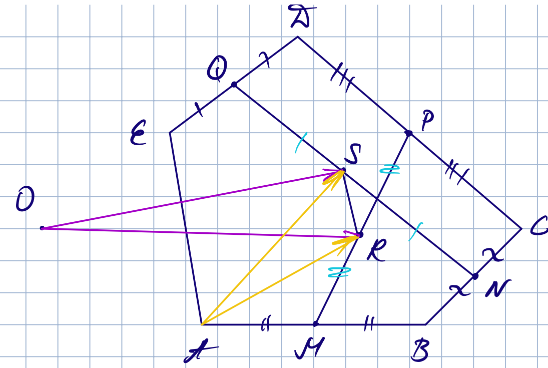
[1] Дадено е $ABCDE$ – произв. петоъг.;

M, N, P, Q – среди на AB, BC, CD, DE;

R, S – среди на MP и NQ;

Ако $RS = 1$, то да се намери AE



I начин:

① Нека точка O е произволна

② $\overrightarrow{OR} \overset{T31}{=} \frac{1}{2}(\overrightarrow{OM}+\overrightarrow{OP}) \overset{T31}{=} \frac{1}{2}\left(\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB})+\frac{1}{2}(\overrightarrow{OC}+\overrightarrow{OD})\right) = \frac{1}{4}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD})$

③ $\overrightarrow{OS} \overset{T31}{=} \frac{1}{2}(\overrightarrow{OQ}+\overrightarrow{ON}) \overset{T31}{=} \frac{1}{2}\left(\frac{1}{2}(\overrightarrow{OD}+\overrightarrow{OE})+\frac{1}{2}(\overrightarrow{OB}+\overrightarrow{OC})\right) = \frac{1}{4}(\overrightarrow{OD}+\overrightarrow{OE}+\overrightarrow{OB}+\overrightarrow{OC})$

④ $\overrightarrow{RS} = \overrightarrow{RO}+\overrightarrow{OS} = -\overrightarrow{OR}+\overrightarrow{OS} = \frac{1}{4}(\cancel{\overrightarrow{OD}}+\overrightarrow{OE}+\cancel{\overrightarrow{OB}}+\cancel{\overrightarrow{OC}}-\overrightarrow{OA}-\cancel{\overrightarrow{OB}}-\cancel{\overrightarrow{OC}}-\cancel{\overrightarrow{OD}}) = \frac{1}{4}(\overrightarrow{OE}-\overrightarrow{OA}) =$

$= \frac{1}{4}(\overrightarrow{OE}+\overrightarrow{AO}) = \frac{1}{4}\overrightarrow{AE} \longrightarrow RS \parallel AE,\ \overrightarrow{RS} \uparrow\uparrow \overrightarrow{AE}$ и $RS = \frac{1}{4}AE \Longrightarrow AE = 4RS = 4$

II начин:

① Нека вземем точка A (или която и да е друг връх)

② $\overrightarrow{RS} = \overrightarrow{RA}+\overrightarrow{AS} = -\overrightarrow{AR}+\overrightarrow{AS} \overset{T31}{=} \frac{1}{2}(\overrightarrow{AN}+\overrightarrow{AQ})-\frac{1}{2}(\overrightarrow{AM}+\overrightarrow{AP}) \overset{T31}{=}$

$= \frac{1}{2}\left(\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AC})+\frac{1}{2}(\overrightarrow{AE}+\overrightarrow{AD})\right)-\frac{1}{2}\left(\frac{1}{2}(\overrightarrow{AA}+\overrightarrow{AB})+\frac{1}{2}(\overrightarrow{AC}+\overrightarrow{AD})\right) =$

$= \frac{1}{4}(\cancel{\overrightarrow{AB}}+\cancel{\overrightarrow{AC}}+\overrightarrow{AE}+\cancel{\overrightarrow{AD}}-\overrightarrow{AA}-\cancel{\overrightarrow{AB}}-\cancel{\overrightarrow{AC}}-\cancel{\overrightarrow{AD}}) = \frac{1}{4}\overrightarrow{AE}$

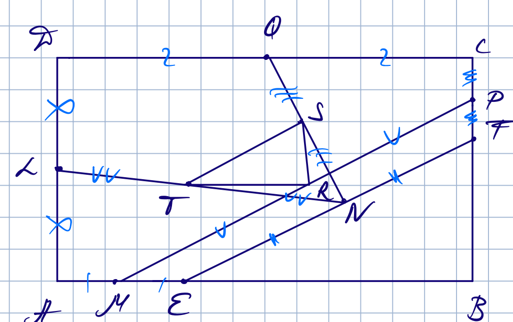
[2] Дадено е $ABCD$ – правоъгълник;

$E \in AB$, $F \in BC$;

M, N, P, Q среди на AB, BC, CD, DA;

R, S, T са среди на MP, NQ, NL.

Ако $S_{ABCD} = 1$, то да се намери S_{RST}.



① $\overrightarrow{BT} \overset{T31}{=} \frac{1}{2}(\overrightarrow{BN}+\overrightarrow{BL}) \overset{T31}{=} \frac{1}{2}\left(\frac{1}{2}(\overrightarrow{BE}+\overrightarrow{BF})+\frac{1}{2}(\overrightarrow{BA}+\overrightarrow{BD})\right) = \frac{1}{4}(\overrightarrow{BE}+\overrightarrow{BF}+\overrightarrow{BA}+\overrightarrow{BD})$

$\overrightarrow{BR} \overset{T31}{=} \frac{1}{2}(\overrightarrow{BM}+\overrightarrow{BP}) \overset{T31}{=} \frac{1}{2}\left(\frac{1}{2}(\overrightarrow{BA}+\overrightarrow{BE})+\frac{1}{2}(\overrightarrow{BF}+\overrightarrow{BC})\right) = \frac{1}{4}(\overrightarrow{BA}+\overrightarrow{BE}+\overrightarrow{BF}+\overrightarrow{BC})$

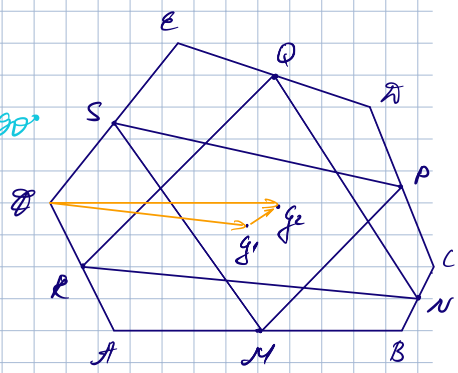
$\overrightarrow{BS} \overset{T31}{=} \frac{1}{2}(\overrightarrow{BN}+\overrightarrow{BQ}) \overset{T31}{=} \frac{1}{2}\left(\frac{1}{2}(\overrightarrow{BE}+\overrightarrow{BF})+\frac{1}{2}(\overrightarrow{BC}+\overrightarrow{BD})\right) = \frac{1}{4}(\overrightarrow{BE}+\overrightarrow{BF}+\overrightarrow{BC}+\overrightarrow{BD})$

$\overrightarrow{RS} = \overrightarrow{RB}+\overrightarrow{BS} = \overrightarrow{BS}-\overrightarrow{BR} = \frac{1}{4}(\cancel{\overrightarrow{BE}}+\cancel{\overrightarrow{BF}}+\cancel{\overrightarrow{BA}}+\overrightarrow{BD}-\cancel{\overrightarrow{BA}}-\cancel{\overrightarrow{BE}}-\cancel{\overrightarrow{BF}}-\overrightarrow{BC}) = \frac{1}{4}(\overrightarrow{AB}+\overrightarrow{BD}) = \frac{1}{4}\overrightarrow{AD}$

$\longrightarrow \overrightarrow{RS} = \frac{1}{4}\overrightarrow{AD} \Longrightarrow RS \parallel AD,\ \overrightarrow{RS} \uparrow\uparrow \overrightarrow{AD},\ RS = \frac{1}{4}AD$ (1)

② $\overrightarrow{RT} = (\overrightarrow{RB}+\overrightarrow{BT}) = \overrightarrow{BT}-\overrightarrow{BR} = \frac{1}{4}(\cancel{\overrightarrow{BE}}+\cancel{\overrightarrow{BF}}+\cancel{\overrightarrow{BA}}+\overrightarrow{BD}-\cancel{\overrightarrow{BA}}-\cancel{\overrightarrow{BE}}-\cancel{\overrightarrow{BF}}-\overrightarrow{BC}) = \frac{1}{4}(\overrightarrow{CB}+\overrightarrow{BD}) = \frac{1}{4}\overrightarrow{CD}$

$\Rightarrow \vec{RT} = \frac{1}{4}\vec{CD} \Rightarrow \vec{RT} \parallel \vec{CD}$, $\vec{RT} \uparrow\uparrow \vec{CD}$, $RT = \frac{1}{4}CD$ (2)



От (1) следва, че $RS \parallel AD$
От (2) следва, че $RT \parallel CD$ $\Big\} \Rightarrow \sphericalangle(RS, RT) = \sphericalangle(AD, CD) = 90^\circ$

$\Rightarrow S_{RTS} = \frac{RT.RS}{2} = \frac{1}{2} \cdot \frac{1}{4} CD \cdot \frac{1}{4} AD = \frac{S_{ABCD}}{32} = \frac{1}{32}$

3) $ABCDEF$ – произв. шестоъгълник;

M, N, P, R, Q, S са среди на AB, BC, CD, DE, EF, FA;

Да се докаже, че медицентровете на $\triangle MPR$ и $\triangle MQS$ съвпадат.

① Нека G_1 и G_2 са медицентровете на $\triangle MPR$ и $\triangle NQS$

② $\vec{G_1G_2} = \vec{G_1F} + \vec{FG_2} = -\vec{FG_1} + \vec{FG_2}$

③ $\vec{FG_1} \overset{T3.2}{=} \frac{1}{3}(\vec{FM} + \vec{FP} + \vec{FR}) \overset{T3.1}{=} \frac{1}{3}\left(\frac{1}{2}(\vec{FA} + \vec{FB}) + \frac{1}{2}(\vec{FC} + \vec{FD}) + \frac{1}{2}(\vec{FE} + \vec{FF})\right) =$

$= \frac{1}{6}(\vec{FA} + \vec{FB} + \vec{FC} + \vec{FD} + \vec{FE} + \vec{FF})$

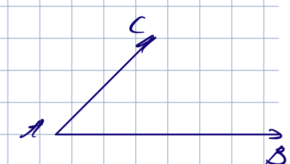
$\vec{FG_2} \overset{T3.2}{=} \frac{1}{3}(\vec{FN} + \vec{FQ} + \vec{FS}) \overset{T3.1}{=} \frac{1}{3}\left(\frac{1}{2}(\vec{FB} + \vec{FC}) + \frac{1}{2}(\vec{FD} + \vec{FE}) + \frac{1}{2}(\underset{\vec{0}}{\vec{FF}} + \vec{FA})\right) =$

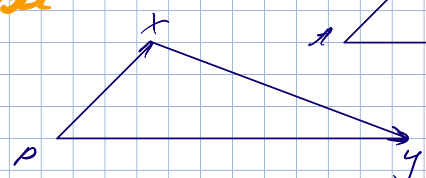
$= \frac{1}{6}(\vec{FB} + \vec{FC} + \vec{FD} + \vec{FE} + \vec{FF} + \vec{FA})$

$\Rightarrow \vec{G_1G_2} = \frac{1}{6}(\cancel{\vec{FB}} + \cancel{\vec{FC}} + \cancel{\vec{FD}} + \cancel{\vec{FE}} + \vec{FF} + \cancel{\vec{FA}} - \cancel{\vec{FA}} - \cancel{\vec{FB}} - \cancel{\vec{FC}} - \cancel{\vec{FD}} - \cancel{\vec{FE}}) = \frac{1}{6}\vec{FF} = \vec{FF} = \vec{0}$

$\Rightarrow G_1$ и G_2 съвпадат, $G_1 \equiv G_2$

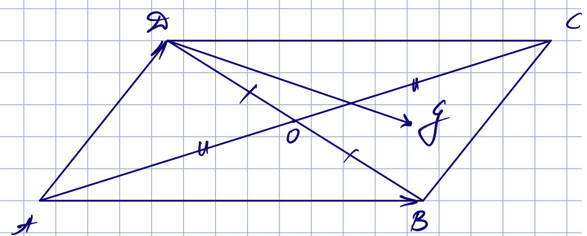
1. Векторна база





Нека точка P да е такава, че $XP \parallel AC$ и $YP \parallel AB \Rightarrow \vec{AC}$ и $\vec{PX}$ са колинеарни ($\vec{AC} \parallel \vec{PX}$)

$\Rightarrow \vec{PX} = e_1\vec{AC}$, $e_1 \in \mathbb{R}$, също $\vec{PY} \parallel \vec{AB}$
$\Rightarrow \vec{PY} = e_2 AB$, $e_2 \in \mathbb{R}$ $\Big\}$ $\vec{XY} = \vec{XP} + \vec{PY} = -\vec{PX} + \vec{PY} = -e_1\vec{AC} + e_2\vec{AB}$



1) $ABCD$ е успоредник,

$AC \cap BD = O$, G е медицентър на $\triangle BCO$

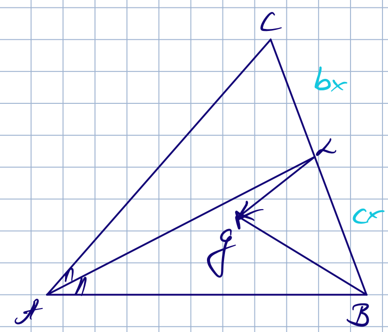
Да се изразят векторите $\vec{AO}$ и $\vec{DG}$

чрез векторите $\overrightarrow{AB}$ и $\overrightarrow{AD}$

① $\overrightarrow{AO} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AD})$

$\overrightarrow{DG} \overset{T32}{=} \frac{1}{3}(\overrightarrow{DB}+\overrightarrow{DO}+\overrightarrow{DC})$

$$\left.\begin{aligned}
&\overrightarrow{DB} = \overrightarrow{DA}+\overrightarrow{AB} = -\overrightarrow{AD}+\overrightarrow{AB}\\
&\overrightarrow{DO} = \tfrac{1}{2}\overrightarrow{DB} = -\tfrac{1}{2}\overrightarrow{AD}+\tfrac{1}{2}\overrightarrow{AB}\\
&\overrightarrow{DC} = \overrightarrow{AB}
\end{aligned}\right\} \; DG = \tfrac{1}{3}(-\overrightarrow{AD}+\overrightarrow{AB}-\tfrac{1}{2}\overrightarrow{AD}+\tfrac{1}{2}\overrightarrow{AB}+\overrightarrow{AB}) = -\tfrac{1}{2}\overrightarrow{AD}+\tfrac{5}{6}\overrightarrow{AB}$$



$\boxed{2}$ Даден е триъгълник ABC;

$AB = c$, $BC = a$, $AC = b$;

G е медицентър на $\triangle ABC$;

AL ($L \in BC$) е ъглополовяща.

Да се изразят векторите $\overrightarrow{BG}$, $\overrightarrow{AL}$ и $\overrightarrow{LG}$ чрез базата $\overrightarrow{CA}$ и $\overrightarrow{CB}$

① $\overrightarrow{BG} \overset{T32}{=} \frac{1}{3}(\overrightarrow{BB}+\overrightarrow{BC}+\overrightarrow{BA}) = \frac{1}{3}(\overrightarrow{BC}+\overrightarrow{BC}+\overrightarrow{CA}) = -\frac{2}{3}\overrightarrow{CB}+\frac{1}{3}\overrightarrow{CA}$

$\overrightarrow{AL} = \overrightarrow{AC}+\overrightarrow{CL}$ $\longrightarrow$ свойство на ъглополовящата: $\frac{CL}{LB} = \frac{AC}{AB} = \frac{b}{c} \Longrightarrow \frac{CL}{CB} = \frac{b}{b+c}$

$\Longrightarrow \frac{BL}{CB} = \frac{c}{b+c}$

$\Longrightarrow \overrightarrow{CL} = \frac{b}{b+c}\overrightarrow{CB}$

$\Longrightarrow \overrightarrow{AL} = -\overrightarrow{CA} + \frac{b}{b+c}\overrightarrow{CB}$

② $LG = LB + BG = \frac{c}{b+c}\cdot\overrightarrow{CB} - \frac{2}{3}\overrightarrow{CB} + \frac{1}{3}\overrightarrow{CA} = \frac{c-2b}{3(b+c)}\overrightarrow{CB} + \frac{1}{3}\overrightarrow{CA}$