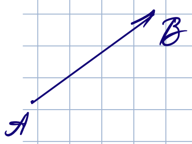
1. Вектори



$|\overrightarrow{AB}|$ - дължина на $\overrightarrow{AB}$

$|\overrightarrow{AB}| = AB$

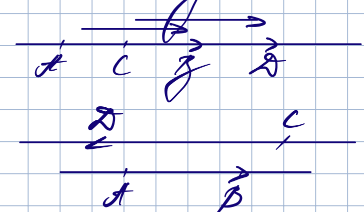
Вектори, които не лежат на една права или на успоредни — не могат да се сравняват.

$\frac{\overrightarrow{AB}}{\overrightarrow{CD}} = k \iff \overrightarrow{AB} = k\overrightarrow{CD}$

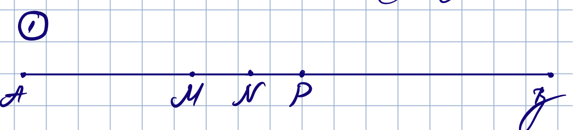
Твърдение: ако векторите $\overrightarrow{AB}$ и $\overrightarrow{CD}$ лежат на една права или на успоредни (т.е. ако $\overrightarrow{AB}$ и $\overrightarrow{CD}$ са колинеарни) и ако са различни от нулевия вектор, то:

1) $\frac{\overrightarrow{AB}}{\overrightarrow{CD}} = +\frac{|\overrightarrow{AB}|}{|\overrightarrow{CD}|}$, когато $\overrightarrow{AB} \uparrow\uparrow \overrightarrow{CD}$

2) $\frac{\overrightarrow{AB}}{\overrightarrow{CD}} = -\frac{|\overrightarrow{AB}|}{|\overrightarrow{CD}|}$, когато $\overrightarrow{AB} \uparrow\downarrow \overrightarrow{CD}$



$\boxed{1}$ Дадена е отсечка AB и точки M, N и P върху нея, за които $\frac{AM}{MB} = \frac{1}{3}$, $\frac{AN}{AB} = \frac{2}{5}$ и $\frac{AP}{PB} = \frac{0,9}{1,1} = \frac{9}{11}$. Да се сравнят двойките вектори $\overrightarrow{AP}$ и $\overrightarrow{AB}$, $\overrightarrow{BP}$ и $\overrightarrow{AB}$, $\overrightarrow{BA}$ и $\overrightarrow{AB}$, $\overrightarrow{MN}$ и $\overrightarrow{BA}$, $\overrightarrow{PM}$ и $\overrightarrow{AB}$, $\overrightarrow{NP}$ и $\overrightarrow{AB}$

① 

Тъй като: $\frac{AN}{AB} = \frac{2}{5} \rightarrow \frac{AN}{NB} = \frac{2}{3}$; $\frac{9}{11}$ $\frac{2}{3}$ $\frac{27}{33} > \frac{22}{33}$

② $\frac{AM}{MB} = \frac{1}{3} \Rightarrow AM = x,\ MB = 3x,\ x \in \mathbb{R}$ } $AB = AM + MB = 4x$

③ $\frac{AN}{AB} = \frac{2}{5} \Rightarrow AN = 2y,\ AB = 5y,\ y \in \mathbb{R}$ } $AB = AN + NB = 7y$

$4x = 5y \Rightarrow y = (4/5)x$

$\Rightarrow AN = (8/5)x$

$NB = AB - AN = 4x - (8/5)x = (12/5)x$

④ $MN = AN - AM = (8/5)x - x = (3/5)x$

$\frac{AP}{PB} = \frac{9}{11} \Rightarrow AP = 9z,\ PB = 11z,\ z \in \mathbb{R}$ } $AB = AP + PB = 20z$

$\Rightarrow 20z = 4x \Rightarrow z = (1/5)x$

$\Rightarrow AP = (9/5)x$ и $PB = (11/5)x$

⑤ $PM = AP - AM = (9/5)x - x = (4/5)x$

$NB = AP - AN = (9/5)x - (8/5)x = (1/5)x$

⑥ $\frac{\overrightarrow{AB}}{\overrightarrow{BA}} = -\frac{|\overrightarrow{AB}|}{|\overrightarrow{BA}|} = -\frac{AB}{AB} = -1 \Rightarrow \overrightarrow{AB} = -\overrightarrow{BA}$

$\frac{\overrightarrow{AP}}{\overrightarrow{AB}} = +\frac{|\overrightarrow{AP}|}{|\overrightarrow{AB}|} = \frac{AP}{AB} = \frac{(9/5)\cancel{x}}{4\cancel{x}} = \frac{9}{20} \Rightarrow \overrightarrow{AP} = (9/20)\overrightarrow{AB}$

$\frac{\overrightarrow{BP}}{\overrightarrow{AB}} = -\frac{|\overrightarrow{BP}|}{|\overrightarrow{AB}|} = -\frac{BP}{AB} = -\frac{(11/5)\cancel{x}}{4\cancel{x}} = -\frac{11}{20} \Rightarrow \overrightarrow{BP} = -(11/20)\overrightarrow{AB}$

$$\frac{\overrightarrow{MN}}{\overrightarrow{BA}} = -\frac{|\overrightarrow{MN}|}{|\overrightarrow{BA}|} = -\frac{MN}{BA} = -\frac{(3/5)x}{4x} = -\frac{3}{20} \Longrightarrow \overrightarrow{MN} = -(3/20)\overrightarrow{BA}$$

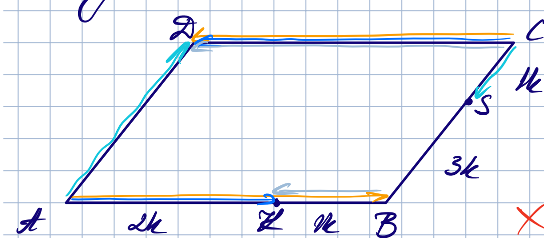
$$\frac{\overrightarrow{PM}}{\overrightarrow{AB}} = -\frac{|\overrightarrow{PM}|}{|\overrightarrow{AB}|} = -\frac{PM}{AB} = -\frac{(4/5)x}{4x} = -\frac{4}{20} \Longrightarrow \overrightarrow{PM} = -(1/5)\overrightarrow{AB}$$

* Да сравним векторите $\overrightarrow{NP}$ и $\overrightarrow{NM}$:

$$\frac{\overrightarrow{NP}}{\overrightarrow{NM}} = -\frac{MP}{NM} = -\frac{(4/5)x}{(3/5)x} = -\frac{4}{3} \Longrightarrow \overrightarrow{MP} = -(4/3)\overrightarrow{NM}$$

$$\frac{\overrightarrow{NP}}{\overrightarrow{AB}} = +\frac{NP}{AB} = \frac{(1/5)x}{4x} = \frac{1}{20} \Longrightarrow \overrightarrow{NP} = (1/20)\overrightarrow{AB}$$

$\boxed{2}$ $ABCD$ е успоредник; т. $K \in AB$: $AK : KB = 2$; т. $S \in BC$: $CS : BC = 1:4$. Да се сравнят двойките вектори $\overrightarrow{AB}$ и $\overrightarrow{CD}$, $\overrightarrow{AK}$ и $\overrightarrow{CD}$, ~~$\overrightarrow{AS}$ и $\overrightarrow{AD}$~~, $\overrightarrow{BK}$ и $\overrightarrow{CD}$, $\overrightarrow{AD}$ и $\overrightarrow{CS}$, ~~$\overrightarrow{BS}$ и $\overrightarrow{AB}$~~



$$\frac{\overrightarrow{AB}}{\overrightarrow{CD}} = -\frac{AB}{CD} \Longrightarrow \overrightarrow{AB} = -\overrightarrow{CD}, \quad \overrightarrow{AB} = \overrightarrow{DC}$$

$$\frac{\overrightarrow{AK}}{\overrightarrow{CD}} = -\frac{AK}{CD} = -\frac{(2/3)AB}{CD} = -(2/3) \Longrightarrow \overrightarrow{AK} = -(2/3)\overrightarrow{CD}$$

X – не могат да се сравнят, защото не са колинеар.

$$\frac{\overrightarrow{BK}}{\overrightarrow{CD}} = \frac{BK}{CD} = \frac{(1/3)AB}{CD} = (1/3) \Longrightarrow \overrightarrow{BK} = (1/3)\overrightarrow{CD}$$

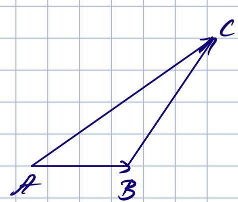
$$\frac{\overrightarrow{AD}}{\overrightarrow{CS}} = -\frac{AD}{CS} = \frac{AD}{(1/4)CB} = 4 \Longrightarrow \overrightarrow{AD} = 4\overrightarrow{CS}$$

Събиране на вектори:

1. Събиране по правило на триъгълника

Пример: $\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{BC} + \overrightarrow{DA} =$

$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{0}$



$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

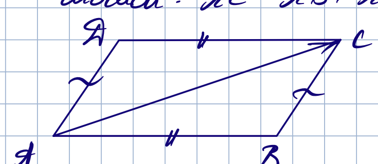
$\overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BC}$

$\overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{CB}$

$-\overrightarrow{CA} + \overrightarrow{AB}$

$-\overrightarrow{CA} + \overrightarrow{CB} = \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$

2. Правило на успоредника: $ABCD$ е успоредник, тогава: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$



Доказателство: $\triangle ABC$: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

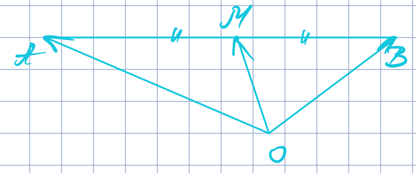
$\triangle ADC$: $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$ (+)

$\overrightarrow{AC} + \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AD} + \overrightarrow{DC}$

$2\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{AD} + \overrightarrow{AB} = 2\overrightarrow{AB} + 2\overrightarrow{AD}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$

Помощна задача: Дадена е отсечка AB със среда M. O е произволна точка. Тогава
$$\vec{OM} = \frac{1}{2}(\vec{OA}+\vec{OB})$$



Доказателство: $\triangle OAM: \vec{OM} = \vec{OA} + \vec{AM}$

$$\begin{aligned} &\triangle OBM: \vec{OM} = \vec{OB} + \vec{BM} \quad (+) \\ \hline &2\vec{OM} = \vec{OA} + \vec{OB} + \vec{AM} + \vec{BM} \end{aligned}$$

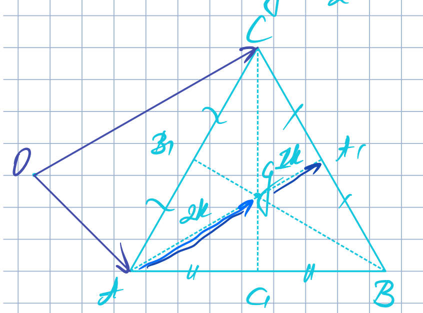
$2\vec{OM} = \vec{OA} + \vec{OB} \quad /:2$

$\vec{OM} = \frac{1}{2}(\vec{OA}+\vec{OB})$

$\vec{AM} \uparrow\downarrow \vec{BM} \Rightarrow \vec{AM} = -\vec{BM}$

$\vec{AM} + \vec{BM} = \vec{0}$

Още една помощна задача: Даден е $\triangle ABC$ с медицентър G, т. O е произволна – тогава
$$\vec{OG} = \frac{1}{2}(\vec{OA}+\vec{OB}+\vec{OC})$$



① AA_1, BB_1, CC_1 – медиани

②
$$\begin{aligned} \vec{OG} &= \vec{OA} + \vec{AG} \\ \vec{OG} &= \vec{OB} + \vec{BG} \quad (+) \\ \vec{OG} &= \vec{OC} + \vec{CG} \\ \hline 3\vec{OG} &= \vec{OA} + \vec{AG} + \vec{OB} + \vec{BG} + \vec{OC} + \vec{CG} \end{aligned}$$

(*) $\underbrace{\vec{AG} + \vec{BG} + \vec{CG} = \left(\tfrac{2}{3}\right)\vec{AA_1} + \left(\tfrac{2}{3}\right)\vec{BB_1} + \left(\tfrac{2}{3}\right)\vec{CC_1}}$

③ $\vec{AA_1} \;\underline{\underline{\text{т.3.1 при } O\equiv A}}\; \frac{1}{2}(\vec{AC}+\vec{AB})$

$\vec{BB_1} \;\underline{\underline{\text{т.3.1 при } O\equiv B}}\; \frac{1}{2}(\vec{BA}+\vec{BC})$

$\vec{CC_1} \;\underline{\underline{\text{т.3.1 при } O\equiv C}}\; \frac{1}{2}(\vec{CA}+\vec{CB})$

④ $\vec{AG}+\vec{BG}+\vec{CG} = \frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{2}}(\cancel{\vec{AC}}+\cancel{\vec{AB}}+\cancel{\vec{BA}}+\cancel{\vec{BC}}+\cancel{\vec{CA}}+\cancel{\vec{CB}}) = \vec{0}$

⑤ $\Rightarrow$ (*) $= 3\vec{OG} = \vec{OA} + \vec{OB} + \vec{OC} \quad /:3$

$\vec{OG} = \frac{1}{3}($