# Master UAB: **NSs, BHs and GWs**

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**Due date**: June 15th, 2021

Please submit your solutions electronically to the above email address by the given deadline. Note that to receive full credit for the following questions, you must show your work, i.e., provide calculations and/or explain your thinking. It is NOT necessary to type up your answers with LaTeX provided that your hand-written and photographed (scanned) solutions are legible. Solutions for all questions will be provided online after the submission deadline.

## Geodesics in General Relativity

In our lectures, we saw that the paths of free test particles in General Relativity are governed by the affine geodesic equation, generalising the concept of a 'straight line' to curved spacetime. The equation reads

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}u^2} + \Gamma^a{}_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}u} \frac{\mathrm{d}x^c}{\mathrm{d}u} = 0, \tag{1}$$

where $u$ denotes an affine parameter and $\Gamma^a{}_{bc}$ are the Christoffel symbols, which are symmetric in the lower two indices. Note that this notation employs the Einstein summation convention. We will use this equation to study the motion of particles around non-rotating black holes.

### 1. Non-affine geodesics (4 points)

While it is common to parametrise the timelike geodesic of a massive particle by the proper time $\tau$, not all parameters are affine parameters and satisfy the above equation. In particular, the coordinate time $t$ is not an affine parameter. To highlight the difference between the two times, starting from Eq. (1), *show that the equation of motion for a massive particle in terms of $t = x^0$ (we use $c = 1$) is given by*

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}t^2} + \Gamma^a{}_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}t} \frac{\mathrm{d}x^c}{\mathrm{d}t} = \Gamma^0{}_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}t} \frac{\mathrm{d}x^c}{\mathrm{d}t} \frac{\mathrm{d}x^a}{\mathrm{d}t}. \tag{2}$$

### 2. Equations of motion in Schwarzschild spacetime (5 points)

For the remainder, we focus on a specific type of solution to the vacuum Einstein Equations, the Schwarzschild metric, which characterises the curvature of spacetime around a non-rotating black hole. Using spherical Schwarzschild coordinates $x^a = (t, r, \theta, \phi)$, the corresponding line element is

$$\mathrm{d}s^2 = (1 - 2m/r)\mathrm{d}t^2 - (1 - 2m/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{3}$$

where $m$ denotes the geometric mass. The non-zero Christoffel symbols are given by (I will not ask you to calculate these for this exercise, but you should make sure that you understand how they are derived)

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{m}{r(r-2m)}, \quad \Gamma^1_{00} = \frac{m(r-2m)}{r^3}, \quad \Gamma^1_{11} = \frac{m}{r(2m-r)}, \quad \Gamma^1_{22} = 2m - r,$$

$$\Gamma^1_{33} = (2m - r)\sin^2\theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^2_{33} = -\frac{\sin 2\theta}{2}, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{32} = \Gamma^3_{23} = \frac{1}{\tan\theta}.$$

*Using Eq.* (1)*, show that the equations of motion for a massive particle in the equatorial plane read*

$$\ddot{t} = -\frac{2m}{r(r-2m)}\,\dot{r}\dot{t}, \tag{4}$$

$$\ddot{r} = -\frac{m(r-2m)}{r^3}\,\dot{t}^2 - \frac{m}{r(2m-r)}\,\dot{r}^2 - (2m-r)\dot{\phi}^2, \tag{5}$$

$$\ddot{\phi} = -\frac{2}{r}\,\dot{r}\dot{\phi}, \tag{6}$$

where a dot represents the derivative w.r.t. $\tau$.

### 3. Integrals of motion (2 points)

These equations of motion are characterised by two integrals of motion, i.e., quantities that stay constant along the particle's trajectory. *Show that these two integrals of motion are given by*

$$\epsilon = (1 - 2m/r)\dot{t}, \qquad l = r^2\dot{\phi}, \tag{7}$$

where $\epsilon$ and $l$ are constants.

### 4. 'Energy equation' (3 points)

In analogy with Newtonian physics, the two integrals of motion are typically identified as the total energy and angular momentum per unit mass of the particle, respectively. Using the Schwarzschild line element, the relation $\mathrm{d}s = \mathrm{d}\tau$, as well as the definitions of $\epsilon$ and $l$, *derive the following 'energy equation' for a massive particle trajectory in the Schwarzschild spacetime:*

$$\dot{r}^2 = \epsilon^2 - \left(1 - \frac{2m}{r}\right)\left(1 + \frac{l^2}{r^2}\right). \tag{8}$$

### 5. A radially infalling particle (6 points)

Equation (8) is exactly the relation, we discussed in our BH Lecture when addressing that the Schwarzschild radius $r_s = 2m$ is an apparent singularity and that a local observer would not notice crossing this spacetime point. Only for a distant observer, measuring events in coordinate time $t$, does the Schwarzschild radius appear special. A massive particle, which would be at rest at infinity, indeed reaches the physical singularity at $r = 0$ in finite proper time $\tau_0$ when falling from $r_0$ radially into a non-rotating black hole.

*Show that this time is given by*[1]

$$\tau_0 = \frac{2}{3}\sqrt{\frac{r_0^3}{2m}}. \tag{9}$$

[1]Hint: You might want to use the following result for a function $f(x)$ and $c, d =$ constant:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = -\frac{c}{\sqrt{f(x)}} \qquad \Rightarrow \qquad f(x) = \left(\frac{3}{2}\right)^{2/3}(-cx + d)^{2/3}.$$