Analytical 3d - Introduction

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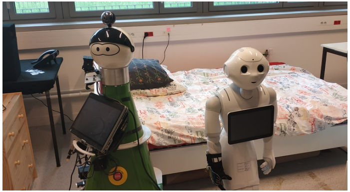
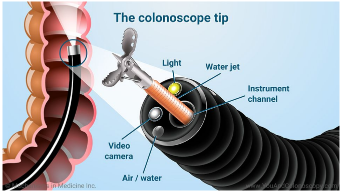
3d Analytique 1 : Géométrie projective, Matrice caméra, Homographies, Stéréovision idéale

Motivations: 3d Reconstruction from Videos

Reconstructing the scene geometry from videos is useful in many applications: Robot navigation (obstacle detection), Metrology, 3d Cartography, Medicine...





+ It is a cheap and flexible approach: One single passive camera, Adaptive baseline,...
− It strongly relies on scene structure (texture) and precise camera positioning.

Presentation Outline

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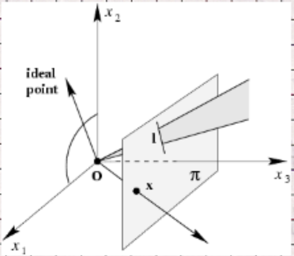
1. Projective Geometry and Camera Matrices
   - Projective Geometry in $\mathbb{P}^2$
   - 2d Projective transformations
   - Projective Geometry in $\mathbb{P}^3$
2. Homographies: Practical cases
   - Rotation around the optical centre
   - Plane viewed from different poses
3. Estimation of a homography

Projective Geometry in $\mathbb{P}^2$

- Homogeneous coordinates $\rightarrow$ additional component $\rightarrow$ non injective representation
- Affine transformations represented by linear functions $\rightarrow$ simpler operations
- Points and lines at infinity represented with finite coordinates

Projective Geometry in $\mathbb{P}^2$



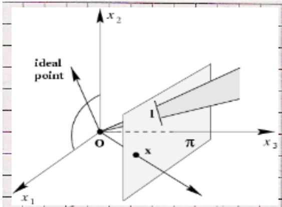
$$\mathbb{R}^2 \qquad\qquad \mathbb{P}^2$$

$$\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}x\\y\\1\end{pmatrix}$$

$$\begin{pmatrix}u/w\\v/w\end{pmatrix} \longleftarrow \begin{pmatrix}u\\v\\w\end{pmatrix}$$

- Equivalence Classes: $\forall \lambda \neq 0 \quad \lambda x \equiv x$
- Duality point / line:
  $m = (x, y, 1)^t \qquad \ell = (a, b, c)^t$
- Ideal points: $(x, y, 0)^t$
- Line at infinity: $(0, 0, 1)^t$

Projective Geometry in $\mathbb{P}^2$



Point m belongs to line ℓ

$\Updownarrow$

$$\boxed{m^t \ell = 0}$$

Point m is at the intersection of lines ℓ and ℓ':

$$\boxed{\ell \times \ell' = m}$$

Line ℓ passes through points m and m':

$$\boxed{m \times m' = \ell}$$

Notation:

pre-vector product: $[U]_\times = \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix}$

with $U = (u, v, w)^t$

Then:

$$\boxed{U \times U' = [U]_\times U'}$$

Projective transformations

- A projective transformation h of the plane is characterized by the fact that: if three point m_1, m_2 and m_3 are aligned, $h(m_1)$, $h(m_2)$ and $h(m_3)$ are aligned too.
- A function $h : \mathbb{P}^2 \mapsto \mathbb{P}^2$ is a projective transformation if and only if there exists a non singular 3×3 matrix H such that $\forall m \in \mathbb{P}^2, h(m) = Hm$.

Projective transformations 1: Translations

$$H = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- with $\mathbf{t} = \left(t_x \;\; t_y\right)^T$ translation vector
- 2 degrees of freedom

Projective transformations 2: Isometries

$$H = \begin{pmatrix} \cos(\theta) & -\varepsilon \sin(\theta) & t_x \\ \sin(\theta) & \varepsilon \cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- with $\mathbf{t} = (t_x \ t_y)^T$ translation vector
- θ rotation angle
- $\varepsilon = \pm 1 \rightarrow$ direct / indirect isometry
- 3 degrees of freedom
- *preserves:* angles, lengths, areas

Projective transformations 3: Similarities

$$H = \begin{pmatrix} s\cos(\theta) & -s\sin(\theta) & t_x \\ s\sin(\theta) & s\cos(\theta) & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

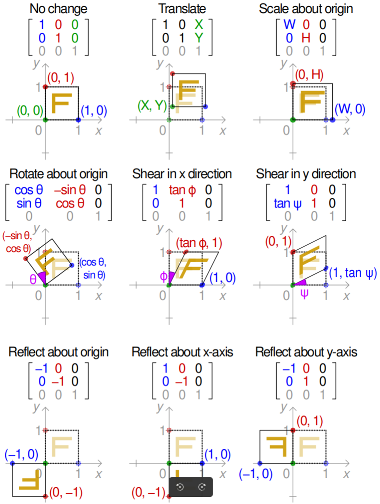
- with $\mathbf{t} = (t_x \;\; t_y)^T$ translation vector
- θ rotation angle
- s homothety factor
- 4 degrees of freedom
- *preserves:* angles, ratios of lengths/areas, parallel lines

Projective transformations 4: Affine transformations

$$H = \begin{pmatrix} a_1^1 & a_1^2 & t_x \\ a_2^1 & a_2^2 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- 6 degrees of freedom
- *preserves:* ratios of areas, parallel lines

(Figure from Wikipedia)



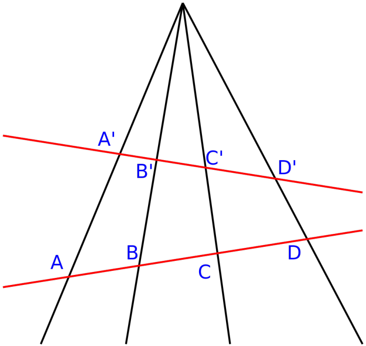
Projective transformations 5: Homographies

$$H = \begin{pmatrix} a_1^1 & a_1^2 & t_x \\ a_2^1 & a_2^2 & t_y \\ v_1 & v_2 & 1 \end{pmatrix}$$

- $\mathbf{v} = (v_1 \;\; v_2)^T$ relates to the action on points/lines at infinity
- 8 degrees of freedom
- *preserves:* cross-ratios of four points on a line:

$$\frac{AC \times BD}{BD \times AC} = \frac{A'C' \times B'D'}{B'D' \times A'C'}$$

(Figure from Wikipedia)



Homographies on points/lines at infinity

Consider a line at infinity $\mathbf{l}_\infty = (l_1 \;\; l_2 \;\; 0)^T$
When applied an affine transformation:

$$\begin{pmatrix} a_1^1 & a_1^2 & t_x \\ a_2^1 & a_2^2 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 a_1^1 + l_2 a_1^2 \\ l_1 a_2^1 + l_2 a_2^2 \\ 0 \end{pmatrix}$$

A line at infinity remains at infinity!
When applied a general homography:

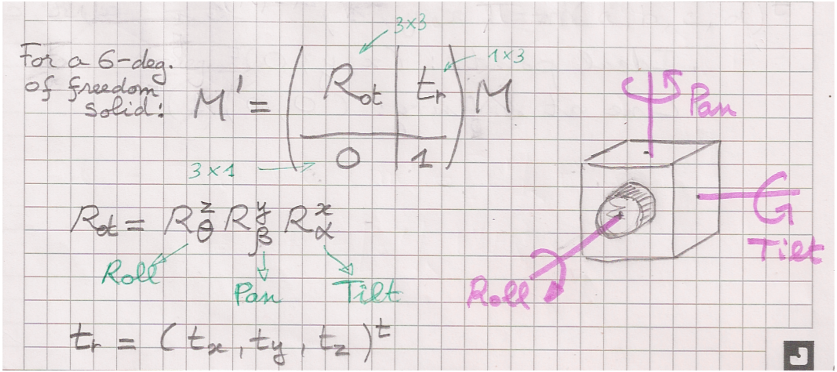
$$\begin{pmatrix} a_1^1 & a_1^2 & t_x \\ a_2^1 & a_2^2 & t_y \\ v_1 & v_2 & 1 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 a_1^1 + l_2 a_1^2 \\ l_1 a_2^1 + l_2 a_2^2 \\ l_1 v_1 + l_2 v_2 \end{pmatrix}$$

A line at infinity becomes finite!
This allows to observe vanishing points and horizon lines.

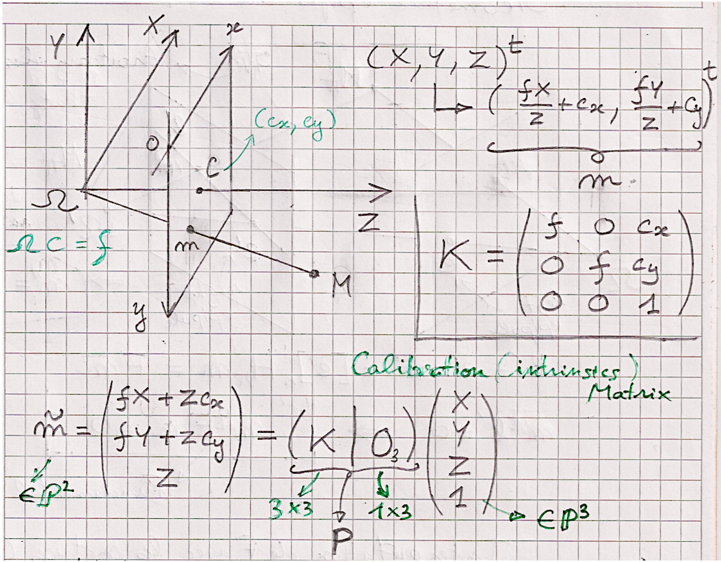
Projective Geometry in $\mathbb{P}^3$

- $\mathbb{R}^3 \leftrightarrow \mathbb{P}^3$: $(X, Y, Z) \to (X, Y, Z, 1)$; $(u/h, v/h, w/h) \leftarrow (u, v, w, h)$
- Duality point / plane: $M = (X, Y, Z, 1)^t$ / $\Pi = (a, b, c, d)$.
- Lines are defined from 2 points or from 2 planes!

$\mathbb{P}^3$ allows to express linearly affine transformations:



Camera (Calibration) Matrix: Intrinsics



Projection and Back-Projection Matrices

$M = (X, Y, Z)^t \in \mathbb{R}^3$
$m = (x, y)^t \in \mathbb{R}^2$, and $\tilde{m} = (x, y, 1)^t \in \mathbb{P}^2$

Camera (Projection) Matrix

$$m = \pi(M) = \left(f\frac{X}{Z} + c_x, f\frac{X}{Z} + c_x\right)$$

Equivalent to:

$$\tilde{m} = KM$$

with: $K = \begin{pmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{pmatrix}$

Back-Projection Matrix

$$M = \pi^{-1}(m, Z) = \left(Z\frac{x-c_x}{f}, Z\frac{y-c_y}{f}, Z\right)$$

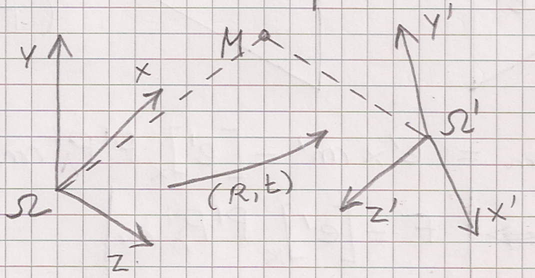
Equivalent to:

$$M = \underbrace{Z}_{\text{Depth}}\underbrace{K^{-1}\tilde{m}}_{\text{Direction}}$$

with: $K^{-1} = \begin{pmatrix} \frac{1}{f} & 0 & -\frac{c_x}{f} \\ 0 & \frac{1}{f} & -\frac{c_y}{f} \\ 0 & 0 & 1 \end{pmatrix}$

Displacement Matrix: Extrinsics

$$\tilde{m}' = \underbrace{\left(K \mid 0_3\right)\begin{pmatrix} R & -Rt \\ 0_3^t & 1 \end{pmatrix}}_{P'} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$



$$m = PM$$

$$m' = P'M$$

Presentation Outline

Homographies

- **Homography** $\rightarrow$ Most general case of 2d projective transformation

$$\tilde{m}' = H\tilde{m}$$

- 8 degrees of freedom $\rightarrow$ At least four non colinear 2d points!
- Corresponds to 2 particular cases of image pairs:
  - 3d scene viewed under pure rotation around the optical centre ($\mathbf{t} = O_3$).
  - Same plane viewed under two different 3d poses.

Rotation around the optical centre

In the case of a pure rotation around the optical centre ($\mathbf{t} = O_3$), the projected image transformation is a homography:

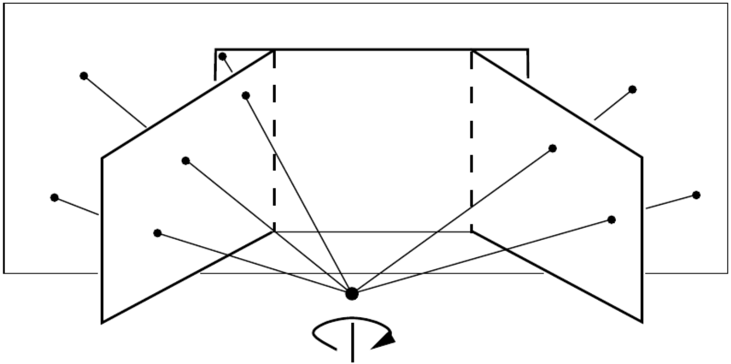


Figure from [**Hartley and Zisserman 2004**]

Rotation around the optical centre

Since $\mathbf{t} = O_3$ we get:

$$\tilde{m} = \left(\begin{array}{c|c} K & O_3 \end{array} \right) \tilde{M}$$

$$\tilde{m}' = \left(\begin{array}{c|c} K & O_3 \end{array} \right) \left(\begin{array}{c|c} R & \color{red}{O_3} \\ \hline O_3^t & 1 \end{array} \right) \tilde{M}$$

which can be written more simply:

$$\begin{array}{rcl} \tilde{m} & = & KM \\ \tilde{m}' & = & KRM = \underbrace{KRK^{-1}}_{H} \tilde{m} \end{array}$$

Rotation around the optical centre

Note the difference between rotation around the optical centre ((a) to (b)), and translation ((a) to (c)):

a

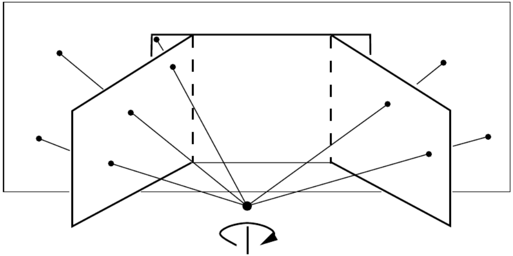
b

c

Images from [**Hartley and Zisserman 2004**]

Rotation around the optical centre

Since there is no parallax, the images can be stitched to form a mosaic:

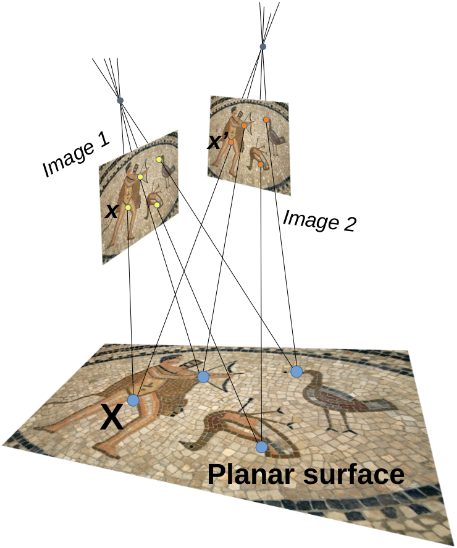


Plane viewed from different poses

$$\tilde{x} = H_{\pi,1} X$$

$$\tilde{x}' = H_{\pi,2} X$$

$$\tilde{x}' = H_{\pi,2} H_{\pi,1}^{-1} \tilde{x} = H_{\pi} \tilde{x}$$



Plane viewed from different poses

Let us first assume that $K = I_3$ (i.e. $f = 1, c_x = c_y = 0$). Then if the pose of the right camera is given by rotation matrix R and translation vector $\mathbf{t}$, we get:

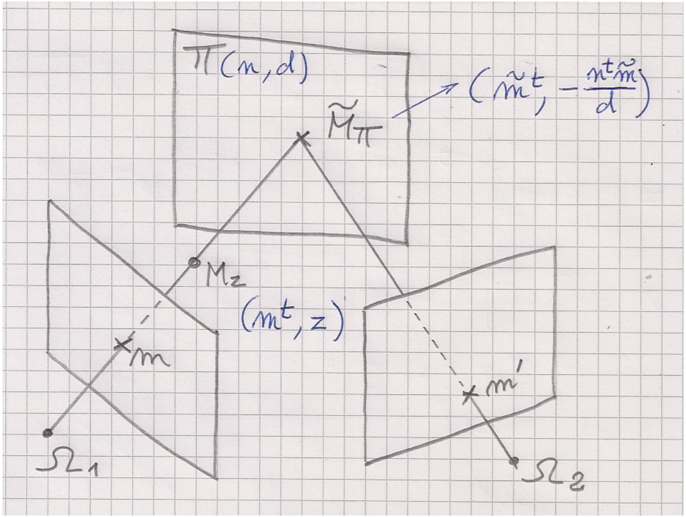
$$\tilde{m} = P\tilde{M} = \left(\begin{array}{c|c} I_3 & O_3 \end{array} \right) \tilde{M}$$

$$\tilde{m}' = P'\tilde{M} = \left(\begin{array}{c|c} R & \mathbf{t} \end{array} \right) \tilde{M}$$

Every point on the ray $M_z = (m^t, z)$ (parameterized by z) projects on m.
If the point M_z is on the plane π, it must satisfy: $\pi^t.\tilde{M}_z = 0$.
If the coordinates of the plane are given as $\pi = (\mathbf{n}^t, d)^t$, so that for points M on the plane, we have: $\mathbf{n}^t M + d = 0$,
then the point of the ray backprojected from m and intersecting plane π is:

$$\tilde{M}_\pi = \left(\tilde{m}^t, -\frac{\mathbf{n}^t \tilde{m}}{d} \right)^t$$

Plane viewed from different poses



Plane viewed from different poses

The point of the ray backprojected from m and intersecting plane π is:

$$\tilde{M}_\pi = \left(\tilde{m}^t, -\frac{\mathbf{n}^t \tilde{m}}{d} \right)^t$$

And then:

$$\begin{aligned}
\tilde{m}' &= P'\tilde{M}_\pi = \left(\begin{array}{c|c} R & \mathbf{t} \end{array} \right) \tilde{M}_\pi \\
&= R\tilde{m} - \frac{\mathbf{t}\mathbf{n}^t}{d}\tilde{m} \\
&= \underbrace{\left(R - \frac{\mathbf{t}\mathbf{n}^t}{d} \right)}_{H_\pi} \tilde{m}
\end{aligned}$$

Finally, by considering the internal parameter matrix K of a single camera moved with rotation R and translation $\mathbf{t}$, the homography related to the plane $\pi = (\mathbf{n}^t, d)^t$ is given by:

$$H = K \left(R - \frac{\mathbf{t}\mathbf{n}^t}{d} \right) K^{-1}$$

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3. Estimation of a homography

Estimation of a Homography

Now we wish to estimate the parameters of a homography using a set of correspondances from a pair of images:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 & t_x \\ a_2^1 & a_2^2 & t_y \\ v_1 & v_2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- In the following practical session we will use a the Direct Linear Transform (DLT) resolved by Singular Values Decomposition (SVD).
- The next slides are stolen from **Gianni Franchi**'s 2022 course.

Estimation by Direct Linear Transformation (DLT)

Let us rearrange the equation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

we use auxiliary variables A,B and C.

$$\mathbf{x'} = \begin{bmatrix} \mathbf{A}^t \\ \mathbf{B}^t \\ \mathbf{C}^t \end{bmatrix} \mathbf{x}$$

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} \mathbf{A}^t\mathbf{x} \\ \mathbf{B}^t\mathbf{x} \\ \mathbf{C}^t\mathbf{x} \end{bmatrix}$$

Estimation by Direct Linear Transformation (DLT)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} \mathbf{A}^t\mathbf{x} \\ \mathbf{B}^t\mathbf{x} \\ \mathbf{C}^t\mathbf{x} \end{bmatrix}$$

$$x' = \frac{u'}{w'} = \frac{\mathbf{A}^t\mathbf{x}}{\mathbf{C}^t\mathbf{x}}$$

$$y' = \frac{v'}{w'} = \frac{\mathbf{B}^t\mathbf{x}}{\mathbf{C}^t\mathbf{x}}$$

Estimation by Direct Linear Transformation (DLT)

We can rewrite the equations:

$$\begin{cases} -\mathbf{A}^t\mathbf{x} & & +x'\mathbf{x}\mathbf{C}^t = 0 \\ & -\mathbf{B}^t\mathbf{x} & +y'\mathbf{x}\mathbf{C}^t = 0 \end{cases}$$

we want to estimate A, B and C

Estimation by Direct Linear Transformation (DLT)

Let us write $\mathbf{P} = \begin{bmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{bmatrix}^t$. P is a vector of size 9×1.
We can rewrite the previous system with $\mathbf{P}$.

$$\begin{cases} \mathbf{a}_x^t \mathbf{P} = 0 \\ \mathbf{a}_y^t \mathbf{P} = 0 \end{cases}$$

with

$$\mathbf{a}_x^t = \begin{bmatrix}-\mathbf{x}^t & \mathbf{0}^t & x'\mathbf{x}^t\end{bmatrix}$$

$$\mathbf{a}_x^t = \begin{bmatrix}-x & -y & -1 & 0 & 0 & 0 & x'x & x'y & x'\end{bmatrix}$$

$$\mathbf{a}_y^t = \begin{bmatrix}\mathbf{0}^t & -\mathbf{x}^t & y'\mathbf{x}^t\end{bmatrix}$$

$$\mathbf{a}_y^t = \begin{bmatrix}0 & 0 & 0 & -x & -y & -1 & y'x & y'y & y'\end{bmatrix}$$

Estimation by Direct Linear Transformation (DLT)

Now let us consider that we have multiple paires of points indexed by i

$$\mathbf{a}_{x_i}^t = \begin{bmatrix} -\mathbf{x}_i^t & \mathbf{0}^t & x_i'\mathbf{x}_i^t \end{bmatrix}$$

$$\mathbf{a}_{y_i}^t = \begin{bmatrix} \mathbf{0}^t & -\mathbf{x}_i^t & y_i'\mathbf{x}_i^t \end{bmatrix}$$

We can rewrite the previous system for the N paires of points :

$$\begin{cases} \mathbf{a}_{x_1}^t \mathbf{P} = 0 \\ \mathbf{a}_{y_1}^t \mathbf{P} = 0 \\ \vdots \\ \mathbf{a}_{x_N}^t \mathbf{P} = 0 \\ \mathbf{a}_{y_N}^t \mathbf{P} = 0 \end{cases}$$

Collecting everything together we have:

M es la matriz A en el TP

$$\underbrace{\mathbf{M}}_{2N\times 9}\underbrace{\mathbf{P}}_{9\times 1} = \underbrace{\mathbf{0}}_{9\times 1}$$

M es lo que nos acaba de dar
P inconnu (Vector formado por los desconocidos)

Estimation by Direct Linear Transformation (DLT)

- if we use $N = 4$ then we have an exact solution
- if we use $N > 4$ then we have an **over-determined solution**. There are no exact solution, hence we need to find approximate solution. Additional constraint needed to avoid 0, e.g. $\|P\|_2^2 = 1$

Estimation of P

In the case of redundant observations we get constradicitons (due to the noise).
Let us write $\mathbf{MP} = \mathbf{w}$.
Our goal is to find $\mathbf{P}$ such that:

$$\hat{\mathbf{P}} = \arg\min_{\mathbf{P}} \mathbf{w}^t\mathbf{w}$$

$$\hat{\mathbf{P}} = \arg\min_{\mathbf{P}} \mathbf{P}^t\mathbf{M}^t\mathbf{MP}$$

with $\|P\|_2^2 = \sum_i p_i^2 = 1$
How do we minimize the loss?

Estimation of P

The eigenvector belonging to the smallest eigenvalue of M solves the system of linear equations.

$$\underbrace{\mathbf{M}}_{2N\times 9} = \underbrace{\mathbf{U}}_{2N\times 9}\underbrace{\mathbf{S}}_{9\times 9}\underbrace{\mathbf{V}}_{9\times 9} = \sum_{i=1}^{9} s_i \mathbf{u}_i \mathbf{v}_i^t$$

with $\mathbf{U}^t\mathbf{U} = \mathbf{I}_9$ and $\mathbf{V}^t\mathbf{V} = \mathbf{I}_9$
The vector v_i are orthonormal since

$$v_i v_j^t = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

So, $\mathbf{P}$ is equal to v_9 with s_9 the smallest eigen value.

El valor propio mas pequeño debe ser realmente pequeño. Esta composicion nos da la fiabilidad de los resultamos que optenemos

S matriz diagonal de valores singulares (valores propios)
V matriz ortonormal donde las columnas seran un vector propio
La ultima columna es la mas chiquita ...

Estimation of P

The estimate of **P** i given by

$$\hat{\mathbf{P}} = \begin{bmatrix} \hat{\mathbf{A}} \\ \hat{\mathbf{B}} \\ \hat{\mathbf{C}} \end{bmatrix} = v_9$$

This leads to the estimated projection matrix.
No solution if all point x are on a line.

DLT algorithm

Objective:
Given $N \geq 4$ 2D to 2D point correspondences $(\mathbf{x}_i, \mathbf{x}'_i)$, determine the 2D homography matrix H such that $\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$.
Algorithm:

- For each correspondence $(\mathbf{x}_i, \mathbf{x}'_i)$ compute $\mathbf{M}_i$. Usually only two first rows needed.
- Assemble N 2×9 matrices $\mathbf{M}_i$ into a single $2N \times 9$ matrix $\mathbf{M}$
- Obtain SVD of $\mathbf{M}$. Solution for h is the last column of $\mathbf{V}$
- Determine $\mathbf{H}$ from h

Estimation of P

$$\mathbf{M}_x^t = \begin{array}{c}\begin{bmatrix} -x & -y & -1 & 0 & 0 & 0 & x'x & x'y & x' \\ 0 & 0 & 0 & -x & -y & -1 & y'x & y'y & y' \end{bmatrix} \\ \begin{array}{ccccccccc} 10^2 & 10^2 & 1 & 10^2 & 10^2 & 1 & 10^4 & 10^4 & 10^2 \end{array}\end{array}$$

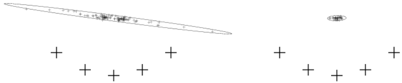
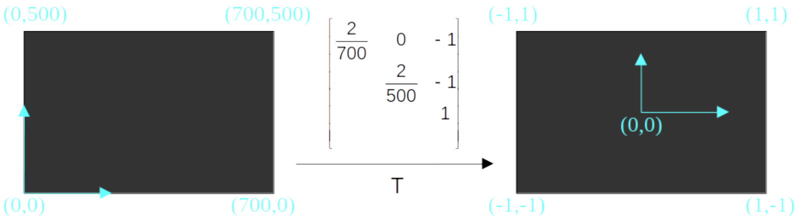


Illustration of distributed errors whose repartition respectively depends, and does depends on the dimensions on the left and the right image.

How do we transform all the coordinates so that the coordinate are between $[-1, 1]$?

Estimation of P



Normalized DLT algorithm

Objective:
Given $N \geq 4$ 2D to 2D point correspondences $(\mathbf{x}_i, \mathbf{x}'_i)$, determine the 2D homography matrix H such that $\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$.
Algorithm:

- Apply the normalization $\tilde{\mathbf{x}}_i = \mathbf{T}_{\text{norm}}\mathbf{x}_i$ and $\tilde{\mathbf{x}}'_i = \mathbf{T}_{\text{norm}}\mathbf{x}'_i$
- apply DLT with $(\tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}'_i)$
- Denomalize the homography: $\mathbf{H} = \mathbf{T}^{-1}_{\text{norm}}\tilde{\mathbf{H}}\mathbf{T}_{\text{norm}}$