Numerical Analysis Homework 2

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Problem I.

Solution:

First, recall that the linear interpolation $p_1(f;x)$ at points x_0 and x_1 is given by Newton's formula:

$$p_1(f;x) = f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0),$$

where $f_0 = f(x_0) = 1$ and $f_1 = f(x_1) = \frac{1}{2}$. Substituting the values into the formula, we get:

$$p_1(f;x) = 1 + \frac{\frac{1}{2} - 1}{2 - 1}(x - 1) = 1 - \frac{1}{2}(x - 1) = -\frac{1}{2}x + \frac{3}{2}.$$

Next, the second derivative of $f(x)$ is:

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}.$$

Substituting these into the interpolation formula:

$$\frac{1}{x} - \left(-\frac{1}{2}x + \frac{3}{2}\right) = \frac{f''(\xi(x))}{2}(x-1)(x-2),$$

we simplify the left-hand side:

$$\frac{1}{x} - \left(-\frac{1}{2}x + \frac{3}{2}\right) = \frac{2-x}{2x}.$$

Thus, the interpolation formula becomes:

$$\frac{2-x}{2x} = \frac{1}{\xi(x)^3}(x-1)(x-2).$$

Finally, we obtain:

$$\boxed{\xi(x) = \sqrt[3]{2x}.}$$

Since $\xi(x)$ is a monotonically increasing function on the interval $x \in [1, 2]$, we can find the minimum and maximum values of $\xi(x)$ on this interval:

$$\boxed{\min \xi(x) = \xi(1) = \sqrt[3]{2}}, \quad \boxed{\max \xi(x) = \xi(2) = \sqrt[3]{4}.}$$

Finally, the maximum value of $f''(\xi(x))$ occurs when $\xi(x)$ is minimized. Since $f''(x) = \frac{2}{x^3}$, we find:

$$\max f''(\xi(x)) = f''(\min \xi(x)) = f''\left(\sqrt[3]{2}\right) = \frac{2}{\left(\sqrt[3]{2}\right)^3} = 1.$$

Thus, $\boxed{\max f''(\xi(x)) = 1}$.

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Problem II.

Solution:
Let

$$\ell_k(x) = \prod_{i=0, i\neq k}^{n} \frac{(x-x_i)^2}{(x_k-x_i)^2}.$$

Clearly, $\ell_k(x) \in \mathcal{P}_{2n}^{+}$. And for every $i \neq k$, we have $\ell_k(x_i) = 0$ and $\ell_k(x_k) = 1$.
Let

$$P(x) = \sum_{k=0}^{n} f_k \ell_k(x),$$

where $P(x) \in \mathcal{P}_{2n}^{+}$, and we can check that $p(x_i) = f_i$, for $i = 1, 2, \ldots, n$.

$\boxed{\mathbb{Q}.\mathbb{E}.\mathbb{D}.}$

Problem III.

Solution:
For $n = 0$, clearly we have:

$$f[t] = e^t.$$

Assume the statement holds for $n-1$. By the induction hypothesis, we have:

$$f[t, t+1, \ldots, t+n] = \frac{f[t+1, t+2, \ldots, t+n] - f[t, t+1, \ldots, t+n-1]}{n}.$$

Substituting the inductive assumption:

$$f[t+1, t+2, \ldots, t+n] = \frac{(e-1)^{n-1}}{(n-1)!} e^{t+1}, \quad f[t, t+1, \ldots, t+n-1] = \frac{(e-1)^{n-1}}{(n-1)!} e^t,$$

we get:

$$f[t, t+1, \ldots, t+n] = \frac{\frac{(e-1)^{n-1}}{(n-1)!} e^{t+1} - \frac{(e-1)^{n-1}}{(n-1)!} e^t}{n} = \frac{(e-1)^n}{n!} e^t.$$

Thus, the formula holds for all n by induction.

$\boxed{\mathbb{Q}.\mathbb{E}.\mathbb{D}.}$

Now, substitute $t = 0$, we have:

$$f[0, 1, \ldots, n] = \frac{(e-1)^n}{n!}.$$

From Corollary 2.22, we know that:

$$f[0, 1, \ldots, n] = \frac{f^{(n)}(\xi)}{n!},$$

which implies:

$$\xi = n \ln(e-1) \approx 0.541n.$$

Since $0.541n > \frac{n}{2}$, ξ is located to the $\boxed{\textit{right}}$ of the midpoint $\frac{n}{2}$.

Problem IV.

Solution: The table of divided differences is as follows:

0	5			
1	3	-2		
3	5	1	1	
4	12	7	2	$\frac{1}{4}$

By Newton's formula, we have:

$$p_3(f; x) = 5 - 2x + (x)(x-1) + \frac{1}{4}(x)(x-1)(x-3) = \boxed{0.25x^3 - 2.25x + 5}.$$

To find the approximate location of the minimum x_{min}, we differentiate $p_3(f;x)$:

$$p_3'(f;x) = 0.75x^2 - 2.25$$

Setting $p_3'(f;x) = 0$ gives:

$$\frac{3}{4}(x^2 - 3) = 0$$

Thus, solving for x, we find:

$$x = \sqrt{3} \quad \text{(since } x \text{ is limited within } (1,3)\text{).}$$

So the approximate minimum value is:

$$\boxed{x_{\text{min}} \approx \sqrt{3}}.$$

Problem V.

Solution:

Make the following table:

$$\begin{array}{c|cccccc}
0 & 0 \\
1 & 1 & 1 \\
1 & 1 & 7 & 6 \\
1 & 1 & 7 & 21 & 15 \\
2 & 128 & 127 & 120 & 99 & 42 \\
2 & 128 & 448 & 321 & 201 & 102 & 30
\end{array}$$

Thus, we have $\boxed{f[0,1,1,1,2,2] = 30}$.

Next, we know that this divided difference can be expressed in terms of the 5th derivative of f evaluated at some $\xi \in (0,2)$.

Since $f(x) = x^7$, we compute the 5th derivative:

$$f^{(5)}(x) = 7 \times 6 \times 5 \times 4 \times 3 \times x^2 = 2520x^2.$$

Now we set up the equation:

$$2520\xi^2 = 30.$$

Solving for ξ^2:

$$\xi^2 = \frac{30}{2520} = \frac{1}{84}.$$

Taking the square root:

$$\xi = \sqrt{\frac{1}{84}} \approx 0.1091.$$

Thus, we find that $\boxed{\xi \approx 0.1091}$, which is located in the interval $(0,2)$.

Problem VI.

Solution:

The divided differences table is constructed as follows:

$$\begin{array}{c|ccccc}
0 & 0 \\
1 & 2 & 1 \\
1 & 2 & -1 & 2 \\
3 & 0 & -1 & 0 & \frac{2}{3} \\
3 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{5}{36}
\end{array}$$

This leads us to the Hermite polynomial expressed as:

$$p(x) = 1 + x - 2x(x-1) + \frac{2}{3}x(x-1)^2 - \frac{5}{36}x(x-1)^2(x-3).$$

Thus, we can approximate $f(2)$ using $p(2)$, yielding:

$$\boxed{f(2) \approx p(2) = \frac{11}{18}}.$$

According to Theorem 2.37, when substituting $x = 2$, we obtain:

$$\boxed{|f(x) - p(x)| = \left|\frac{f^{(5)}(\xi)}{5!}x(x-1)^2(x-3)^2\right| = \left|\frac{f^{(5)}(\xi)}{60}\right| \leq \frac{M}{60}}.$$

Problem VII.

Proof:

When $n = 1$, it is clear that

$$\Delta^1 f(x) = \Delta f(x) = f(x+h) - f(x) = 1!h^1 \frac{f(x+h) - f(x)}{h} = 1!h^1 f[x, x+h].$$

Assuming that the statement holds for $n = k$ where $k \in \mathbb{N}^*$, i.e.,

$$\Delta^k f(x) = k!h^k f[x, x+h, \ldots, x+kh],$$

we can derive the case for $n = k+1$:

$$\begin{aligned}
\Delta^{k+1} f(x) &= \Delta^k f(x+h) - \Delta^k f(x) \\
&= k!h^k f[x+h, x+2h, \ldots, x+(k+1)h] - k!h^k f[x, x+h, \ldots, x+kh] \\
&= k!h^k \left(f[x+h, x+2h, \ldots, x+(k+1)h] - f[x, x+h, \ldots, x+kh]\right) \\
&= k!h^k (k+1)hf[x, x+h, \ldots, x+(k+1)h] \\
&= (k+1)!h^{k+1} f[x, x+h, \ldots, x+(k+1)h].
\end{aligned}$$

Thus, when $n = k+1$, the result holds. By mathematical induction, we conclude that

$$\Delta^n f(x) = n!h^n f[x, x+h, \ldots, x+nh].$$

Similarly, when $n = 1$, it is clear that

$$\nabla^1 f(x) = \nabla f(x) = f(x) - f(x-h) = 1!h^1 \frac{f(x) - f(x-h)}{h} = 1!h^1 f[x, x-h].$$

Assuming that the statement holds for $n = k$ where $k \in \mathbb{N}^*$, i.e.,

$$\nabla^k f(x) = k!h^k f[x, x-h, \ldots, x-kh],$$

we can similarly derive the case for $n = k+1$:

$$\begin{aligned}
\nabla^{k+1} f(x) &= \nabla^k f(x) - \nabla^k f(x-h) \\
&= k!h^k f[x, x-h, \ldots, x-kh] - k!h^k f[x-h, x-2h, \ldots, x-(k+1)h] \\
&= k!h^k \left(f[x, x-h, \ldots, x-kh] - f[x-h, x-2h, \ldots, x-(k+1)h]\right) \\
&= k!h^k (k+1)hf[x, x-h, \ldots, x-(k+1)h] \\
&= (k+1)!h^{k+1} f[x, x-h, \ldots, x-(k+1)h].
\end{aligned}$$

Thus, when $n = k+1$, the result holds. By mathematical induction, we conclude that

$$\nabla^n f(x) = n!h^n f[x, x-h, \ldots, x-nh].$$

$\boxed{\mathbb{Q}.\mathbb{E}.\mathbb{D}.}$

Problem VIII.

Proof:

We start by using the definition of divided differences and their continuity:

$$\begin{aligned}\frac{\partial}{\partial x_0} f[x_0, x_1, \ldots, x_n] &= \lim_{h \to 0} \frac{f[x_0 + h, x_1, \ldots, x_n] - f[x_0, x_1, \ldots, x_n]}{h} \\ &= \lim_{h \to 0} f[x_0, x_0 + h, x_1, \ldots, x_n] \\ &= f[x_0, x_0, x_1, \ldots, x_n].\end{aligned}$$

This shows that taking the partial derivative of the divided difference with respect to x_0 introduces an additional x_0 into the sequence.

For the partial derivative with respect to any other variable, say x_i for $i \neq 0$, the result follows a similar logic due to the symmetry of divided differences:

$$\frac{\partial}{\partial x_i} f[x_0, x_1, \ldots, x_n] = f[x_0, \ldots, x_i, x_i, \ldots, x_n].$$

This means that differentiating with respect to any variable x_i duplicates that variable within the divided difference expression.

$\boxed{\mathbb{Q}.\mathbb{E}.\mathbb{D}.}$

Problem IX.

Proof:

We are tasked with solving the following min-max problem. For $n \in \mathbb{N}^+$ and a fixed $a_0 \neq 0$, we need to find:

$$\min_{\{a_i \in \mathbb{R}\}} \max_{x \in [a,b]} \left| a_0 x^n + a_1 x^{n-1} + \cdots + a_n \right|.$$

We start by making the substitution:

$$x = \frac{b-a}{2} x' + \frac{a+b}{2},$$

so that $x' \in [-1, 1]$. This transforms the original expression into:

$$\min \max_{x' \in [-1,1]} \left| a_0' x'^n + \cdots + a_n' \right|.$$

From this, using Corollary 2.47, we deduce the solution to be:

$$\min \max_{x \in [a,b]} |a_0 x^n + \cdots + a_n| = \frac{1}{2^{n-1}} |a_0|.$$

$\boxed{\mathbb{Q}.\mathbb{E}.\mathbb{D}.}$

Problem X.

Proof:

We are asked to prove that for the rescaled Chebyshev polynomial $\hat{p}_n(x)$, the following inequality holds:

$$\forall p \in P_n^a, \quad \|\hat{p}_n\|_\infty \leq \|p\|_\infty.$$

We begin by considering the infinity norm of the Chebyshev polynomial:

$$\|P_n(z)\|_\infty = \frac{|f_n(z)|_\infty}{|T_n(x)|_\infty} = \frac{1}{|T_n(x)|}.$$

Suppose, for contradiction, that there exists a polynomial P such that $\|P\|_\infty < \|P_n\|_\infty$. This would imply:

$$\|P\|_\infty < \frac{1}{|T_n(x)|}.$$

Define the difference between P and P_n as:

$$r(n) = P(n) - P_n(n),$$

which gives:

$$r(x) = P(x) - P_n(x).$$

Since $r(x) = 0$ at $n+1$ points, this leads to a contradiction. Hence, for all $P \in P_n^a$, we conclude:

$$\|\hat{p}_n\|_\infty \leq \|p\|_\infty.$$

$\boxed{\mathbb{Q}.\mathbb{E}.\mathbb{D}.}$

Problem XI.

Proof:

The Bernstein base polynomial $b_{n-1,k}(t)$ can be expressed as a linear combination of Bernstein polynomials of higher degrees. We begin by recalling the recursive definition of Bernstein polynomials:

$$b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad t \in [0,1].$$

Next, we expand $b_{n-1,k}(t)$ as follows:

$$b_{n-1,k}(t) = \binom{n-1}{k} t^k (1-t)^{n-1-k}.$$

By using the identity for binomial coefficients:

$$\binom{n-1}{k} = \frac{n-k}{n}\binom{n}{k},$$

we rewrite $b_{n-1,k}(t)$ as:

$$b_{n-1,k}(t) = \frac{n-k}{n} b_{n,k}(t).$$

Similarly, for $b_{n,k+1}(t)$, we use:

$$\binom{n-1}{k+1} = \frac{k+1}{n}\binom{n}{k+1},$$

which gives:

$$b_{n-1,k}(t) = \frac{n-k}{n} b_{n,k}(t) + \frac{k+1}{n} b_{n,k+1}(t).$$

Thus, we have expressed $b_{n-1,k}(t)$ as a linear combination of $b_{n,k}(t)$ and $b_{n,k+1}(t)$, completing the proof:

$$b_{n-1,k}(t) = \frac{n-k}{n} b_{n,k}(t) + \frac{k+1}{n} b_{n,k+1}(t).$$

$\boxed{\mathbb{Q}.\mathbb{E}.\mathbb{D}.}$

Problem XII.

Proof:

We are tasked with proving that the integral of a Bernstein base polynomial $b_{n,k}(t)$ over the interval $[0,1]$ depends only on its degree n, i.e.,

$$\forall k = 0, 1, \ldots, n, \quad \int_0^1 b_{n,k}(t)\, dt = \frac{1}{n+1}.$$

Recall that the Bernstein base polynomial is defined as:

$$b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad t \in [0,1],\ k = 0, 1, \ldots, n.$$

We aim to compute the integral:

$$I_{n,k} = \int_0^1 b_{n,k}(t)\,dt = \int_0^1 \binom{n}{k} t^k (1-t)^{n-k}\,dt.$$

Since the binomial coefficient $\binom{n}{k}$ is constant with respect to t, we factor it out of the integral:

$$I_{n,k} = \binom{n}{k} \int_0^1 t^k (1-t)^{n-k}\,dt.$$

This integral is a standard Beta function, $B(k+1, n-k+1)$, which is defined as:

$$B(k+1, n-k+1) = \int_0^1 t^k (1-t)^{n-k}\,dt.$$

The Beta function has the following property in terms of Gamma functions:

$$B(k+1, n-k+1) = \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)}.$$

Since $\Gamma(k+1) = k!$ and $\Gamma(n-k+1) = (n-k)!$, we get:

$$B(k+1, n-k+1) = \frac{k!(n-k)!}{(n+1)!}.$$

Thus, the integral becomes:

$$I_{n,k} = \binom{n}{k} \cdot \frac{k!(n-k)!}{(n+1)!}.$$

Simplifying using $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, we obtain:

$$I_{n,k} = \frac{n!}{k!(n-k)!} \cdot \frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1}.$$

Hence, the integral of a Bernstein base polynomial is independent of k and is given by:

$$\int_0^1 b_{n,k}(t)\,dt = \frac{1}{n+1}, \quad k = 0, 1, \ldots, n.$$

$\boxed{\mathbb{Q}.\mathbb{E}.\mathbb{D}.}$

References

- handoutsNumPDEs
- ChatGPT, AI Language Model, OpenAI Platform, 2024.