Tampere University

AUT.360 Distributed Control and Optimization of Cyber-Physical Systems

Homework 8

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Problem 1

We are given a network in figure 1. Let us call this network graph G = (V, E). The goal is to add six links to the graph to maximize the algebraic connectivity of the graph which can also be presented as the following function:

$$max\ \lambda_2(L + \Delta L) \quad (1)$$

Where ΔL is the Laplacian of the graph $\bar{G} = (V, E_{add})$ with $E_{add} \subseteq E^c$ and E^c is the set of links corresponding to the complement of the original graph G.

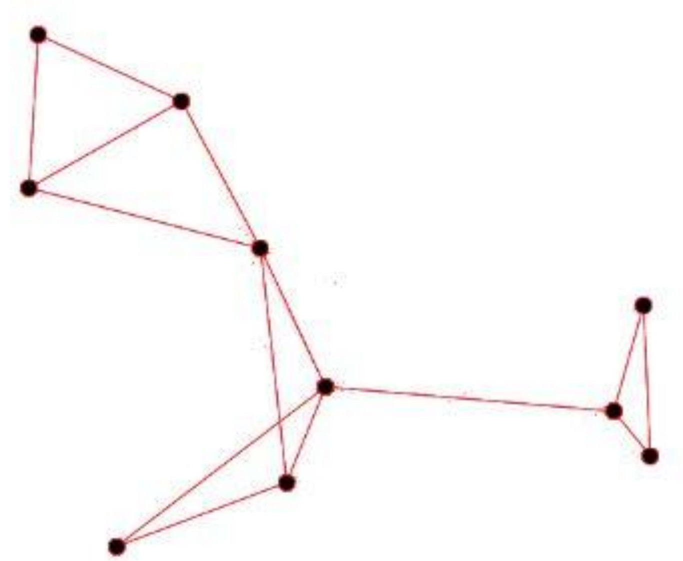

Figure 1: A network with 10 nodes and 14 edges.

We need to find the complement graph for the network and it is shown in figure 2. The original networks its λ_2 is 0.2604 and our task is to improve it as much as we can by adding six links to the network.

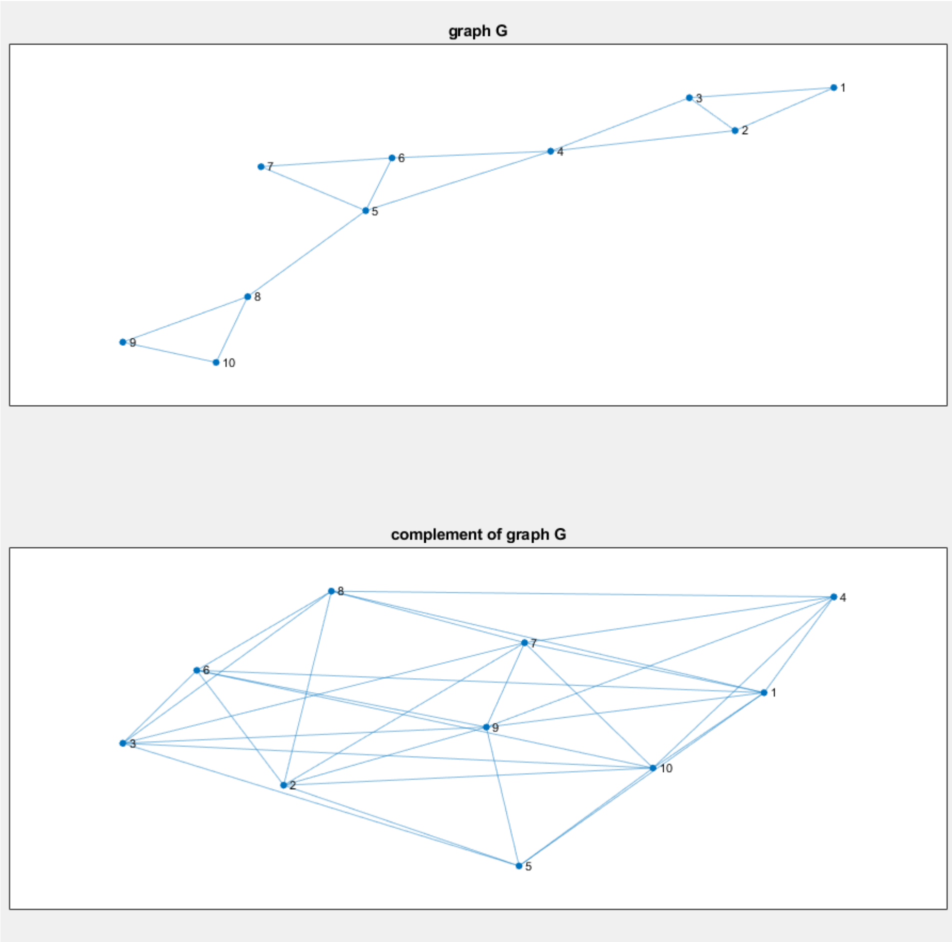

Figure 2: Original network and its compliment plotted with MATLAB.

1. First, we try the brute force method, adding all possible different combinations of 6 links at a time and comparing the results then taking that which gave the maximum lambda value. We are computing $\lambda_2(L + \Delta L)$ for every possible set of links.

   With brute-force searching the best links to add were

   1-7, 1-9, 2-8, 3-10, 6-10, 7-9

   Adding these edges to the graph improved its λ_2 to 2.2679 which is a huge improvement. The brute force method found the best solution, but it was slow and taxing on the computing resource.

2. To reduce this computing complexity, we will use a method based on matrix perturbation theorem

$$\lambda_2(L + \Delta L) = \lambda_2(L) + v_2^T \Delta L v_2 + H.O.T \quad (3)$$

from which the higher order term reduces close to zero and $v_2^T \Delta L v_2$ can be approximated as $\Delta\lambda_2$. With this in mind we can write the equation 3 as:

$$\lambda_2(L + \Delta L) = \lambda_2(L) + \Delta\lambda_2 \quad (4)$$

And so, the original problem in equation 1 can be reformulated as:

$$\max_{y_{l_c}} \Delta\lambda_2 = \max_{y_{l_c}} \sum_{l_c=1}^{|E^c|} y_{l_c}(v_{2,i} - v_{2,j})^2 \quad (5)$$

Where $y_{l_c} \in \{0,1\}$ and $y = [y_1, \ldots, y_{|E^c|}]^T$

We will add two links at a time using the equation 5 to determine which two links to add at which iteration. The graph will be updated at each iteration and the new graph used to determine which two links are to be added next. These steps are repeated until we have added all six links.

On the first iteration we add links

2-8, 7-8

And on the second iteration we add

1-8, 1-10

And on the third and final iteration we add

6-9, 3-7

After adding all six links the λ_2 has increased to 1.7271.

3. The problem, theory and methodology are the same as in the previous part, but now we are adding three links at a time.

On the first iteration we add links

2-8, 7-8, 3-8

And on the second iteration we add

4-8, 1-4, 1-10

After adding all the links our λ_2 has increased to 1.1814.

4. The problem, theory and methodology are the same as in the previous part, but now we are adding all six links at a time.

   The links we add are

   2-8, 7-8, 3-8, 6-8, 4-8, 1-7

   After adding the links our λ_2 has increased to 0.9457.

5. All the resulting graphs are plotted below in figures 3-6.

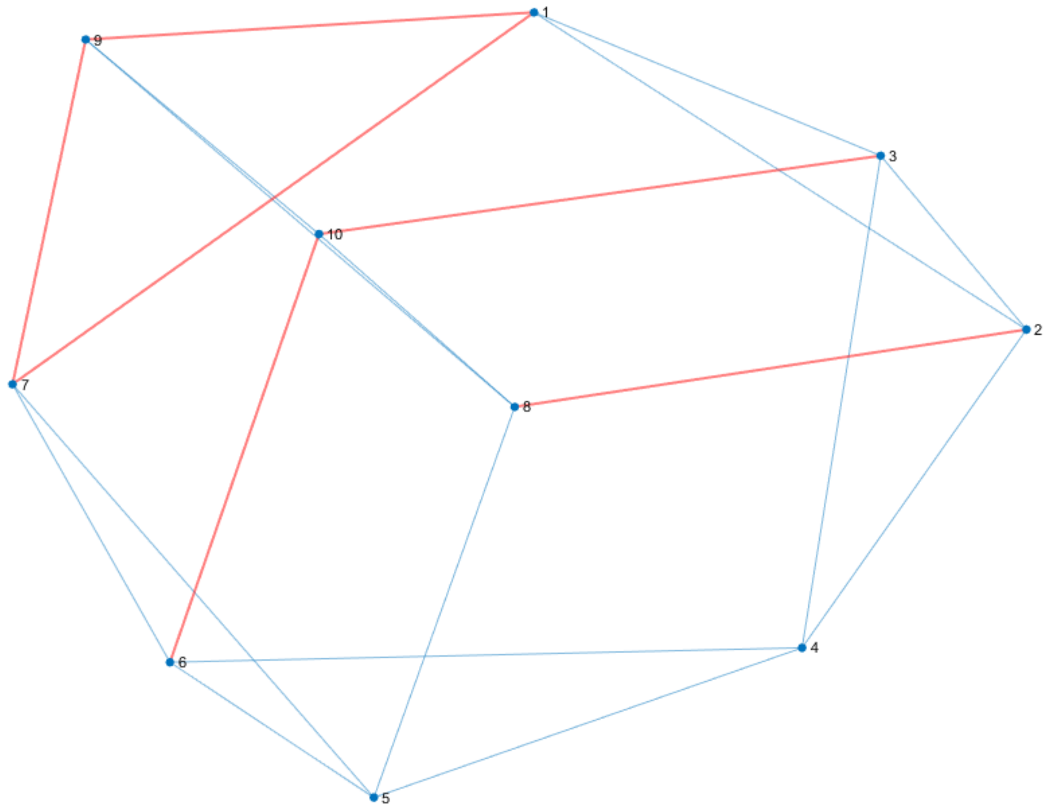

Figure 3: The new graph with brute-force search

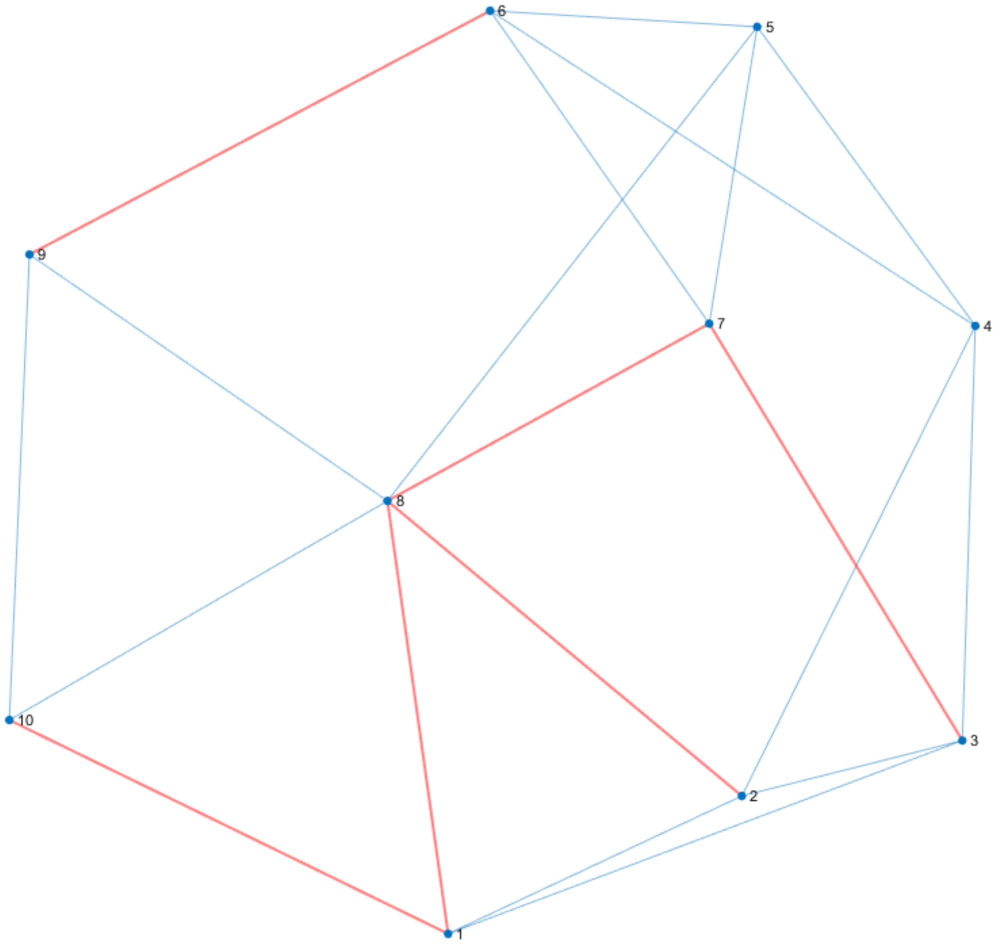

Figure 4: Two links at a time. The added links are in red.

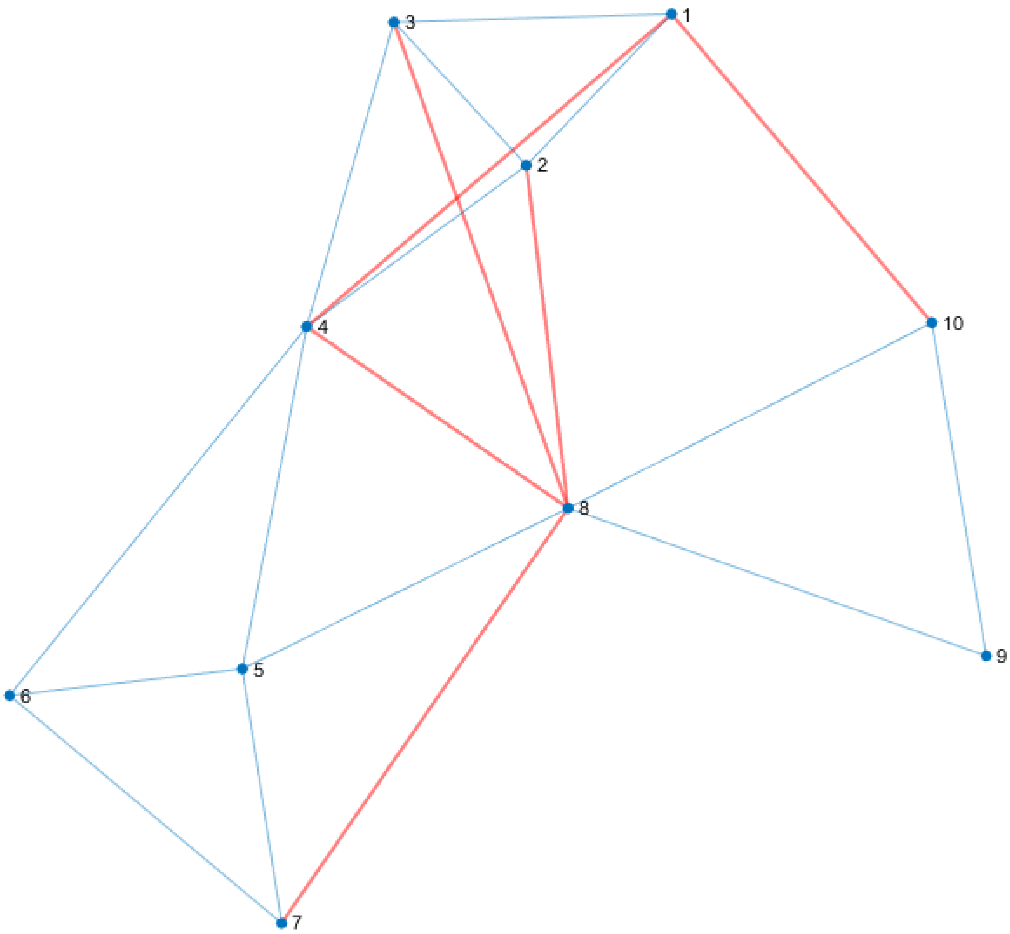

Figure 5: Three links at a time. The added links are in red.

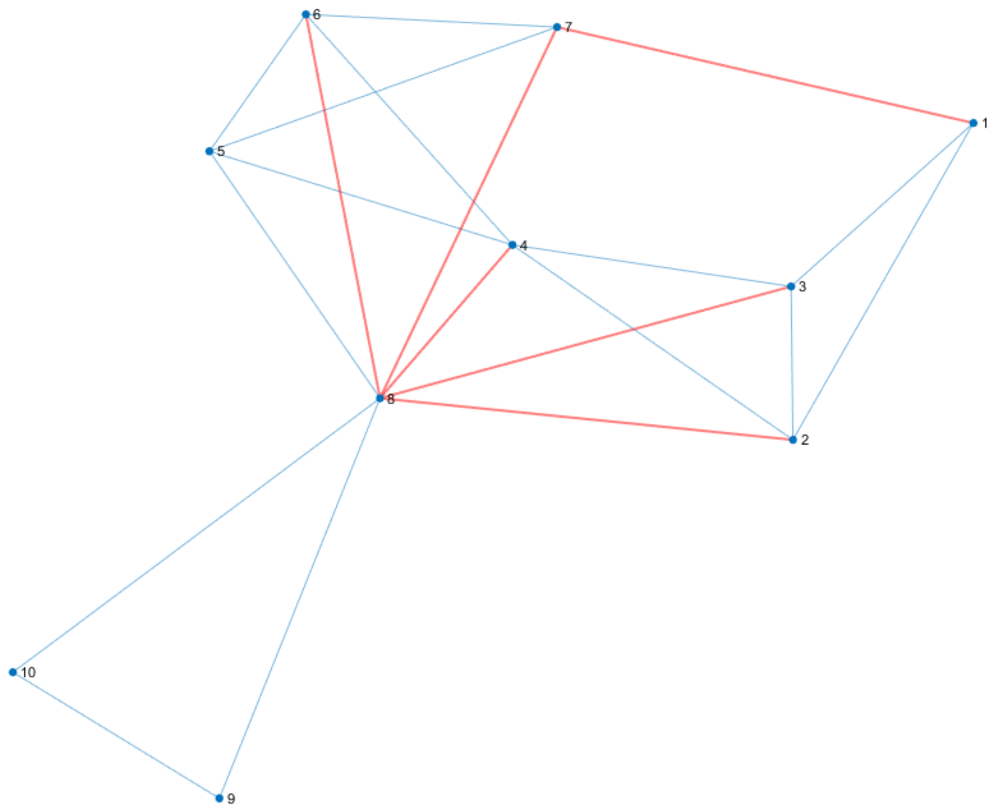

Figure 6: Six links at a time. The added links are in red.

i) For the method using matrix perturbation we can clearly see that adding fewer links at a time provides better results than adding many links at a time. This is not surprising as adding fewer links at a time forces us to do more iterations and calculate the best position for the next link to be added to the updated graph. From the results in this paper the best results with matrix perturbation were from when only two links were added at a time. Which is also clear from the λ_2 values shown earlier.

ii) Comparison between the matrix perturbation and brute-force search, we can see that brute force gives the best connectivity, but also takes a lot of time as it goes through every possibility of possible link combinations. Matrix perturbation is faster, but the solutions are not quite as connected.

Sources

(1) Homework 8, M. Iqbal, A. Gusrialdi, March 24, 2023
https://moodle.tuni.fi/pluginfile.php/3198499/mod_assign/introattachment/0/HW8.pdf?forcedownload=1

(2) Lecture 9 slides, A. Gusrialdi, March 24, 2023.
https://moodle.tuni.fi/mod/resource/view.php?id=2284740