Numerical Analysis Homework 1

Chen Wanqi 3220102895 *

Information and Computer Science 2201, Zhejiang University

Due time: September 20, 2024

I. Consider the bisection method starting with the initial interval [1.5, 3.5].

I-a.

Width of the interval at the nth step

The bisection method halves the interval at each step.

For the interval $[1.5, 3.5]$, the initial width is $3.5 - 1.5 = 2$. Therefore, the width at the n-th step is:

$$W_n = \frac{2}{2^n} = \frac{1}{2^{n-1}}$$

I-b.

Supremum of the distance between the root r and the midpoint of the interval

The distance between the root r and the midpoint of the interval after n steps, denoted as D_n, is always less than or equal to half the width of the interval. In the case of the interval $[1.5, 3.5]$, this becomes:

$$D_n = \frac{2}{2^{n+1}} = \frac{1}{2^n}$$

II. Proof of accuracy with relative error ϵ

We want to determine the number of steps n such that the relative error of the approximation to the root is no greater than ϵ. Specifically, we need to show that this goal is achieved if:

$$n \geq \frac{\log(b_0 - a_0) - \log(\epsilon) - \log(a_0)}{\log(2)} - 1$$

Proof:

In the bisection method, the width of the interval after n steps is given by:

$$W_n = \frac{b_0 - a_0}{2^n}$$

This width bounds the absolute error of the root approximation. The midpoint of the interval is the best approximation of the root, and the error in this approximation is at most half the interval width:

$$\mathrm{E}_n = \frac{W_n}{2} = \frac{b_0 - a_0}{2^{n+1}}$$

To achieve a relative error no greater than ϵ, the following condition must hold:

$$\frac{\mathrm{E}_n}{r} \leq \epsilon$$

*Electronic address: 3220102895@zju.edu.cn

Since the root r lies in the interval $[a_0, b_0]$, and $r \geq a_0$, we have:

$$\frac{\frac{b_0-a_0}{2^{n+1}}}{a_0} \leq \epsilon$$

Simplifying this:

$$\frac{b_0 - a_0}{2^{n+1}a_0} \leq \epsilon$$

Multiplying both sides by $2^{n+1}a_0$:

$$b_0 - a_0 \leq 2^{n+1}\epsilon a_0$$

Taking the logarithm of both sides:

$$\log(b_0 - a_0) \leq \log(2^{n+1}\epsilon a_0)$$

Using logarithm properties:

$$\log(b_0 - a_0) \leq (n + 1)\log(2) + \log(\epsilon) + \log(a_0)$$

Rearranging to solve for n:

$$n + 1 \geq \frac{\log(b_0 - a_0) - \log(\epsilon) - \log(a_0)}{\log(2)}$$

Subtracting 1 from both sides gives:

$$n \geq \frac{\log(b_0 - a_0) - \log(\epsilon) - \log(a_0)}{\log(2)} - 1$$

Thus, the number of steps $n \geq \frac{\log(b_0-a_0)-\log(\epsilon)-\log(a_0)}{\log(2)} - 1$ guarantees that the relative error in the root approximation is no greater than ϵ.

III. Perform four iterations of Newton's method for the polynomial equation

The polynomial equation is $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$.

Newton's method is defined by the iterative formula:

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}$$

Given the polynomial $p(x) = 4x^3 - 2x^2 + 3$, its derivative is:

$$p'(x) = 12x^2 - 4x$$

Starting with $x_0 = -1$, we apply the Newton's method formula for four iterations:

$$x_{n+1} = x_n - \frac{4x_n^3 - 2x_n^2 + 3}{12x_n^2 - 4x_n}$$

The iterations are organized in the following table:

n	x_n	$p(x_n)$	$p'(x_n)$
0	-1.00000	-3.00000	16.00000
1	-0.81250	-0.46582	11.17188
2	-0.77080	-0.02014	10.21289
3	-0.76883	-0.00004	10.16857
4	-0.76883	-0.00000	10.16847

IV. Consider a variation of Newton's method

In this variation of Newton's method, only the derivative at x_0 is used for all iterations. The update rule is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}$$

Let α be the true root of the function $f(x)$. The error at step n is defined as:

$$e_n = x_n - \alpha$$

To analyze the convergence behavior of this method, we aim to find constants C and s such that:

$$e_{n+1} = Ce_n^s$$

where e_{n+1} is the error at step $n+1$, s is a constant, and C may depend on x_n, the true root α, and the derivative of $f(x)$.

Solve:

Assume $f(x)$ is sufficiently smooth and can be expanded in a Taylor series around α. Thus, for x_n near α, we can write:

$$f(x_n) = f(\alpha) + f'(\alpha)(x_n - \alpha) + \frac{f''(\alpha)}{2}(x_n - \alpha)^2 + O((x_n - \alpha)^3)$$

Since α is the true root, we know that $f(\alpha) = 0$, so this simplifies to:

$$f(x_n) = f'(\alpha)(x_n - \alpha) + \frac{f''(\alpha)}{2}(x_n - \alpha)^2 + O((x_n - \alpha)^3)$$

Using the modified Newton's method iteration rule:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}$$

Substitute the Taylor expansion of $f(x_n)$:

$$x_{n+1} = x_n - \frac{f'(\alpha)(x_n - \alpha) + \frac{f''(\alpha)}{2}(x_n - \alpha)^2 + O(x_n - \alpha)^3}{f'(x_0)}$$

Let $e_n = x_n - \alpha$. Then the update equation becomes:

$$e_{n+1} = e_n - \frac{f'(\alpha)e_n + \frac{f''(\alpha)}{2}e_n^2 + O(e_n)^3}{f'(x_0)}$$

For small e_n, the quadratic term e_n^2 dominates. Therefore, the error at step $n+1$ can be approximated by:

$$e_{n+1} \approx -\frac{f''(\alpha)}{2f'(x_0)}e_n^2$$

This shows that the error follows a quadratic convergence pattern, with:

$$e_{n+1} = Ce_n^2$$

where $C = -\frac{f''(\alpha)}{2f'(x_0)}$.

Acknowledgement

I would like to express my gratitude to my course instructor for their guidance and the resources provided during the course.

it is acceptable to put a table here for your references.