 0 \end{array} \right\} M_0(1,1,1) \quad M_0 \in \pi$$

③ $M'(m_1', m_2', m_3')$ и M_0 е среда на MM' $\Rightarrow$

$$\left| \begin{array}{l} 1 = \frac{2+m_1'}{2} \\ 1 = \frac{2+m_2'}{2} \\ 1 = \frac{2+m_3'}{2} \end{array} \right\} M'(0,0,0)$$

$\boxed{6}$ Дадена е т. $M(7,8,0)$ и правата $g: \begin{cases} x = 3+3\lambda \\ y = -1+4\lambda \\ z = 4-\lambda \end{cases}$. Да се намерят ортогоналната проекция M_0 на M върху g и симетричната т. M' на M относно g.



① $M_0 \in g \Rightarrow M_0(3+3\lambda_0, -1-4\lambda_0, 4-\lambda_0)$

за някое $\lambda_0 \in \mathbb{R} \Rightarrow \overrightarrow{MM_0}(3\lambda_0-4,\ 4\lambda_0-9,\ 4-\lambda_0)$

② $\vec{v}(3,4,-1) \parallel g$, но $g \perp \overrightarrow{MM_0} \Rightarrow \overrightarrow{MM_0} \perp \vec{v} \Rightarrow \langle \overrightarrow{MM_0}, \vec{v} \rangle = 0$

③ $3(3\lambda_0-4)+4(4\lambda_0-9)-(4-\lambda_0)=0$; $\quad 26\lambda_0-52=0$, $\underline{\lambda_0=2} \Rightarrow$

④ $M_0(9,7,2)$ и нека $M'(m_1, m_2, m_3)$, а M_0 е среда на $MM' \Rightarrow$

$$\left| \begin{array}{l} 9 = \dfrac{7+m_1}{2} \\ 7 = \dfrac{8+m_2}{2} \\ 2 = \dfrac{0+m_3}{2} \end{array} \right\} M'(11, 6, 4)$$

$\boxed{7}$ Да се намери разстоянието от т. M до равнината π, ако:

а) $M(-1,0,1)$ и $\pi: 2x-3y-2z=0$

б) $M(6,2,-5)$ и $\pi: x-y+9z-7=0$

Формула: ако $M(m_1, m_2, m_3)$ и $\pi: Ax+By+Cz+D=0$, то за разстоянието $d(M,\pi)$ от т. M до π имаме:

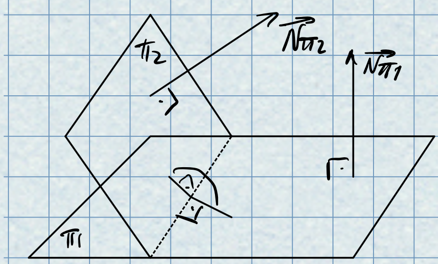
$$d(M,\pi) = \frac{|Am_1+Bm_2+Cm_3+D|}{\sqrt{A^2+B^2+C^2}}$$

а) $d(M,\pi) = \dfrac{|2(-1)-3.0-2.1|}{\sqrt{2^2+(-3)^2+(-2)^2}} = \dfrac{4}{\sqrt{17}} = \dfrac{4\sqrt{17}}{17}$

б) $d(M,\pi) = \dfrac{|6-2+9(-5)-7|}{\sqrt{1^2+(-1)^2+9^2}} = \dfrac{48}{\sqrt{83}} = \dfrac{48\sqrt{83}}{83}$

$\boxed{8}$ Да се намери ъгълът между равнините $\pi_1: x+y+z-6=0$ и π_2:

$2x-2y+z+5=0$



① $\sphericalangle(\pi_1, \pi_2) = \sphericalangle(\vec{N}_{\pi_1}, \vec{N}_{\pi_2})$

$\Rightarrow \cos\sphericalangle(\pi_1, \pi_2) = \cos\sphericalangle(\vec{N}_{\pi_1}, \vec{N}_{\pi_2})$, където $N_{\pi_1}(1,1,1)$, а $\vec{N}_{\pi_2}(2,-2,1)$

$$\Rightarrow \cos\sphericalangle(\vec{N}_{\pi_1}, \vec{N}_{\pi_2}) = \frac{\langle \vec{N}_{\pi_1}, \vec{N}_{\pi_2} \rangle}{|\vec{N}_{\pi_1}| . |\vec{N}_{\pi_2}|} = \frac{1.2 + 1(-2) + 1.1}{\sqrt{3.1^2}\,\sqrt{2^2 + (-2)^2 + 1^2}} =$$

$$= \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{9} \Rightarrow \arccos\frac{\sqrt{3}}{9}$$