ALERT Geomaterials School 2016

Fundamentals of bifurcation theory and stability analysis

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Objectives

- Understand what we call bifurcation of a (dynamical) system.
- Understand the concept of stability.
- Gather different concepts under ONE theoretical framework.
- Perform bifurcation and stability analysis of simple systems.
- Understand strain localization as a bifurcation and stability problem.

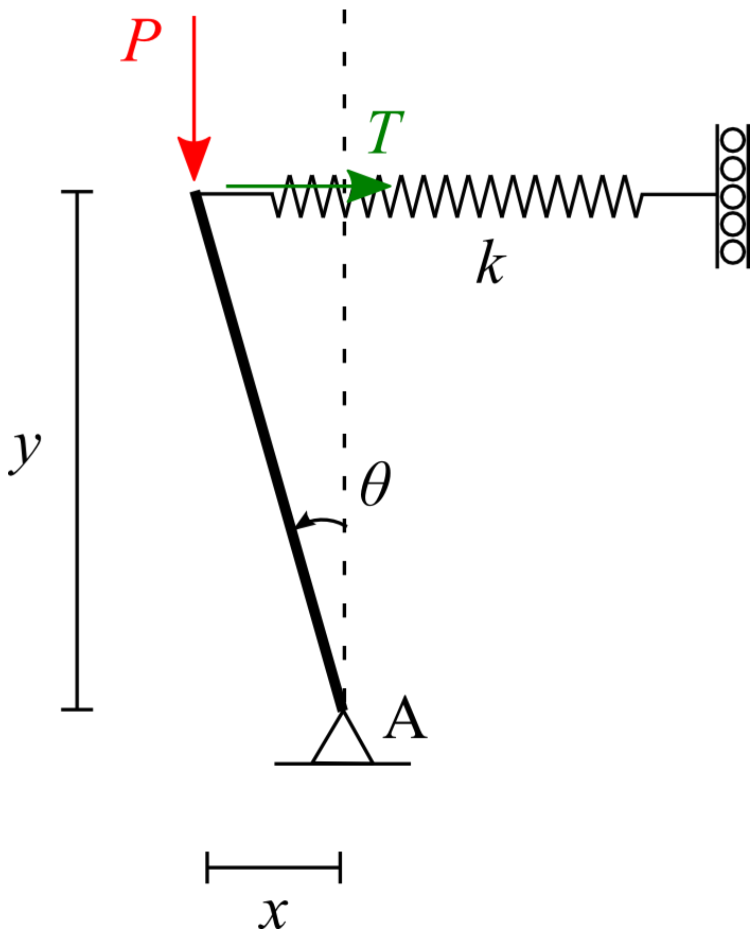
Prerequisites

- Ordinary Differential Equations (ODE's), notions
- Partial Differential Equations (PDE's), notions
- A bit of tensor calculus for calculus
- Studying…

Basic concepts

Exercise #1

-> *Find all the equilibrium points (angles ϑ) of the system*:



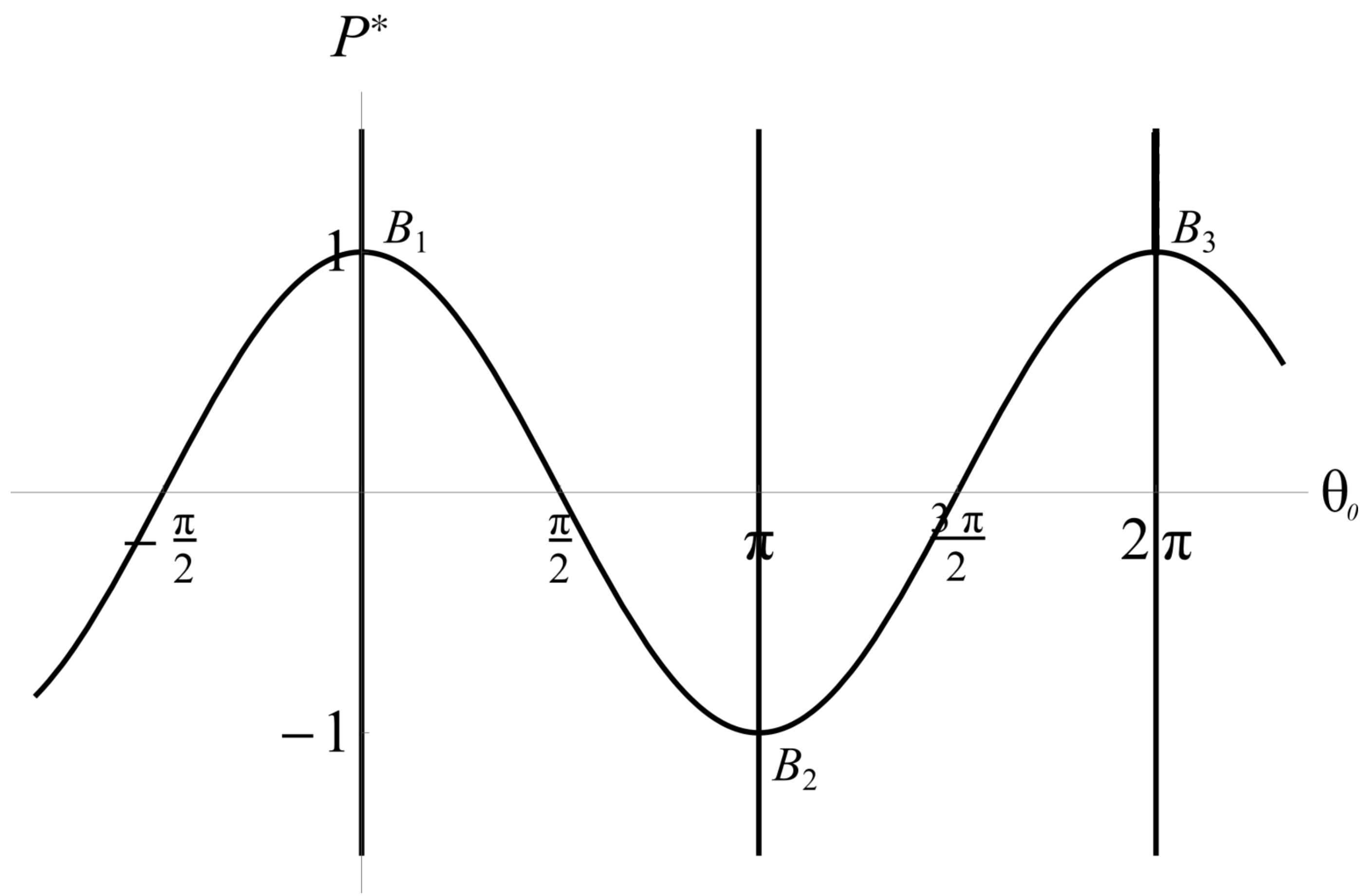
$$I_A \ddot{\theta} = \Sigma M_A = P\,x - T\,y$$

$$T = k\,x$$

$$x = \ell \sin\theta$$

$$y = \ell \cos\theta$$

Equilibrium diagram



It is called also **bifurcation diagram** because at points B_1, B_2, B_3... the equilibrium diagram bifurcates!


A 4
E 50 E 25
LUXEMBOURG
NANCY
PARIS
A 314
METZ
Bifurcation de
Lauvallières
2000m

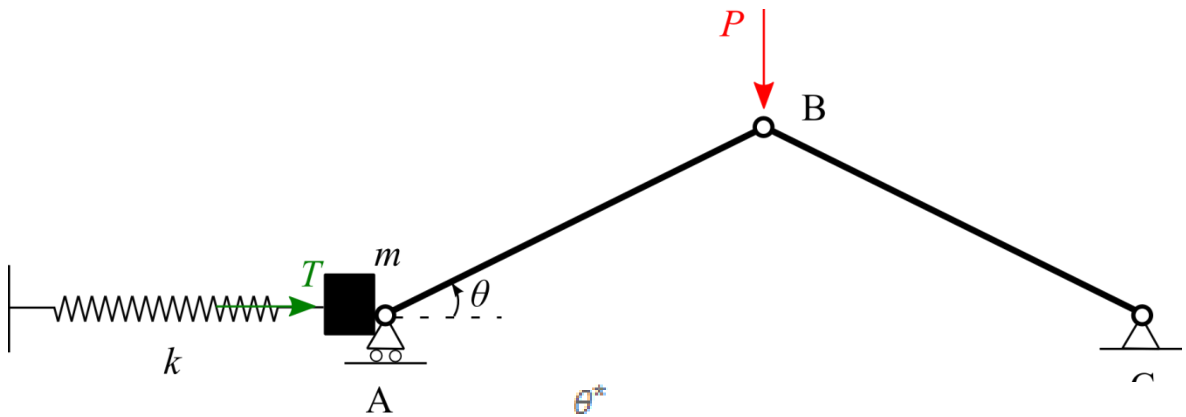
Bifurcation diagram

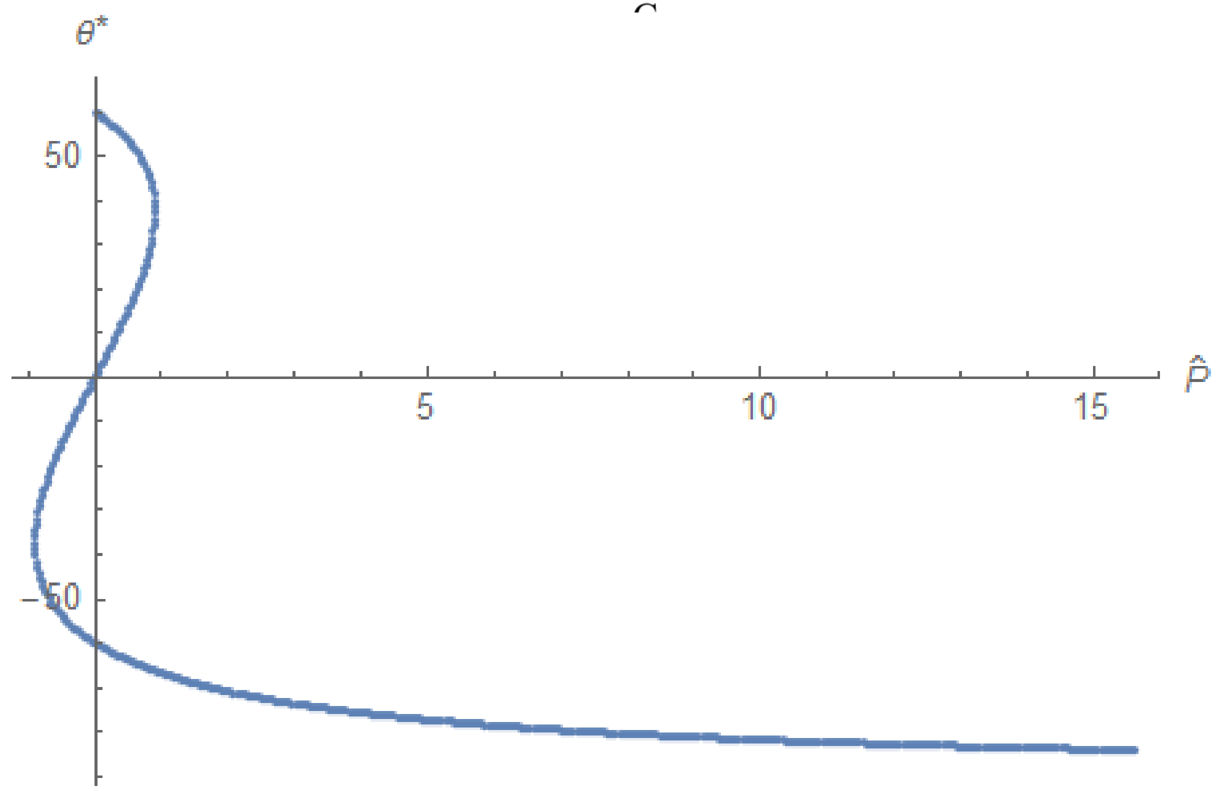
The **diagram of the steady state (or equilibrium) solutions** of a dynamical system in terms of one or more parameters is usually called bifurcation diagram.

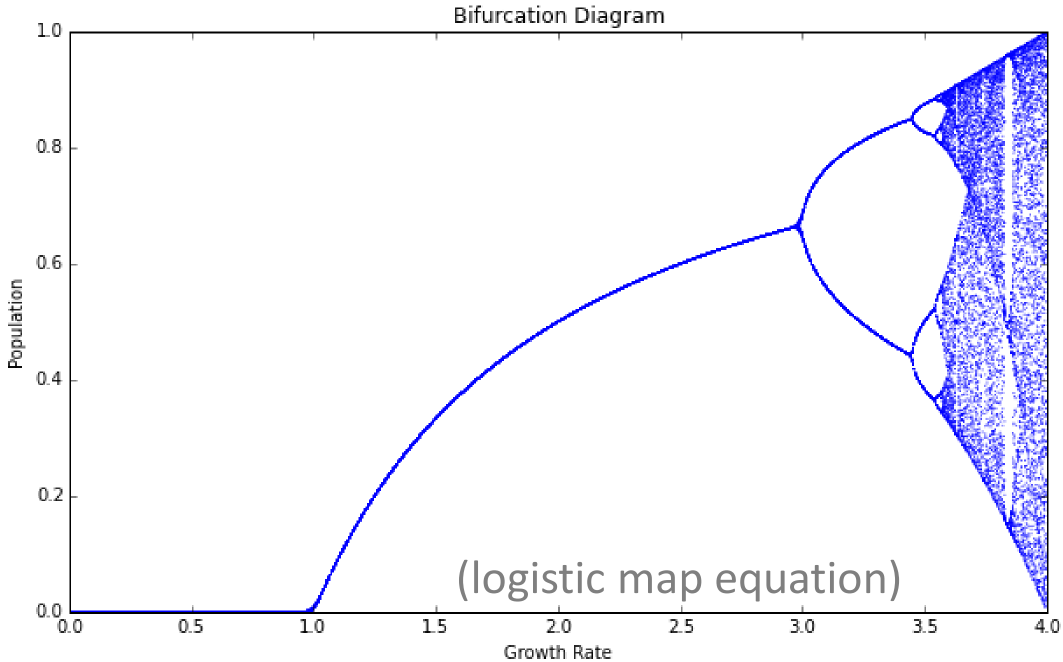
These parameters are called **bifurcation parameters**.

It is important because it represents the appearance (or disappearance) of a *qualitatively different* (equilibrium) solution for a nonlinear system as some parameter is varied.

Another example:

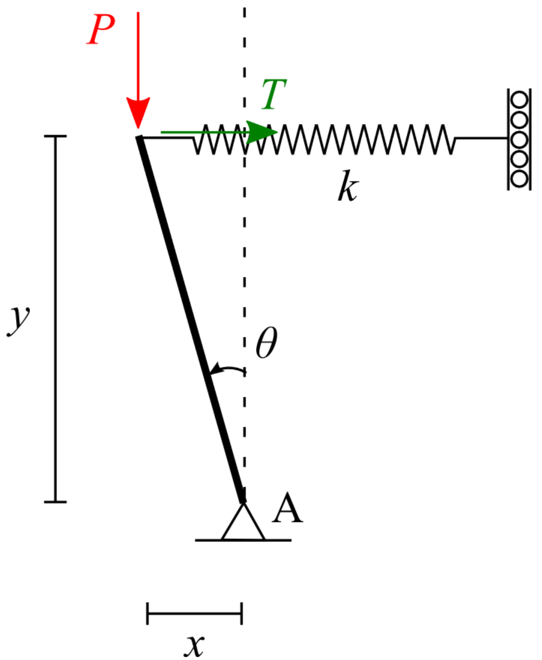


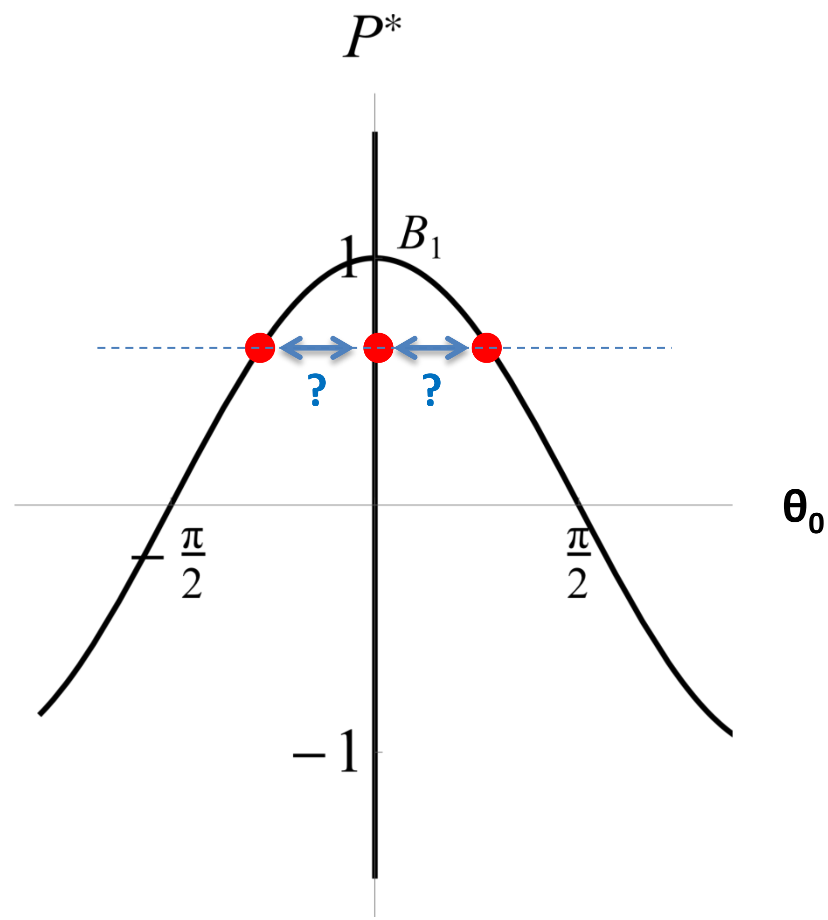


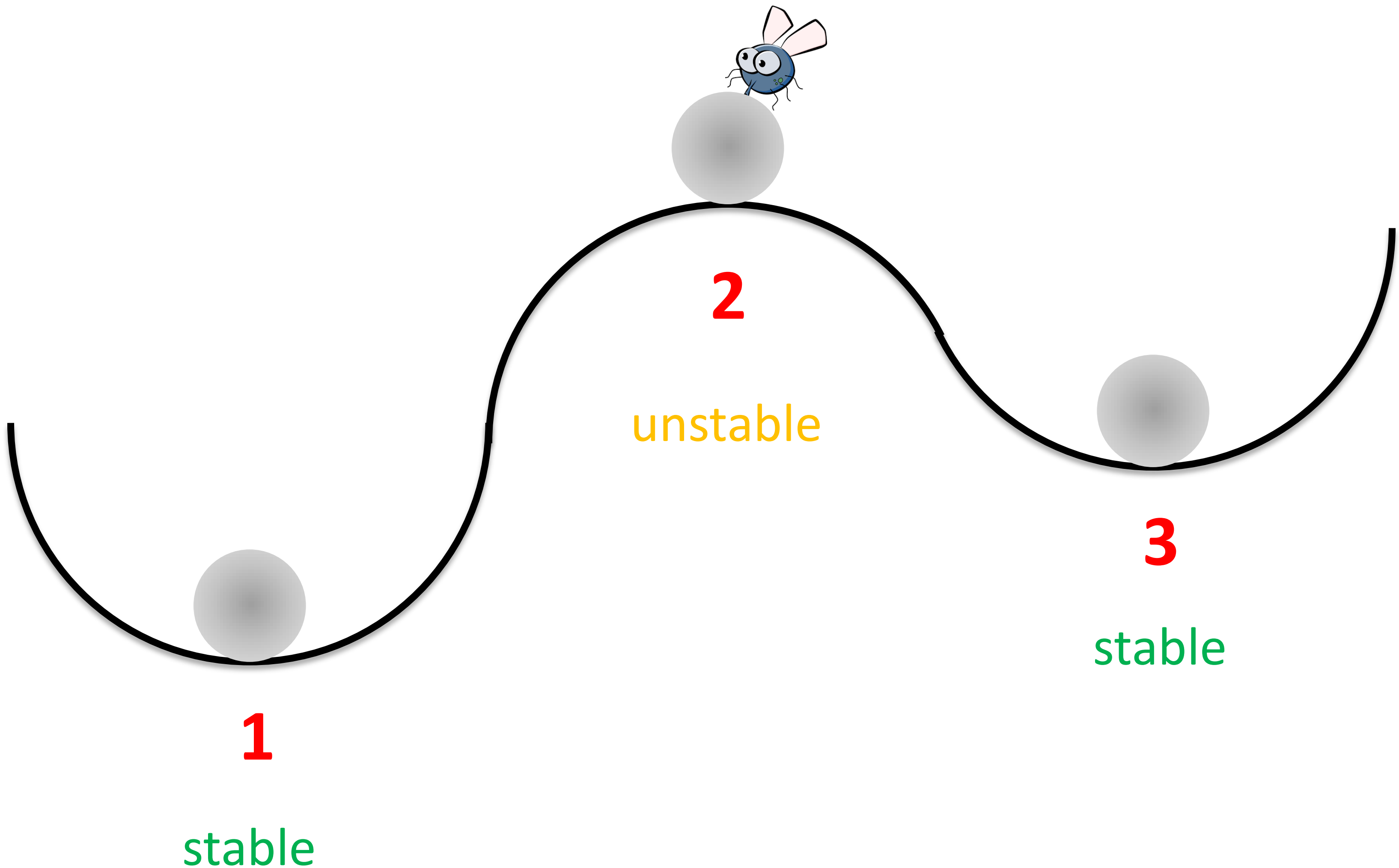


It might be simple or complicated… but the idea is the same.

How the system decides where to go?







1
stable
2
unstable
3
stable

The notion of (Lyapunov) stability

If we apply a **small perturbation** (the fly!) and the system **stays close or returns** back to its equilibrium

↓

Stable equilibrium

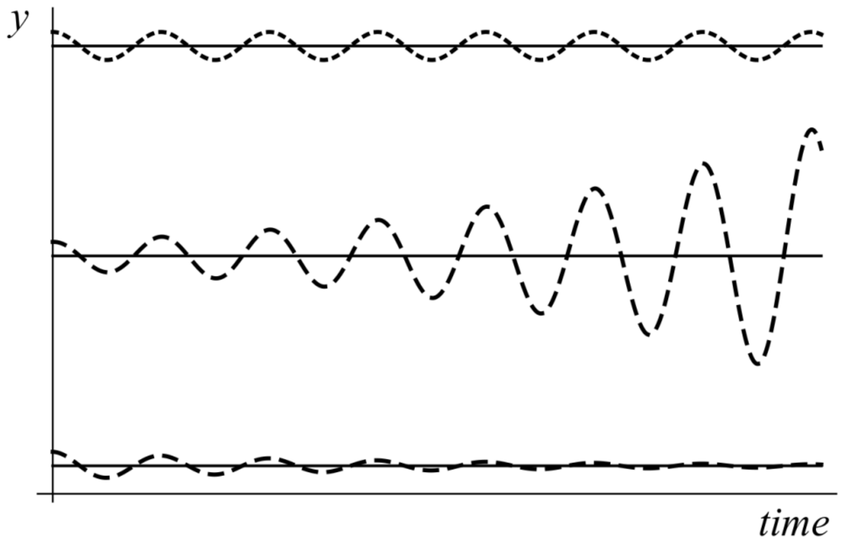
If we apply a small perturbation and the system **moves away** from its equilibrium

↓

Unstable equilibrium

Time… is central even if we forget it or not consider it directly in our analyses.

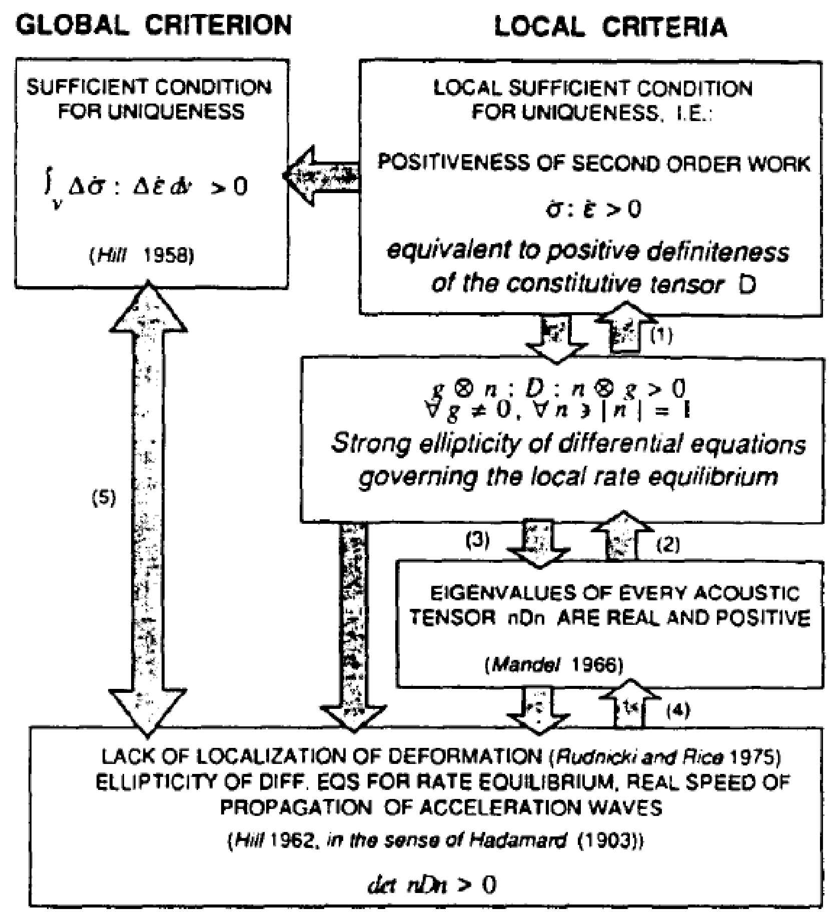
Stability theory was formulated in 1892 by A.M.Lyapunov (1857-1917).



Other stability postulates

- Loss of uniqueness
- Second order work
- Hill's stability
- Mandel's stability
- Loss of ellipticity
- Loss of controllability
- …

Confused?



(1) for plane isochoric motion.
(2) for associative plasticity.
(3) only if all eigenvalues of tensor nDn are real.
(4) for associative plasticity.
(5) for associative plasticity, constrained kinematics over the whole boundary of the body and homogeneous conditions before bifurcation (Rice 1976)

Fig. 1. Relationships between criteria for uniqueness, second order work, strong ellipticity, Mandel's stability and localization.

Bigoni & Hueckel, 1991

A couple of nice papers that **clarify** the applicability and (in)adequacy of many other “stability” postulates are:

Chambon, R., D. Caillerie, and G. Viggiani (2004), Loss of uniqueness and bifurcation vs instability: some remarks, *Rev. Française Génie Civ.*, *8*(5–6), 517–535.
(ALERT School 2004)

Bigoni, D., and T. Hueckel (1991), Uniqueness and localization—I. Associative and non-associative elastoplasticity, *Int. J. Solids Struct.*, *28*(2), 197–213.

Loss of uniqueness

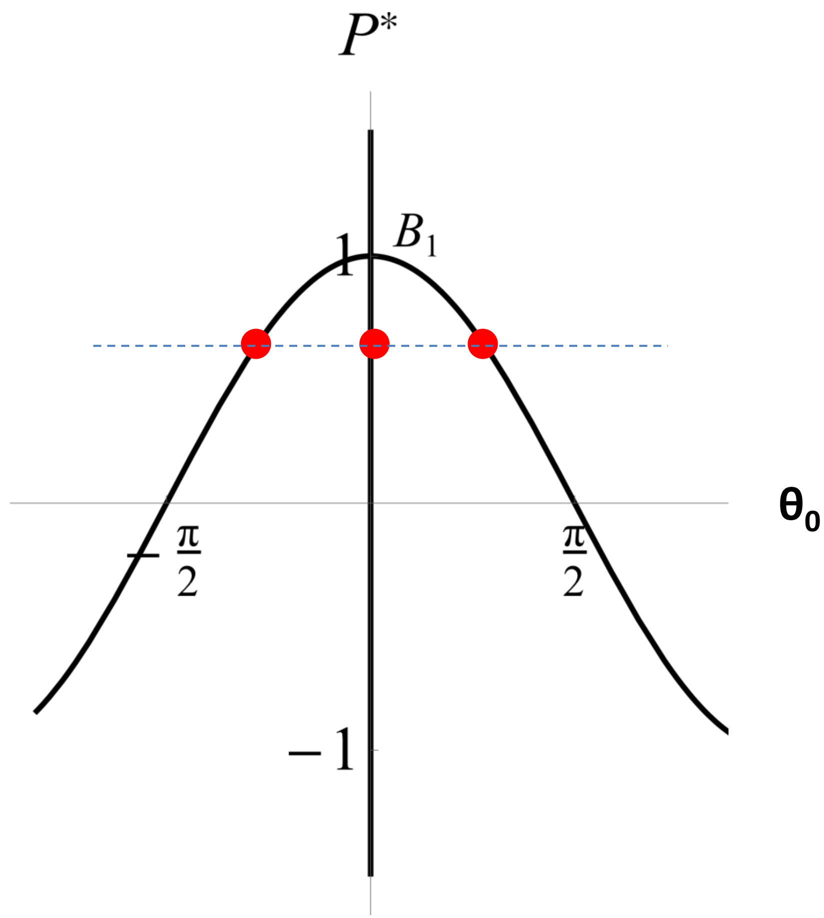
= existence of more than one (equilibrium or steady state) solutions

$\neq$

Bifurcation

$\neq$

Instability


P^*
B_1
1
−1
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
θ_0

Theory

... in mathematical terms

A physical system:

$$\underline{\dot{y}} = \underline{f}(\underline{y})$$

$\underline{y}$ is a vector of *n* components that contains the various quantities that determine the evolution of the physical system (e.g. temperature, displacement, velocity…)

The dot represents the time derivative and $\underline{f}$ is a vector function that does not depend explicitly on the independent variable which is the time -> **Autonomous system**

This is *an initial value problem* and it *has a unique solution* $\underline{y}(t)$
if: $\underline{f} \in C^1(D)$ for a given set of initial conditions.

BUT this does not mean that it has a unique equilibrium $\underline{y}_0$ point (fixed point):

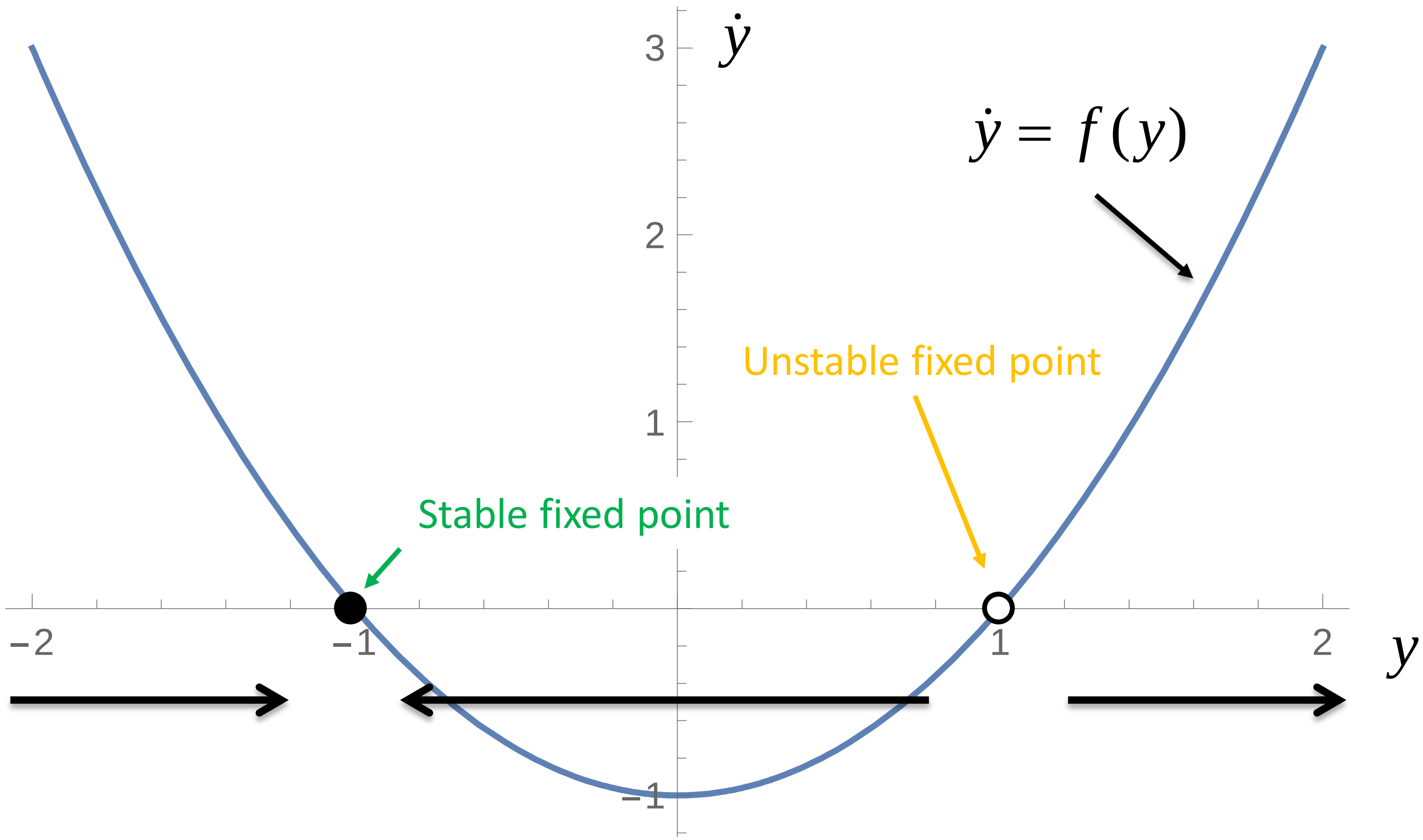
$$\underline{f}(\underline{y}_0) = 0$$

Example:

$$\dot{y} = f(y) = (y-1)(y+1)$$

Two equilibrium points: $y_0 = +1$ or $y_0 = -1$

Plot of $\dot{y} = f(y) = (y-1)(y+1)$ (phase portrait)



Let's say that y is a displacement. Then $\dot{y}$ is a velocity.

Lyapunov stability

The important question is if *an equilibrium is stable or not*.

In other words, if at time t_0 we are in equilibrium ($\dot{y}_0 = f(y_0) = 0$)

and a **tiny perturbation** $\underline{\tilde{\psi}}$ takes place such as $\underline{y} = \underline{\psi} = \underline{y}_0 + \underline{\tilde{\psi}}$,

do we return to the initial equilibrium,y_0 , or the system

diverges to another position?

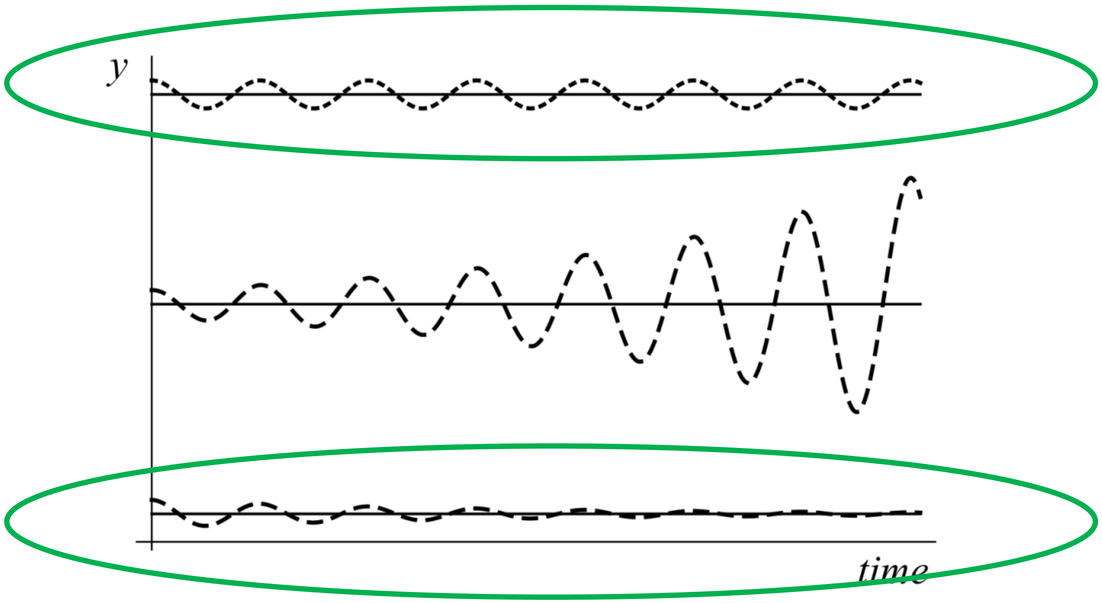
Lyapunov introduced the following definitions to answer this question:

Definition 1 (stable equilibrium)

The equilibrium solution $\underline{y}_0$ is said to be **stable** if for each number $\varepsilon > 0$ we can find a number $\delta > 0$ (depending on ε) such that if $\underline{\psi}(t)$ is any solution of $\dot{\underline{y}} = \underline{f}(\underline{y})$ $\boxed{\text{with } \left\|\underline{\psi}(t_0) - \underline{y}_0\right\| < \delta,}$ then the solution $\underline{\psi}(t)$ exists for all $t \geq t_0$ and $\boxed{\left\|\underline{\psi}(t) - \underline{y}_0\right\| < \varepsilon \text{ for } t \geq t_0}$.

the perturbation is small

the growth of the perturbation is bounded

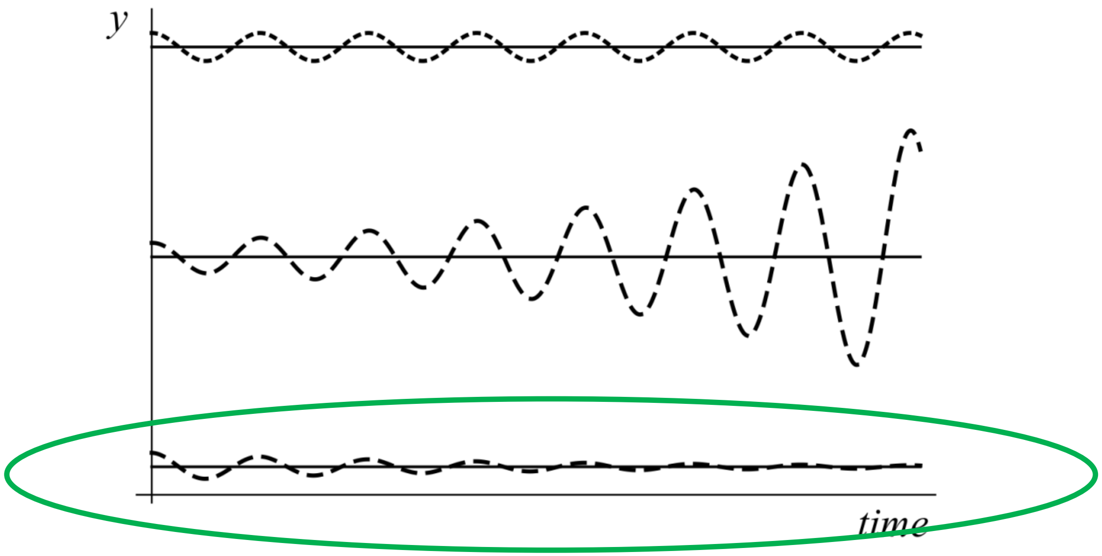

y
time

Definition 2 (asymptotically stable equilibrium)

The equilibrium solution $\underline{y}_0$ is said to be **asymptotically stable** is if it is stable and if there exists a number $\delta_0 > 0$ such that if $\underline{\psi}(t)$ is any solution of $\dot{\underline{y}} = \underline{f}(\underline{y})$ with $\boxed{\left\| \underline{\psi}(t_0) - \underline{y}_0 \right\| < \delta_0}$ then $\boxed{\lim_{t \to +\infty} \underline{\psi}(t) = \underline{y}_0}$.

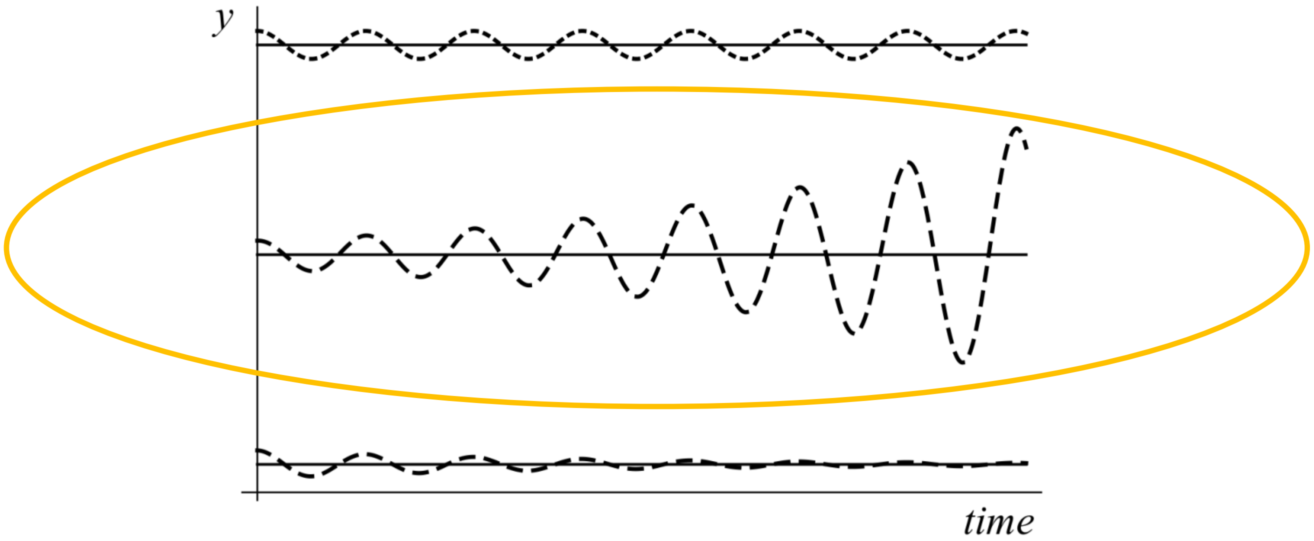
the perturbation is small

asymptotic convergence


y
time

Definition 3 (unstable equilibrium)

The equilibrium solution $\underline{y}_0$ is said to be **unstable** if it is not stable.



Nothing was said about $\underline{f}$.

a. Linear ODE’s

$$\dot{\underline{y}} = \underline{\underline{A}}\,\underline{y}$$

$\underline{\underline{A}}$ is *nxn* matrix with real constant coefficients.

Equilibrium (fixed) point: $\underline{y}_0 = 0$

If the eigenvalues of the system are **distinct** (no repeated eigenvalues, called *simple eigenvalues*) the general solution of this ODE system is:

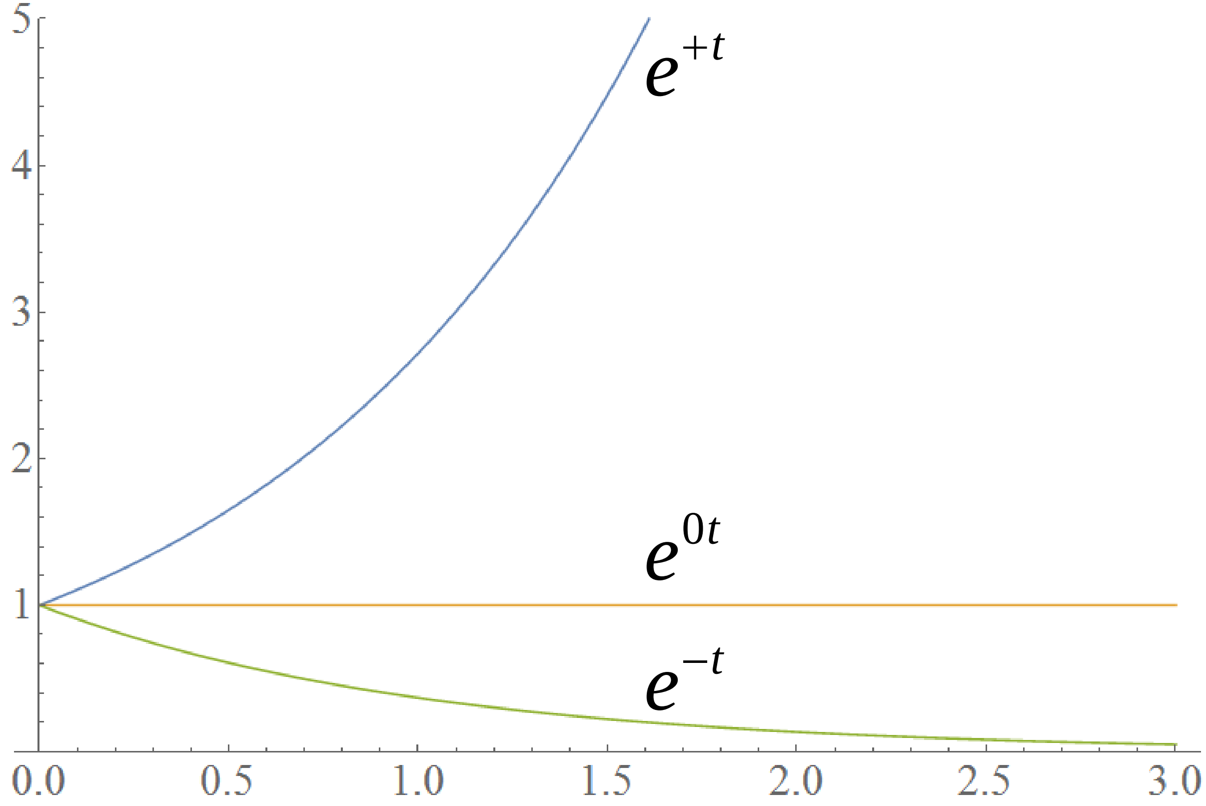
$$\underline{y}(t) = \sum_{i=1}^{n} c_i \underline{\eta}^{(i)} e^{s^{(i)}t}$$

$\underline{\eta}^{(i)}$ is the i^{th} eigenvector of $\underline{\underline{A}}$

$s^{(i)}$ is the i^{th} eigenvalue of $\underline{\underline{A}}$

c_i are constants determined by the initial conditions.

$$\underline{y}(t) \sim e^{\mathrm{Re}[s^{(i)}]t}$$



warning:

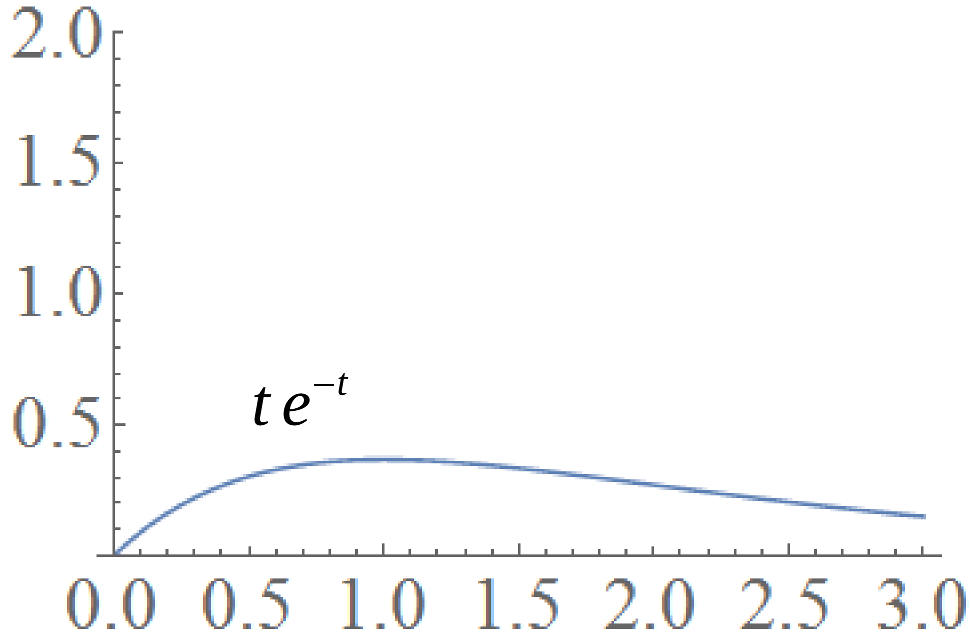
If $\underline{\underline{A}}$ has p distinct eigenvalues $s^{(i)}$ ($1 \leq i \leq p$), with multiplicity $m^{(i)}$ each one (if the eigenvalue k is simple, then $m^{(k)} = 1$), and associated eigenvectors $\underline{\eta}^{(i)}$, then the general solution of the ODE system is:

$$\underline{y}(t) = \sum_{i=1}^{p} \sum_{j=1}^{m^{(i)}} c_{i,j} \underline{\eta}^{(i)} \, t^{j-1} \, e^{s^{(i)} t}$$

Example: System with n=3 and two distinct eigenvalues (one of them of multiplicity 2). Then its general solution is:

$$\underline{y}(t) = c_{1,1}\underline{\eta}^{(1)}\, e^{s^{(1)}t} + c_{2,1}\underline{\eta}^{(i)}\, e^{s^{(2)}t} + c_{2,2}\underline{\eta}^{(i)}\, t\, e^{s^{(2)}t}$$

Notice, the term $t\, e^{s^{(2)}t}$, is strictly increasing in a region close to $t = 0$ even if $s^{(2)} \le 0$:



Theorem 1

- If **all** eigenvalues of $\underline{\underline{A}}$ have **non-positive real** parts and **all those eigenvalues with zero real parts are simple**, then the zero solution of $\underline{\dot{y}} = \underline{\underline{A}}\,\underline{y}$ is **stable**.

- If (and only if) **all** eigenvalues of $\underline{\underline{A}}$ have **negative real** parts, then the zero solution of $\underline{\dot{y}} = \underline{\underline{A}}\,\underline{y}$ is **asymptotically stable**.

- If **one or more** eigenvalues of $\underline{\underline{A}}$ have a **positive real part**, the zero solution of $\underline{\dot{y}} = \underline{\underline{A}}\,\underline{y}$ is **unstable**.

-> The stability of the equilibrium state of a linear system is investigated by simply studying the eigenvalues of the coefficients matrix.

b. Non-linear ODE's

$$\dot{\underline{y}} = \underline{f}(\underline{y})$$

Let $\underline{\psi}(t) = \underline{y}_0 + \tilde{\underline{\psi}}(t)$ the solution of the system, where $\underline{y}_0$ is one of the equilibrium solutions. Then if we can linearize around $\underline{y}_0$ we obtain:

$$\dot{\tilde{\underline{\psi}}}(t) = \underline{\underline{A}}\,\tilde{\underline{\psi}} + \underline{p}(\tilde{\underline{\psi}})$$

where: $\underline{\underline{A}} = \underline{\underline{J}}(\underline{y}_0) = \left\{ \left. \frac{\partial f_i}{\partial y_j} \right|_{\underline{y}=\underline{y}_0} \right\}$

Theorem 2

Suppose that $\underline{p}(\underline{\tilde{\psi}})$ is continuous, $\left\|\underline{\tilde{\psi}}\right\| < k$, where k is a constant,

and $\underline{p}(\underline{\tilde{\psi}})$ is small in the sense that $\lim_{\underline{\psi}\to 0} \dfrac{\left\|\underline{p}(\underline{\tilde{\psi}})\right\|}{\left\|\underline{\tilde{\psi}}\right\|} = 0$ and $\underline{p}(0) = 0$,

Then:

- If **all** eigenvalues of $\underline{\underline{A}}$ have **negative real parts**, then the solution $\underline{\tilde{\psi}} = 0$ of $\underline{\dot{\tilde{\psi}}}(t) = \underline{\underline{A}}\,\underline{\tilde{\psi}} + \underline{p}(\underline{\tilde{\psi}})$ is **asymptotically stable**.

- If **one or more** eigenvalues of $\underline{\underline{A}}$ have **a positive real part**, then the solution $\underline{\tilde{\psi}} = 0$ of $\underline{\dot{\tilde{\psi}}}(t) = \underline{\underline{A}}\,\underline{\tilde{\psi}} + \underline{p}(\underline{\tilde{\psi}})$ is **unstable**.

Linear Stability Analysis (LSA) **is based on the above theorem.**

Remarks:

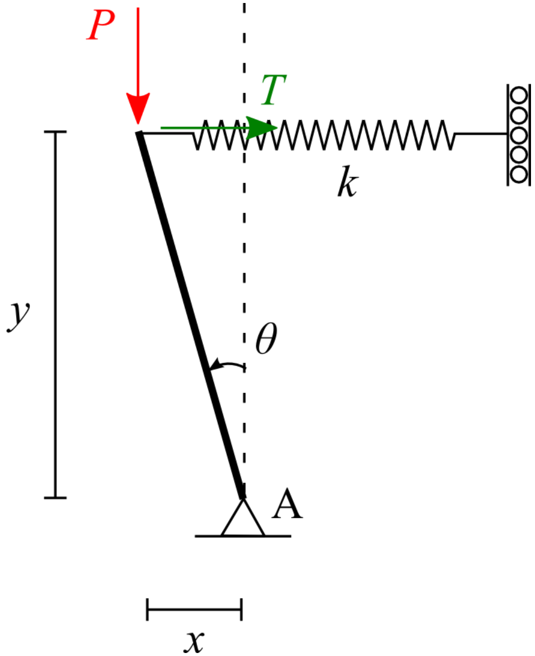
> If the second derivative of $\underline{f}$ exists then $\underline{p}$ is the reminder of a Taylor expansion of $\underline{f}$ around $\underline{y}_0$.

> If all eigenvalues of $\underline{\underline{A}}$ **have non-positive real parts** and **there exists at least one eigenvalue with zero real part** then the dynamics of the linearized system do not represent the dynamics of the non-linear system and **no conclusion can be safely derived for the stability of the non-linear system** (deficiency of linearization).

> In the special case of **conservative** (systems where a conserved quantity exists, e.g. the total energy) or **reversible systems** (systems with time reversal symmetry) it can be proven that when **all the eigenvalues of $\underline{\underline{A}}$ have non-positive real parts and there exists at least one eigenvalue with zero real part, then all orbits close to a fixed point are closed**. In this case the (isolated) fixed point is called non-linear center and it is **stable** in the Lyapunov sense (but not asymptotically stable).

Example 1

inverted pendulum



$$I_{\mathrm{A}}\ddot{\theta} = k\ell^2 \sin\theta\left(\frac{P}{k\ell} - \cos\theta\right)$$

$$\begin{cases} \dot{\theta} = \omega \\ I_{\mathrm{A}}\dot{\omega} = k\ell^2 \sin\theta\left(\dfrac{P}{k\ell} - \cos\theta\right) \end{cases}$$

Or in the form $\dot{\underline{y}} = \underline{f}(\underline{y})$:

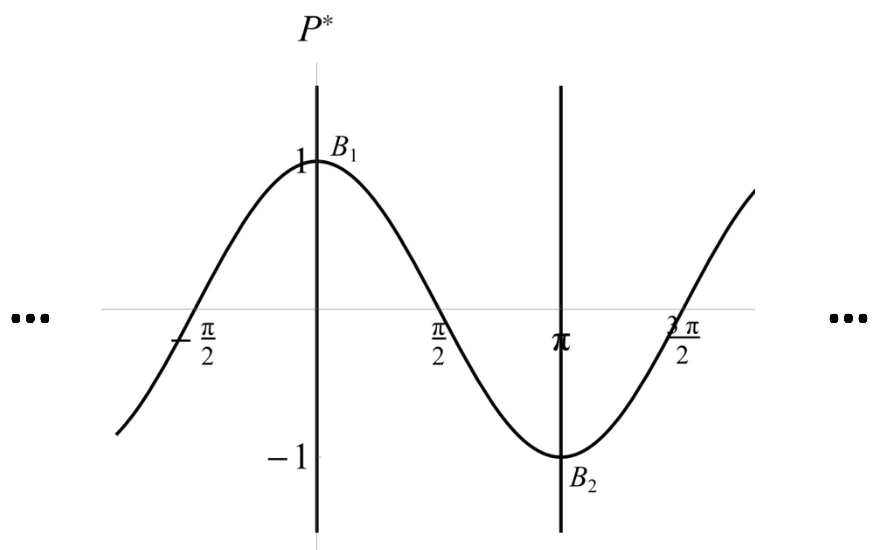
$$\underline{y} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

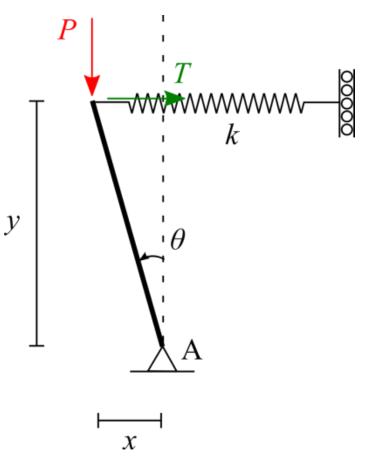
$$\underline{f} = \begin{bmatrix} \omega \\ \dfrac{k\ell^2}{I_{\mathrm{A}}}\sin\theta\left(\dfrac{P}{k\ell} - \cos\theta\right) \end{bmatrix}$$

Equilibrium point(s):

$$\underline{f} = \begin{bmatrix} \omega \\ \dfrac{k\ell^2}{I_A}\sin\theta\left(\dfrac{P}{k\ell}-\cos\theta\right) \end{bmatrix} = 0$$

$$\omega_0 = \dot{\theta}_0 = 0 \text{ and } \frac{k\ell^2}{I_A}\sin\theta_0\left(\frac{P}{k\ell}-\cos\theta_0\right) = 0 \overset{P^*=\frac{P}{k\ell}}{\Longleftrightarrow} \begin{cases} P^* - \cos\theta_0 = 0 \\ or \\ \sin\theta_0 = 0 \end{cases}$$





Linear Stability Analysis (LSA):

Perturbing the equilibrium solution we replace $\underline{y}(t)$ by $\underline{\psi}(t) = \underline{y}_0 + \underline{\tilde{\psi}}(t)$

Performing a Taylor expansion of $\underline{f}$ up to the first order around the point $\underline{y} = \underline{y}_0$ we retrieve a linear equation of the form

$\dot{\underline{\tilde{\psi}}}(t) = \underline{\underline{A}}\,\underline{\tilde{\psi}} + \underline{p}(\underline{\tilde{\psi}})$ (see slide 33), where:

$$\underline{\underline{A}} = \underline{\underline{J}}(\underline{y}_0) = \left\{ \left.\frac{\partial f_i}{\partial y_j}\right|_{\underline{y}=\underline{y}_0} \right\} = \begin{bmatrix} 0 & 1 \\ \dfrac{k\ell^2}{I_\mathrm{A}}\left(\dfrac{P}{k\ell} - \cos\theta_0\right)\cos\theta_0 + \dfrac{k\ell^2}{I_\mathrm{A}}\sin^2\theta_0 & 0 \end{bmatrix}$$

The characteristic polynomial of the eigenvalue problem is:

$$s^2 - \frac{k\ell^2}{I_{\mathrm{A}}}\left(\frac{P}{k\ell} - \cos\theta_0\right)\cos\theta_0 - \frac{k\ell^2}{I_{\mathrm{A}}}\sin^2\theta_0 = 0$$

which leads to 2 distinct eigenvalues:

$$s_{1,2} = \pm\sqrt{\frac{k\ell^2}{I_{\mathrm{A}}}\left(\frac{P}{k\ell} - \cos\theta_0\right)\cos\theta_0 + \frac{k\ell^2}{I_{\mathrm{A}}}\sin^2\theta_0}$$

$$s_{1,2} = \pm\sqrt{\frac{k\ell^2}{I_A}\left(\frac{P}{k\ell} - \cos\theta_0\right)\cos\theta_0 + \frac{k\ell^2}{I_A}\sin^2\theta_0}$$

- Branch: $\sin\theta_0 = 0$ $\Rightarrow$ $s_{1,2} = \pm\sqrt{\frac{k\ell^2}{I_A}\left(\frac{P}{k\ell} - 1\right)}$

*If $\frac{P}{k\ell} - 1 > 0$ there exists one positive eigenvalue with positive real part (imaginary part is zero) → **UNSTABLE**

*If $\frac{P}{k\ell} - 1 < 0$ both eigenvalues are imaginary with zero real part.

In this case no conclusion can be drawn in general, but the system is conservative (see slide 35) → **STABLE**

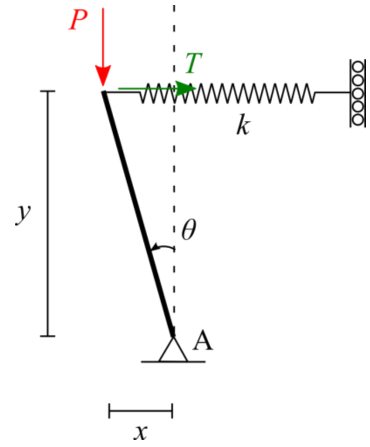
$$s_{1,2} = \pm\sqrt{\frac{k\ell^2}{I_A}\left(\frac{P}{k\ell} - \cos\theta_0\right)\cos\theta_0 + \frac{k\ell^2}{I_A}\sin^2\theta_0}$$

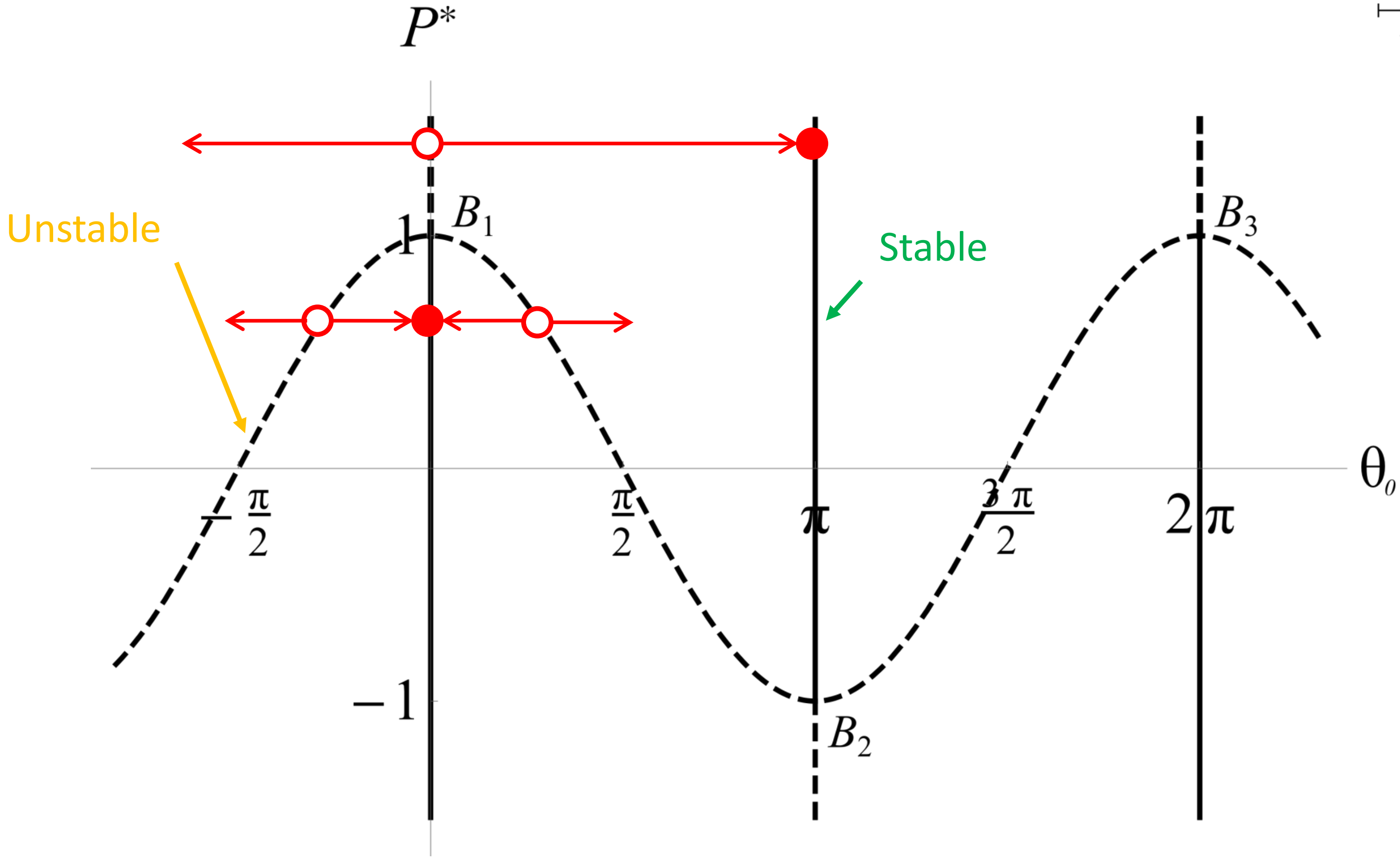
- Branch: $\frac{P}{k\ell} - \cos\theta_0 = 0 \Rightarrow s_{1,2} = \pm\left|\sin\theta_0\right|\sqrt{\frac{k\ell^2}{I_A}}$

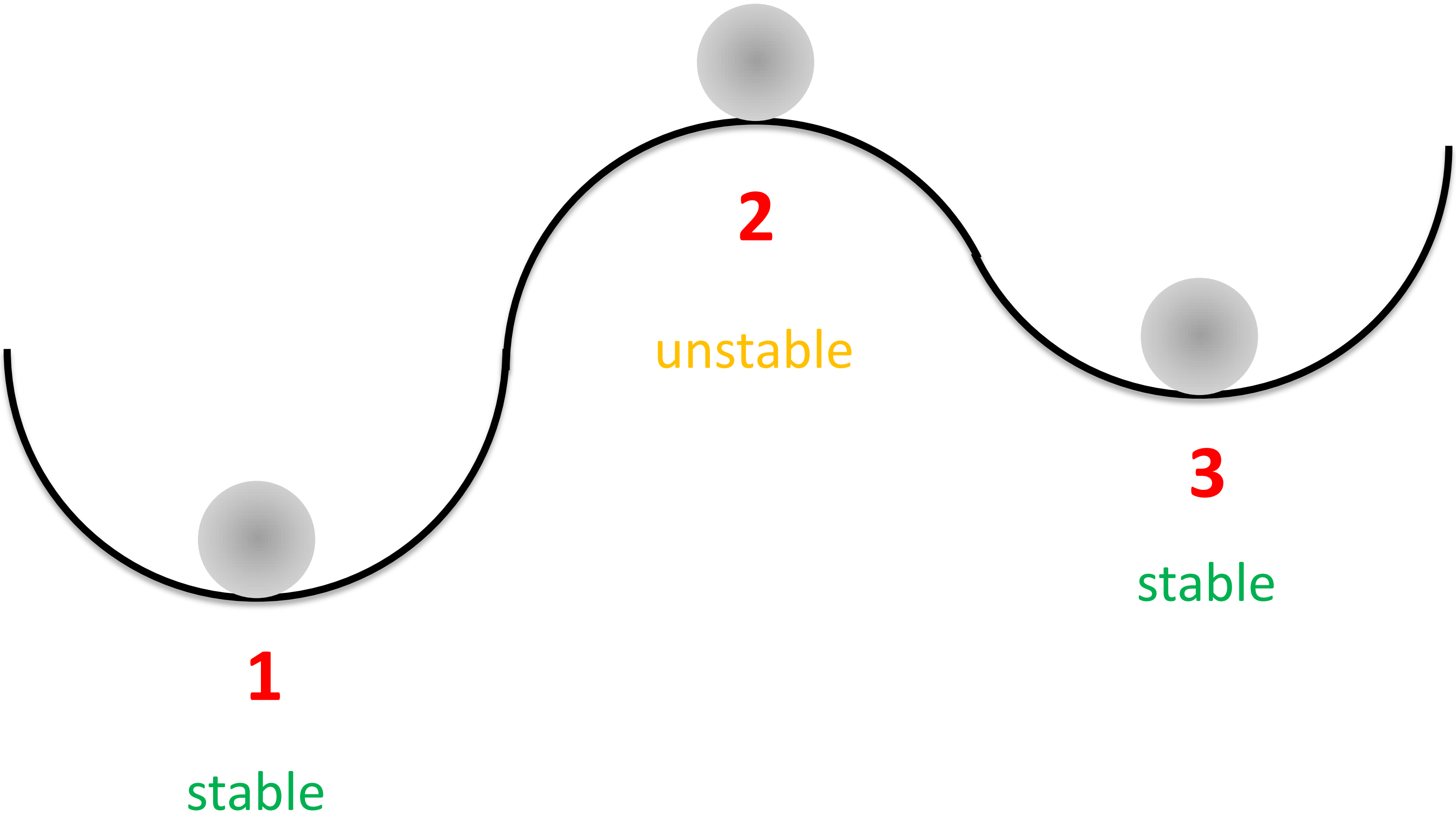
There exists one positive eigenvalue with positive real part

(imaginary part is zero) if $\sin\theta_0 \neq 0$ → **UNSTABLE**

Summarizing:






2
unstable
3
stable
1
stable

Example 2

love affairs

Love is an instability!!

$$\dot{R} = m(R, J)$$
$$\dot{J} = w(R, J)$$

R is the love of Romeo for Juliette and *J* the love of Juliette for Romeo.

A simpler form:

$$\dot{R} = a\,R + b\,J$$
$$\dot{J} = c\,R + d\,J$$

(Strogatz, 1988,2004)

a and *d* express fear to love (<0) or enthusiasm (>0)

b and *c* express fear of being loved (<0) or enthusiasm (>0)

J, R > 0 mean love, while *J, R < 0* hate.

The state of mutual indifference is the state of no love or hate: *R=J=0* and it is an equilibrium point (fixed point).

The coefficient matrix is: $\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

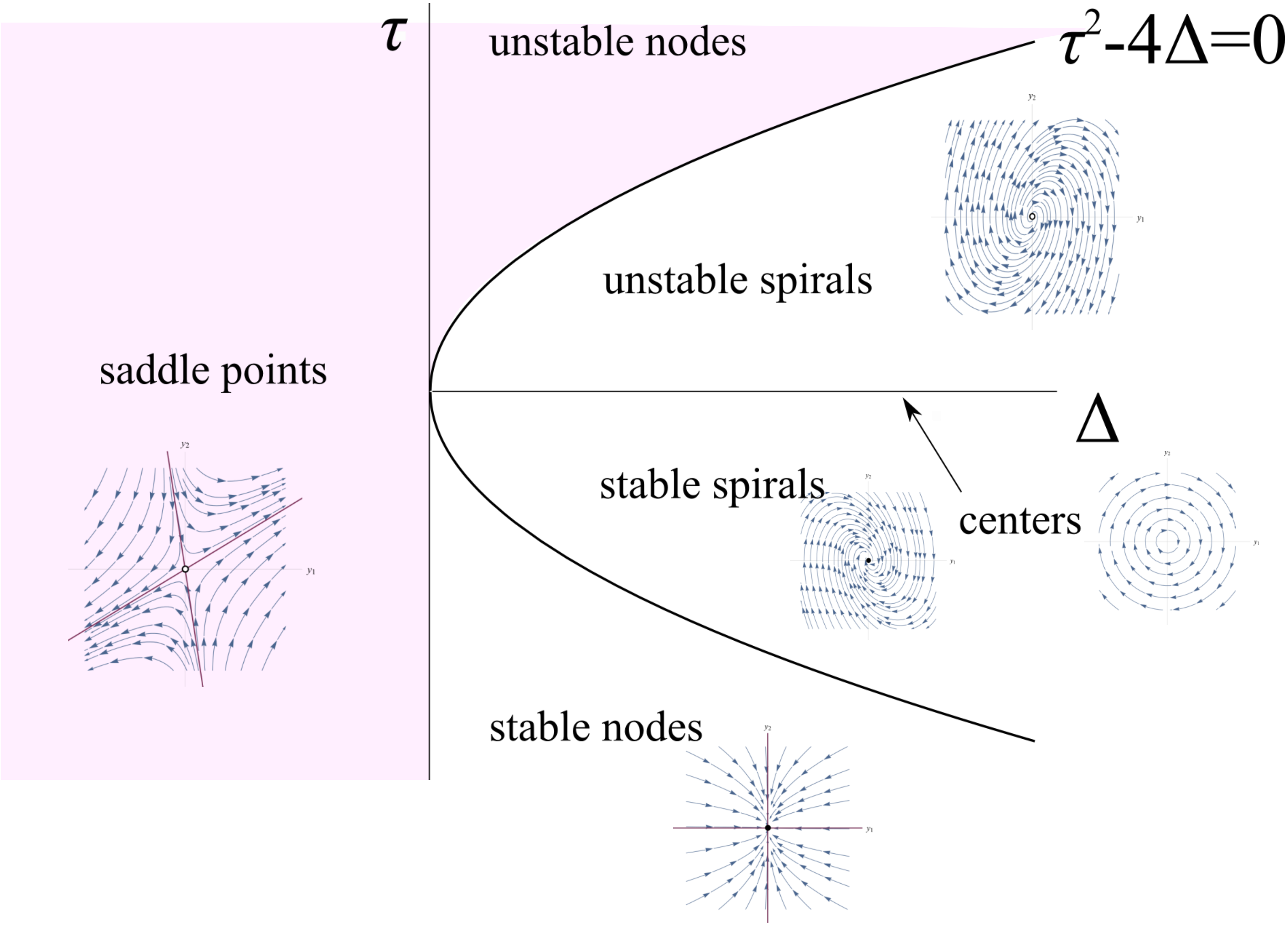
The characteristic polynomial: $s^2 - \tau s + \Delta = 0$

The eigenvalues: $s_1 = \dfrac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}, \qquad s_2 = \dfrac{\tau - \sqrt{\tau^2 - 4\Delta}}{2},$

$$\tau = a + d$$

$$\Delta = ad - bc$$

Classification of fixed points



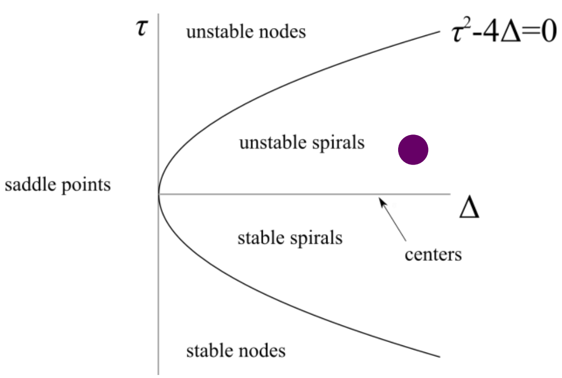
Scenario?

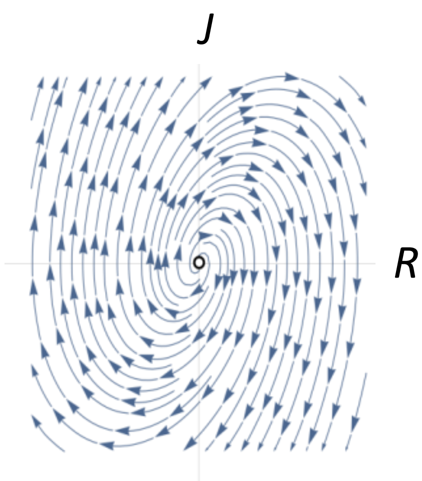
$$\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let's say that Romeo is not afraid to love Juliette *a>0* and it is an enthusiastic lover *b>0*, but Juliette is cautious *c<0*.

What will happen?

$$\underline{\underline{A}} = \begin{bmatrix} 0.1 & 1 \\ -1 & 0 \end{bmatrix} \qquad \begin{aligned} \tau &= a + d = 0.1 > 0 \\ \Delta &= ad - bc = 1 > 0 \end{aligned} \Longrightarrow \tau^2 - 4\Delta < 0$$





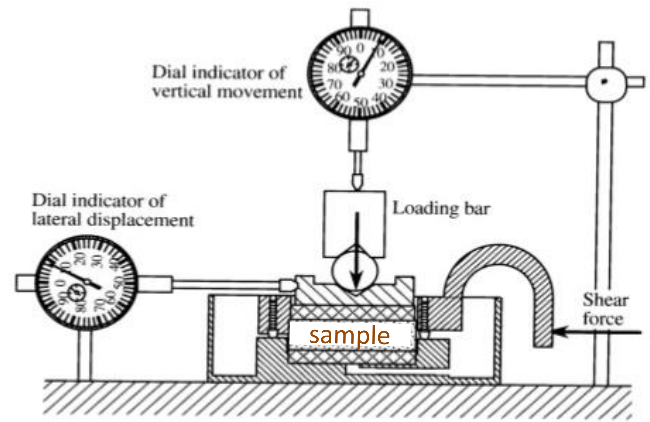
The more Romeo loves Juliette the more she is afraid and takes her distance, which makes Romeo to loose his interest, but then Juliette finds him again attractive, Romeo's love is rekindled and… so on…

Example 2

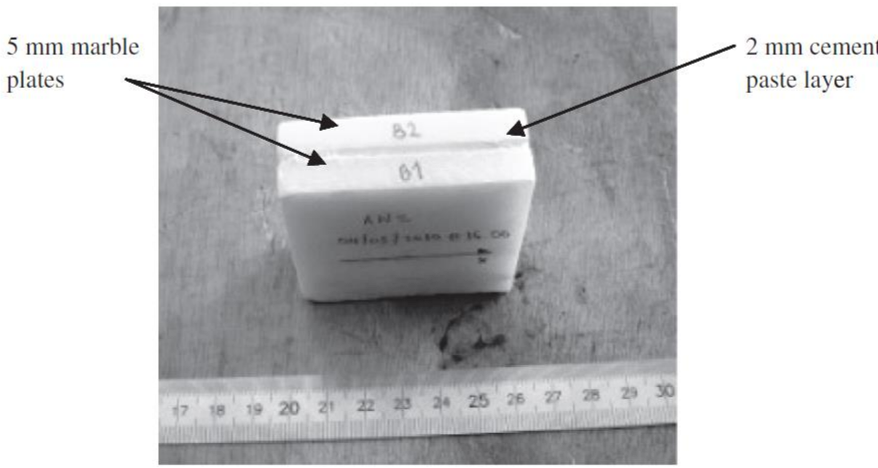
geotechnical testing

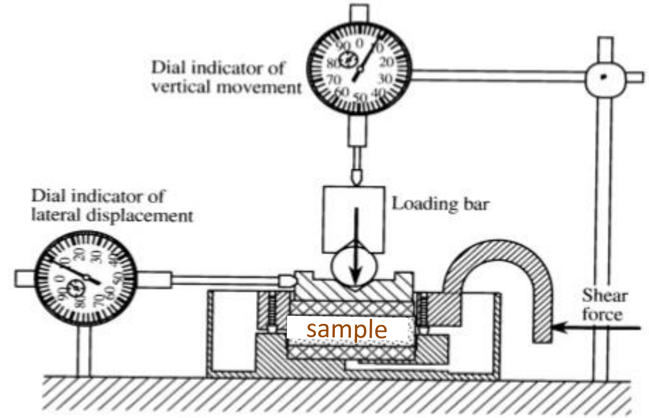
The direct shear test… “earthquakes” in the lab?

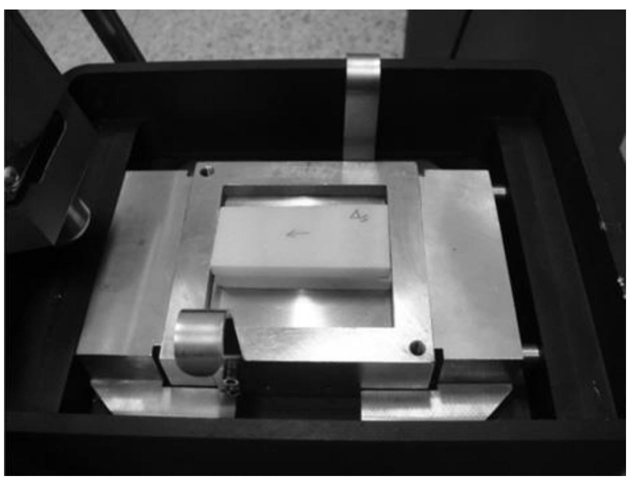
(courtesy Aimil ltd.)



Testing interfaces

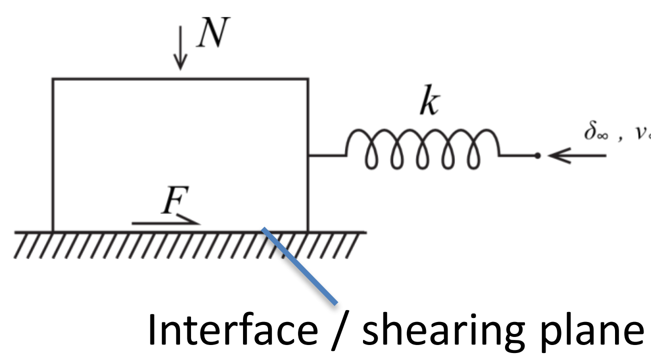




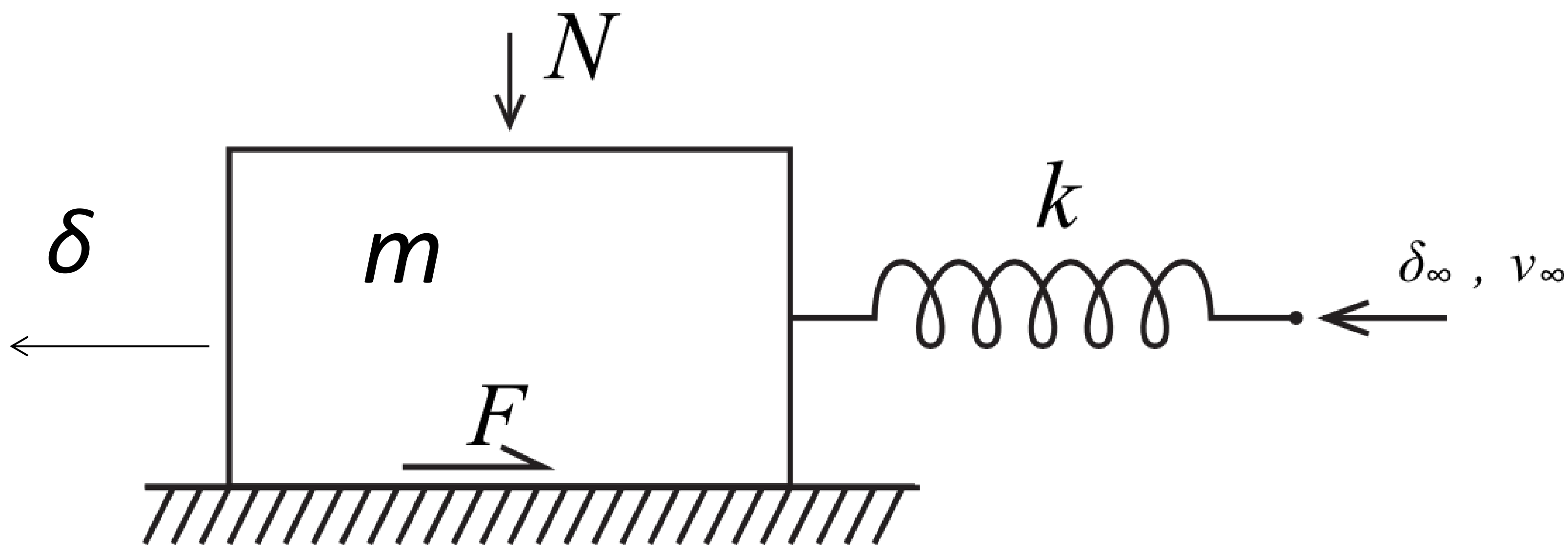


(I.Stefanou et al., 2010)

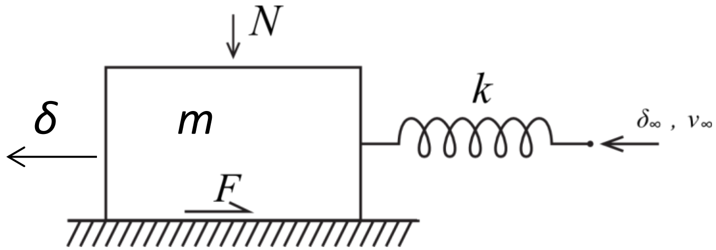
Equivalent model:



k is the equivalent stiffness of the apparatus
v_∞ is the applied displacement rate



$$m\ddot{\delta} = ?$$



$$m\ddot{\delta} = k(v_\infty t - \delta) - F(\delta) \quad \overset{\dot{\delta}=v}{\Leftrightarrow} \quad \begin{cases} \dot{w} = \dfrac{k}{m}(v_\infty - v) - \dfrac{1}{m}\dfrac{dF}{d\delta}v \\ \dot{v} = w \end{cases}$$

Steady state (fixed point): $\dot{w} = \dot{v} = 0 \quad \Leftrightarrow \quad \begin{cases} v = \dfrac{k}{k + \dfrac{dF}{d\delta}} v_\infty \\ w = 0 \end{cases}$

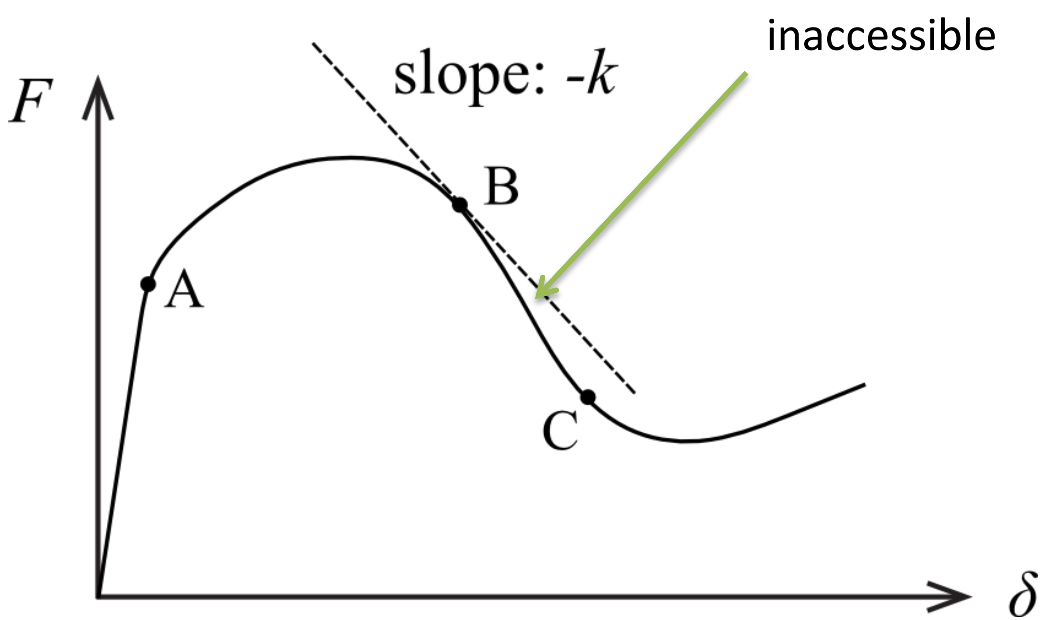
Coef. matrix:
(slide 28&39)

$$\begin{bmatrix} \dot{w} \\ \dot{v} \end{bmatrix} = \underline{\underline{A}} \begin{bmatrix} w \\ v \end{bmatrix}$$

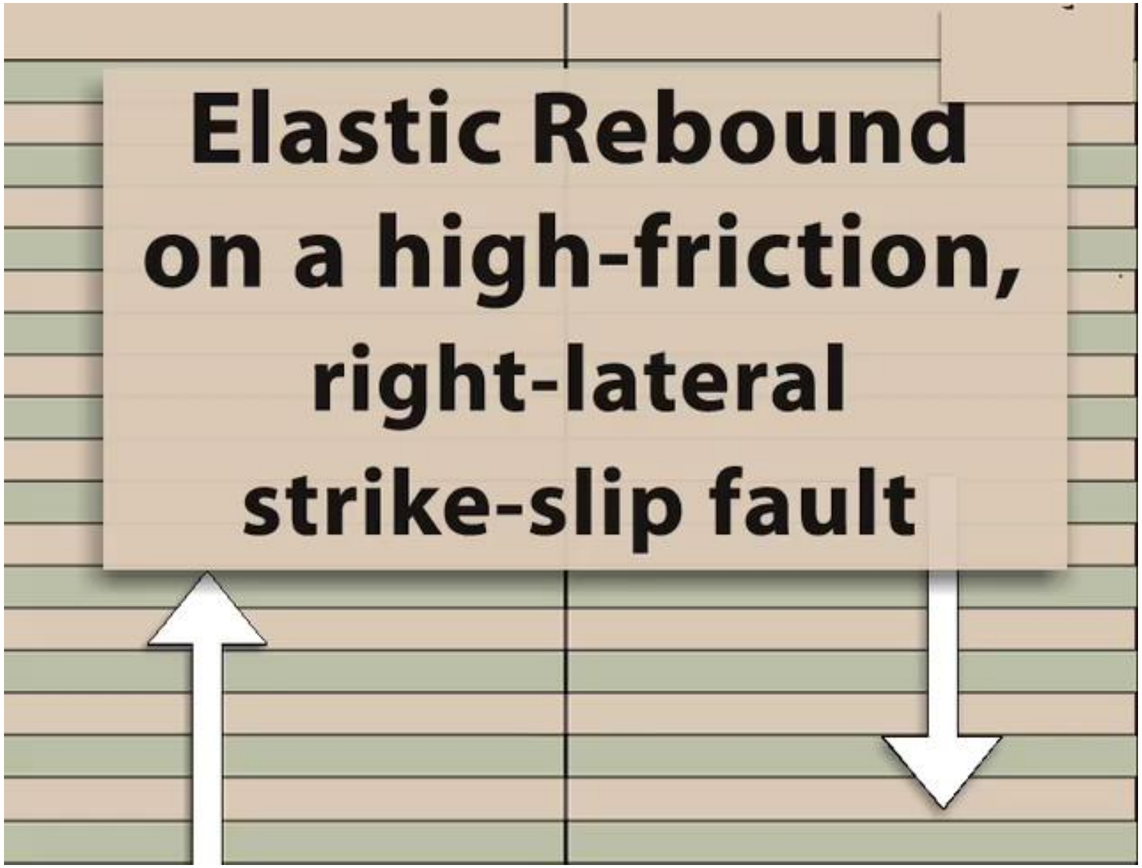
$$\underline{\underline{A}} = \begin{bmatrix} 0 & -\dfrac{1}{m}\left(k + \dfrac{dF}{d\delta}\right) \\ 1 & 0 \end{bmatrix}$$

Characteristic polynomial: $s^2 = -\frac{1}{m}\left(k + \frac{dF}{d\delta}\right)$

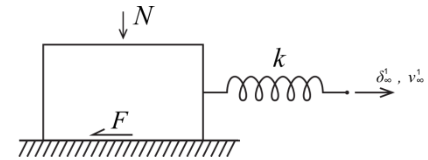
If $k + \frac{dF}{d\delta} < 0 \iff \frac{dF}{d\delta} < -k$ → **UNSTABLE**



Earthquakes



(IRIS, Incorporated Research Institution for Seismology)



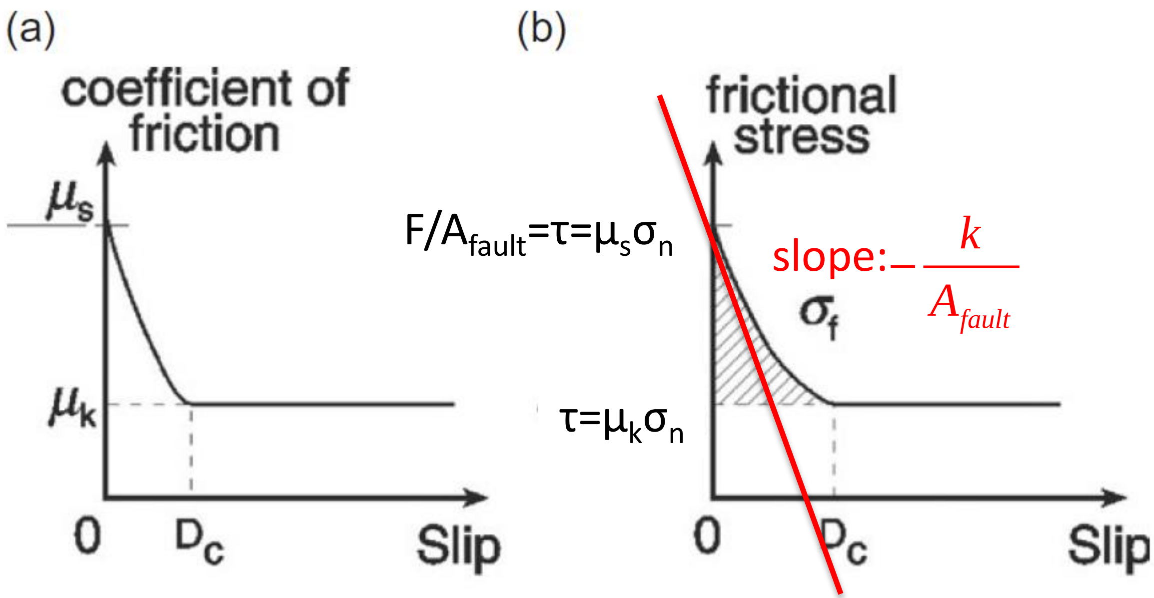


Figure 16. Static and kinetic friction. (*a*) The coefficient of friction. In the simple model, μ drops from μ_s to μ_k instantly, but in general, it drops to μ_k after a slip D_c. (*b*) The frictional stress. D_c is the critical slip. σ_f is the frictional stress.

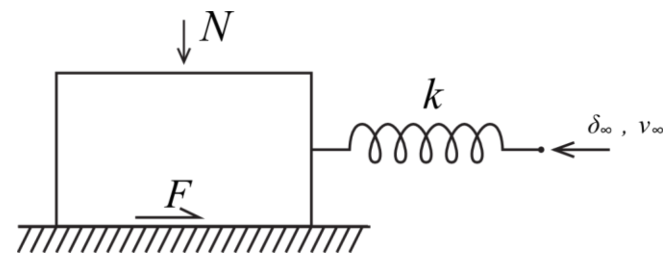
(Kanamori & Brodsky, 2004)

$$\frac{dF}{d\delta} = \frac{\mu_k - \mu_s}{D_c} \left|\sigma_n\right| A_{fault} < -k$$

Exercise #2

-> *Repeat the same analysis for a rate dependent*

interface: $F = F(\delta, v)$.



What changes?
Are there any similarities with Romeo and Juliette?
When is the system unstable?
Are spirals possible?

Clarifying bifurcation

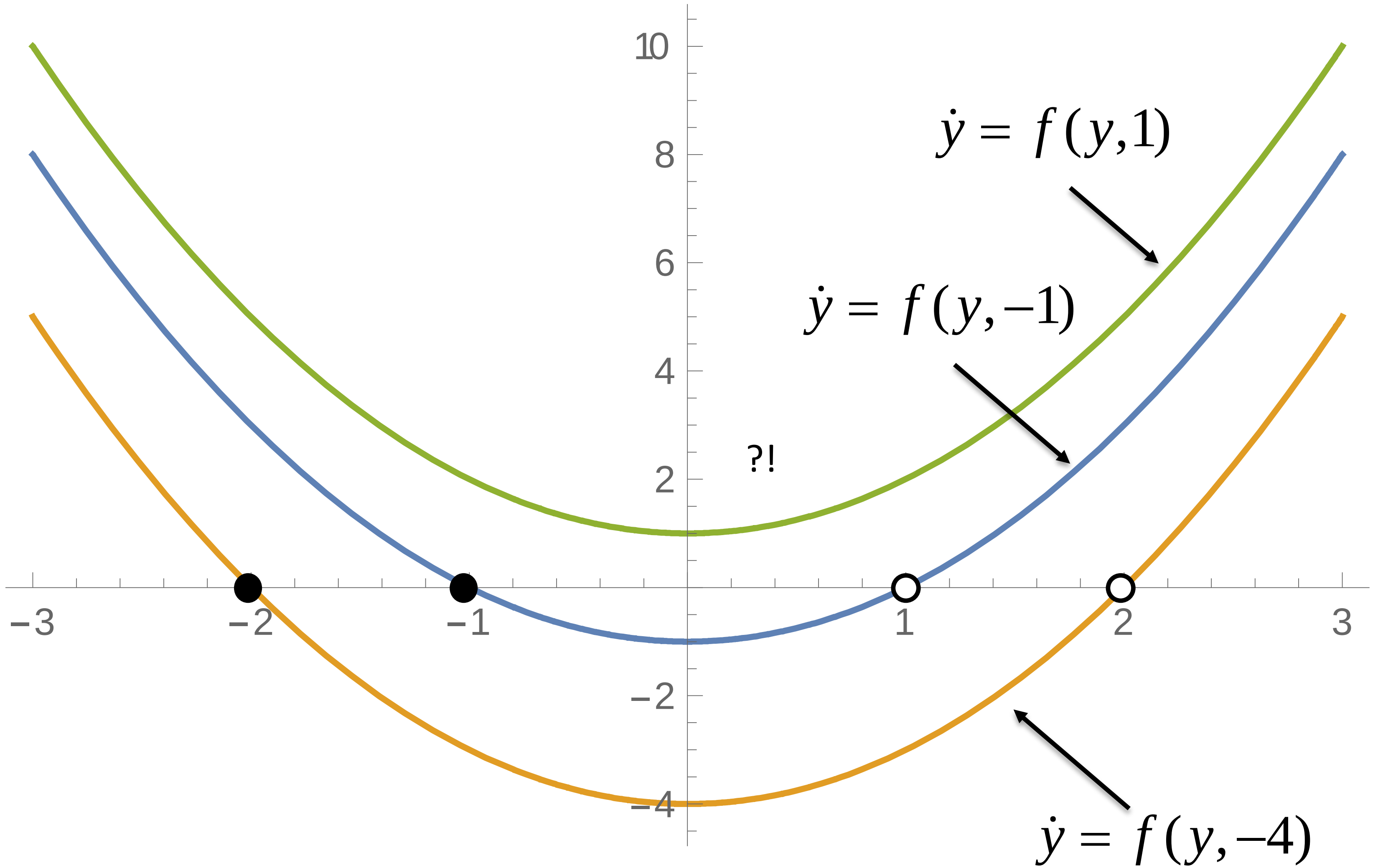
Stability, bifurcation and uniqueness

- ✓ We talked about stability of equilibria (or steady states)
- ✓ We have seen that for non-linear systems, equilibrium points are not unique but not necessarily each one unstable
- ✓ We have seen what bifurcation points are

When we are talking about bifurcation, we are interested in the **change of the equilibrium** (or steady state) solutions of a (dynamical) system **in terms of a parameter** μ, which is called bifurcation parameter:

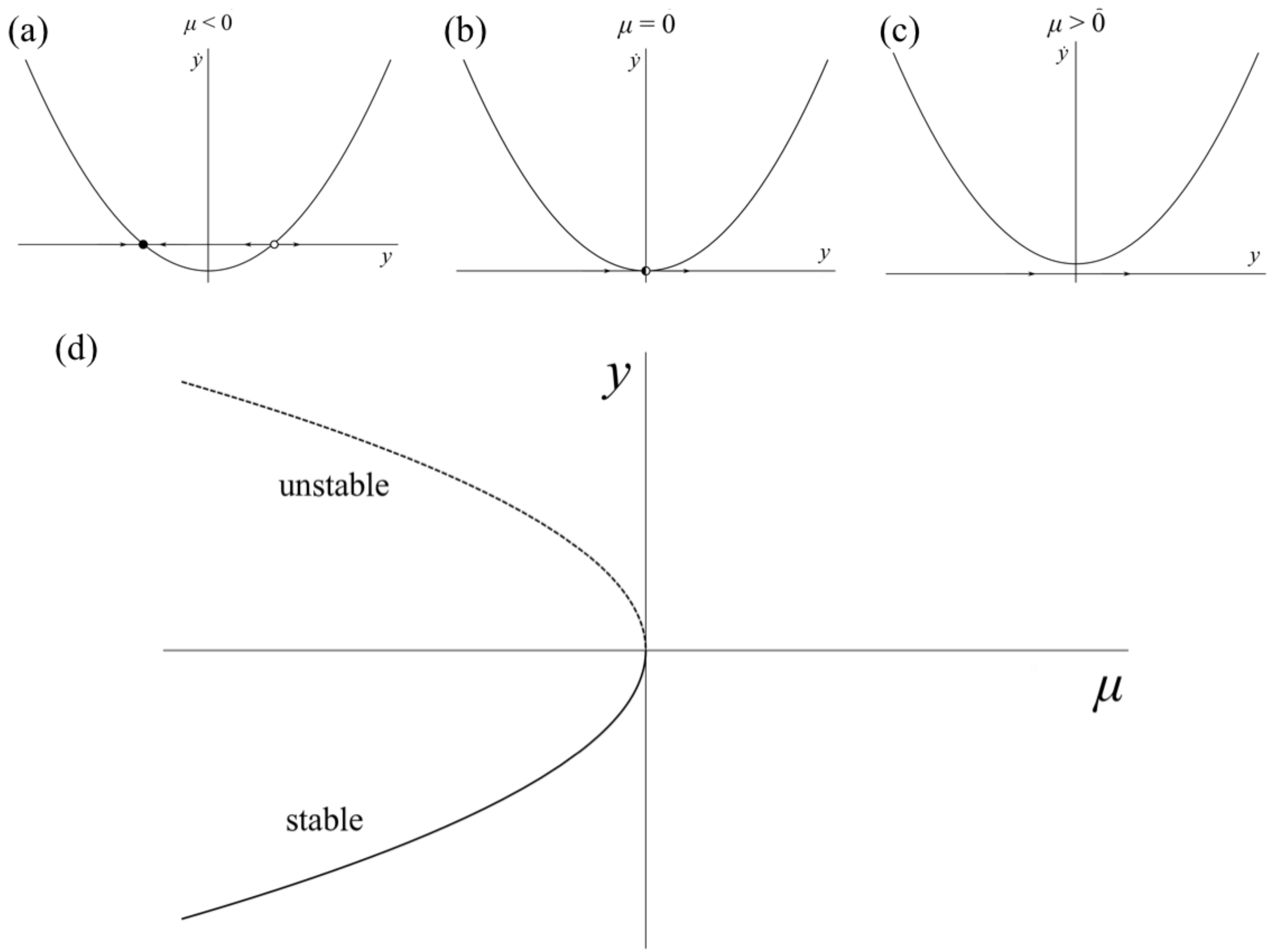
$$\dot{\underline{y}} = f(\underline{y}, \mu) = 0$$

Plot of $\dot{y} = f(y,\mu) = y^2 + \mu$ (phase portrait)



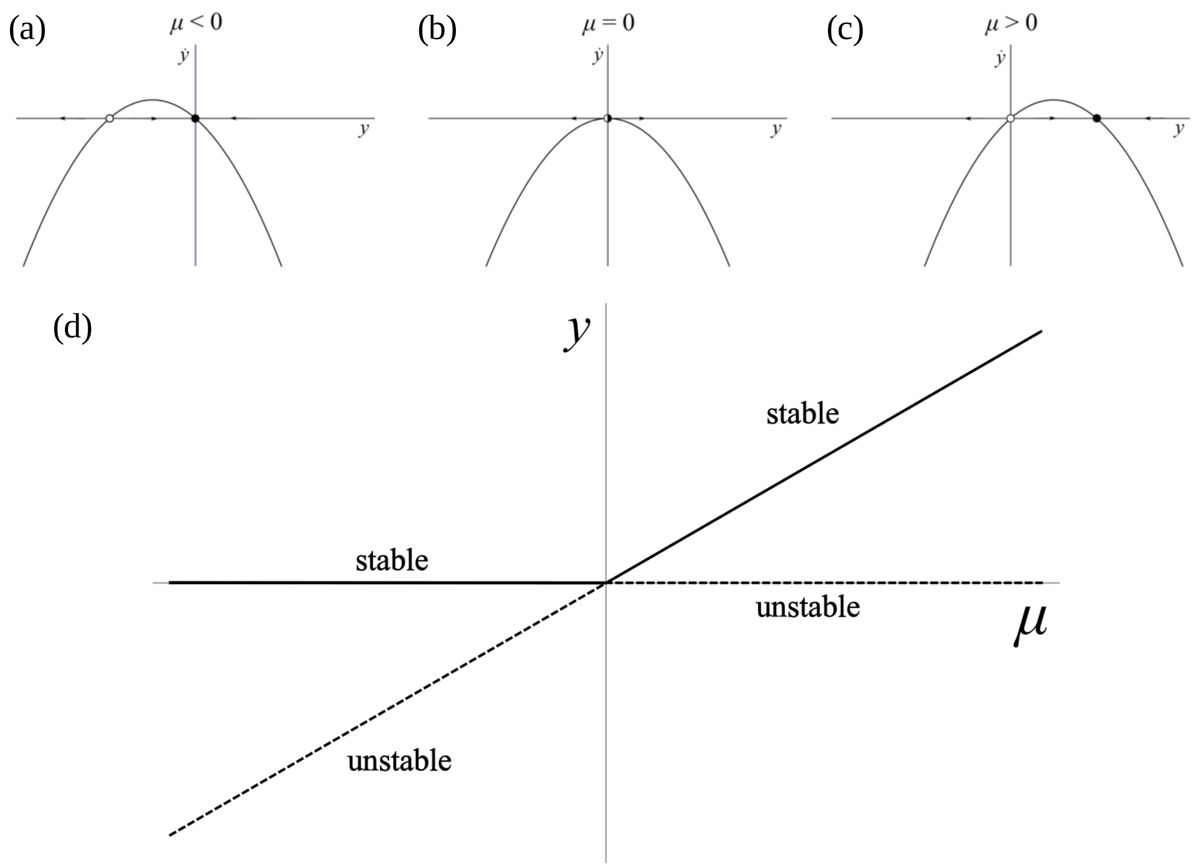
Saddle-node bifurcation

Normal form: $\dot{y} = y^2 + \mu$



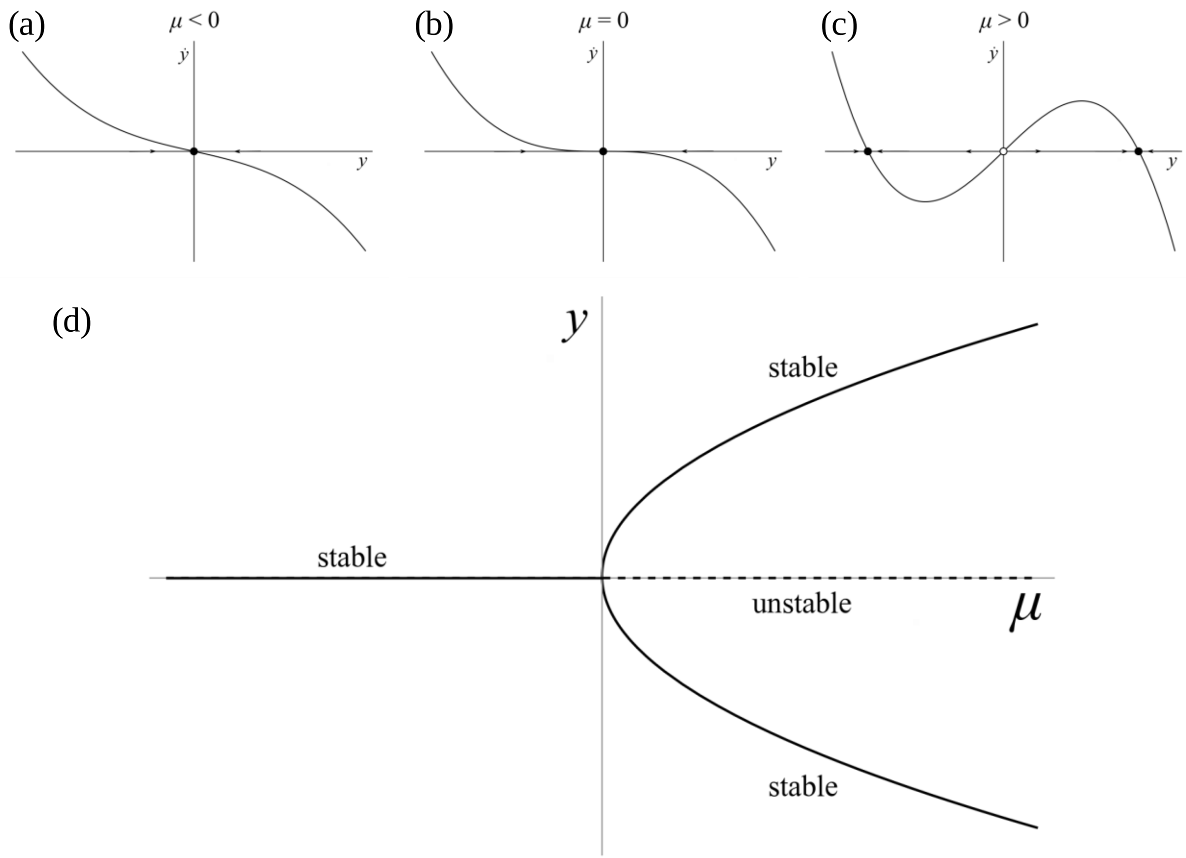
Transcritical bifurcation

Normal form: $\dot{y} = \mu y - y^2 = y(\mu - y)$



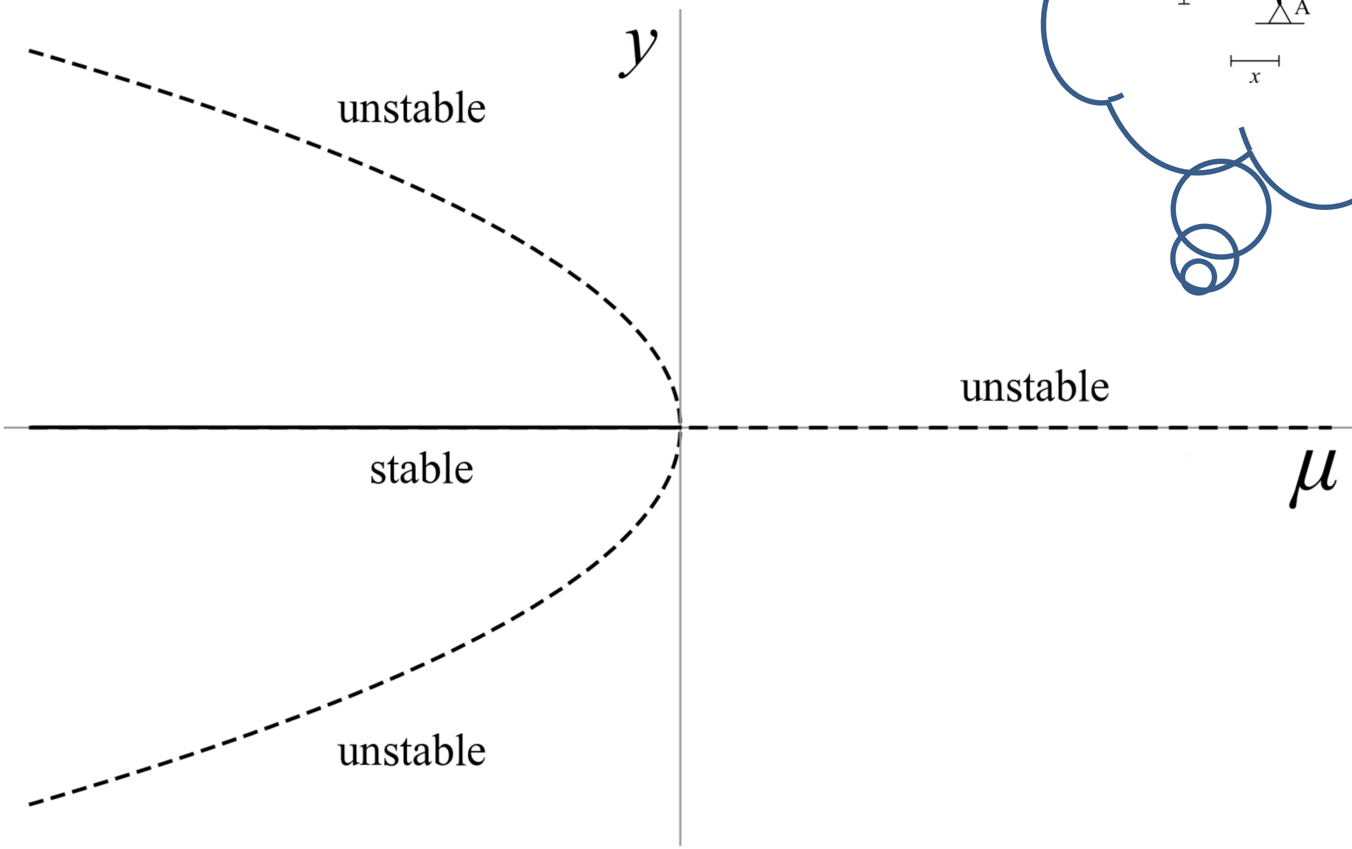
Supercritical pitchfork bifurcation

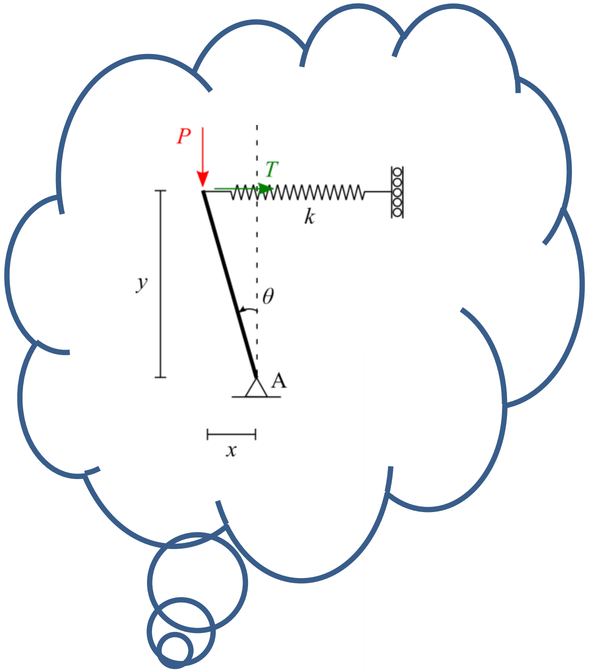
Normal form: $\dot{y} = \mu y - y^3$



Subcritical pitchfork bifurcation

Normal form: $\dot{y} = \mu y + y^3$





“Higher order bifurcations”
&
Limit cycles

$$\dot{r} = r\left(1 - r^2\right)$$

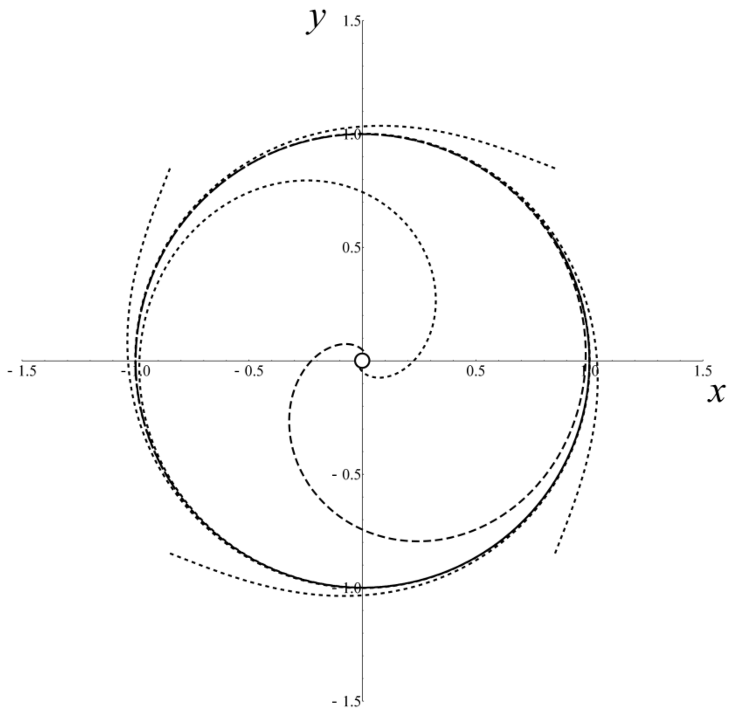
$$\dot{\theta} = 1 \qquad\qquad r \geq 0$$

It is easy to identify that the two equations are uncoupled and that the first one **if treated alone**, has two fixed points for $r \geq 0$, namely $r = 0$ (unstable) and $r = 1$(stable).

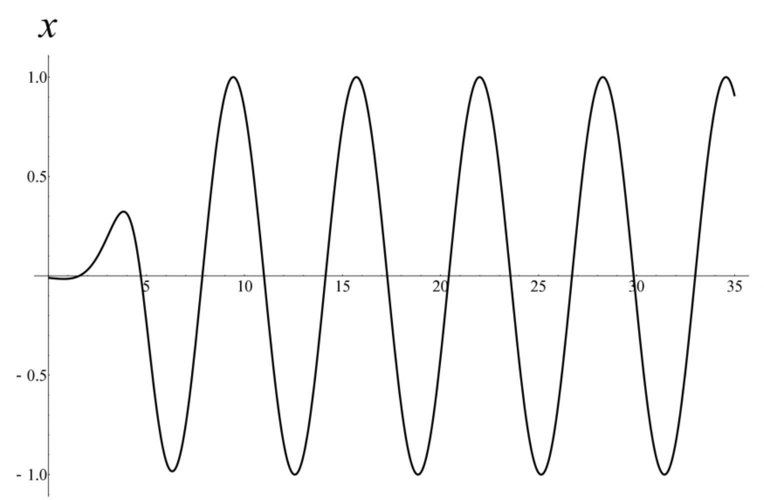
The system of two equations (2D) has no fixed points at all because $\dot{\theta} = 1 \neq 0$ (constant angular velocity).

All trajectories on the phase plane are approaching the unit circle ($r = 1$) monotonically.

This can be visualized if we revert again to Cartesian coordinates:



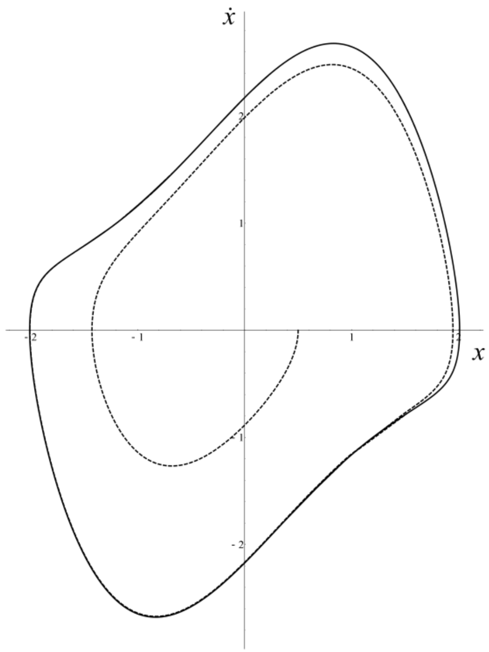
Stable closed orbit.



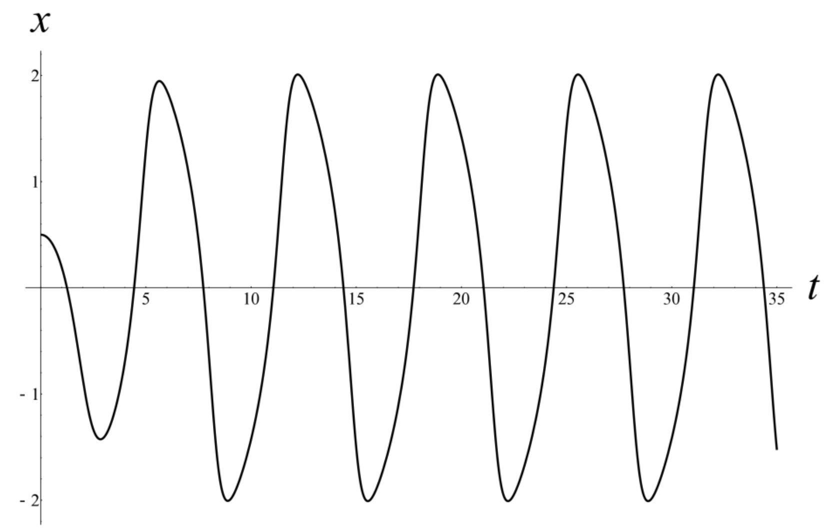
No fixed points… but periodic behavior.

van der Pol equation

$$\ddot{y} + \mu\left(y^2 - 1\right)\dot{y} + y = 0 \qquad \mu \geq 0$$



Stable closed orbit.



Periodic behavior.

Non-linear dynamical systems of order higher than one can present perfectly periodic solutions.

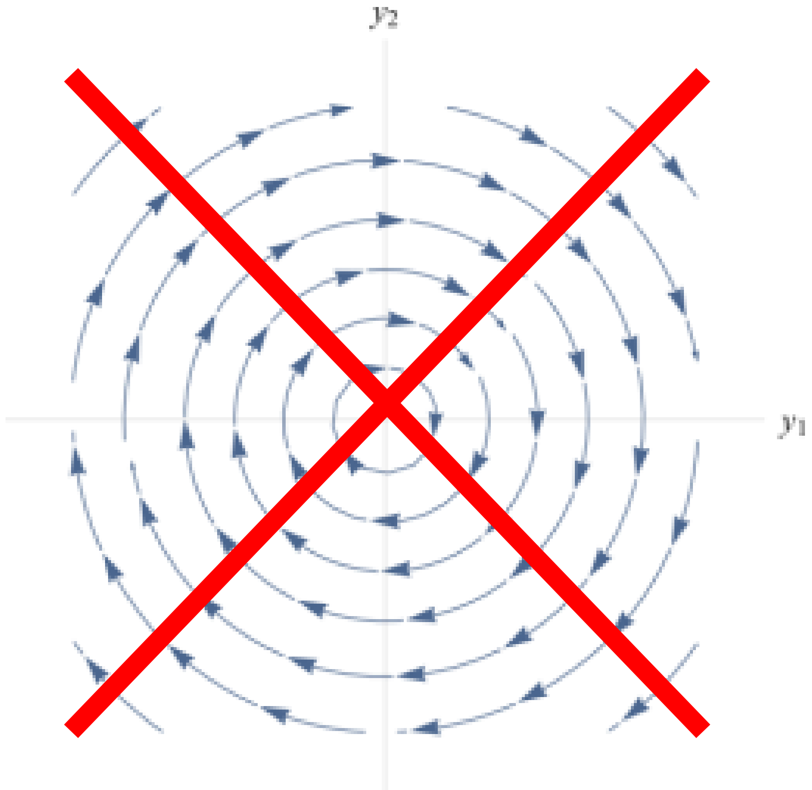
Such solutions appear on the phase space as isolated closed orbits, which can attract or repel all neighboring trajectories, much like the fixed points.

These orbits are called *limit cycles*. Limit cycles are an inherent phenomenon of two or higher dimensional systems that are non-linear.

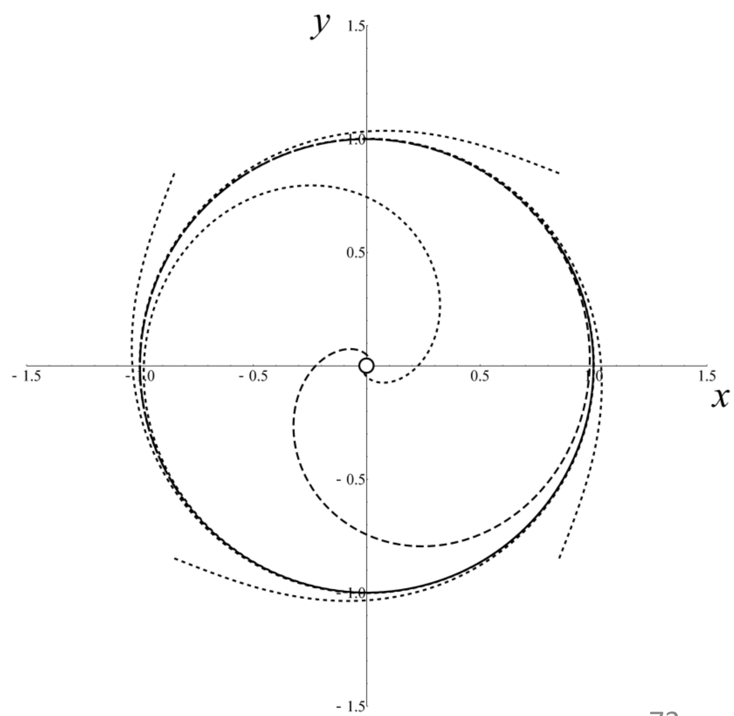
When a fixed point looses stability involving the creation or destruction of a limit cycle around it, we have a *Hopf bifurcation*.

Even though, linear systems can present closed orbits, when the fixed point is a stable center (neutral stability), such solutions are non-isolated, i.e. if $x(t)$ is a periodic solution, then $c\,x(t)$ is also a periodic solution for all $c \in \mathbb{R}^*$.

Limit-cycles are a non-linear effects.



$\neq$



Non-linear systems with *n≥3* can have trajectories that might be in an open, bounded domain, yet, they can move freely inside it without settling into a fixed point or a closed orbit.

They can be attracted to topological manifolds (called *stable manifolds*) or even to **complex geometric objects that are called *strange attractors* or *fractals***.

The passage to chaos…

From ODE’s to PDE’s

Continuous systems

All the above concepts and techniques are transferred to the study of Partial Differential Equations too.

Solid mechanics (infinitesimal deformations):

Localization in solid mechanics

Dynamic equations
of a Cauchy continuum: $$\sigma_{ij,j} = \rho \ddot{u}_i$$

Equilibrium point: $$\sigma^{*}_{ij,j} = 0$$

Let's assume that we are in a state of homogeneous deformation.

We want to investigate the possibility of non-homogeneous deformations such as compaction, shear and dilation bands.

Let's assume that we are in a state of homogeneous deformation.

Example:

Successive equilibria
for increasing P:

$$\sigma^*_{ij,j} = 0$$

Are they stable?

Considering the class of materials that we can linearize σ (hypothesis of equivalent material/linear comparison solid):

$$\sigma_{ij} = \sigma_{ij}^{*} + \Delta\sigma_{ij} = \sigma_{ij}^{*} + L_{ijkl}\Delta u_{k,l}$$

(Rice, 1976)

Δu_i is a perturbation from the reference, homogeneous, equilibrium configuration such that: $\Delta u_i = u_i - u_i^{*}$

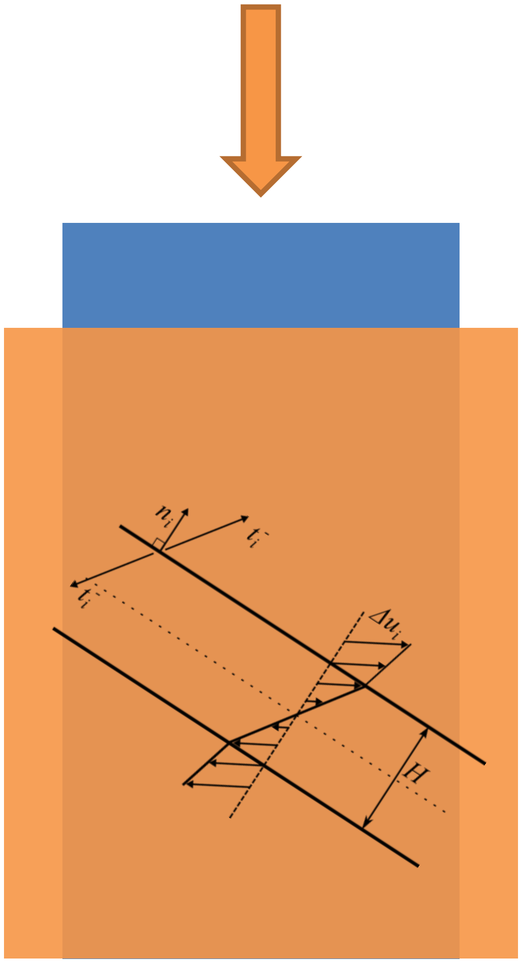
Replacing: $$L_{ijkl}\Delta u_{k,lj} = \rho \ddot{\Delta u}_i$$

Separation of variables:

$$\Delta u_i = X(x_p)U_i(t) \quad \rightarrow \quad L_{ijkl} X_{,lj}\, U_k(t) = \rho X\, \ddot{U}_i(t)$$

Allowing plane wave solutions for *X* that satisfy the *BC's*

$$X(x_p) = e^{i\frac{2\pi}{\lambda}n_p x_p}$$



we finally obtain the system of ODE's:

$$\dot{V}_i = -\frac{1}{\rho}\left(\frac{2\pi}{\lambda}\right)^2 n_j L_{ijkl} n_l U_k$$

$$\dot{U}_i = V_i$$

which can be studied as before!

Stability analysis leads to the following eigenvalue problem:

$$\left[-n_j L_{ijkl} n_l - \rho \left(\frac{\lambda s}{2\pi} \right)^2 \delta_{ik} \right] g_k = 0$$

L_{ijkl} depends on the constitutive law of the material.

$$\left[-n_j L_{ijkl} n_l - \rho \left(\frac{\lambda s}{2\pi} \right)^2 \delta_{ik} \right] g_k = 0$$

If the real part of $\rho \left(\frac{\lambda s}{2\pi} \right)^2 = sth > 0$ is positive then the

homogeneous solution is **unstable and the system could bifurcate to a non-uniform solution** (which we do not need to find). Due to the form of X the non-uniform solution will be a band, with direction n_i.

The type of the deformation band (compaction, shear or dilation band) is determined by the product $g_i n_i$.

The above condition is independent of the specific constitutive law, provided that it is rate-independent.

$$\rho\left(\frac{\lambda s}{2\pi}\right)^2 = sth > 0 \quad \Rightarrow \quad s = \frac{2\pi}{\lambda}\sqrt{\frac{sth}{\rho}}$$

$$\Delta u_i = U_i e^{st + i\frac{2\pi}{\lambda} n_p x_p}$$

The perturbation that propagates the fastest in the medium maximizes s and therefore minimizes λ.

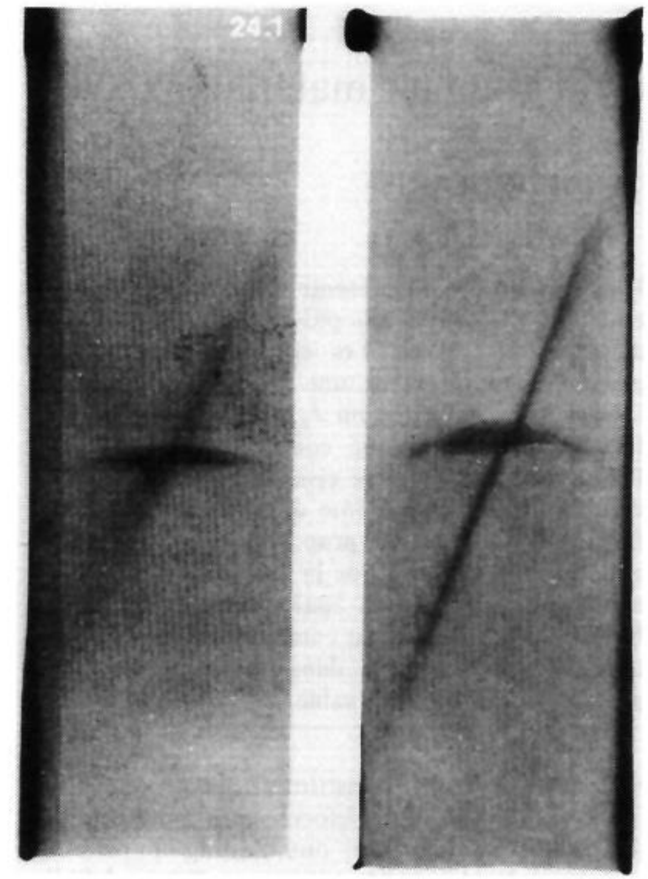
Localization happens on a mathematical plane (λ->0).

But this is not in accordance with experiments, which show that deformation bands have a **finite thickness**, controllable by the grain size (at least).

These experiments are very slow for the material to show any rate dependent sensitivity (Zheng et Zhao et al., 2013). So it seems not to be related to viscous effects, at least at 1st order.

The reason seems to be the **absence of internal lengths in Cauchy medium**.

Higher order micromorphic continua, Cosserat (microstructure), temperature etc. are some approaches to put more physics in the problem leading to finite band thickness.



(Mühlhaus & Vardoulakis, 1987)

Exercise #3

Repeat the previous analysis for $\sigma_{ij} = \sigma_{ij}(\varepsilon_{pq}, \dot{\varepsilon}_{pq})$.

Any differences?
What about the band thickness? Is it again zero?
If not, its thickness depends on?

Onset of localization

$$\left[-n_j L_{ijkl} n_l - \rho\left(\frac{\lambda s}{2\pi}\right)^2 \delta_{ik}\right] g_k = 0$$

At the onset of localization *s->0⁺* :

$$\left[n_j L_{ijkl} n_l\right] g_k = 0$$

$\Gamma_{ik} = n_j L_{ijkl} n_l$ is called acoustic tensor.

Travelling wave equation

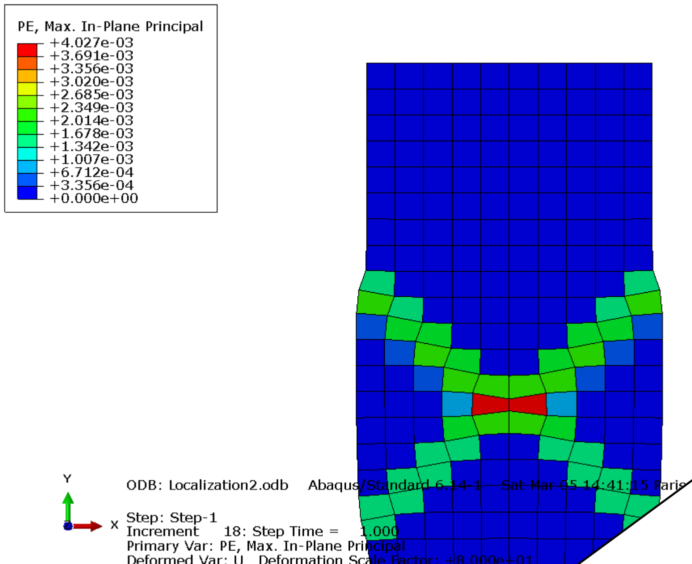
$$\Delta u_i = U_i e^{i\frac{2\pi}{\lambda} n_p x_p + st} = e^{i\frac{2\pi}{\lambda}\left(n_p x_p - i\frac{s\lambda}{2\pi}t\right)} = e^{i\frac{2\pi}{\lambda}\left(n_p x_p - ct\right)}$$

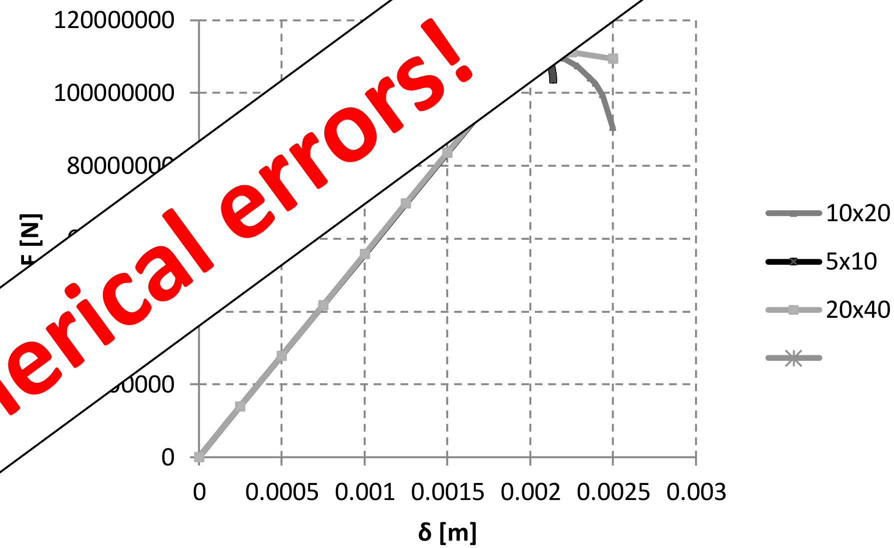
c is the wave speed of the travelling (assumed planar) perturbation.

$$\left[n_j L_{ijkl} n_l - \rho c^2 \delta_{ik} \right] g_k = 0$$

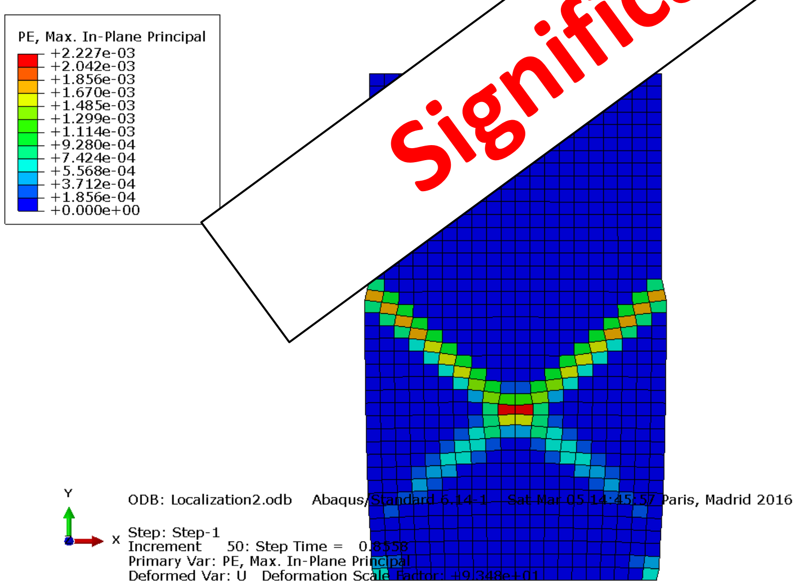
For *c=0* standing waves -> localization (no elastic waves propagation)

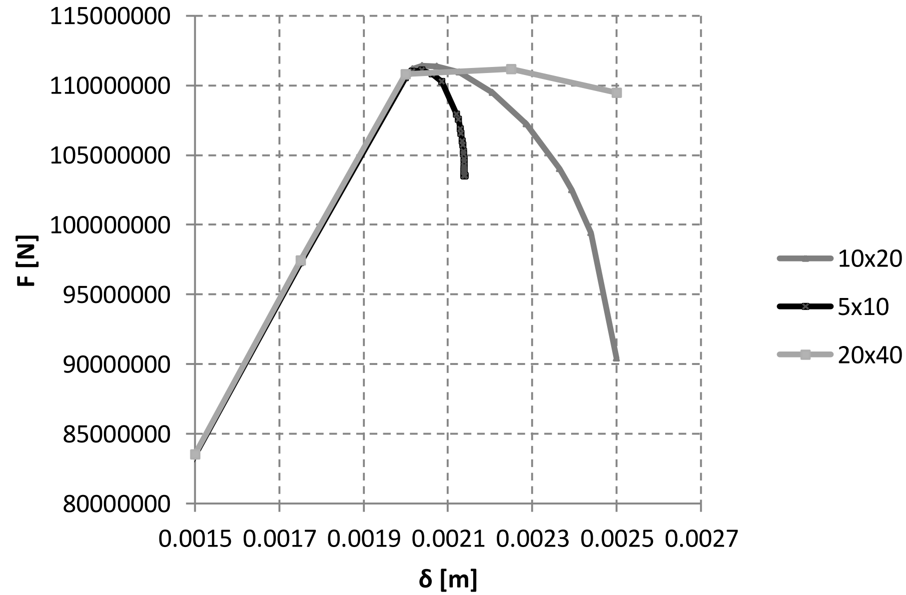
Modeling with Cauchy?



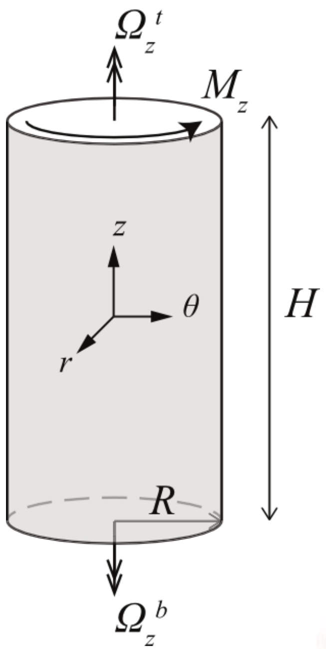


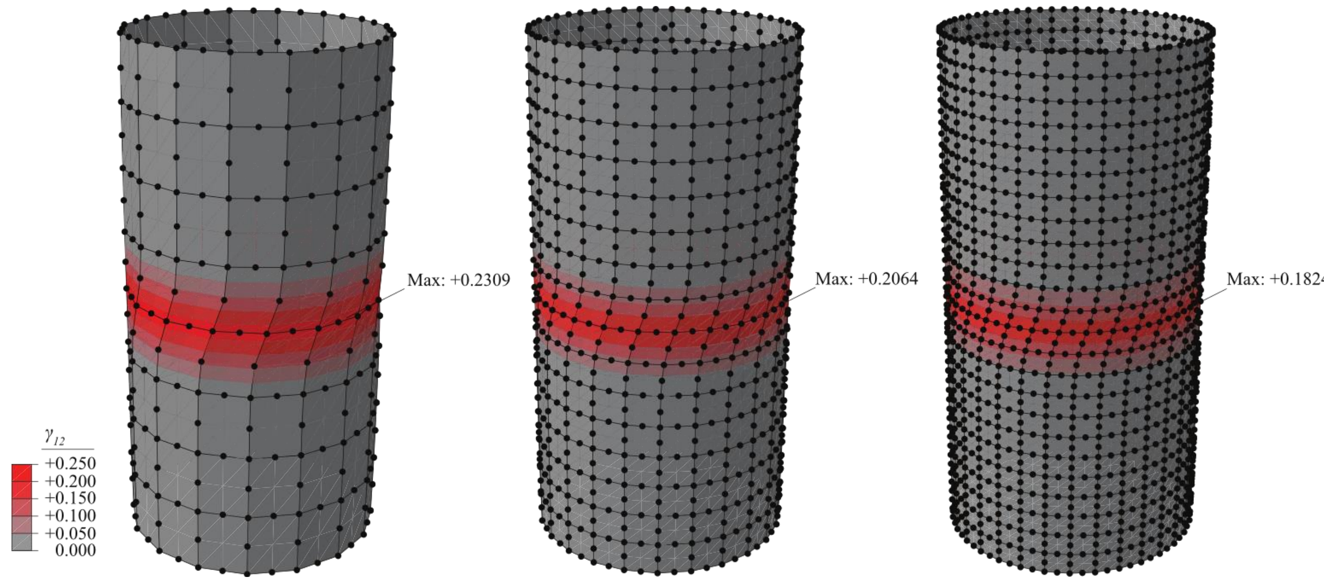
Significant numerical errors!

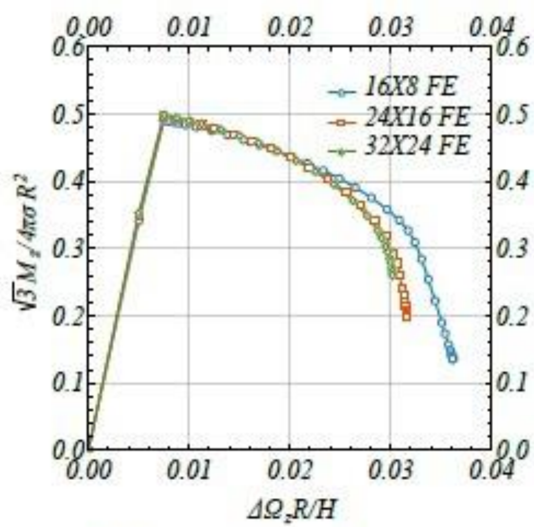




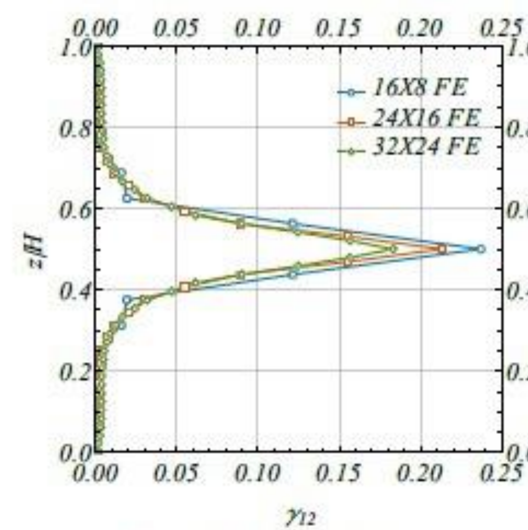
Modeling with Cosserat







Moment-rotation diagram



Shear strain localisation

(Godio et al., 2015,2016)

Summary

- Clarified the concepts of uniqueness ≠ bifurcation ≠ stability
- Set ONE established theoretical framework
- Perform bifurcation and stability analysis
- Show that ODE’s and PDE’s are treated similarly
- Acoustic tensor as a stability problem
- Do the exercises…

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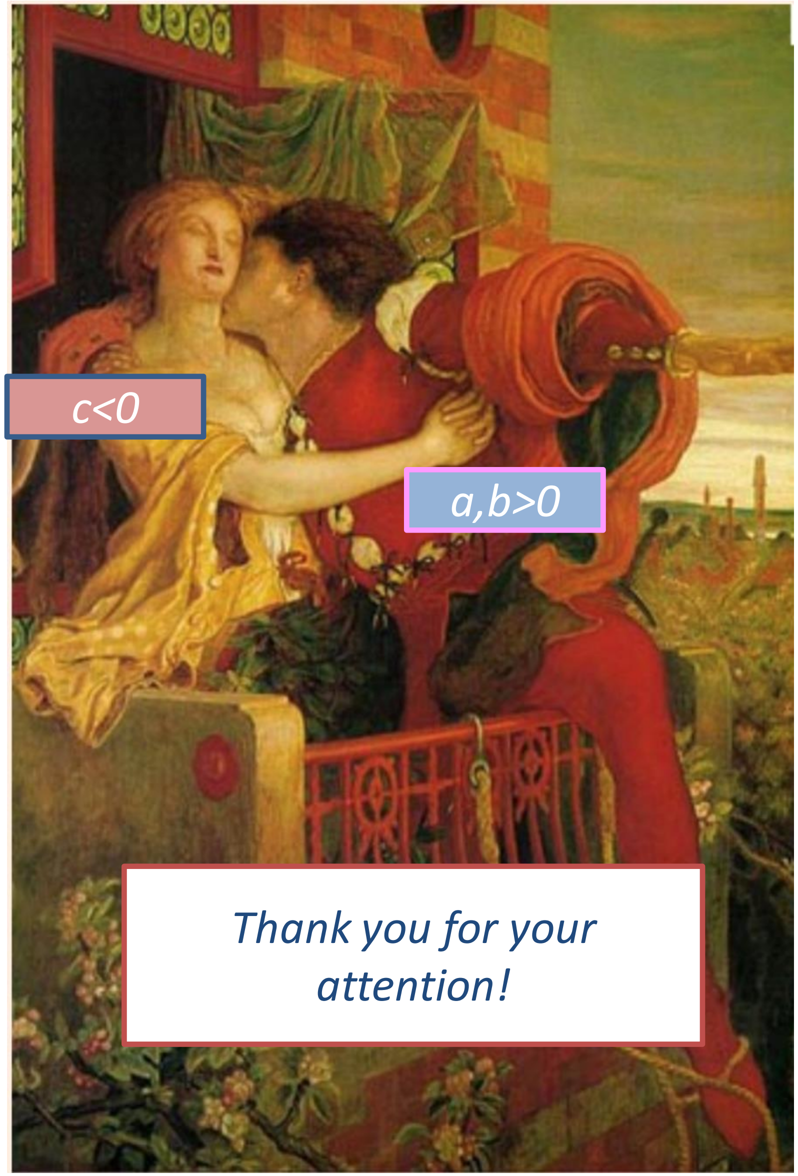

(courtesy: T. Hueckel)