Master UAB - *High Energy Physics, Astrophysics and Cosmology*

# NSs, BHs and GWs

# EINSTEIN'S THEORY OF GRAVITY

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# Summary I:

### Covered last lecture: special relativity

- Einstein combined earlier results to develop a **new theory of (special) relativity** based on two postulates: i) all inertial frames are equivalent, ii) the speed of light is constant.
- In SR, the laws of physics are invariant under **Lorentz transformations**, which couples space & time into **spacetime**.
- Special relativity **extends Newtonian physics** to those cases where speeds are close to that of light (but gravity negligible).

# Summary II:

## Covered last lecture: tensor calculus

- To simplify the SR formalism and eventually appreciate the beauty of GR, we make use of **tensor calculus**. We will use tensors to write equations in **coordinate independent** form.
- Tensors are objects satisfying certain properties under **coordinate transformations**. We distinguish scalars (mass), contravariant (tangent vector) and covariant (gradient) tensors.
- Using the formalism, we can encode information about a manifold's **curvature** and determine **geodesics** and **distances**.

# Overview:

## Covered so far: special relativity, tensor calculus

1. Equivalence principles
2. Einstein equations
3. Why gravity is not a force!
4. Schwarzschild solution

# Overview:

## Covered so far: special relativity, tensor calculus

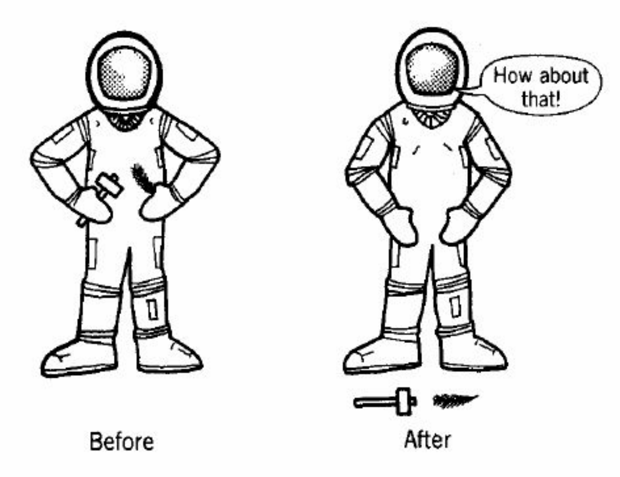
1. **Equivalence principles (EPs)**
2. Einstein equations
3. Why gravity is not a force!
4. Schwarzschild solution

# 1.1 EPs - Mass in Newtonian theory

- So far, we haven't specified what the **mass** of a body actually is. In principle, one can distinguish 3 different masses:
    - **inertial mass $m^i$:** measure of a body's resistance to a change in motion; mass in Newton's second law $\mathbf{F} = m^i\mathbf{a}$.
    - **passive gravitational mass $m^p$:** measure of a body's response to a gravitational field, $\mathbf{F} = -m^p \,\mathrm{grad}\, \phi$.
    - **active gravitational mass $m^a$:** measure of a body's strength to produce a gravitational field, $\phi = -Gm^a/r$.
- **Galileo** already realised in ~1610 that two bodies (of different mass) dropped from some height **reach** the ground **together**.

# 1.2 Galileo’s or the weak EP (WEP)

- Galileo’s observation suggests: *all particles* (regardless of mass and composition) *fall with the same acceleration when placed in the same gravitational field*. This is **one form** of the WEP. It implies that **free fall is universal**.




- This principle combined with Newton’s laws suggests further that the **two gravitational masses** are **identica**l and also **equivalent** to the **inertial mass**, i.e., $m = m^i = m^p = m^a$.

# 1.3 EPs - Mass equivalence

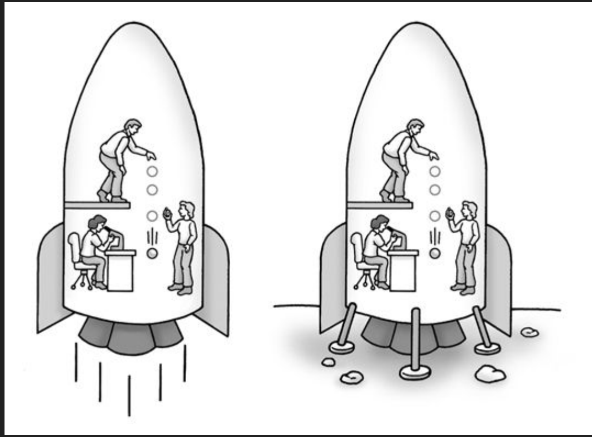
- Equivalence of inertial and gravitational mass is one of the most **accurately tested principles** in physics. Experiments typically measure the **Eötvös parameter** for two test masses A and B:

$$\eta(A,B) = 2\frac{\left(\frac{m^g}{m^i}\right)_A - \left(\frac{m^g}{m^i}\right)_B}{\left(\frac{m^g}{m^i}\right)_A + \left(\frac{m^g}{m^i}\right)_B}$$

- The best constraint comes from the MICROSCOPE experiment onboard a satellite, confirming the universality of free fall with a **precision of $10^{-15}$**, a factor 100 better than tests on Earth.

# 1.4 EPs - Einstein's contribution

- Einstein realised that no body can be shielded from a gravitational field, but we can **locally remove effects of gravity** (& recover SR) by considering a **free falling reference frame**.
- Someone falling from a roof feels **weightless**, so locally there is not gravitational field and we have an **inertial frame**. Free falling observers are **inertial observer**!
- A frame **linearly accelerating** in empty space is locally identical to a **frame at rest** in a gravitational field.



# 1.5 Einstein's EP

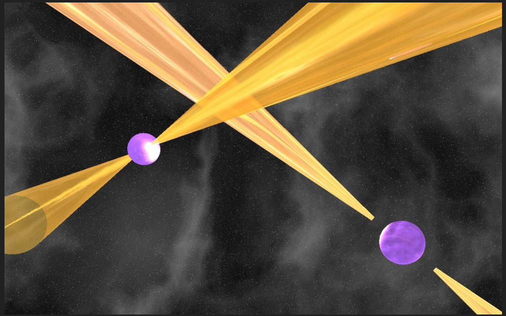
- *We cannot distinguish between gravitational & inertial forces (accelerations) with any local experiment using test particles*:
  - **Gravitational forces** can be described **like inertial forces**!

$$\underbrace{\frac{\mathrm{d}^2 x^a}{\mathrm{d}^2 u} = 0,}_{\text{inertial}} \qquad \underbrace{\frac{\mathrm{d}^2 x^a}{\mathrm{d}^2 u} + \Gamma^a{}_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}u} \frac{\mathrm{d}x^c}{\mathrm{d}u} = 0}_{\text{non-inertial test particle}}$$

  - When gravitational accelerations are present, then space cannot be flat: a **gravitational field curves spacetime**!
  - If gravity is present, **no inertial frames** can exist: there are no special frames & we have to use general coordinates!

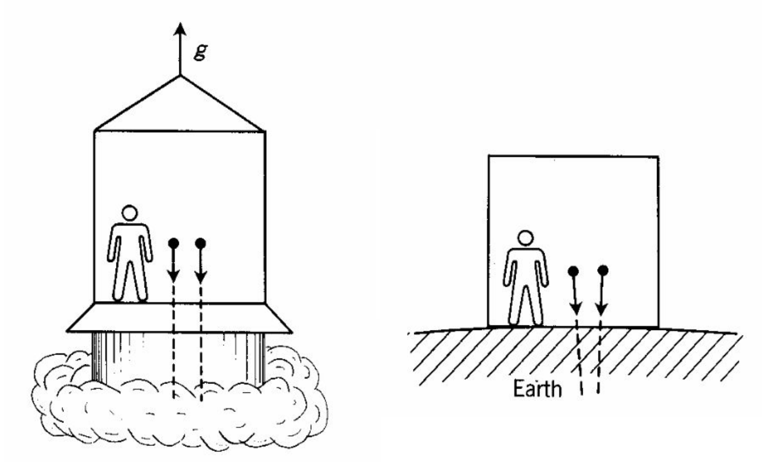
# 1.6 The strong EP (SEP)

- The WEP is only applicable to **test particles** and local but **non-gravitational experiments**. A stronger statement is: *The WEP also holds for self-gravitating bodies (like stars) and any type of experiment (gravitational or non-gravitational).*
- This form is **more restrictive** than the WEP: **only GR** seems to satisfy the SEP. Other gravity theories violate it to some level.
- **SEP tests** involve searching for **variations in G** or **compact binaries** (with at least one pulsar to 'measure' effects) as they strongly affect the spacetime.



# 1.7 EPs - Locality vs. non-locality

- **Local experiments** not only imply that an observer **cannot look outside** of their laboratory / spacecraft, but also that the lab / spacecraft is small enough so that **tidal effects** (due to a gradient in the gravitational field) are not detected.




- If their **rocket** is **wide** enough and the observer's **equipment sensitive** to detect **changes in a gravitational field**, they could distinguish the two cases illustrated on the left.

# 1.8 Questions

- Go to www.menti.com & enter 6689 6774.
    - 1. Does the universality of free fall imply that $m^i = m^g$?
        - Yes
        - No
    - 2. A (non-rotating) observer is able to perform a local experiment to determine whether they are freely falling in a gravitational field or moving at v=const in empty space?
        - Yes
        - No

# 1.8 Answers

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    - No

# Overview:

## Covered so far: special relativity, tensor calculus, equivalence principles

1. Equivalence principles
2. **Einstein equations (EEs)**
3. Why gravity is not a force!
4. Schwarzschild solution

# 2.1 EEs - Minkowski spacetime

- As we have seen now, the **flat spacetime** we encountered in the Special Theory of Relativity plays an important role when gravity is absent or local inertial observers are considered.
- We can now write the **line element** in tensorial form using the **Minkowski metric** $g_{ab} = \eta_{ab}$:

$$ds^2 = \eta_{ab} dx^a dx^b$$

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Considering the **length** (norm) $X^2 = X_a X^a$ of a vector, we distinguish **timelike** ($X^2>0$), **spacelike** ($X^2<0$) or **lightlike / null** ($X^2=0$) vectors.

# 2.2 EEs - Exercise

- Let's define a **timelike geodesic** as a geodesic, whose tangent vector is timelike everywhere. Show that the so-called **proper time τ** (time measured by a clock following this geodesic, with $d\tau = ds/c$) between $t_1$ and $t_2$ is given by

$$\tau = \int_{t_1}^{t_2} \frac{\mathrm{d}t}{\gamma(t)}$$

  - The following relations will help:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

$$\gamma = 1/\sqrt{1 - v^2/c^2}$$

$$\boldsymbol{v} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)$$

# 2.2 EEs - Exercise

- Solution:

$$\tau = \int_{t_1}^{t_2} \mathrm{d}\tau = \int_{t_1}^{t_2} \frac{\mathrm{d}s}{c} = \int_{t_1}^{t_2} \frac{1}{c}\sqrt{c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2}$$

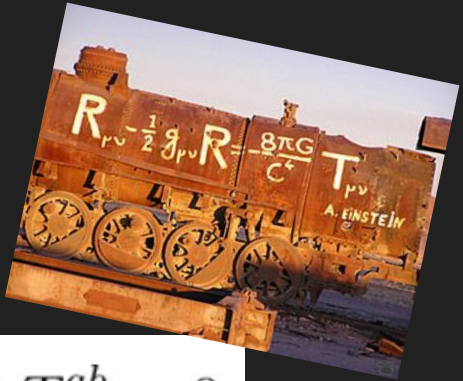
$$= \int_{t_1}^{t_2} \mathrm{d}t\sqrt{1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]}$$

$$= \int_{t_1}^{t_2} \mathrm{d}t\sqrt{1 - \frac{v(t)^2}{c^2}} = \int_{t_1}^{t_2} \frac{\mathrm{d}t}{\gamma(t)}$$

# 2.3 EEs - The ideas

- All our considerations helped Einstein develop a **range of arguments** that determine how the equations that completely control the **dynamics of gravitation** will have to look like:
  - The *correct Newtonian limit* has to be recovered.
  - The equations should have *tensorial form*.
  - Mass and energy are sources of the gravitational field, so the *energy-momentum tensor* $T_{ab}$ should appear.
  - Energy-momentum has to be *conserved*.
  - *Second derivatives* of the metric combined into some order 2 tensor $A_{ab}$ should enter the equations.

## 2.4 Einstein equations




- Combining these suggests the following:

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

$$\nabla_a T^{ab} = 0$$

- *These equations (famously) describe (i) the way that matter and energy curve spacetime and (ii) how curved spacetime controls the motion of matter and energy.*
- In **vacuum (empty space)**, these reduce to the simple case

$$T_{ab} = 0, \quad R_{ab} = 0$$

# 2.5 EEs - Energy-momentum tensor

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}$$

- A generalisation of the Newtonian stress tensor, $T_{ab}$ describes the density and flux of energy and momentum in spacetime.

- In GR, the tensor is **symmetric** $T_{ab} = T_{ba}$ and its **precise form** depends on the **type of matter/energy** considered, e.g.,
  - Non-interacting matter (dust): $$T^{ab} = \rho_0 u^a u^b$$
  - Perfect fluid: $$T^{ab} = (\rho_0 + p/c^2) u^a u^b - p g^{ab}$$

- From the **conservation law** $\nabla_a T^{ab} = 0$ we recover conservation equations, e.g., continuity / Navier-Stokes /

# 2.6 EEs - Geodesics part II

- We saw that a **free particle** (no external forces acting) follows a geodesic. These curves are typically parameterised by the **proper time**, i.e., u = τ. Thus, for a timelike geodesic:

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}^2 \tau} + \Gamma^a{}_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}\tau} \frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0$$

- τ is an **affine parameter** (the tangent vector remains parallel or equivalently the acceleration is perpendicular to the curve / velocity vector). Note that the **coordinate time** t is not an affine parameter and the **equation of motion** would **differ**.

# 2.7 EEs - Newtonian limit I

- If our **spacetime is almost flat**, we expect the metric to differ only slightly from the Minkowski metric, i.e.,

$$g_{ab} = \eta_{ab} + \epsilon h_{ab} + \mathcal{O}(\epsilon^2) \quad \text{with} \quad \epsilon \sim v/c \ll 1$$

- In the **slow-motion limit**, we can also assume that derivatives w.r.t $x^0 = ct$ are smaller than the spatial ones w.r.t. $x^\alpha = (x,y,z)$:

$$\mathrm{d}x^\alpha \sim \epsilon c\,\mathrm{d}t$$

- From $c\,d\tau = ds$, we find to lowest order in ε that **τ~t**, while the definition of the **connections** suggest that

$$\Gamma^a{}_{bc} = \mathcal{O}(\epsilon)$$

# 2.8 EEs - Newtonian limit II

- We are now interested in the **geodesic equation**, specifically the spatial components. To first order in ε, we find

$$
\begin{aligned}
0 &= \frac{1}{c^2}\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}^2 t} + \frac{1}{c^2}\Gamma^\alpha{}_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}t}\frac{\mathrm{d}x^c}{\mathrm{d}t} \sim \frac{1}{c^2}\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}^2 t} + \Gamma^\alpha{}_{00} \\
&= \frac{1}{c^2}\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}^2 t} - \frac{1}{2}\epsilon\left(2\frac{\partial h_{0\alpha}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^\alpha}\right) \sim \frac{1}{c^2}\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}^2 t} + \frac{1}{2}\epsilon\frac{\partial h_{00}}{\partial x^\alpha}
\end{aligned}
$$

- Analogy with the Newtonian equation gives **weak-field limit**

$$
\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}^2 t} = -\frac{\partial \phi}{\partial x^\alpha}, \quad \Rightarrow \quad g_{00} = 1 + 2\frac{\phi}{c^2} + \mathcal{O}\left(\frac{v}{c}\right)
$$

# 2.9 Questions

- Go to www.menti.com & enter 6904 3933.
  - 1. Which of these statements does not hold for the EEs?
    - Second derivatives of $g_{ab}$ should appear.
    - The equations have scalar form.
    - Newtonian physics need to be recovered.
  - 2. Free particles move along geodesics. If external forces are present, we can account for these by adding extra force terms $f^a$ to the rhs of our geodesic equation?
    - Yes
    - No

# 2.9 Answers

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    - 1. Which of these statements does not hold for the EEs?
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        - Yes
        - No

# Overview:

## Covered so far: special relativity, tensor calculus, equivalence principles, Einstein equations

1. Equivalence principles
2. Einstein equations
3. **Why gravity is not a force!**
4. Schwarzschild solution

# 3.1 Video

- To make sure that we have all understood the main concepts discussed in class so far, we are going to watch the following **Youtube video**. It summarises all the elements we have introduced last week and today (in ~15 min!!):
  - **Why Gravity is NOT a Force** by *Veritasium*

    https://tinyurl.com/a3ean3tc

  - To break things up, we will watch in **two blocks** and have 2 rounds of questions.

# 3.2 <u>Questions</u> https://tinyurl.com/a3ean3tc

- Watch **until 6:55** & go to [www.menti.com](http://www.menti.com) & enter 1792 886.
    - 1. An observer in a spacecraft approaching a planet will not notice any difference but an external observer will see the spacecraft move on a bent path due to spacetime curvature?
        - Incorrect
        - Correct
    - 2. Why can the bent sheet or rubber sheet analogy of GR be somewhat misleading? Please type out your answer in **1 or 2** sentences, but not more.

# 3.3 Questions https://tinyurl.com/a3ean3tc

- Watch **the rest** & go to www.menti.com & enter 9342 1837.
  - 3. How can you be at rest (on the Earth's surface) if you are a non-inertial observer and 'accelerating upwards'? Please type out your answer in **1 or 2** sentences, but not more.
  - 4. GR predicts that in an accelerating frame of reference & similarly in curved spacetime the paths of light are bent.
    - Correct
    - Incorrect

# 3.2 Answers

- Watch **until 6:55** & go to [www.menti.com](http://www.menti.com) & enter 1792 886.
  - 1. An observer in a spacecraft approaching a planet will not notice any difference but an external observer will see the spacecraft move on a bent path due to spacetime curvature?
    - Incorrect
    - Correct
  - 2. The object's motion around the central mass is closer to the analogy of an object falling into a well due to a gravitational force. But in GR gravity is not a force and the test mass travels on a straight path through curved spacetime.

# 3.3 Answers

- Watch the rest & go to [www.menti.com](http://www.menti.com) & enter 9342 1837.
  - 3. A stationary position is possible if our acceleration is exactly cancelled by the product of a curvature term times the velocity squared. We can mathematically show this by adding force/acceleration terms to our geodesic equation.
  - 4. GR predicts that in an accelerating frame of reference & similarly in curved spacetime the paths of light are bent.
    - Correct
    - Incorrect

# 3.4 And more questions


- Are there any more **questions from your sid**e so far?

# Overview:

## Covered so far: special relativity, tensor calculus, equivalence principles, Einstein equations

1. Equivalence principles
2. Einstein equations
3. Why gravity is not a force!
4. **Schwarzschild solution (SS)**

# 4.1 SS - Solving EEs

- EEs are very **complicated** and difficult to solve, but there are a **few cases** where we can take advantage of special properties of the gravitational system and find **analytical solutions**.

- One such case is the **Schwarzschild solution** for which we will assume that the following statements hold:
  - a.) Spacetime is *spherically symmetry*.
  - b.) Spacetime is *static*.
  - c.) Spacetime is *empty*.
  - d.) Spacetime is *asymptotically flat*.

# 4.2 SS - General solution for a. & b.

- In broad terms, a **static spacetime** is one that does not change over time and is also irrotational. This implies that
    - i.) all *metric components* $g_{ab}$ are independent of $x^0$.
    - ii.) the *line element* $ds^2$ is invariant under $x^0 \rightarrow -x^0$.
- A metric that only satisfies i.) is called **stationary**. If both hold, $ds^2$ can only depend on rotational invariants of $x^\alpha$ and their differentials, which implies the metric is **isotropic**. The most **general spherically symmetric line element** is given by

$$\mathrm{d}s^2 = \mathrm{e}^{\nu(t,r)}\mathrm{d}t^2 - \mathrm{e}^{\lambda(t,r)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

# 4.3 SS - Schwarzschild metric

- What remains is the **determination of** the **two functions** $\nu$ and $\lambda$. To do this, we would have to consider the vacuum Einstein Equations and calculate $R_{ab} = 0$ for our isotropic metric

$$g_{ab} = \mathrm{diag}(\mathrm{e}^{\nu}, -\mathrm{e}^{\lambda}, -r^2, -r^2 \sin^2\theta)$$

- One would find that $\lambda = \lambda(r)$ & $\nu = \nu(r)$ as well as $\nu(r) = -\lambda(r)$, and explicitly for the **Schwarzschild line element**

$$\mathrm{d}s^2 = (1 - 2m/r)\mathrm{d}t^2 - (1 - 2m/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

# 4.4 SS - Observations

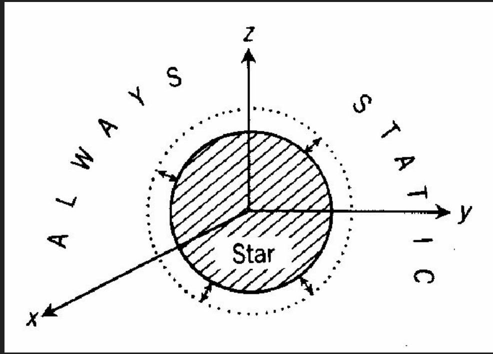
- In the **asymptotic limit** $r \to \infty$, we recover the flat metric in spherical coordinates, i.e., the SS metric is asymptotically flat.
- We can learn more about the **quantity m** in the line element, by considering the **weak-field limit**. On slide 2.8, we saw that in this case $g_{00} \sim 1+2\phi/c^2$. For a point mass M at the origin, **Newtonian theory** gives rise to a potential $\phi = -GM/r$. Thus

$$1 + 2\phi/c^2 = 1 - 2GM/c^2 r \overset{!}{=} 1 - 2m/r \quad \Rightarrow \quad m = GM/c^2$$

- We interpret the SS solution as due to a point particle at the origin, with m as the **(geometric) mass** in relativistic units.

# 4.5 SS - Birkhoff's theorem

- While we have seen that the SS metric is a static & spherically symmetric solution of the vacuum Einstein equations, it is possible to show that this solution is **unique**. This implies that
  - *any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat.*
  - *spacetime outside of a spherical, nonrotating, gravitating body (exterior solution) MUST be given by the SS metric.*
- A **spherically pulsating star** has a static exterior and **cannot emit GWs**.



# 4.6 Questions

- Go to www.menti.com & enter 7148 0271.
  - 1. A static solution to the Einstein Equations is always stationary, but the opposite is not automatically true.
    - Correct
    - Incorrect
  - 2. We interpret the Schwarzschild solution to the vacuum Einstein Equations as due to a point particle at the origin.
    - No
    - Yes

# 4.6 Answers

- Go to www.menti.com & enter 7148 0271.
  - 1. A static solution to the Einstein Equations is always stationary, but the opposite is not automatically true.
    - Correct
    - Incorrect
  - 2. We interpret the Schwarzschild solution to the vacuum Einstein Equations as due to a point particle at the origin.
    - No
    - Yes

# Summary I:

## Covered today: equivalence principles, Einstein equations, Schwarzschild solution

- The **weak equivalence principle** has multiple forms, reflecting the fact that we cannot distinguish between gravitational & inertial forces with any local experiment using test particles.
- The **strong equivalence principle** generalises the WEP to self-gravitating bodies and gravitational experiments.
- The **Einstein equations** are a set of tensor equations that completely control the **dynamics of gravitation**.

# Summary II:

## Covered today: equivalence principles, Einstein equations, Schwarzschild solution

- The essence of the Einstein equations is that “Spacetime tells matter how to move, while matter tells spacetime how to curve.” as summarised by John Wheeler.
- In General Relativity, **gravitational phenomena** arise not from forces or fields but from the curvature of spacetime itself.
- The (unique) **spherically symmetric & static solution** of the vacuum Einstein Equations is **Schwarzschild’s solution**.

# Final impressions:

## Covered today: equivalence principles, Einstein equations, Schwarzschild solution

- Go to www.menti.com & enter 1882 8492.
  - If you could use **one word** to describe today's class, what would that word be?