MMF1941H Stochastic Analysis - Assignment 1

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Problem 1

Solution

In order to show

$$\mathbb{E}((X-K)^+) = \varphi(K) - K(1-\Phi(K))$$

we need to introduce an indicator function.

$$\mathbb{1}(X) = \begin{cases} 1 & \text{if } X \geq K \\ 0 & \text{if } X < K \end{cases}$$

Then, we apply this indicator function into $\mathbb{E}((X-K)^+)$

$$\begin{aligned} \mathbb{E}((X-K)^+) &= \mathbb{E}((X-K)\mathbb{1}_{X\geq K}) \\ &= \mathbb{E}(X\mathbb{1}_{X\geq K}) - K\mathbb{E}(\mathbb{1}_{X\geq K}) \end{aligned}$$

Now, the expectation of the indicator function for event is the cdf of that event.

$$= \mathbb{E}(X\mathbb{1}_{X\geq K}) - K(1-\Phi(X))$$

To find $\mathbb{E}(X)\mathbb{1}_{X\geq K}$, we need to exploit $\varphi^{'}(x) = -x\varphi(x)$

$$\begin{aligned} \mathbb{E}(X\mathbb{1}_{X\geq K}) &= \int_K^\infty X\varphi(X)dX \\ &= \int_K^\infty -\varphi^{'}(X)dX \\ &= -\varphi(X)\Big|_K^\infty \\ &= \varphi(K) \end{aligned}$$

Thus, back to $\mathbb{E}(X\mathbb{1}_{X\geq K}) - K(1-\Phi(X))$

$$\begin{aligned} \mathbb{E}((X-K)^+) &= \mathbb{E}(X\mathbb{1}_{X\geq K}) - K(1-\Phi(X)) \\ &= \varphi(K) - K(1-\Phi(X)) \end{aligned}$$

Alternative Solution

$$\begin{aligned} \mathbb{E}((X-K)^+) &= \int_K^\infty (X-K)\varphi(X)dx \\ &= \int_K^\infty X\varphi(X)dX - \int_K^\infty K\varphi(X)dX \\ &= \int_K^\infty -\varphi^{'}(X)dX + K(1-\Phi(K)) \\ &= \varphi(K) - K(1-\Phi(K)) \end{aligned}$$

Problem 2

Let start with an indicator function.

$$\mathbb{1}(X) = \begin{cases} 1 & \text{if } X \geq \frac{K-\mu_X}{\sigma_X} \\ 0 & \text{if } X < \frac{K-\mu_X}{\sigma_X} \end{cases}$$

Like we did in the previous problem, we apply this indicator function into v.

$$\begin{aligned}
v &= \mathbb{E}((\mu_X + \sigma_X X - K)^+) \\
&= \mathbb{E}((\sigma_X X + \mu_X - K)^+) \\
&= \mathbb{E}((\sigma_X X + \mu_X - K)\mathbb{1}_{X \geq \frac{K-\mu_X}{\sigma_X}})) \\
&= \mathbb{E}(\sigma_X X \mathbb{1}_{X \geq \frac{K-\mu_X}{\sigma_X}}) + \mathbb{E}(\mu_X - K))\mathbb{1}_{X \geq \frac{K-\mu_X}{\sigma_X}} \\
&= \sigma_X \mathbb{E}(X \mathbb{1}_{X \geq \frac{K-\mu_X}{\sigma_X}}) + (\mu_X - K)\mathbb{E}(\mathbb{1}_{X \geq \frac{K-\mu_X}{\sigma_X}}) \\
&= \sigma_X \varphi(\frac{K-\mu_X}{\sigma_X}) + (\mu_X - K)\left(1 - \Phi(\frac{K-\mu_X}{\sigma_X})\right) \\
&= \sigma_X \varphi(\frac{K-\mu_X}{\sigma_X}) - (K - \mu_X)\left(1 - \Phi(\frac{K-\mu_X}{\sigma_X})\right)
\end{aligned}$$

Problem 3

For a parameter $a \in \mathbb{R}$, we define a measure $\mathbb{Q}_a$ via the definition

$$\mathbb{Q}_a(A) = \mathbb{E}\left(\mathbb{1}_A \frac{e^{ax}}{\mathbb{E}(e^{aX})}\right)$$

For $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{Q}_a(X \leq x) &= \mathbb{E}\left(\mathbb{1}_{X \leq x} \frac{e^{ax}}{\mathbb{E}(e^{aX})}\right) \\
&= \int_{-\infty}^{x} \frac{e^{ax}}{\mathbb{E}(e^{aX})} \varphi(x) dx \\
&= \int_{-\infty}^{x} \frac{e^{ax}}{e^{\frac{1}{2}a^2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 + ax - \frac{1}{2}a^2} dx \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2ax + a^2)} dx \\
&= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-a)^2} dx
\end{aligned}$$

$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-a)^2} dx$ is nothing else, but a normal distribution.
Thus, we can conclude that X follows $N(a, 1)$ under $\mathbb{Q}_a$

Problem 4

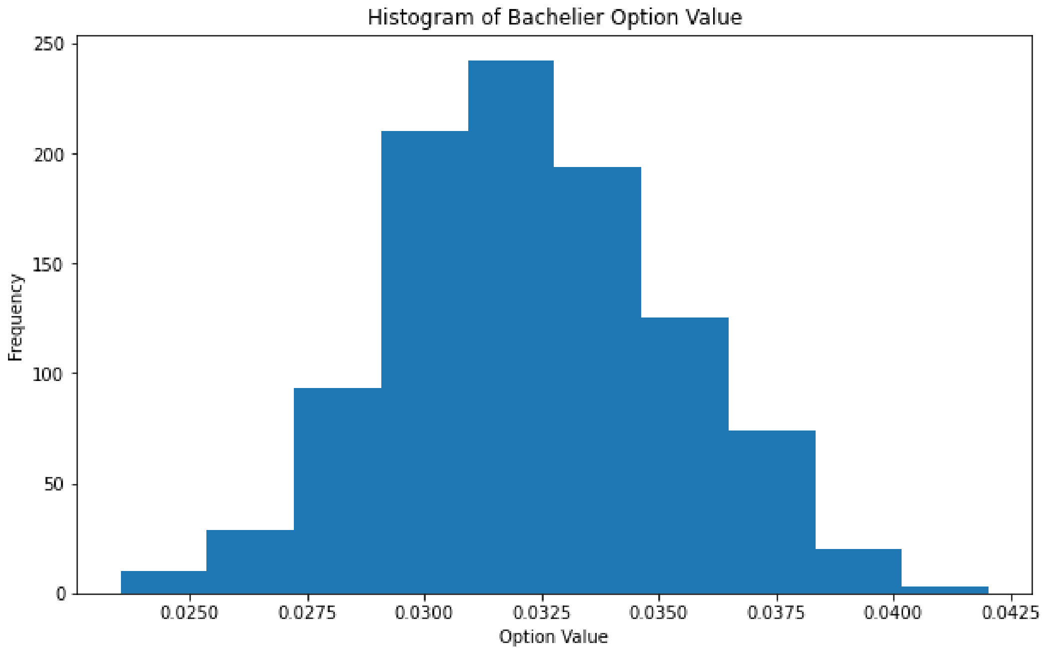

Figure 1: Histogram of Bachelier Option Value ($\mu_X = 2.5, \sigma_X^2 = 4, K = 6$, Sample Size=5000, Simulations=1000)

The exact value is 0.03235.

Problem 5

The sample variance for the previous experiment is 0.0000089826.

Problem 6

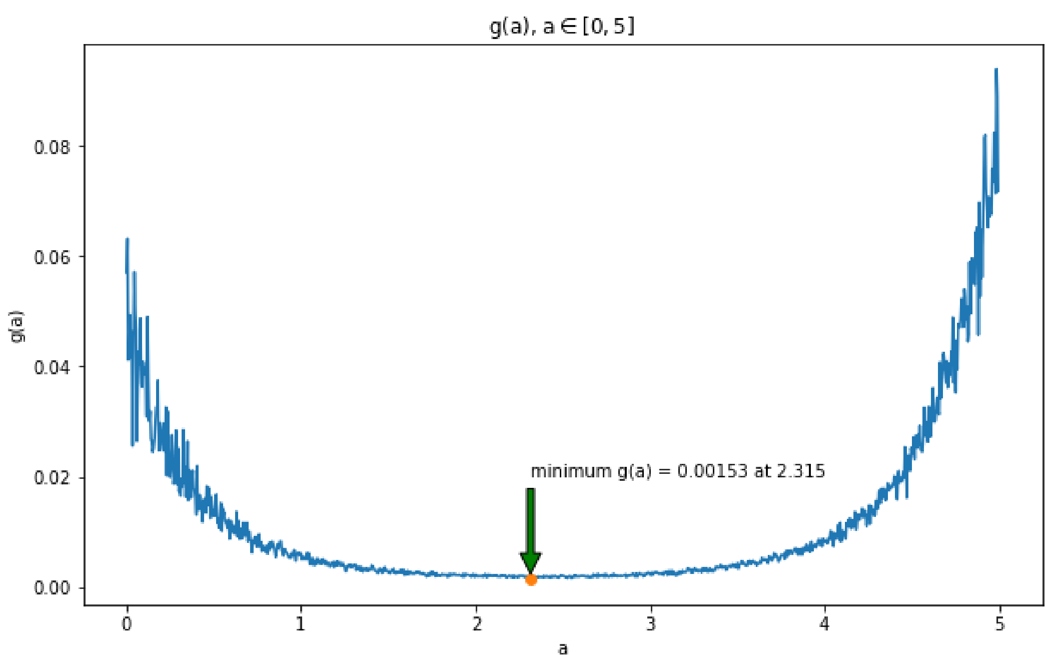

Figure 2: Graph of the Function $g(a)$ where $a \in [0, 5]$

Problem 7

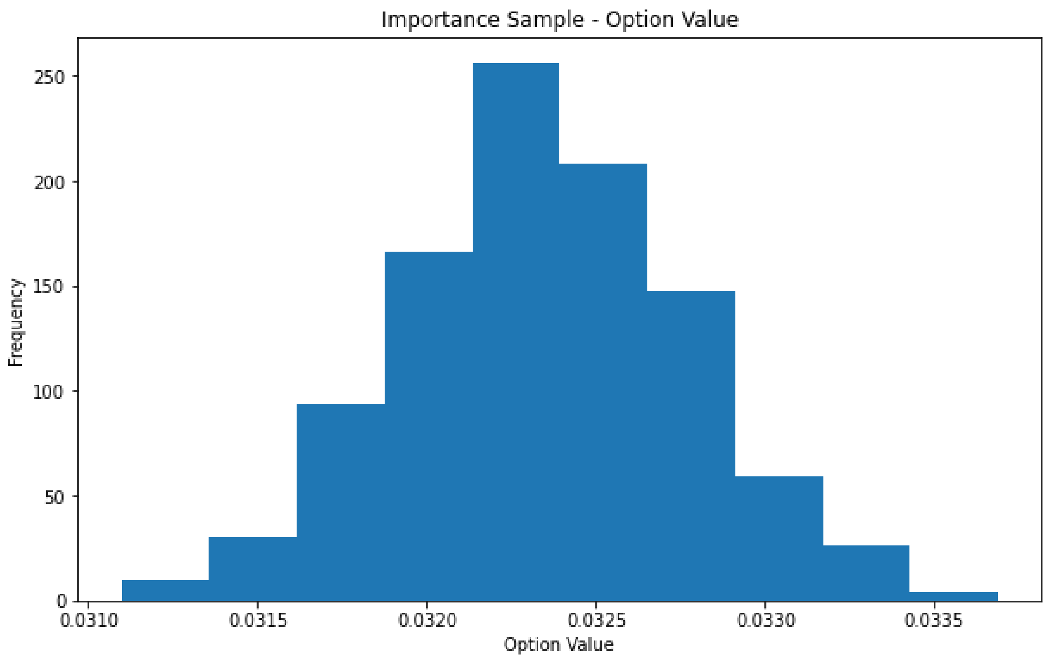

Figure 3: Graph of the Function $g(a)$ where $a \in [0, 5]$

The minimum of the function $g(a)$ is 0.00153 at 2.315. This minimum is also shown in Figure 2. Lastly, the sample variance of the previous experiment is 0.0000001762.