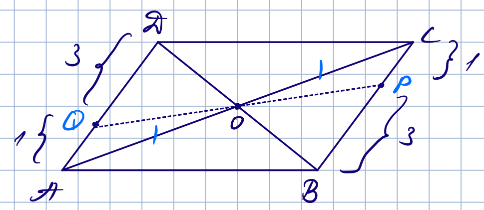
2. Векторна база в равнината

[1] Даден е $ABCD$ - успоредник

$AC \cap BD = O$, $P \in BC : BP : PC = 3 : 1$

$Q \in AD : AQ : QD = 1 : 3$

Да се докаже, че P, Q, O са колинеарни.



① Нека $\overrightarrow{AB}$ и $\overrightarrow{AD}$ е база; $\overrightarrow{AB} = \vec{a}$ и $\overrightarrow{AD} = \vec{b}$

② $\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP} = \frac{1}{2}\overrightarrow{AC} + \frac{1}{4}\overrightarrow{CB} \overset{\text{пу}}{=\!=} \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AD}) + \frac{1}{4}\underset{-\overrightarrow{AD}}{\underset{\parallel}{\overrightarrow{DA}}} = \frac{1}{4}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\vec{a} + \frac{1}{4}\vec{b}$

③ $\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = -\overrightarrow{AO} + \frac{1}{4}\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AC} + \frac{1}{4}\overrightarrow{AD} \overset{\text{пу}}{=\!=} \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AD}) + \frac{1}{4}\overrightarrow{AD} = -\frac{1}{2}\overrightarrow{AB} - \frac{1}{4}\overrightarrow{AD}$

$= -\frac{1}{2}\vec{a} - \frac{1}{4}\vec{b}$

→ Съществува ли реално число k, за което $\overrightarrow{OP} = k.\overrightarrow{OQ}$?

$\frac{1}{2}\vec{a} + \frac{1}{4}\vec{b} \overset{?}{=} k(-\frac{1}{2}\vec{a} - \frac{1}{4}\vec{b}) \longrightarrow \frac{1}{2}\vec{a} + \frac{1}{4}\vec{b} \overset{?}{=} -\frac{1}{2}k\vec{a} - \frac{1}{4}k\vec{b}$

$\Longrightarrow \left| \begin{array}{l} \frac{1}{2} = -\frac{1}{2}k \\ \frac{1}{4} = -\frac{1}{4}k \end{array} \right\}$ Изпълнено за $k = -1$

$\Longrightarrow \overrightarrow{OP} = -\overrightarrow{OQ} \longrightarrow OP \parallel OQ \Longrightarrow OP \equiv OQ \longrightarrow P, Q, O$ лежат на 1 права

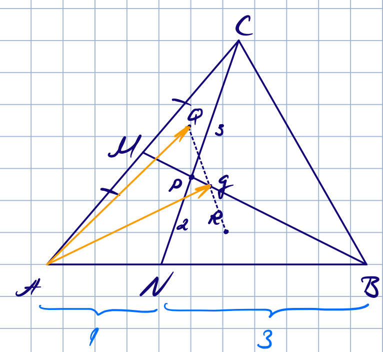
[2] Даден е $\triangle ABC$:

M - среда на AC, G - медицентър;

$N \in AB : AN = \frac{1}{3}AB$; $BM \cap CN = P$; $NP/CP = 2/3$

Q и R - медицентрове на $\triangle CPM$ и $\triangle BNP$.

Да се докаже, че Q, G, R лежат на една права



① Нека $\overrightarrow{AB}$ и $\overrightarrow{AC}$ - база, $\overrightarrow{AB} = \vec{a}$ и $\overrightarrow{AC} = \vec{b}$

② $\overrightarrow{GQ} = \overrightarrow{GA} + \overrightarrow{AQ} = -\overrightarrow{AG} + \overrightarrow{AQ}$

$\overrightarrow{AG} \overset{\text{тзд}}{=\!=} \frac{1}{3}(\overrightarrow{AA} + \overrightarrow{AB} + \overrightarrow{AC}) = \frac{1}{3}\vec{a} + \frac{1}{3}\vec{b}$

$\overrightarrow{AQ} \overset{\text{тзд}}{=\!=} \frac{1}{3}(\overrightarrow{AM} + \overrightarrow{AP} + \overrightarrow{AC}) = \frac{1}{3}(\frac{1}{2}\overrightarrow{AC} + \overrightarrow{AC} + \overrightarrow{CP} + \overrightarrow{AC}) = \frac{1}{3}(\frac{5}{2}\overrightarrow{AC} + \overrightarrow{CP}) = \frac{1}{3}(\frac{5}{2}\overrightarrow{AC} + \frac{3}{5}\overrightarrow{CN}) =$

$= \frac{1}{3}(\frac{5}{2}\overrightarrow{AC} + \frac{3}{5}(\overrightarrow{CA} + \overrightarrow{AN})) = \frac{1}{3}(\frac{5}{2}\overrightarrow{AC} - \frac{3}{5}\overrightarrow{AC} + \frac{3}{5}\overrightarrow{AN}) = \frac{1}{3}(\frac{19}{10}\overrightarrow{AC} + \frac{3}{5}.\frac{1}{3}\overrightarrow{AB}) =$

$= \frac{1}{15}\vec{a} + \frac{19}{30}\vec{b}$

$\Longrightarrow \overrightarrow{GQ} = -\overrightarrow{AG} + \overrightarrow{AQ} = -\frac{1}{3}\vec{a} - \frac{1}{3}\vec{b} + \frac{1}{15}\vec{a} + \frac{19}{30}\vec{b} = -\frac{4}{15}\vec{a} + \frac{3}{10}\vec{b}$

③ $\overrightarrow{GR} = \overrightarrow{GA} + \overrightarrow{AR} = -\overrightarrow{AG} + \overrightarrow{AR}$

$\overrightarrow{AR} \overset{\text{т.32}}{=\!=} \frac{1}{3}(\overrightarrow{AN}+\overrightarrow{AP}+\overrightarrow{AB}) = \frac{1}{3}(\frac{1}{3}\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{CP}+\overrightarrow{AB}) = \frac{1}{3}(\frac{4}{3}\overrightarrow{AB}+\overrightarrow{AC}+\frac{3}{5}\overrightarrow{CN}) =$

$= \frac{1}{3}(\frac{4}{3}\overrightarrow{AB}+\overrightarrow{AC}+\frac{3}{5}(\overrightarrow{CA}+\overrightarrow{AN})) = \frac{1}{3}(\frac{4}{3}\overrightarrow{AB}+\overrightarrow{AC}-\frac{3}{5}\overrightarrow{AC}+\frac{3}{5}.\frac{1}{3}\overrightarrow{AB}) = \frac{23}{45}\overrightarrow{AB}+\frac{2}{15}\overrightarrow{AC} =$

$= \frac{23}{45}\vec{a}+\frac{2}{15}\vec{b}$

$\Longrightarrow \overrightarrow{GR} = -\overrightarrow{AG}+\overrightarrow{AR} = -\frac{1}{3}\vec{a}-\frac{1}{3}\vec{b}+\frac{23}{45}\vec{a}+\frac{2}{15}\vec{b} = \frac{8}{45}\vec{a}-\frac{1}{5}\vec{b}$

$\rightarrow$ Съществува ли $k \in \mathbb{R}$, за което $\overrightarrow{GQ} = k\,\overrightarrow{GR}$?

$$\left.\begin{aligned} -\tfrac{4}{15}\vec{a}+\tfrac{3}{10}\vec{b} &= k(\tfrac{8}{45}\vec{a}-\tfrac{1}{5}\vec{b}) \\ -\tfrac{4}{15}\vec{a}+\tfrac{3}{10}\vec{b} &= \tfrac{8}{45}k\vec{a}-\tfrac{1}{5}k\vec{b} \end{aligned}\right\} k = -\tfrac{3}{2}$$

Домашно: Да се реши задачата, използвайки базата $\overrightarrow{BA}$ и $\overrightarrow{BC}$