Master UAB - *High Energy Physics, Astrophysics and Cosmology*

NSs, BHs and GWs

BLACK HOLES

Feb 25th, 2022

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Recap I:

Covered last lecture: equivalence principles, Einstein equations

- The **weak equivalence principle** has multiple forms, reflecting the fact that we cannot distinguish between gravitational & inertial forces with any local experiment using test particles.
- The **strong equivalence principle** generalises the WEP to self-gravitating bodies and gravitational experiments.
- The **Einstein equations** are a set of tensor equations that completely control the **dynamics of gravitation**.

Recap II:

Covered last lecture: Einstein equations, Schwarzschild solution

- The essence of the Einstein equations is that “Spacetime tells matter how to move, while matter tells spacetime how to curve.” as summarised by John Wheeler.
- In General Relativity, **gravitational phenomena** arise not from forces or fields but from the curvature of spacetime itself.
- The (unique) **spherically symmetric & static solution** of the vacuum Einstein Equations is **Schwarzschild’s solution**.

Overview:

Covered so far: special relativity, tensor calculus, equivalence principles, Einstein equations, Schwarzschild solution

1. A general introduction
2. Non-rotating black holes
3. Charged & rotating black holes

Overview:

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1. **A general introduction**
2. Non-rotating black holes
3. Charged & rotating black holes

1.1 Video

- In today’s lecture, we will start with a little video to provide a general introduction to the topic of black holes. It will set the stage for our more mathematical discussion in the remainder of this lecture.
  - **The Ultimate Guide to Black Holes** by *Kurzgesagt*

    https://tinyurl.com/4vw9xbsz

  - To break things up, we will watch in **two blocks** and have 2 rounds of questions.

1.2 Questions https://tinyurl.com/4vw9xbsz

- Watch **until 5:32** & go to www.menti.com & enter 4638 2609.
  - 1. The majority of BHs form when very **massive stars die** and undergo **gravitational collapse**.
    - Incorrect
    - Correct
  - 2. Although light cannot escape from inside a BH's event horizon, can we observe them indirectly via effects on external particles, e.g., those forming an accretion disk?
    - Yes
    - No

1.3 Questions https://tinyurl.com/4vw9xbsz

- Watch the rest & go to www.menti.com & enter 7116 2416.
  - 3. Which of these quantities completely describe a BH?
    - Its mass, spin and charge.
    - Its event horizon.
    - The amount of matter that crossed its event horizon.
  - 4. An object within the **ergosphere** around a rotating BH cannot appear at rest for a distant observer.
    - Incorrect
    - Correct

1.2 Answers https://tinyurl.com/4vw9xbsz

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Overview:

Covered so far: special relativity, tensor calculus, equivalence principles, Einstein equations, Schwarzschild solution

1. A general introduction
2. **Non-rotating black holes (NRBHs)**
3. Charged & rotating black holes

2.1 NRBHs - Singularities

- Last lecture, we encountered the **Schwarzschild solution** of the vacuum Einstein Equations. Our choice of coordinates, i.e., (t, r, θ, ϕ), reflected the spherical symmetry of the spacetime.

$$ds^2 = (1 - 2m/r)\mathrm{d}t^2 - (1 - 2m/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

- However, our coordinate system does **not cover** the **entire manifold**, e.g., the points $\theta = 0, \pi$ are **not uniquely defined**.

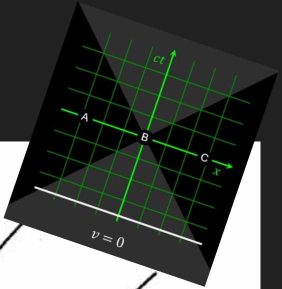
- These are so-called **coordinate singularities**, which are an **artefact** of the choice of coordinate system. They can be **removed** using a different system, e.g., Cartesian coordinates.

2.2 NRBHs - Schwarzschild radius

$$ds^2 = (1 - 2m/r)\mathrm{d}t^2 - (1 - 2m/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

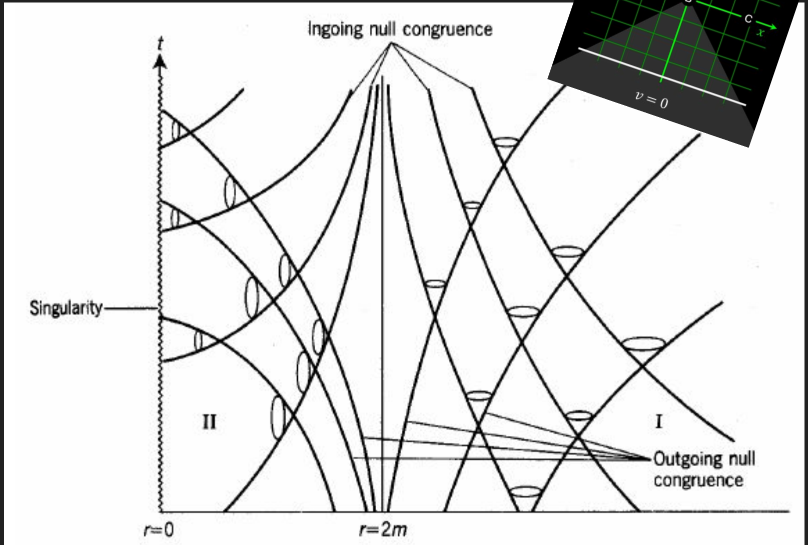
- The SS solution is degenerate at **two other points** $r_1 = 2m$ and $r_2 = 0$. r_2 cannot be removed & is a **true** (physical) **singularity**.
- r_1 is called the **Schwarzschild radius r_s**. It is a removable coordinate singularity but **separates** the **manifold** into two disconnected regions, where **t & r invert** their character:
    - $2m < r < \infty$: the exterior with r = spacelike and t = timelike.
    - $0 < r < 2m$: the interior with r = timelike and t = spacelike.

2.3 NRBHs - Spacetime diagrams



- To illustrate this, we look at the **local light cone** at different points. We construct these via **lightlike** (null) lines, $ds^2 = 0$.

- Fixing θ & ϕ, we recover curves t(r) that satisfy this constraint, i.e., the **null congruences**.



- As **observers** move along **timelike worldlines**, they 'move into' their future light cones. Inside r_s the light cone is flipped and an observer will ALWAYS **fall into the singularity**.

2.4 Questions

- Go to www.menti.com & enter 9650 2435.
  - 1. Which of the following are **coordinate singularities** of the **SS solution**, i.e., removable with a transformation?
    - $r = 0$
    - $r = 2m$
    - $\theta = \pi$
  - 2. There is **one clear problem** with the diagram on the previous page. Can you think of what this might be? Please type out your answer in **1 or 2** sentences, but not more.

2.4 Answers

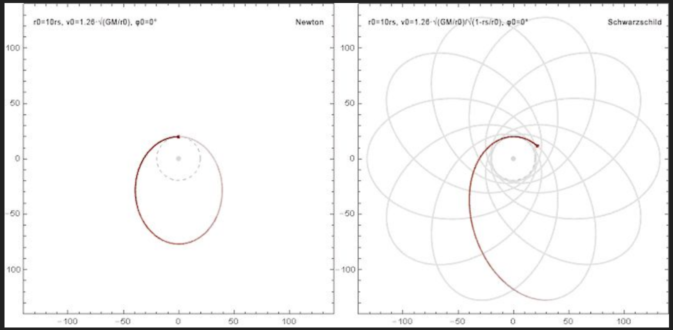
- Go to www.menti.com & enter 9650 2435.
  - 1. Which of the following are **coordinate singularities** of the **SS solution**, i.e., removable with a transformation?
    - $r = 0$
    - $r = 2m$
    - $\theta = \pi$
  - 2. The spacetime diagram seems to suggest that an observer outside the Schwarzschild radius (or an ingoing light ray) would require infinite time to reach r_s and cannot cross it.

2.5 NRBHs - Particle obits

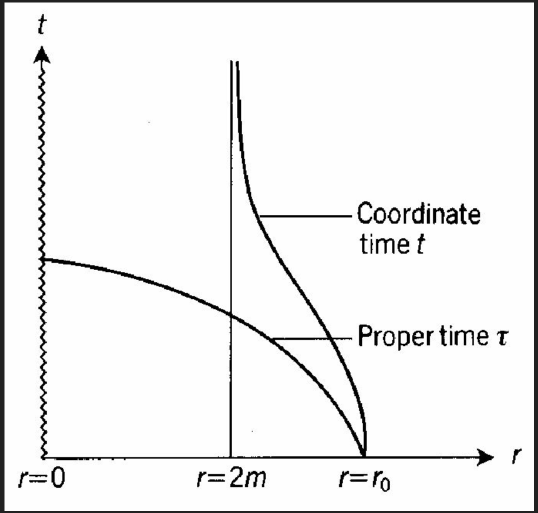
- The motion of a massive particle around a BH is described by the **geodesic equation** parameterised by τ. As the SS metric is symmetric about $\theta = \pi/2$, it's convenient to consider particle motion in the **equatorial plane**.

- As this motion conserves the particle's **total energy E** and its **specific angular momentum l**, the equation of motion is

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{m^2c^2} - \left(1 - \frac{r_\mathrm{s}}{r}\right)\left(c^2 + \frac{l^2}{r^2}\right)$$



2.6 Eddington-Finkelstein coordinates



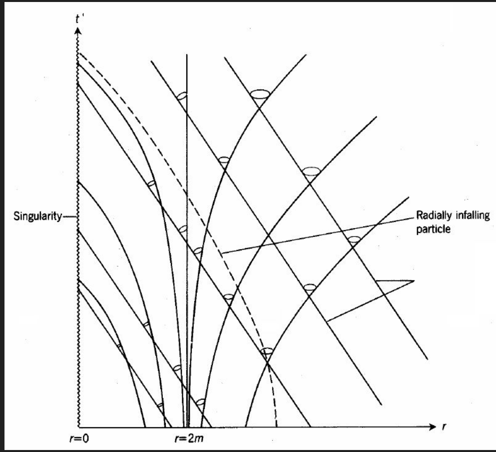
- The issue is that the **motion of a test particle** is not determined by the Schwarzschild coordinate t but instead the **proper time τ**. When determining the trajectory of an infalling particle τ (r), we find that it falls continuously towards r = 0 **in finite time**.

- Ingoing congruences become **straight lines** when transforming t' = t + 2m ln(r - 2m). The Eddington-Finkelst. **line element** is

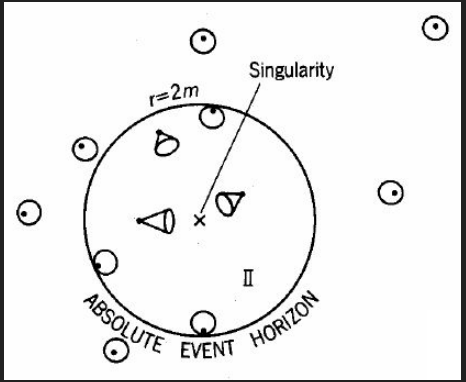
$$ds^2 = (1 - 2m/r)\mathrm{d}t'^2 - 4m/r\,\mathrm{d}t'\mathrm{d}r - (1 + 2m/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

2.7 NRBHs - Event horizons

- The **2D spacetime diagram** in Eddington-Finkelstein coordinates looks now like this:



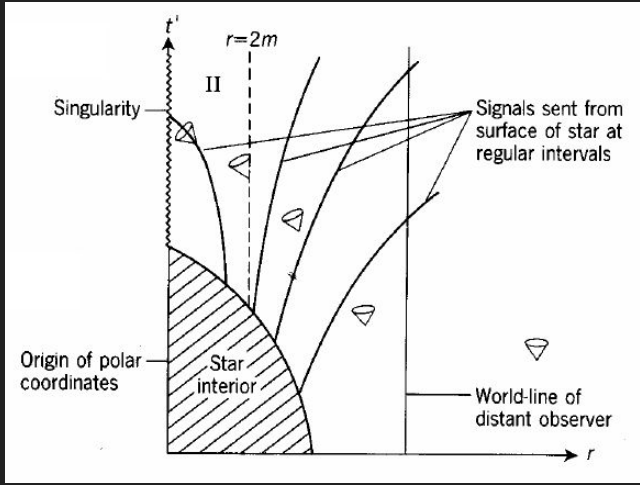
- With angular information, we can use a different perspective and look at the **equatorial plane** to visualise light-cone cross sections.



- Approaching r_s from infinity, the cone apexes (black dots) move from the circle centres to outer boundary. Once we reach r_s, the **cones** start to **tilt** and all timelike & null geodesics point towards $r = 0$. r_s is the **event horizon**.

2.8 The black hole concept

- We introduced the SS solution as a mathematically **abstract concept**. To make this concrete, consider the **gravitational collapse** of a spherically symmetric star. The star will contract until all its matter is contained in the singularity.



- Imagine an observer on the surface sending out **regular light signals**. A **distant observer** sees these with larger and larger time gaps until the surface contracts beyond r_s & no more signals appear. **It becomes 'black'**.

2.9 Exercise

- The idea that light cannot escape a gravitational field has a **classical analogue** (assuming its particle nature). Consider a particle of mass m, moving away radially from a uniform, symmetric matter distribution of mass M and radius R.
- Show that R is the **Schwarzschild radius** for an **escape velocity** (the velocity at the surface of M, so that v → 0 for r → ∞) equal to the speed of light c.

$$E = \frac{1}{2}mv^2 - GMm/r$$

$$r_s = 2GM/c^2$$

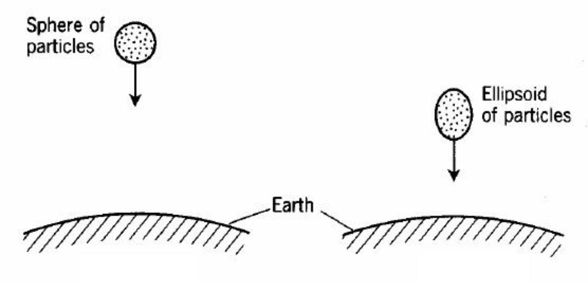
2.9 Exercise

- Solution: The constraint v → 0 for r → ∞ implies that the total energy vanishes, i.e., E = 0. Solving for the velocity, we then find $v^2 = 2\ GM/r$, so that the **escape velocity** is $v'^2 = 2\ GM/R$.
- If a particle at r = R has less than v', it will eventually be pulled back by the gravitational attraction of the mass distribution. Now if we want a photon escape velocity at infinity that is equal to the speed of light, we require $c^2 = 2\ GM/R$.
- Rearranging leads exactly to

$$r_s = 2GM/c^2$$

2.10 NRBHs - Tidal effects

- If we go beyond point particles in space-time to objects with **extended mass distribution**, spacetime curvature will result in tidal effects.



- These will cause **elongation** in the direction of motion & **compression** in the transverse direction. In the case of a BH, this **effect** becomes **infinite** as we reach the singularity.

- An astronaut falling into a black hole will thus experience ***spaghettification.***

2.11 NRBHs - Orbital stability

- Earlier, we considered purely radial motion of massive particles. Similarly, we could ask how a particle moves along **circular orbits** in the equatorial plane, i.e., r = const but $\phi = \phi(\tau \text{ or } r)$.
- From the resulting equations, it's possible to **deduce that**
  - $r_{ISCO} = 3\, r_s$: this is the smallest marginally stable circular orbit in which a test particle can stably orbit a SS BH; it's called the innermost stable circular orbit (**ISCO**).
  - $2\, r_s < r < r_{ISCO}$: particles can still be bound but they are unstable; $2\, r_s$ is also called the **marginally bound orbit**.

2.12 NRBHs - Photon orbits

- Orbits for massless particles cannot be parameterised by the proper time. Opting for a **general affine parameter** u, the equation of motion that follows from the geodesic equations is

$$\left(\frac{dr}{du}\right)^2 = \frac{E^2}{m^2c^2} - \left(1 - \frac{r_s}{r}\right)\frac{l^2}{r^2}$$

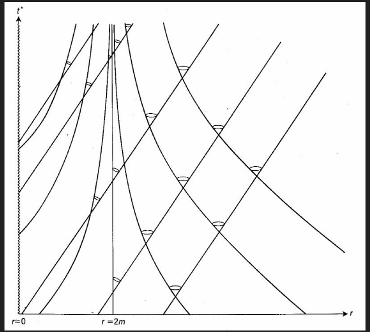
- Looking at **circular orbits**, we find a single possible (although unstable) solution at $r_{ph} = 1.5\ r_s$. This orbit is referred to as the **photon sphere**. Gravity is so strong that light is forced onto a circle. This effect does not have a Newtonian analogue.

2.13 NRBHs - White holes

$4m/r\,dt'dr$

- While the SS solution in Eddington-Finkelstein coordinates highlights that r_s is not a true singularity, our redefinition of $t \to t'$ causes the solution to **no longer** be **time-symmetric**.

- In principle, we can introduce the **time-reversed (retarded) solution** by making another transition $t^* = t - 2m \ln(r - 2m)$. In this case, the outgoing congruences would become straight.



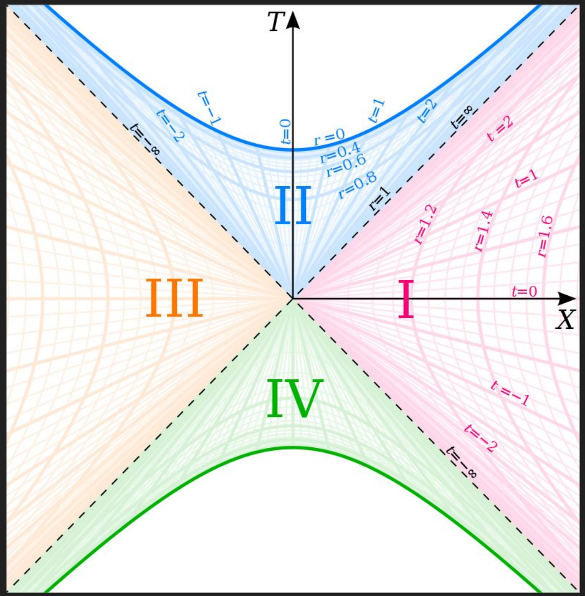
- The solution is again **regular** for $0 < r < \infty$, with $r = 2m$ acting as a special surface. BUT now only past-directed timelike or null lines can cross. This is essentially a **white hole**.

2.14 NRBHs - Kruskal & beyond

- It is possible to extend the EF solutions in such a way that all geodesics can either be extended to ∞ or terminate at a true singularity. A spacetime that satisfies this is called **maximal**.

- For this, a **new** choice of coordinates is **needed.** The **maximal extension** of the SS metric is the **Kruskal metric**:

$$ds^2 = \frac{16m^2}{r} e^{-r/2m}(\mathrm{d}T^2 - \mathrm{d}X^2) - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$



2.15 Questions

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  - 3. The apparent issue of SS coordinates, i.e., test particles taking infinite time to cross the event horizon, is resolved by considering the object's motion with proper time τ.
    - Correct
    - Incorrect
  - 4. Are white holes, the (hypothetical) ‘inverse’ of BHs, mathematically valid solutions of the Einstein Equations?
    - No
    - Yes

2.15 Answers

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Overview:

Covered so far: special relativity, tensor calculus, equivalence principles, Einstein equations, Schwarzschild solution, non-rotating BH

1. A general introduction
2. Non-rotating black holes
3. **Charged & rotating black holes (CRBHs)**

3.1 CRBHs - Charged BHs

- To go beyond the simple non-rotating BH picture, we will first consider **electrically charged solutions**, although BHs of large charge are **unlikely to exist** in nature.
- In contrast to the Schwarzschild solution, which is derived from the vacuum Einstein Equations, charged BHs cannot satisfy the same equations. Presence of an electric field causes $\mathbf{T_{ab} \neq 0}$.
- We again look for a **static, asymptotically flat & spherically symmetric solution** to the Einstein-Maxwell equations:

$$ds^2 = \mathrm{e}^{\nu(t,r)}\mathrm{d}t^2 - \mathrm{e}^{\lambda(t,r)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$$

3.2 CRBHs - Reissner-Nordstøm solution

- The full solution obtained from Einstein's equations has **two integration constants**, m & q, (which we interpret as geometric **mass** & **charge** located at r = 0). The line element reads

$$ds^2 = (1 - 2m/r + q^2/r^2)\mathrm{d}t^2 - (1 - 2m/r + q^2/r^2)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2$$

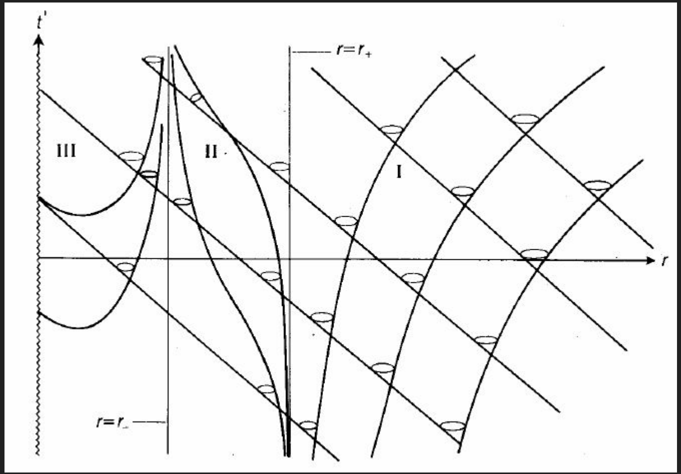
- Looking at the **element g_{tt}**, we can use the Newtonian limit to determine q as well as deriving the **horizon** of a charged BH:

$$q^2 = \frac{1}{4\pi\varepsilon_0}\frac{G}{c^4}Q^2$$

$$r_\pm = m \pm \sqrt{m^2 - q^2}$$

3.3 CRBHs - Observations

- The physical spacetime realisations depend on the roots in $r_{\pm}$. For **q > m** (super-extremal BHs), we cannot have physical solutions as no event horizon exists, while for **q = m** (extremal BHs) the horizons are degenerate. We typically focus on **q < m**.
- We distinguish **three regions** in which the metric remains **regular**: (I) $r_+ < r < \infty$, (II) $r_- < r < r_+$, (III) $0 < r < r_-$. The situation at r_+ is similar to that at r_s. However, the existence of r_- alters the role of singularity at $r = 0$.



3.4 CRBHs - Rotating BHs

- As BHs form via gravitational collapse of rotating massive stars the study of rotating BH is **important for astrophysics.**
- However, the solution of the **vacuum EEs** for rotating BHs is rather **tedious** (it was only discovered in 1963), so we will only refer to results here. The line element is often expressed in two equivalent forms. In **Boyer-Lindquist coordinates** it reads

$$ds^2 = \frac{\Delta}{\rho^2}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right)^2 - \frac{\sin^2\theta}{\rho^2}\Big[\left(r^2 + a^2\right)\,\mathrm{d}\phi - a\,\mathrm{d}t\Big]^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\,\mathrm{d}\theta^2$$

$$\Delta = r^2 - 2mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta$$

3.5 CRBHs - Kerr solution

- In Boyer-Lindquist form, (r, θ, ϕ) are standard oblate spheroidal coordinates, which are related to Cartesian coordinates (x, y, z). Because of this, it's possible to rewrite the line element in terms of Cartesian-type coordinates as originally done by Kerr.

- We can rewrite the solution in a **more convenient way** to directly read off the metric:

$$A = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta$$

$$ds^2 = \frac{(\Delta - a^2\sin^2\theta)}{\rho^2}\,\mathrm{d}t^2 + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 - \frac{A\sin^2\theta}{\rho^2}\,\mathrm{d}\phi^2 + \frac{4ma}{\rho^2}\,r\sin^2\theta\mathrm{d}\phi\mathrm{d}t$$

3.6 CRBHs - Basic properties

- We observe that the Kerr solution depends on two parameters **m and a**, where the latter is the so-called **Kerr parameter** $a = J/Mc$ representing **angular momentum** of the BH. For $a = 0$ ($r \rightarrow \infty$), we recover the SS (Minkowski) solution.
- As the metric coefficients are independent of t & ϕ, the Kerr solution is **stationary & axially symmetric** but **not static**, i.e., $t \rightarrow -t$ does not reproduce the same solution. However, ds^2 is **invariant** under **simultaneous inversion** of t & ϕ.
- We also note the final term, $dt \times d\phi$, implies the **coupling** between time and motion in the rotational plane for $a \neq 0$.

3.7 CRBHs - Singularities & horizons

- The Kerr solution has one **physical singularity** at $\rho = 0$. This occurs for $x^2 + y^2 = a^2$ and $z = 0$, suggesting that the singularity is not a point but a **ring** of radius a in the equatorial plane.
- When $\Delta = 0$, the g_{rr} component of the metric diverges. We recover the following **two roots**:

$$\frac{\rho^2}{\Delta} dr^2$$

$$r_\pm = m \pm \sqrt{m^2 - a^2}$$

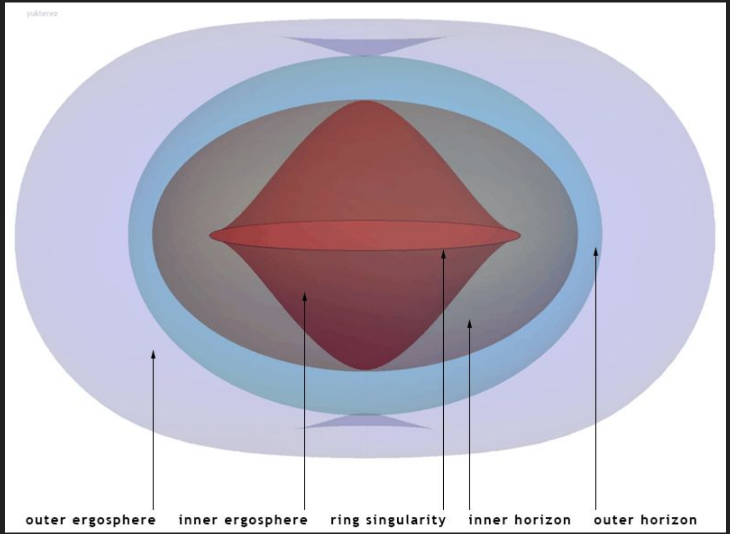
- These have the same structure as the RN solution, so we conclude that $a > m$ is unphysical & physical solutions correspond to $a \leq m$. We again recover **3 regular regions** of the Kerr solution with coordinate singularities / **event horizons at $r_\pm$**.

3.8 CRBHs - Ergosphere I

$$\frac{(\Delta - a^2 \sin^2\theta)}{\rho^2}\,dt^2$$

- In addition, we observe **special behaviour** arising from the g_{tt} metric element and obtain two **apparent singularities** of the form

$$r_{S\pm} = m \pm \sqrt{m^2 - a^2\cos^2\theta}$$



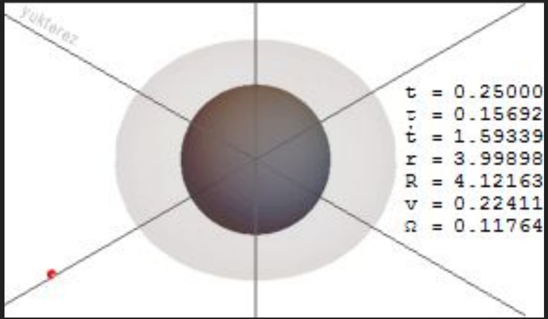
- At the outer surface (**surface of infinite redshift**), the dt^2 term changes from timelike (outside) to spacelike (inside).

- The region between r_+ and r_{s+} is called the **ergosphere**.

3.9 CRBHs - Ergosphere II

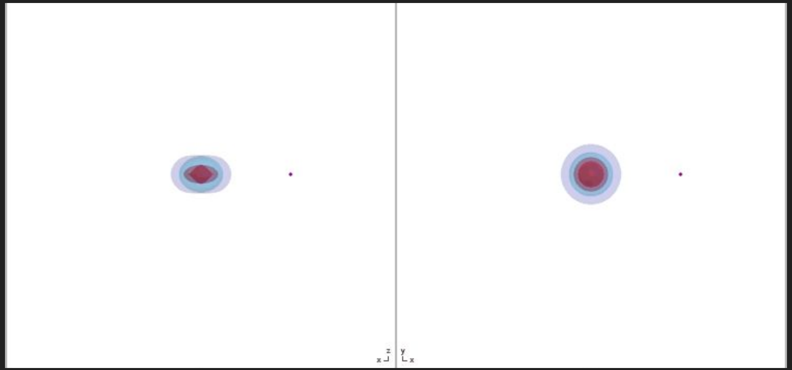
$$+\frac{4ma}{\rho^2} r \sin^2\theta \mathrm{d}\phi \mathrm{d}t$$

- Within the ergosphere, the rotating black hole affects the surrounding spacetime so much that inertial reference frames are entrained by it. This is an extreme form of **frame dragging**.
- As a result, an object inside the ergosphere **cannot appear stationary** for a distant observer (unless it could move faster than the speed of light w.r.t to the local spacetime). It has to **co-rotate** with the central BH.
- As particles are outside r_+, they could escape the ergosphere. It is possible to extract energy via **Penrose process**.



3.10 CRBHs - Orbits

- All these considerations can be properly investigated looking at the motion of test particles like we did for the non-rotating BH.
- The equations of motion depend on a **third integral of motion** related to the BH’s angular momentum L, leading to a generalised form of the equation we saw on slide 2.5.
- One finds that particles are no longer confined to planes but orbit in a **torus-like region** around the central black hole.



3.11 Questions

- Go to www.menti.com & enter 9498 1243.
  - 1. Which of the following metrics is not a solution of the Einstein Equations in vacuum?
    - Kerr metric
    - Reissner-Nordstrøm metric
    - Schwarzschild metric
  - 2. The Kerr solution is the one most relevant for astrophysical applications.
    - Correct
    - Inforrect

3.11 Answers

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Summary I:

Covered today: non-rotating BHs

- The SS solution of the vacuum EEs describes the nature of non-rotating BHs. The characteristics of the metric (apparent or real **singularities**) determine the spacetime.
- By constructing spacetime diagrams for different coordinate systems, we uncovered the existence of an **event horizon** and the fate of everything falling into a non-rotating black hole.
- A full study of the SS spacetime hinted at the (mathematical) existence of **white holes** and the possibility of wormholes.

Summary II:

Covered today: charged/rotating BHs

- We then looked at **charged BHs** (although unlikely to exist in nature) characterised by the **Reissner-Nordstrøm metric**.
- The Kerr solution describes the (astrophysically relevant) case of rotating BHs. Rotation introduces new effects like frame dragging and specifically the concept of the **ergosphere**.
- This concludes our 'classical' GR discussion of BHs, as all solutions of the Einstein-Maxwell equations are described by three parameters (mass, charge, spin) due to the **no-hair theorem**.

Final impressions:

Covered today: non-rotating/charged/ rotating BHs

- To recap, which concepts that we covered so far are represented in this image?

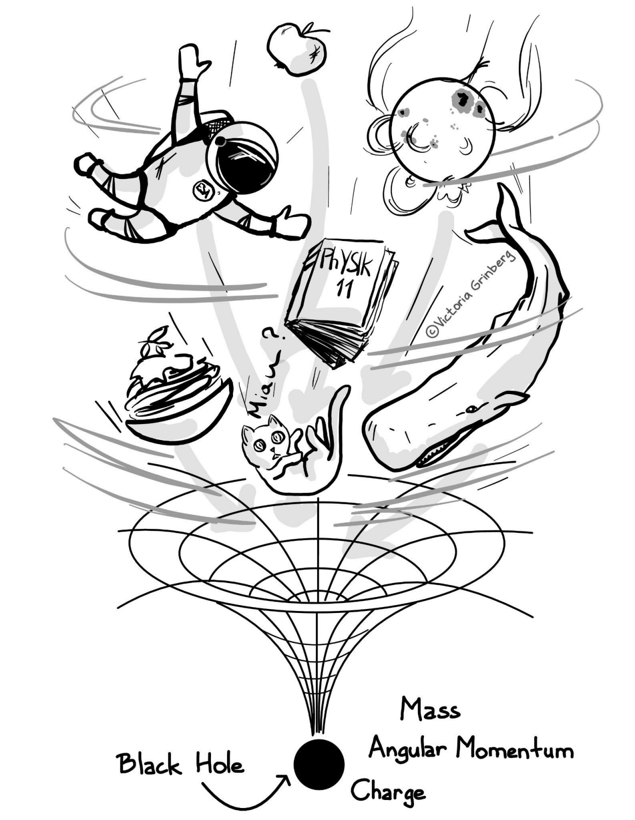

Image credit: Victoria Grinberg @vicgrinberg