Numerical Analysis Homework 3

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Problem I.

Solution.
To satisfy the continuity and smoothness conditions at $x = 1$, the polynomial $p(x)$ must meet the following constraints:

$$p(0) = 0, \quad p(1) = 1, \quad p'(1) = -3, \quad p''(1) = 6.$$

Using Hermite interpolation, we find that

$$\boxed{p(x) = 7x^3 - 18x^2 + 12x}.$$

To determine if $s(x)$ is a natural cubic spline, we check the boundary condition for the second derivative at $x = 0$. For $s(x)$ to be a natural cubic spline, $s''(0)$ must equal zero. However, we compute

$$s''(0) = -36 \neq 0,$$

so $s(x)$ is $\boxed{\textit{not}}$ a natural cubic spline.

Problem II.

(a)

Proof.
Since $p_i = s|_{[x_i, x_{i+1}]} \in \mathbb{P}_2$ for each interval $[x_i, x_{i+1}]$, there are $3(n-1)$ unknown coefficients for $p_1, \ldots, p_{n-1}$. The conditions

$$p_i(x_i) = f_i, \quad p_i(x_{i+1}) = f_{i+1} \quad \text{for } i = 1, \ldots, n-1$$

yield $2(n-1)$ equations. Additionally, the continuity of the first derivative at each interior point x_{i+1} provides

$$p_i'(x_{i+1}) = p_{i+1}'(x_{i+1}) \quad \text{for } i = 1, \ldots, n-2,$$

giving $n-2$ more equations. In total, there are $3(n-1)$ unknowns and $3(n-1)-1$ equations, so one additional condition is needed for a unique solution.

$\boxed{\mathbb{Q.E.D.}}$

(b)

Solution.
Let $p_i(x) = a_i x^2 + b_i x + c_i$ on $[x_i, x_{i+1}]$. The conditions on p_i give:

$$\begin{cases} a_i x_i^2 + b_i x_i + c_i = f_i, \\ a_i x_{i+1}^2 + b_i x_{i+1} + c_i = f_{i+1}, \\ 2a_i x_i + b_i = m_i. \end{cases}$$

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Solving these equations, we find:

$$a_i = \frac{f_{i+1} - f_i}{(x_{i+1} - x_i)^2} - \frac{m_i}{x_{i+1} - x_i},$$
$$b_i = \frac{m_i(x_{i+1} + x_i)}{x_{i+1} - x_i} - \frac{2x_i(f_{i+1} - f_i)}{(x_{i+1} - x_i)^2},$$
$$c_i = f_i + \frac{x_i^2(f_{i+1} - f_i)}{(x_{i+1} - x_i)^2} - \frac{m_i x_i x_{i+1}}{x_{i+1} - x_i}.$$

Finally we get,

$$\boxed{p_i(x) = [\frac{f_{i+1} - f_i}{(x_{i+1} - x_i)^2} - \frac{m_i}{x_{i+1} - x_i}]x^2 + [\frac{m_i(x_{i+1} + x_i)}{x_{i+1} - x_i} - \frac{2x_i(f_{i+1} - f_i)}{(x_{i+1} - x_i)^2}]x + [f_i + \frac{x_i^2(f_{i+1} - f_i)}{(x_{i+1} - x_i)^2} - \frac{m_i x_i x_{i+1}}{x_{i+1} - x_i}]}$$

(c)

Solution.

Starting with p_1 calculated using f_1, f_2, and m_1, we set $m_2 = p_1'(x_2)$.
With m_2, we compute p_2 using f_2, f_3, and set $m_3 = p_2'(x_3)$.
Repeating this process through p_{n-1} with f_{n-1}, f_n, and m_{n-1} completes the calculations.

Problem III.

Solution.

Consider $s_2(x) = \alpha x^3 + \beta x^2 + \gamma x + \theta$. The following conditions must be satisfied:

$$s_2(0) = s_1(0) = 1 + c, \quad s_2'(0) = s_1'(0) = 3c, \quad s_2''(0) = s_1''(0) = 6c, \quad s_2(1) = s(1) = -1, \quad s_2''(1) = 0.$$

These conditions lead to the system:

$$\begin{cases} \theta = 1 + c, \\ \gamma = 3c, \\ 2\beta = 6c, \\ \alpha + \beta + \gamma + \theta = -1, \\ 6\alpha + 2\beta = 0 \end{cases} .$$

Solving this system yields $\boxed{c = -\frac{1}{3}}$.

Problem IV.

Solution.

(a)

To construct the natural cubic spline $s(x)$ that interpolates $f(x) = \cos\left(\frac{\pi}{2}x\right)$ at the knots -1, 0, and 1, we divide the interval $[-1, 1]$ into two segments, $[-1, 0]$ and $[0, 1]$, and define $s(x)$ piecewise over these intervals.

The natural cubic spline $s(x)$ must satisfy the following conditions:

$$s(-1) = f(-1) = 0, \quad s(0) = f(0) = 1, \quad s(1) = f(1) = 0.$$

$$s_1'(0) = s_2'(0).$$

$$s_1''(0) = s_2''(0).$$

$$s''(-1) = 0, \quad s''(1) = 0.$$

Using these conditions, we find that the natural cubic spline interpolant $s(x)$ is given by:

$$s(x) = \begin{cases} -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & \text{for } x \in [-1, 0], \\ \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & \text{for } x \in [0, 1]. \end{cases}$$

(b)

The bending energy of s is calculated as:

$$\int_{-1}^{1} [s''(x)]^2\,dx = \int_{-1}^{0} (-3x-3)^2\,dx + \int_{0}^{1} (3x-3)^2\,dx = 6.$$

The quadratic polynomial interpolating f at -1, 0, and 1 is:

$$p(x) = -x^2 + 1.$$

Its bending energy is:

$$\int_{-1}^{1} [p''(x)]^2\,dx = \int_{-1}^{1} 4\,dx = 8 > 6.$$

The bending energy of f is:

$$\int_{-1}^{1} [f''(x)]^2\,dx = \int_{-1}^{1} \left[-\frac{\pi^2}{4}\cos\left(\frac{\pi}{2}x\right)\right]^2 \approx \frac{\pi^4}{16} \approx 6.0881 > 6.$$

Thus, the natural cubic spline $s(x)$ has minimal bending energy among the functions considered.

Problem V.

Solution.

(a)

See that

$$B_i^1(x) = \begin{cases} \frac{x-t_{i-1}}{t_i-t_{i-1}} & x \in (t_{i-1}, t_i], \\ \frac{t_{i+1}-x}{t_{i+1}-t_i} & x \in (t_i, t_{i+1}], \\ 0 & \text{other}. \end{cases}$$

$$B_{i+1}^1(x) = \begin{cases} \frac{x-t_i}{t_{i+1}-t_i} & x \in (t_i, t_{i+1}], \\ \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} & x \in (t_{i+1}, t_{i+2}], \\ 0 & \text{other}. \end{cases}$$

By the recursive definition, we have

$$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} B_i^1(x) + \frac{t_{i+2}-x}{t_{i+2}-t_i} B_{i+1}^1(x).$$

For $x \in (t_{i-1}, t_i]$,

$$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \cdot \frac{x-t_{i-1}}{t_i-t_{i-1}} + \frac{t_{i+2}-x}{t_{i+2}-t_i} \cdot 0 = \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}.$$

For $x \in (t_i, t_{i+1}]$,

$$B_i^2(x) = \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}.$$

For $x \in (t_{i+1}, t_{i+2}]$,

$$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \cdot 0 + \frac{t_{i+2}-x}{t_{i+2}-t_i} \cdot \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} = \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}.$$

Hence we derived

$$B_i^2(x) = \begin{cases} \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & x \in (t_{i-1}, t_i], \\ \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} & x \in (t_i, t_{i+1}], \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & x \in (t_{i+1}, t_{i+2}], \\ 0 & \text{other}. \end{cases}$$

(b)

We have

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) = \begin{cases} p_1(x) = \frac{2(x-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & x \in (t_{i-1}, t_i], \\ p_2(x) = \frac{t_{i+1}+t_{i-1}-2x}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_{i+2}+t_i-2x}{(t_{i+2}-t_i)(t_{i+1}-t_i)} & x \in (t_i, t_{i+1}], \\ p_3(x) = \frac{2(x-t_{i+2})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & x \in (t_{i+1}, t_{i+2}], \\ 0 & \text{other.} \end{cases}$$

Next, evaluating the derivatives at the boundary points yields:

$$\begin{aligned} p_1(t_i) &= \frac{2(t_i - t_{i-1})}{(t_{i+1}-t_{i-1})(t_i - t_{i-1})} = \frac{2}{t_{i+1}-t_{i-1}}, \\ p_2(t_i) &= \frac{t_{i+1}+t_{i-1}-2t_i}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_{i+2}+t_i-2t_i}{(t_{i+2}-t_i)(t_{i+1}-t_i)} \\ &= \frac{t_{i-1}-t_i}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{1}{t_{i+1}-t_{i-1}} + \frac{1}{t_{i+1}-t_i} \\ &= \frac{2}{t_{i+1}-t_{i-1}} = p_1(t_i). \end{aligned}$$

This shows that $\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x)$ is continuous at t_i.

Similarly, we find:

$$\begin{aligned} p_3(t_{i+1}) &= \frac{2(t_{i+1}-t_{i+2})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} = -\frac{2}{t_{i+2}-t_i}, \\ p_2(t_{i+1}) &= \frac{t_{i+1}+t_{i-1}-2t_{i+1}}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_{i+2}+t_i-2t_{i+1}}{(t_{i+2}-t_i)(t_{i+1}-t_i)} = -\frac{2}{t_{i+2}-t_i} = p_3(t_{i+1}). \end{aligned}$$

Thus, $\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x)$ is also continuous at t_{i+1}.

(c)

It is established that $\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x)$ maintains continuity and is a linear function across the intervals $(t_{i-1}, t_i]$, $(t_i, t_{i+1}]$, and $(t_{i+1}, t_{i+2}]$. We observe:

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(t_{i-1}) = 0, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}B_i^2(t_i) = \frac{2}{t_{i+1}-t_{i-1}} > 0.$$

This implies that within the interval $(t_{i-1}, t_i]$:

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) > 0.$$

In addition, we find:

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(t_{i+1}) = -\frac{2}{t_{i+2}-t_i} < 0.$$

Therefore, by the behavior of linear functions, there exists a unique point $x^* \in (t_i, t_{i+1})$ where $\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x^*) = 0$. This leads to the equation:

$$\frac{t_{i+1}+t_{i-1}-2x^*}{t_{i+1}-t_{i-1}} + \frac{t_{i+2}+t_i-2x^*}{t_{i+2}-t_i} = 0.$$

Solving this equation gives us:

$$x^* = \frac{t_{i+2}t_{i+1} - t_i t_{i-1}}{(t_{i+2}+t_{i+1}) - (t_i + t_{i-1})}.$$

(d)

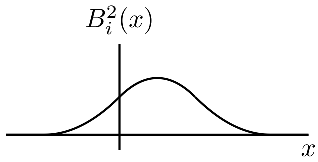
From part (c), we have established:

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) > 0, \quad x \in (t_{i-1}, x^*)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) < 0, \quad x \in (x^*, t_{i+2}).$$

Additionally, we note that $B_i^2(t_{i-1}) = B_i^2(t_{i+2}) = 0$. A simple computation confirms that $B(x^*) < 1$. Therefore, it follows that $B_i^2(x) \in [0, 1)$.

(e)

It is evident that the graphs of $B_i^2(x)$ for different values of i can be generated through translations. Thus, we will illustrate the case for $i = 0$.



Q.E.D.

Problem VI.

Proof.

For $x \in (t_{i-1}, t_i]$, applying Lagrange's interpolation formula yields:

$$\begin{aligned}
[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 =& \frac{(t_i - x)^2}{(t_i - t_{i-1})(t_i - t_{i+1})(t_i - t_{i+2})} + \frac{(t_{i+1} - x)^2}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)(t_{i+1} - t_{i+2})} \\
&+ \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_{i-1})(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} \\
=& \frac{(x - t_{i-1})^2}{(t_{i+2} - t_{i-1})(t_{i+1} - t_{i-1})(t_i - t_{i-1})} = \frac{B_i^2(x)}{t_{i+2} - t_{i-1}}.
\end{aligned}$$

For $x \in (t_i, t_{i+1}]$, we similarly have:

$$\begin{aligned}
[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 =& \frac{(t_{i+1} - x)^2}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)(t_{i+1} - t_{i+2})} + \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_{i-1})(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} \\
=& \frac{B_i^2(x)}{t_{i+2} - t_{i-1}}.
\end{aligned}$$

For $x \in (t_{i+1}, t_{i+2}]$, we find:

$$\begin{aligned}
[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 =& \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_{i-1})(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} \\
=& \frac{B_i^2(x)}{t_{i+2} - t_{i-1}}.
\end{aligned}$$

Thus, we have confirmed that

$$(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = B_i^2(x)$$

holds true within the support of $B_i^2(x)$. Moreover, this equation is evidently valid even when $B_i^2(x)$ is equal to zero.

Q.E.D.

Problem VII.

Solution.

According to the theorem on B-spline derivatives, we have:

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^{n+1}(x) = \frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}} - \frac{(n+1)B_{i+1}^n(x)}{t_{i+n+1}-t_i}, \quad n \geq 1$$

Integrating both sides results in:

$$\int_{t_{i-1}}^{t_{i+n+1}} \frac{\mathrm{d}}{\mathrm{d}x}B_i^{n+1}(x)\,\mathrm{d}x = \int_{t_{i-1}}^{t_{i+n+1}} \left(\frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}} - \frac{(n+1)B_{i+1}^n(x)}{t_{i+n+1}-t_i}\right)\mathrm{d}x$$

For the left-hand side, we have:

$$\text{LHS} = B_i^{n+1}(t_{i+n+1}) - B_i^{n+1}(t_{i-1}) = 0.$$

For the right-hand side, this yields:

$$\text{RHS} = (n+1)\left(\int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n}-t_{i-1}}\,\mathrm{d}x - \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n(x)}{t_{i+n+1}-t_i}\,\mathrm{d}x\right).$$

Therefore, we obtain:

$$\int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n}-t_{i-1}}\,\mathrm{d}x = \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n(x)}{t_{i+n+1}-t_i}\,\mathrm{d}x.$$

Consequently, the scaled integral of $B_i^n(x)$ across its support is independent of the index i.

Problem VIII.

Solution.

(a)

Firstly, we have:

$$\tau_2(x_i, x_{i+1}, x_{i+2}) = x_i^2 + x_{i+1}^2 + x_{i+2}^2 + x_i x_{i+1} + x_i x_{i+2} + x_{i+1}x_{i+2}.$$

Next, we can create a divided difference table as follows:

$$\begin{array}{l|lll}
x_i & x_i^4 & & \\
x_{i+1} & x_{i+1}^4 & (x_{i+1}^2+x_i^2)(x_{i+1}+x_i) & \\
x_{i+2} & x_{i+2}^4 & (x_{i+2}^2+x_{i+1}^2)(x_{i+2}+x_{i+1}) & \frac{(x_{i+2}^2+x_{i+1}^2)(x_{i+2}+x_{i+1})-(x_{i+1}^2+x_i^2)(x_{i+1}+x_i)}{x_{i+2}-x_i}
\end{array}$$

From the above, we derive:

$$\begin{aligned}
&\frac{(x_{i+2}^2+x_{i+1}^2)(x_{i+2}+x_{i+1})-(x_{i+1}^2+x_i^2)(x_{i+1}+x_i)}{x_{i+2}-x_i}\\
=&\frac{(x_{i+2}^3-x_i^3)+x_{i+1}(x_{i+2}^2-x_i^2)+x_{i+1}^2(x_{i+2}-x_i)}{x_{i+2}-x_i}\\
=&(x_{i+2}^2+x_{i+2}x_i+x_i^2)+x_{i+1}(x_{i+2}+x_i)+x_{i+1}^2\\
=&\tau_2(x_i, x_{i+1}, x_{i+2}).
\end{aligned}$$

(b)

According to the properties of complete symmetric polynomials, we can express:

$$\begin{aligned}
&(x_{i+n+1} - x_i)\sigma_{m-n-1}(x_i, \dots, x_{i+n+1}) \\
=&\sigma_{m-n}(x_i, \dots, x_{i+n+1}) - \sigma_{m-n}(x_i, \dots, x_{i+n}) - x_i\sigma_{m-n-1}(x_i, \dots, x_{i+n+1}) \\
=&\sigma_{m-n}(x_{i+1}, \dots, x_{i+n+1}) + x_i\sigma_{m-n-1}(x_i, \dots, x_{i+n+1}) - \sigma_{m-n}(x_i, \dots, x_{i+n}) - x_i\sigma_{m-n-1}(x_i, \dots, x_{i+n+1}) \\
=&\sigma_{m-n}(x_{i+1}, \dots, x_{i+n+1}) - \sigma_{m-n}(x_i, \dots, x_{i+n}).
\end{aligned}$$

Next, we will demonstrate the theorem using mathematical induction. When $n = 0$, it is clear that:

$$\sigma_m(x_i) = [x_i]x^m = x_i^m.$$

Assuming the theorem holds for some $0 \leq n < m$, we now consider the case for $n + 1$:

$$\begin{aligned}
\sigma_{m-n-1}(x_i, \dots, x_{i+n+1}) &= \frac{\sigma_{m-n}(x_{i+1}, \dots, x_{i+n+1}) - \sigma_{m-n}(x_i, \dots, x_{i+n})}{x_{i+n+1} - x_i} \\
&= \frac{[x_{i+1}, \dots, x_{i+n+1}]x^m - [x_i, \dots, x_{i+n}]}{x_{i+n+1} - x_i} \\
&= [x_i, \dots, x_{i+n+1}]x^m.
\end{aligned}$$

This completes the proof of the theorem via induction.

References

- handoutsNumPDEs
- ChatGPT, *AI Language Model*, OpenAI Platform, 2024.