AUT.360 Distributed Control and Optimization of Cyber-Physical Systems

Homework 7

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Problem 1

We have an undirected network of three agents presented in figure 1, where they can share their local information. From ex1data2 file we have split our data to these three agents, and we are designing a consensus-based distributed machine learning algorithm to fit the data.

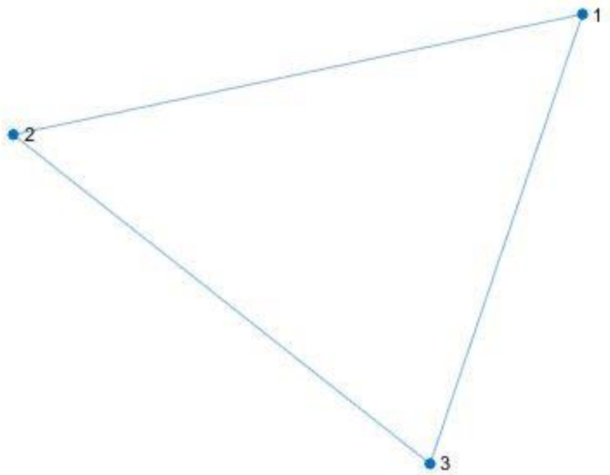

Figure 1: Network of three agents

Firstly, we are going to split the given data into 3 almost equal parts for training data for each of the agents. As we have 47 samples for training, we will split it: 15 samples for the first agent and 16 for each of the remaining two agents. Secondly as the aim is to optimize the gradient descent parameters we need to take an initial guess for them. As we have 3 agents with 3 parameters in each for the 2 features in the given data we need to initialize 9 parameters all together.

We can distribute the data to groups, for group one:

$$h_{\gamma_1}^1(x_1^i) = \gamma_{01} + \gamma_{11} \cdot x_1^i + \gamma_{21} \cdot x_2^i \qquad (1)$$

where x_j^i is features from that group and for groups two and three similarly. To fit the data we need to optimize the γ_{ij} parameters for the system.

Once the data is split, we need to normalize it for the gradient descent algorithm.

The cost function for the whole system is (2)

$$J = \frac{1}{3}(J_1 + J_2 + J_3) \qquad (2)$$

where J_i is split cost function for *i*th agent:

$$J_i = \frac{1}{2n_i} \sum_{j=1}^{n_i} (h_{\gamma_i}^i(x_j^i) - y_j^i)^2 \quad (3)$$

where y is the target values.

We create P matrix $P = I - \epsilon L$.

1/3	1/3	1/3
1/3	1/3	1/3
1/3	1/3	1/3

Table 1: P matrix

and we can calculate, as $\gamma \in \mathbb{R}^3$,

$$\gamma(k+1) = (P \otimes I^{3x3}) \cdot \gamma(k) - \propto (k) \cdot \nabla J$$

For the algorithm it is necessary to use diminishing step size.

Partial derivates for ∇J from (3) we get:

$$\frac{dJ_1}{d\gamma_{01}} = \frac{1}{n_1} \sum_{j=1}^{n_1} (\gamma_{01} + \gamma_{11} \cdot x_1^j + \gamma_{21} \cdot x_2^j - y_1^j)$$

$$\frac{dJ_1}{d\gamma_{11}} = \frac{1}{n_1} \sum_{j=1}^{n_1} (\gamma_{01} + \gamma_{11} \cdot x_1^j + \gamma_{21} \cdot x_2^j - y_1^j) \cdot x_1^j$$

$$\frac{dJ_1}{d\gamma_{21}} = \frac{1}{n_1} \sum_{j=1}^{n_1} (\gamma_{01} + \gamma_{11} \cdot x_1^j + \gamma_{21} \cdot x_2^j - y_1^j) \cdot x_2^j$$

and same for agents two and three.

Calculating for the solutions we get the optimized parameters:

$$\gamma^* = [3.4020,\ 1.0903,\ -0.0475,\ 3.4020,\ 1.0903,\ -0.0475,\ 3.4020,\ 1.0903,\ -0.0475] * 10^5.$$

As we can see all the parameters from the three agents are the same. This means that the system has reached consensus on the parameters and found the optimal solutions for fitting to the data. The evolution of the cost function is presented in figure 2. For the results we used 230000 iterations of the algorithm. But as we can see no major improvements in the cost function happen after the first about 5000 iterations. We can see that the total cost is lowered significantly which is what we were looking for.

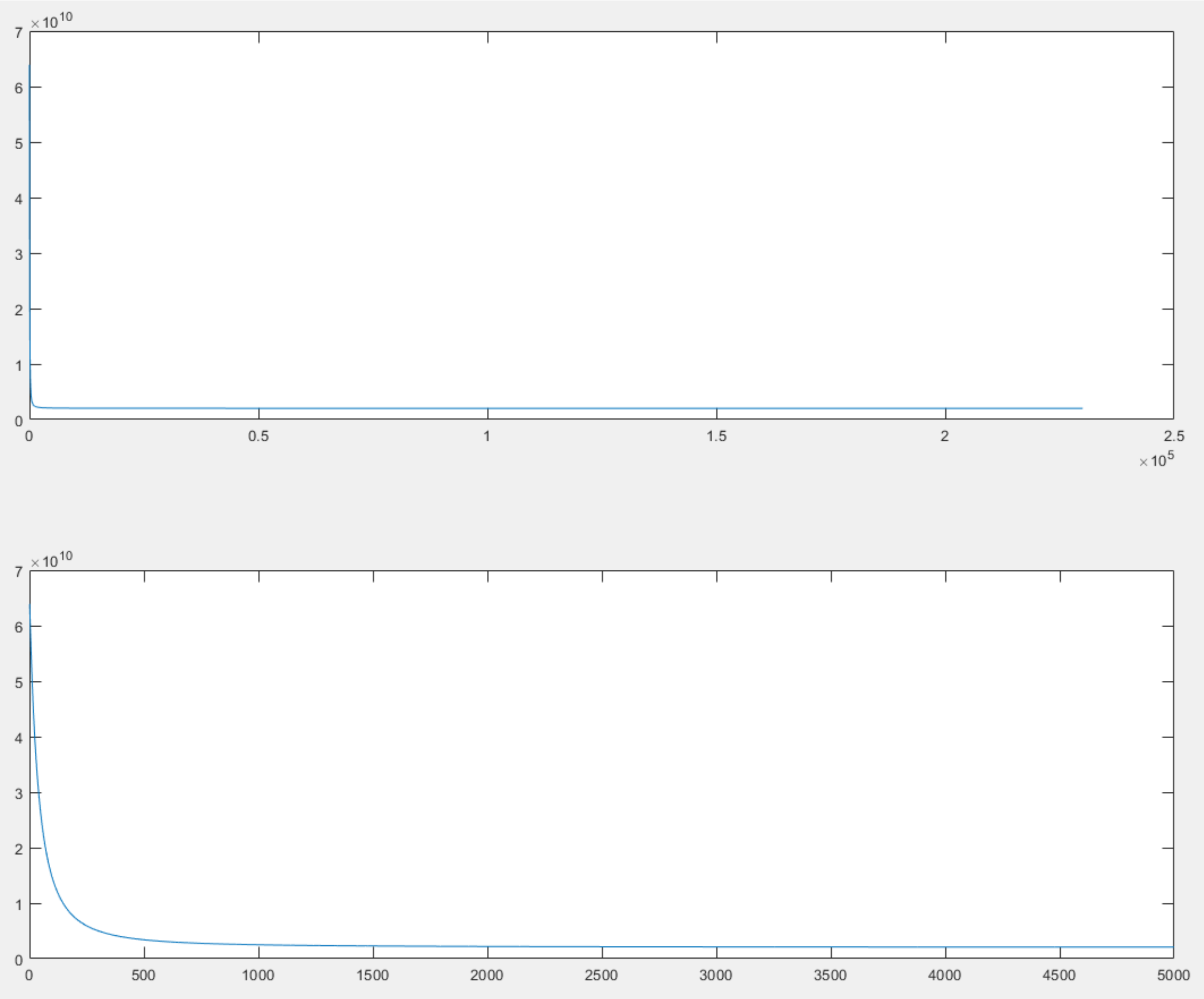

Figure 2: Evolution of the cost function with respect to iterations. Upper is the whole cost function with the 230000 iterations and the lower is the first 5000 iterations to better show the shape.

Problem 2

We are given the network in figure 3. We need to solve the following optimization problem in a distributed way.

$$minimize\ C(P) = \sum_{i=1}^{4} C_i(P_i)\ subject\ to\ \sum_{i=1}^{4} P_i = L$$

Where L is the total load equal to 100 and the $C_i(P_i)$ are the local cost functions for the generators. As we can see in figure 1 there are a total of 4 generators in the network.

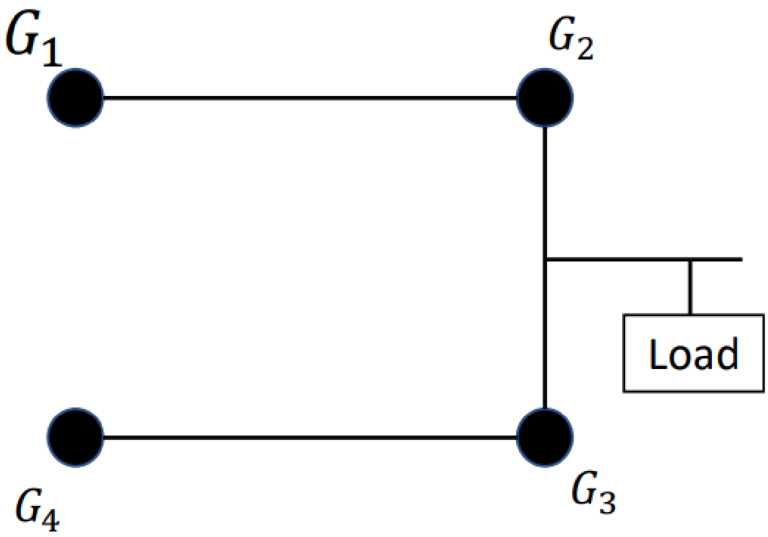

Figure 3: Generator network for problem 2

The local cost functions are:

$$C_1(P_1) = 0.04P_1^2 + 2P_1 + 1$$

$$C_2(P_2) = 0.03P_2^2 + 3P_2 + 3$$

$$C_3(P_3) = 0.035P_1^2 + 4P_3 + 2$$

$$C_4(P_4) = 0.03P_4^2 + 4P_4 + 2$$

where P_i is the power generated by *i*th generator.

To solve the optimization problem in a distributed way we can use the following equation.

$$\hat{P}_i(k+1) = \hat{P}_i(k) - \sum_{j \in N_i} W_{ij}(\dot{C}_i(P_i) - \dot{C}_j(P_j))$$

Where W is a weighted Laplacian matrix for the network. The W matrix can be calculated with the following parameters

$$[W]_{ij} = \begin{cases} -min\left(\frac{1}{|N_i|u_i}, \frac{1}{|N_j|u_j}\right) \\ w_{ii} = -\sum_{j=1}^{n} w_{ij} \end{cases}$$

Where $|N_i|$ is the in-degree of the agent and $|N_j|$ is the in-degree of the *i*th neighbor. And u_i is in this case equal to the second derivative of the cost function of the agent.

With this information we can create the W matrix for the distributed system.

8.3333	-8.3333	0	0
-8.3333	15.4762	-7.1429	0
0	-7.1429	14.2857	-7.1429
0	0	-7.1429	7.1429

Table 2: The W matrix for the network

Now if we iterate long enough the distributed system should come to the optimal solution while still fulfilling the requirements. As long as the original starting values for the generators also fulfil the requirements in other words as long as their sum is equal to L. The results are presented in figure 4.

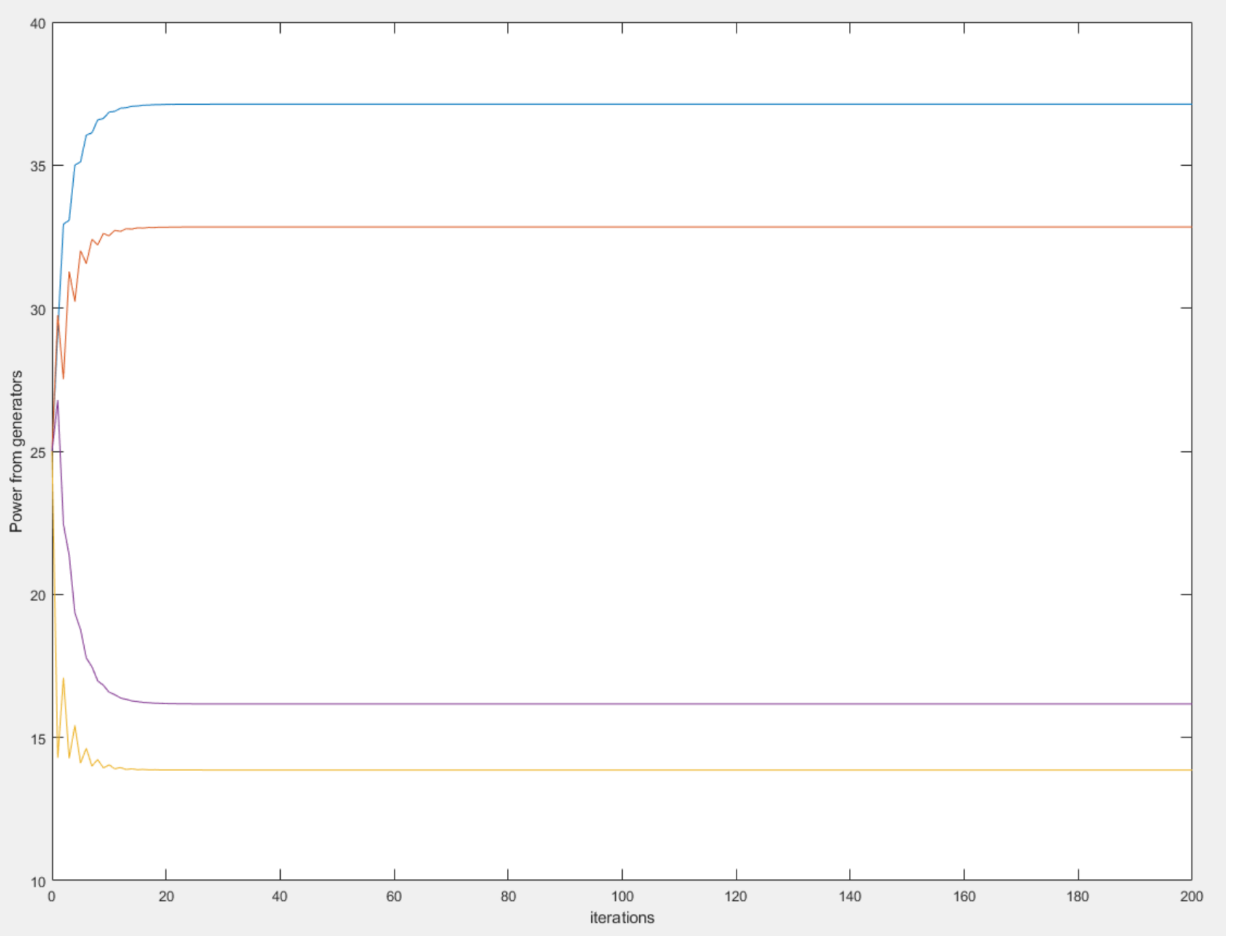

Figure 4: Distributed power generator system

The final values for the generators are:

$$P_1 = 37.1287,\ P_2 = 32.8383,\ P_3 = 13.8614,\ P_4 = 16.1716$$

Total power P = $P_1 + P_2 + P_3 + P_4 = 100$ as it should to fulfil the conditions set to the system.

Finding optimal solution for centralized solution we need the minimum cost function, which we can get from

$$min\ C(P) = \sum_{i=1}^{4} C_i(P_i)$$

which is subject to

$$\sum_{i=1}^{4} P_i = \mathrm{L} \rightarrow\ g(P) = P_1 + P_2 + P_3 + P_4 - L$$

From this we can derive the following condition

$$P_1 + P_2 + P_3 + P_4 = L$$

where L = 100.

$$\nabla C =\ \lambda^{*}\nabla \mathrm{g}(P)$$

$$\begin{bmatrix} 0.08P_1 + 2 \\ 0.06P_2 + 3 \\ 0.07P_3 + 4 \\ 0.06P_4 + 4 \end{bmatrix} = \begin{bmatrix} \lambda^* \\ \lambda^* \\ \lambda^* \\ \lambda^* \end{bmatrix} \rightarrow \begin{bmatrix} 2 * a1 * P_1 + b1 \\ 2 * a2 * P_2 + b2 \\ 2 * a3 * P_3 + b3 \\ 2 * a4 * P_4 + b4 \end{bmatrix} = \begin{bmatrix} \lambda^* \\ \lambda^* \\ \lambda^* \\ \lambda^* \end{bmatrix}$$

from which we can calculate powers for every generator. Using the condition as the fifth equation.

We can get the following equations for the power of the generators

$$\begin{cases} P_1^* = \dfrac{\lambda^* - b_1}{2a_1} \\ P_2^* = \dfrac{\lambda^* - b_2}{2a_2} \\ P_3^* = \dfrac{\lambda^* - b_3}{2a_3} \\ P_4^* = \dfrac{\lambda^* - b_4}{2a_4} \end{cases}$$

We can solve the still unknow λ* with the L and given conditions for the power production

$$P_1 + P_2 + P_3 + P_4 = L$$

$$\frac{\lambda^* - b_1}{2a_1} + \frac{\lambda^* - b_2}{2a_2} + \frac{\lambda^* - b_3}{2a_3} + \frac{\lambda^* - b_4}{2a_4} = L$$

Now by solving for λ* we get

$$\lambda^* = \frac{\frac{b_1}{2a_1} + \frac{b_2}{2a_2} + \frac{b_3}{2a_3} + \frac{b_4}{2a_4} + L}{\frac{1}{2a_1} + \frac{1}{2a_2} + \frac{1}{2a_3} + \frac{1}{2a_4}}$$

From this we get that the $\lambda^* = 4.9703$, from which we can calculate the powers for each of the generators

$$P_1^* = \frac{\lambda^* - b_1}{2a_1} = 37.1287$$
$$P_2^* = \frac{\lambda^* - b_2}{2a_2} = 32.8383$$
$$P_3^* = \frac{\lambda^* - b_3}{2a_3} = 13.8614$$
$$P_4^* = \frac{\lambda^* - b_4}{2a_4} = 16.1717$$

These are the exact same values as the ones we got form the distributed method which is good news for both methods as both succeed in what was required form them.

Sources

(1) Homework 7, M. Iqbal, A. Gusrialdi, March 19, 2023 https://moodle.tuni.fi/pluginfile.php/3187984/mod_assign/introattachment/0/HW7.pdf?forcedownload=1

(2) Lecture 7 slides, A. Gusrialdi, March 19, 2023. https://moodle.tuni.fi/pluginfile.php/3169473/mod_resource/content/3/AUT360_lecture7.pdf