AUT.360 Distributed Control and Optimization of Cyber-Physical Systems

Homework 6

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Problem 1

1. Data from the file ex1data2 is plotted into a 3D graph to represent all the possible axis of the given data. This is plotted in figure 1. Furthermore the same data is then shown as plotted in the 2D with the price as a function of the number of rooms and the size of the apartment and lastly the rooms of the apartment as the function of the size of the apartment. These are shown in figures 2-4.

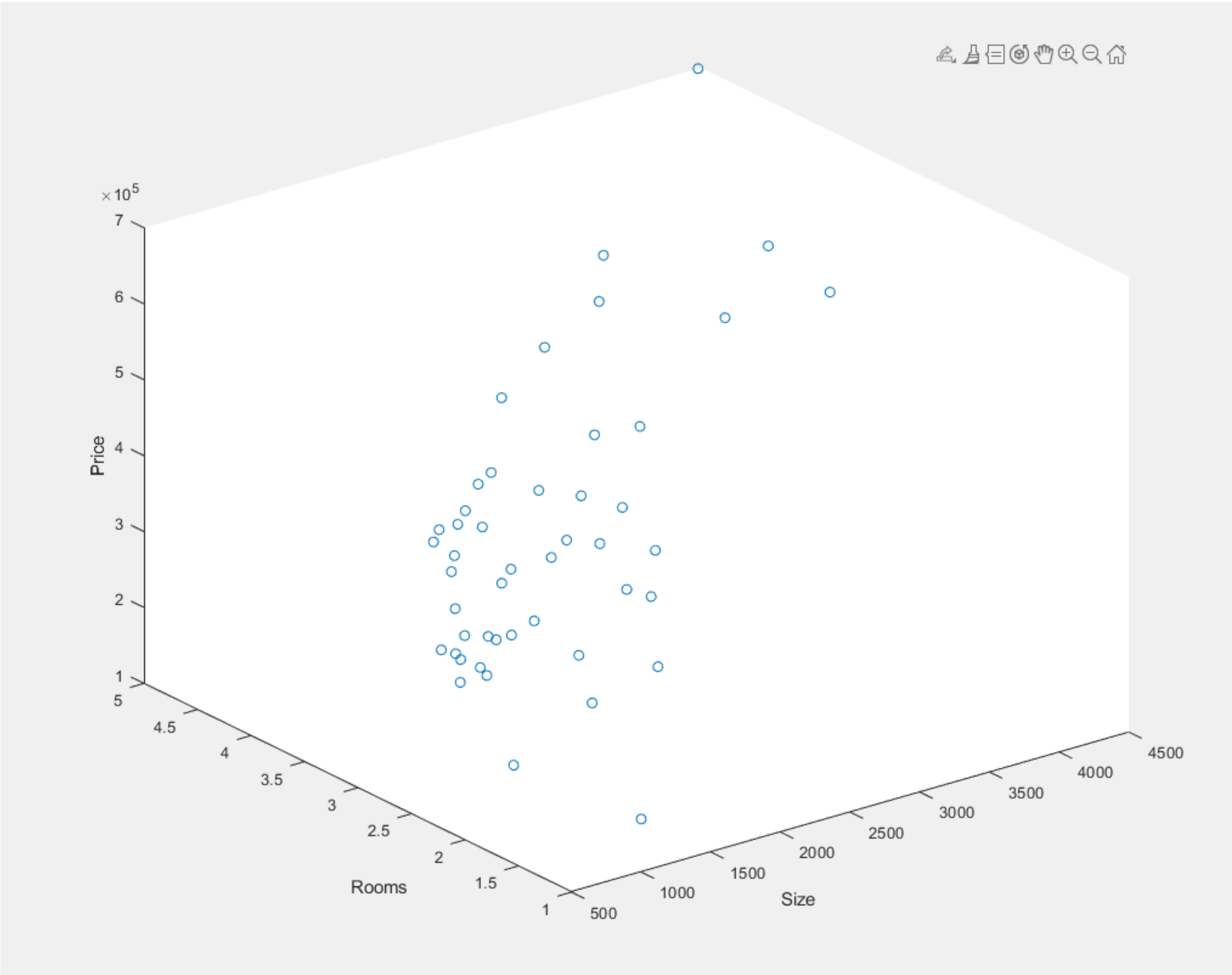

Figure 1: Data from file ex1data2

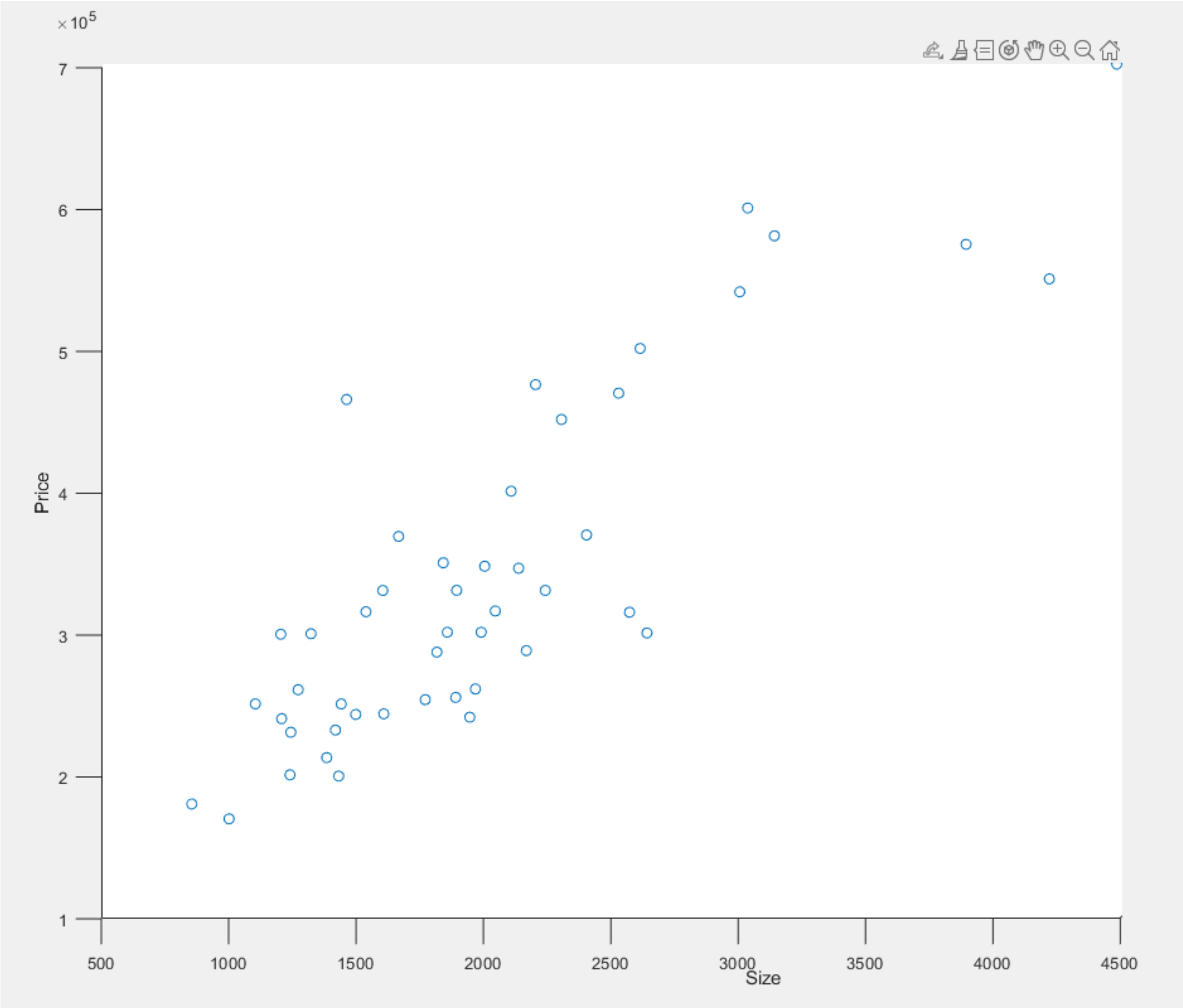

Figure 2: Price as function of size of the apartment

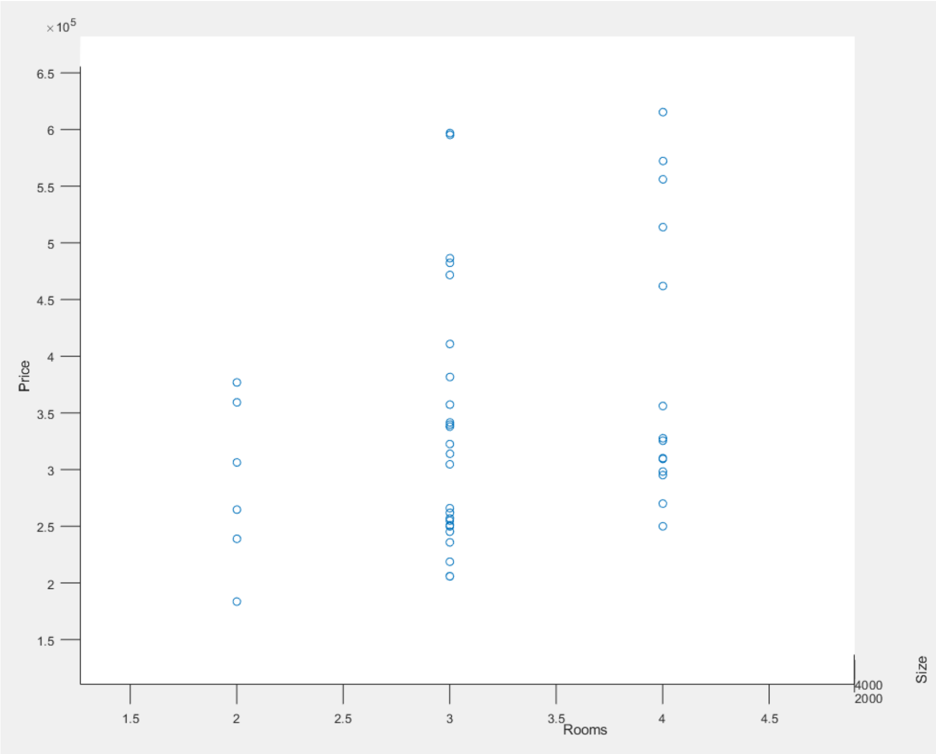

Figure 3: Price as a function of the number of rooms.

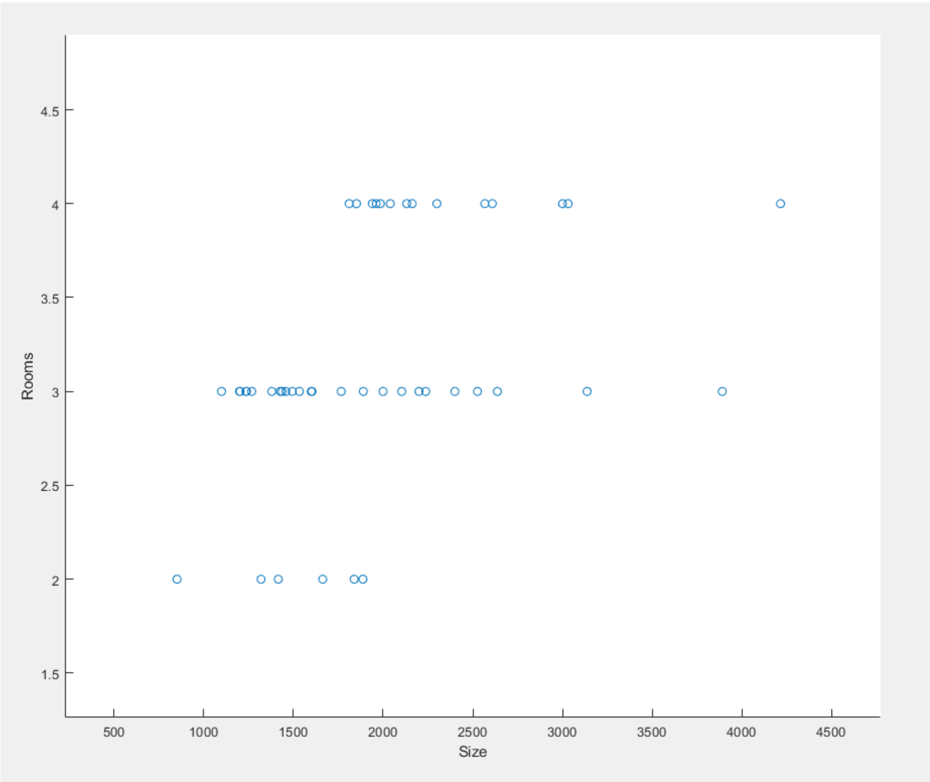

Figure 4: Rooms as a function of the size of the apartment

Least square gives us the function that passes closest to all the given data points. Linear model here would give us an easily digestible function/model that estimates on how the price grows as the size and number of rooms increase. So we would get a simple model that would somewhat give us a reasonable price given the size and room data.

2. We want a linear model fitting into our 3D data. This line should be in the form of

$$Price = \alpha * Size + \beta * Rooms + \gamma \qquad (1)$$

We now need to solve the values of alpha, beta and gamma with the least square method.

We have the features let's call them x and the targets let us call them y from the given data. We also know that we have 2 different features in x. Let us call the size x1 and the rooms x2. Now we can have:

$$X = \begin{bmatrix} 1 & x_1^1 & x_1^2 \\ 1 & x_2^1 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n^1 & x_n^2 \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

We wish to minimize the following function in least square method:

$$f(\gamma_0, \gamma_1, \gamma_2) = \|X\gamma - y\|^2 \ , \qquad (2)$$

where $\gamma = [\gamma_0 \quad \gamma_1 \quad \gamma_2]^T$

To find the minimum the derivative for this function needs to be 0 which gives us the following closed-form solution

$$\gamma^* = (X^T X)^{-1} X^T y \qquad (3)$$

And with these parameters we can then plot the model fit. The plotted least square model is presented in figures 5-7. The parameters we got for the data were:

$$\gamma_0 = 8.9598 * 10^4, \qquad \gamma_1 = 0.0139 * 10^4, \qquad \gamma_2 = -08738 * 10^4$$

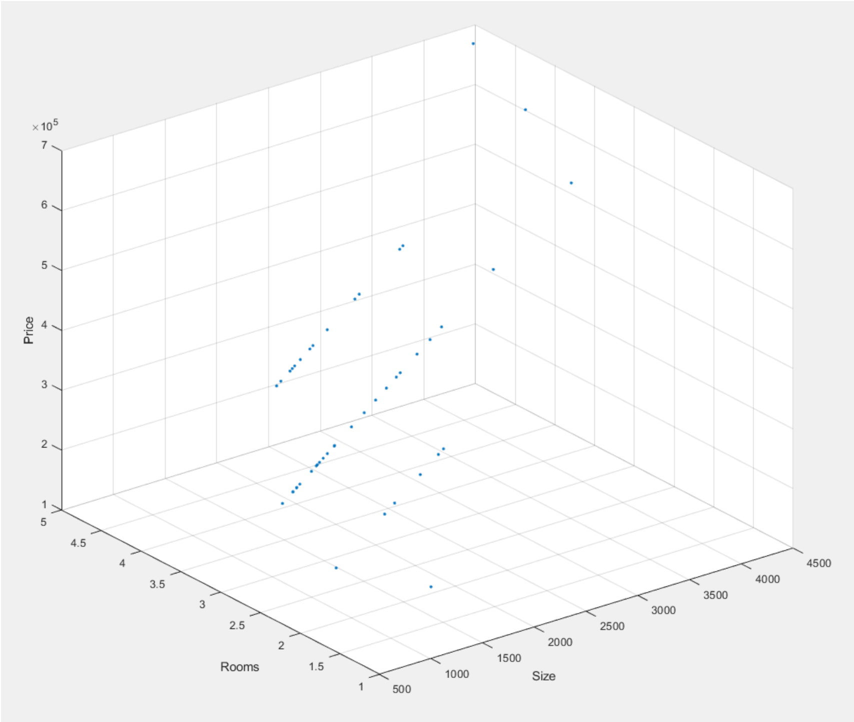

Figure 5: Least square model plot

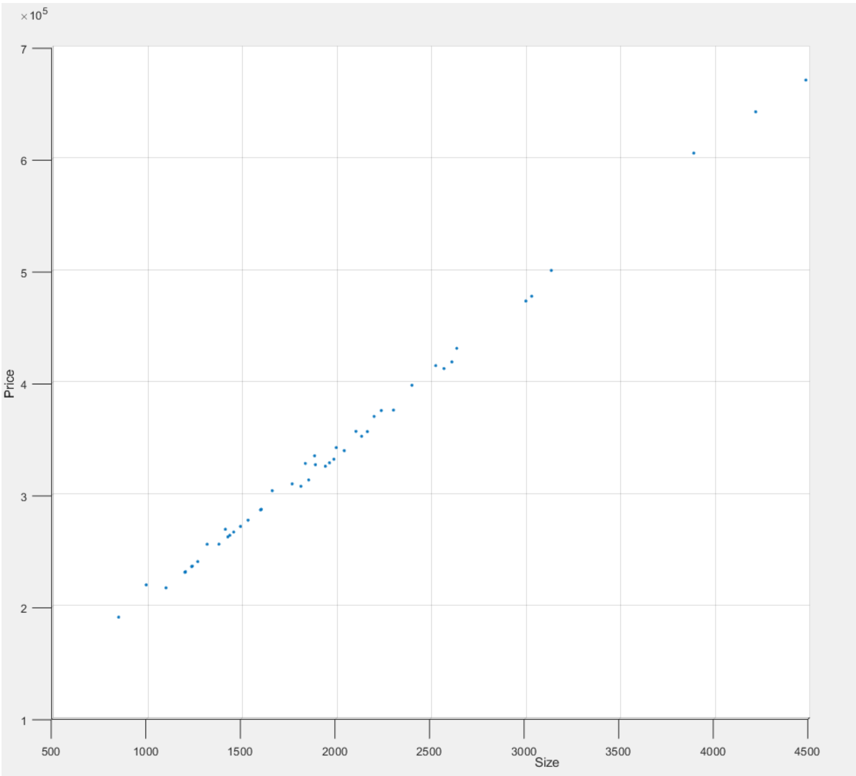

Figure 6: Least square model plot price as a function of size of the apartment

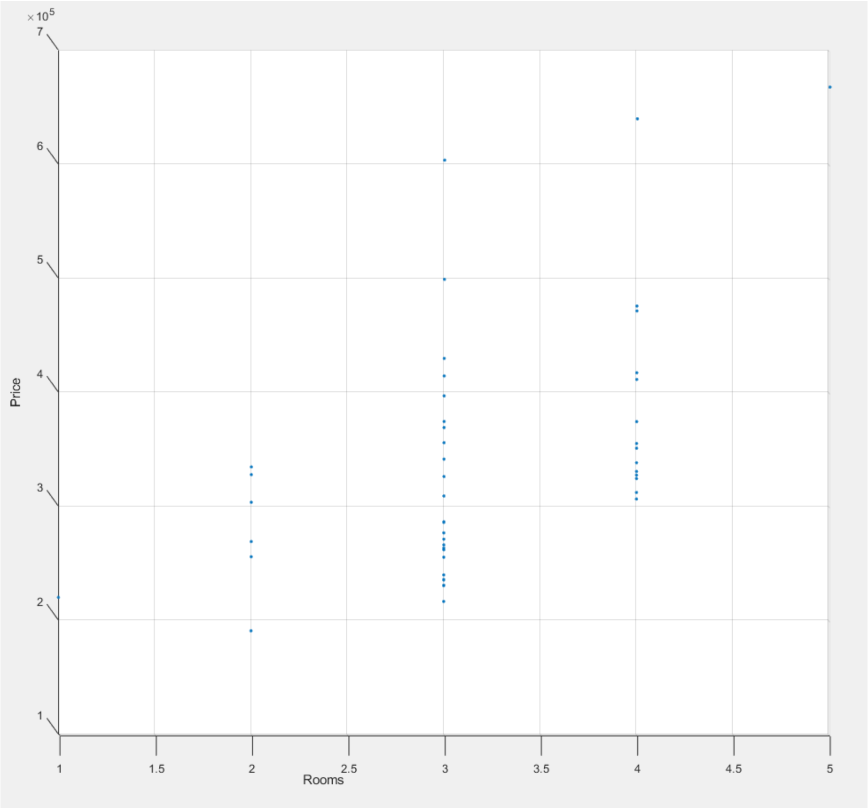

Figure 7: Least square model plot price as a function of the number of rooms

3. Model with gradient descent

Gradient descent is an iterative method for finding the optimal solution for the given function. It requires for the given function to be differentiable. Gradient descent algorithm consists of two steps. We need to start with an initial estimate of the optimal solution and then we update that estimate for the desired number of iterations. So that

$$f\big(x(k+1)\big) < f(x(k)) \tag{4}$$

The updated estimate is given by the algorithm itself:

$$x(k+1) = x(k) - \varepsilon(k)\nabla f(x(k)) \tag{5}$$

Where ε is the step size and the gradient tells the direction for the step. The step size can be either constant or diminishing. In figures 8-10 our model created with gradient descent is presented.

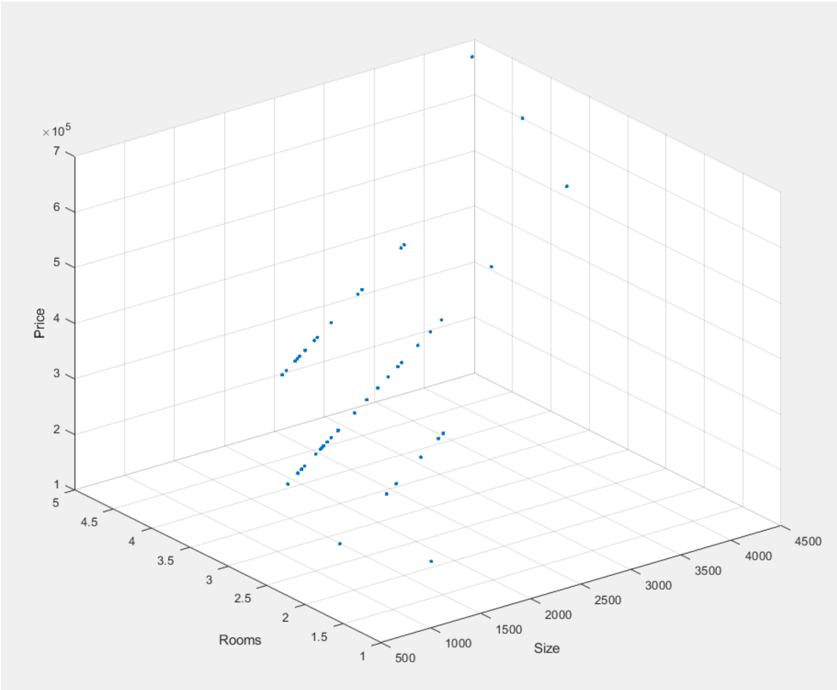

Figure 8: Model with gradient descent

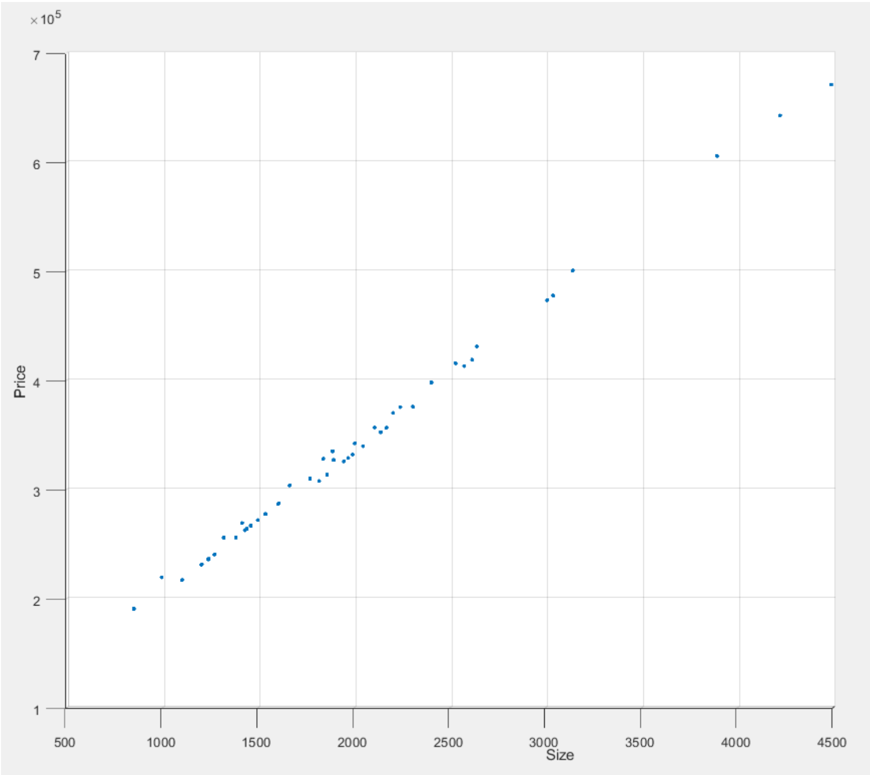

Figure 9: Price as a function of size with gradient descent

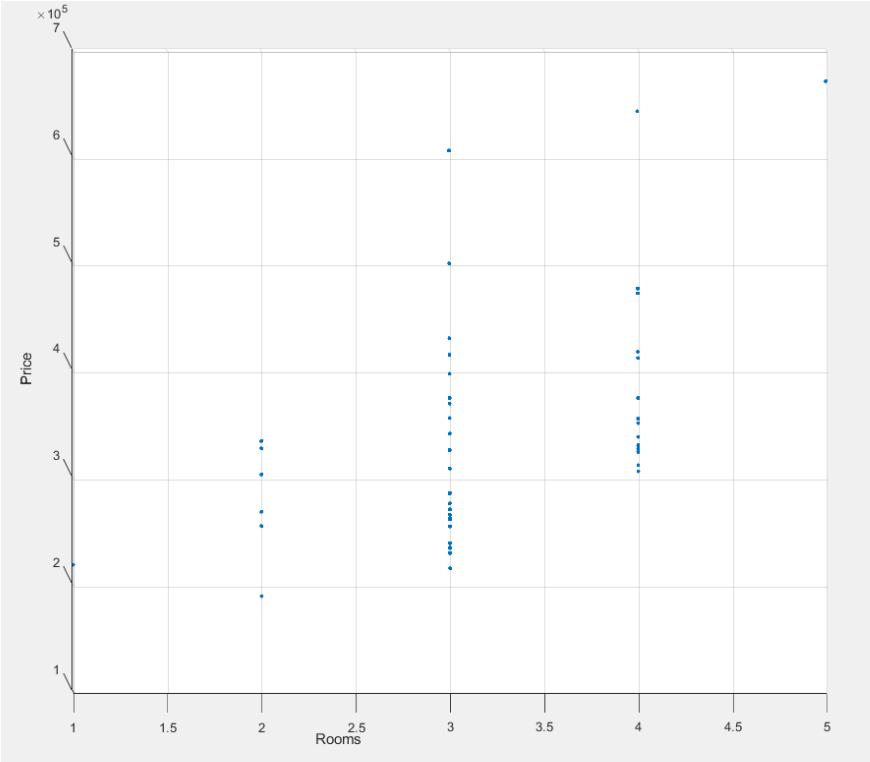

Figure 10: Price as a function of rooms with gradient descent

Parameters for the model with gradient descent

$$\gamma_0 = 3.4041 * 10^5, \qquad \gamma_1 = 1.1063 * 10^4, \qquad \gamma_2 = -0.0665 * 10^5$$

Problem 2

We have an undirected ring network of multiple agents of four agents which can share their local information. The aim is to solve the given optimization problem

$$\min_{x \in \mathbb{R}} f(x) = \frac{1}{4} \sum_{i=1}^{4} f_i(x) \tag{6}$$

Where f_i is a local function available for Agent i. The functions for the agents are:

$$f_1(x_1) = (x_1 - 1)^2,\ f_2(x_2) = {x_2}^2 + 1,\ f_3(x_3) = 4(x_3 - 1)^2,\ f_4(x_4) = (x_4 + 5)^2$$

1. We need to find local minimums for all the given f functions. To do this we can find the minimum values of the derivatives of the functions and then determine if the found value is a minimum or a maximum value for the function.

$$f_1(x_1) = (x_1 - 1)^2, \quad \dot{f}_1(x_1) = 2x_1 - 2\,, \quad \dot{f}_1(x_1) = 0 \; -> \; x_1 = 1$$

$$f_2(x_2) = {x_2}^2 + 1, \quad \dot{f}_2(x_2) = 2x_2\,, \quad \dot{f}_2(x_2) = 0 \; -> \; x_2 = 0$$

$$f_3(x_3) = 4(x_3 - 1)^2, \quad \dot{f}_3(x_3) = 8x_3 - 8\,, \quad \dot{f}_3(x_3) = 0 \; -> \; x_3 = 1$$

$$f_4(x_4) = (x_4 + 5)^2, \quad \dot{f}_4(x_4) = 2x_4 + 10\,, \quad \dot{f}_3(x_3) = 0 \; -> \; x_3 = -5$$

Now for example by plotting the functions we can easily determine if the points are minimum or maximum points. This plotting is presented in figure 11 and from it we can say that the found values are global minimums for the local functions.

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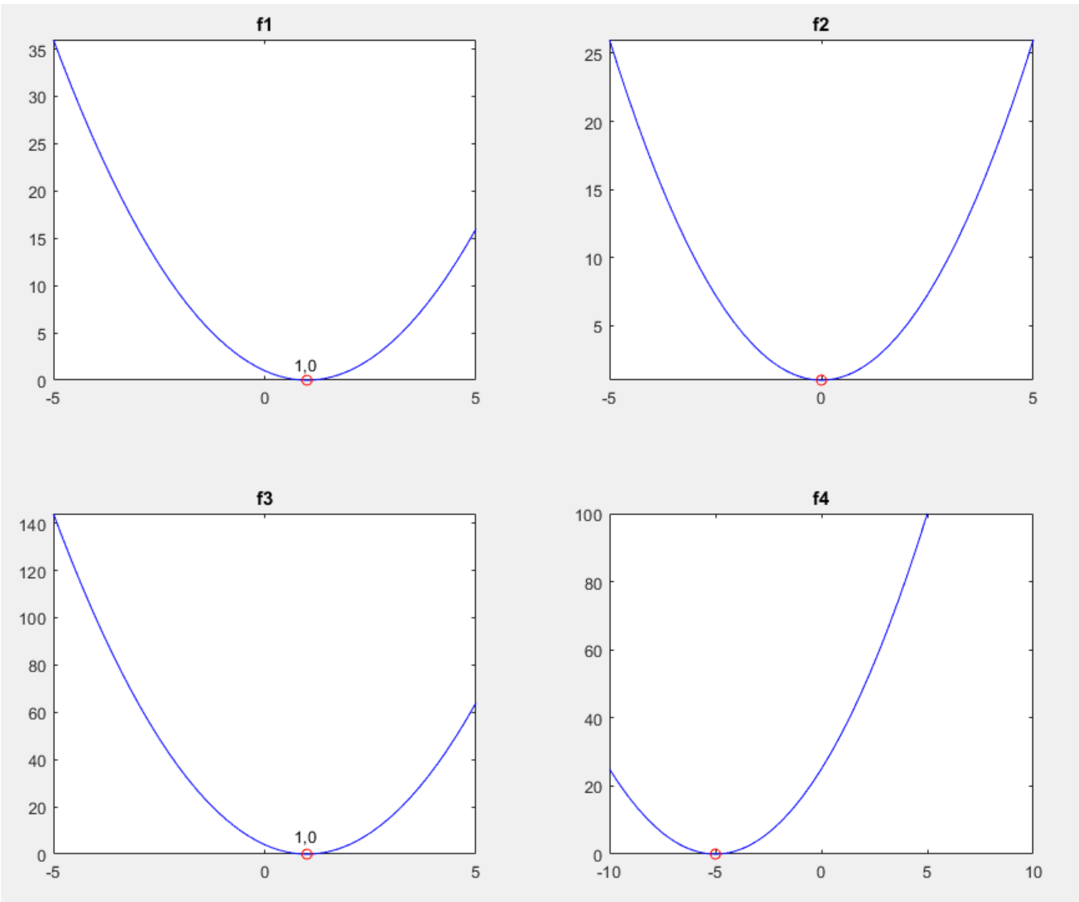

Figure 11: Local functions plotted.

Now we need to compute the minimum for the function 6, which represents the whole system. First let us calculate the system function.

$$f(x) = \frac{1}{4}\sum_{i=1}^{4} f_i(x) = \frac{1}{4} * (((x-1)^2) + (x^2+1) + (4(x-1)^2) + ((x+5)^2))$$

$$f(x) = \frac{1}{4} * ((x^2 - 2x + 1) + (x^2+1) + (4x^2 - 8x + 4) + (x^2 + 10x + 25))$$

$$f(x) = \frac{1}{4} * (7x^2 + 31) = \frac{7}{4}x^2 + \frac{31}{4}$$

We can use the same way as before to get the minimum of the function. First, we need to derivate the system function and set it to equal zero, then check if the value corresponds to a global minimum or a maximum.

$$\dot{f}(x) = \frac{7}{2}x, \qquad \dot{f}(x) = 0 -> x = 0$$

We can now plot the function to check if the function is at a minimum point at the gotten value. This is presented in figure 12 and shows us that it in fact is the global minimum point so we can say that the system gets its minimum value when x is equal to zero.

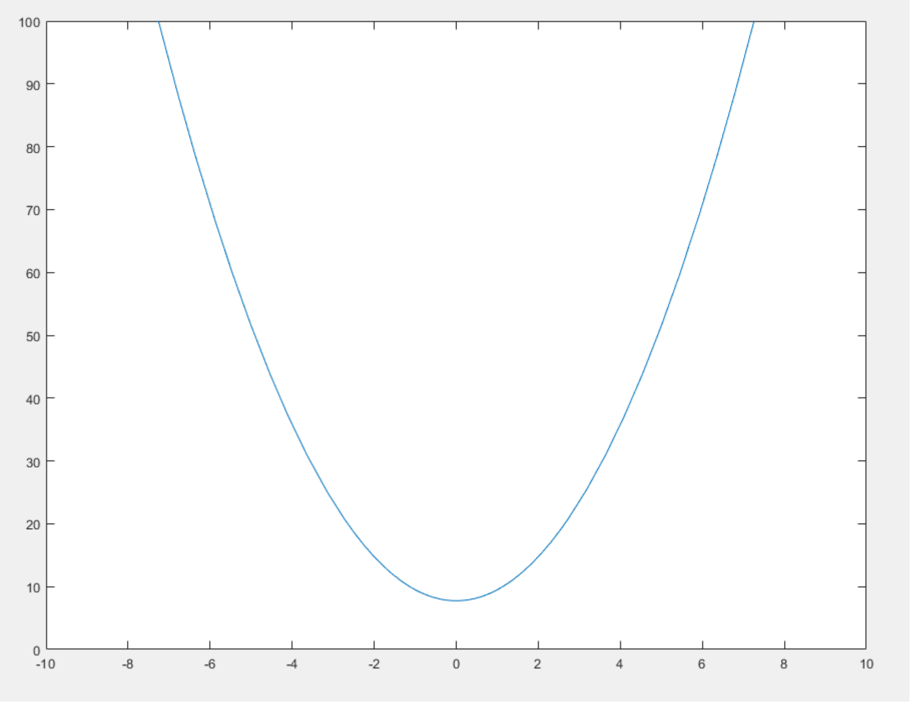

Figure 12: System function plotted

2. In problem 2 we need to design a discrete-time consensus-based distributed algorithm for the function (6). This network is an undirected ring of four agents where they can share their information.

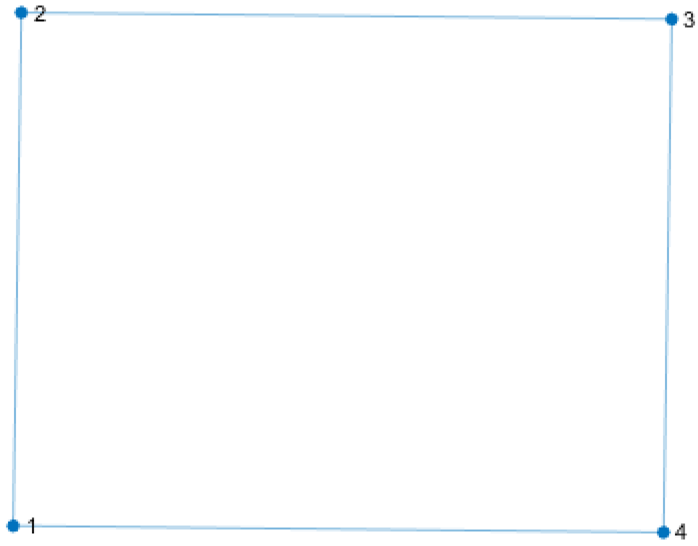

Figure 13: System network topology

We can use consensus-based distributed gradient descent to solve the problem.

$$x(k+1) = \sum_{j=1}^{N} p_{ij} x_j(k) - \epsilon(k) \nabla f_i(x_i(k)) \quad (7)$$

Now we need to get the discrete time system matrix P which is used in the equation 7. For this we need the Laplacian matrix of our system and the step-size.

2	-1	0	-1
-1	2	-1	0
0	-1	2	-1
-1	0	-1	2

Table 1: Laplacian matrix for the network

where step size ϵ is diminishing and $\epsilon \in \left(0, \frac{1}{\Delta}\right)$, $\nabla \mathrm{f(x)}^T < 0$ and largest in-degree $\Delta = 2$. We need to create matrix P which can be calculated from $P = I_n - \epsilon L$. In the simulation step-size diminished by half in every iteration.

0.2	0.4	0	0.4
0.4	0.2	0.4	0
0	0.4	0.2	0.4
0.4	0	0.4	0.2

Table 2: P matrix of the discrete time network

Now by implementing the algorithm 7 in MATLAB we can get the history data for the x_i values. The results are presented in Figure 14 with random starting points. In figure 15 is the cost function of all the agents and the cost function of the whole system plotted as the algorithm progresses.

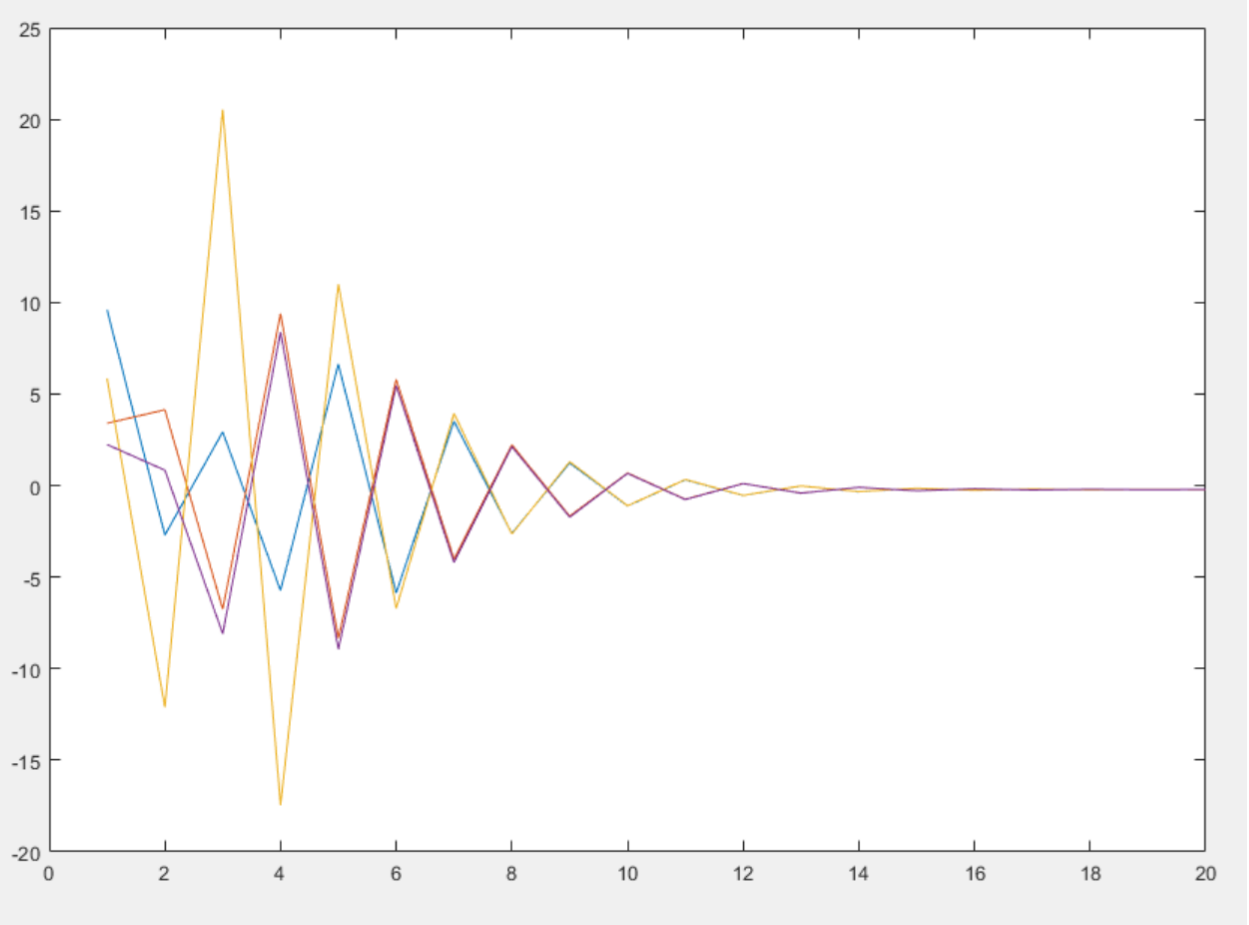

Figure 14: x_i history data

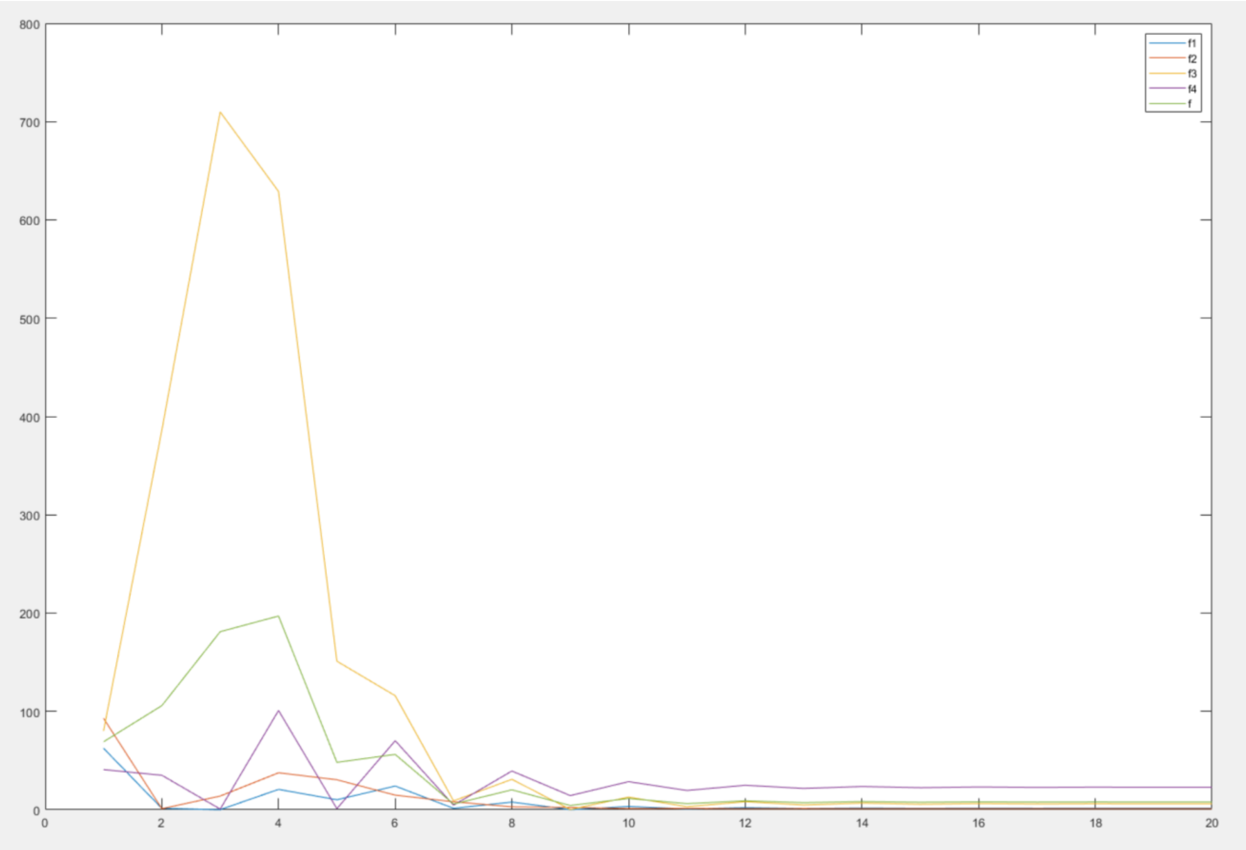

Figure 15: Cost function history data

We can see from the figure 14 that the results converge towards 0, which is same as the decision variable we got in part 1. And we see that the cost functions go to rather low values where they stabilize which indicates that the algorithm works at least somewhat.

Sources

(1) Homework 6, M. Iqbal, A. Gusrialdi, March 13, 2023 https://moodle.tuni.fi/pluginfile.php/3174805/mod_assign/introattachment/0/HW6.pdf?forcedownload=1

(2) Lecture 7 slides, A. Gusrialdi, March 13, 2023. https://moodle.tuni.fi/pluginfile.php/3169473/mod_resource/content/3/AUT360_lecture7.pdf