LINEAR ALGEBRA ASSIGNMENT

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1. Find the eq of the parabola $y = A + Bx + Cx^2$ that passes through 3 points $(1,1), (2,-1)$ & $(3,1)$ using Gaussian Elimination

Ans.
$A + B + C = 1$
$A + 2B + 4C = -1$
$A + 3B + 9C = 1$.

matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$ $x = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Aug matrix $[A\ b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{bmatrix}$

using Gauss elem.

$R_2 \leftarrow R_2 - R_1$
$R_3 \leftarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

$R_3 \leftarrow R_3 - 2R_2$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix}.$$

$\therefore 2C = 4$. $\boxed{C = 2}$

$B + 3C = -2$

$B + 3(2) = -2$

$\boxed{B = -8}$

$A + B + C = 1$

$A - 8 + 2 = 1$

$\boxed{A = 7}$

$\therefore A = 7 \quad B = -8 \quad C = 2$.

Eq of Parabola =

$y = 7 - 8x + 2x^2$.

2. Find the LU decomposition for the matrix.

Ans.
$$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & 29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$$

$R_2 \Leftarrow R_2 - 2R_1$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$L_{21} = -2$

$R_3 \leftarrow R_3 - (-5) R_1$

~~mul~~ $= 5$ L_{31}

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \leftarrow R_4 - 5 R.$

$L_{41} = -5$.

$$E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{41} E_{31} E_{21} A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix}$$

$R_3 \leftarrow R_3 - (-2) R_2$.

$L_{32} = -2$.

$$E_{32} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \leftarrow R_4 - (-2) R_2$.

$L_{42} = -2$.

$$E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

$$E_{42} E_{32} E_{41} E_{31} E_{21} A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix}$$

$R_4 \leftarrow R_1 - 3R_3$.

$L_{43} = -3$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$$E_{43} E_{42} E_{32} E_{41} E_{31} E_{21} A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$LU = A$.

$A = LU$.

$$\begin{bmatrix} 2 & 5 & 2 & -5 \\ 11 & 12 & 3 & -14 \\ -10 & -24 & -5 & 38 \\ 11 & 21 & 21 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

3 $T: R^3 \to R^3$ defined by :

$T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$

i) Find the matrix T relative to the standard basis of R^3.

ii) Find the basis for 4 fundamental subspaces of T

iii) Find the eigen values & eigen vectors of T

iv) Decompose T = QR.

i) bases of $R^3 = (1,0,0)\ (0,1,0)$ & $(0,0,1)$

$T(1,0,0) = (1,0,1)$

$$= 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$T(0,1,0) = (2,1,1)$

$$= 2\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$T(0,0,1) = (-1,1,-2)$

$$(-1)\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + (-2)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

ii) $$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for column space =

$C(T) = \{(1,0,1), (2,1,1)\}$.

null space:

$TX = 0$.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 .$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$R_1 \leftarrow R_1 - 2R_2.$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

let $z = 2.$

$y + z = 0.$

$y = -z.$

$x - 3z = 0 \Rightarrow x = 3z.$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$N(T) = \{(3, -1, 1)\}.$

Row space.

$$(T)^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 + R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \leftarrow R_3 - R_2$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad C(T)^T = \{(1,2,-1), (0,1,1)\}.$$

left null space:

$(T)^T x = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

let $z = z$.

$y - z = 0 \Rightarrow y = z$.

$x + z = 0 \Rightarrow x = -z$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$N(T)^T = \{(-1, 1, 1)\}$

(iii) $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$|T - \lambda I| = 0$.

$$\left| \begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix} \right| = 0$$

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$(1-\lambda)((1-\lambda)(-2-\lambda)-1)+1(2+1-\lambda)=0$

$= -\lambda^3 + 3\lambda = 0$

$\lambda = 0, \sqrt{3}, -\sqrt{3}$

Eigen vectors.

For $\lambda_1 = 0$, solve $(T - \lambda_1 I)X = 0$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$R_3 \leftarrow R_3 - R_1$ & $R_3 \leftarrow R_3 + R_2$.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$$X = z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = \sqrt{3}$,

$(T - \lambda_2)X = 0.$

$$\begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$$R_3 \leftarrow R_3 - \frac{1}{1-\sqrt{3}} R_1$$

$$\begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 0 & \frac{1-2}{1-\sqrt{3}} & -2\sqrt{3}+\frac{1}{1-\sqrt{3}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$$R_3 \leftarrow R_3 + \frac{\sqrt{3}+1}{(1-\sqrt{3})^2} R_2$$

$$\begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 0 & 0 & -2-\sqrt{3}+\frac{1}{1-\sqrt{3}} + \frac{\sqrt{3}+1}{(1-\sqrt{3})^2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0,$$

$z = z.$

$(1-\sqrt{3})y + z = 0.$

$$y = \frac{-1}{1-\sqrt{3}} z.$$

$$(1-\sqrt{3})x + 2\left(\frac{-1}{1-\sqrt{3}} z\right) - z = 0$$

$$x = \frac{z(\sqrt{3})}{1-\sqrt{3}}$$

$$x_2 = \begin{bmatrix} -\sqrt{3}/(1-\sqrt{3}) \\ -1/(1-\sqrt{3}) \\ 1 \end{bmatrix}$$

$\lambda_3 = -\sqrt{2}.$

$(T - \lambda_3 T)\, x = 0$

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_3 \leftarrow R_3 - \frac{1}{1+\sqrt{3}} R_1$$

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 0 & 1-\frac{2}{1+\sqrt{3}} & -2+\sqrt{3}+\frac{1}{1+\sqrt{3}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_3 \leftarrow R_3 - \frac{(\sqrt{3}-1)}{(1+\sqrt{3})} R_2 .$$

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 0 & 0 & -2+\sqrt{3}+\frac{1}{1+\sqrt{3}} - \frac{(\sqrt{3}-1)}{(1+\sqrt{3})^2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

let $z = z$.

$$(1+\sqrt{3})y^{z} = 0 .$$

$$y = \frac{-1}{1+\sqrt{3}} z .$$

$$(1-\sqrt{3})x + 2\left(\frac{-1}{1+\sqrt{3}} z\right) - z = 0 .$$

$x = \frac{\sqrt{3}}{1+\sqrt{3}}$ 2

$$x_2 = 2 \begin{bmatrix} \sqrt{3}/(1+\sqrt{3}) \\ -1/(1+\sqrt{3}) \\ 1 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} \sqrt{3}/(1+\sqrt{3}) \\ -1/(1+\sqrt{3}) \\ 1 \end{bmatrix}$$

$\therefore$ Eigen val are.

$$x_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} -\sqrt{3}/(1-\sqrt{3}) \\ -1/(1-\sqrt{3}) \\ 1 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} \sqrt{3}/(1+\sqrt{3}) \\ -1/(1+\sqrt{3}) \\ 1 \end{bmatrix}$$

eigen vectors by rationlaising the denominators.

$$x_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} \frac{\sqrt{3}+3}{2} \\ \frac{-\sqrt{3}+1}{2} \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} \frac{3-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \\ 1 \end{bmatrix}$$

$\lambda_1 = 0$ $\quad\lambda_2 = \sqrt{3}$ $\quad\lambda_3 = -\sqrt{3}$, IV $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$.

$a = (1, 0, 1)$ $b = (2, 1, 1)$ $c = (-1, 1, -2)$

Gram Schmidt process.

$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}}(1, 0, 1)$.

$\|a\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$.

$q_2 = \frac{B}{\|B\|}$; $B = b - (q_1^T b) q_1$.

$q_1^T b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 3/\sqrt{2}$.

$B = b - (q_1^T b) q_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - 3/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$

$q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{6}}(1, 2, -1)$.

$q_3 = \frac{C}{\|C\|}$, $C = c - (q_2^T c) q_2 - (q_1^T c) q_1$

$q_1^T c = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = -3/\sqrt{2}.$

$q_1^T c = 1/\sqrt{6} \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = 3/\sqrt{6}.$

$c = (-1, 1, -2) - 3/\sqrt{6} \times 1/\sqrt{6}\, (1, 2, -1)$

$\left(-3/\sqrt{2}\right)\left(1/\sqrt{2}\right)(1, 0, 1).$

$c = (-1, 1, -2) + \left(-1/2, -1, 1/2\right) - \left(3/2, 0, 3/2\right)$

$= (0, 0, 0)$

$q_3 = (0, 0, 0).$

QR factorisation

$R = \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{bmatrix}$

$q_1^T a = 1/\sqrt{2} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \sqrt{2}.$

$q_1^T b = 1/\sqrt{6} \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 3/\sqrt{6}.$

$q_3^T c = 0.$

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$$R = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{12} & 3\sqrt{3} & -3\sqrt{3} \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$q_1^T b = 3/\sqrt{2} = \frac{3\sqrt{3}}{\sqrt{6}}$$

$$q_3^T c = -3/\sqrt{2} = \frac{-3\sqrt{3}}{\sqrt{6}}$$

$$q_2^T c = 3/\sqrt{6}$$

Q is $[q_1 \; q_2 \; q_3]$

$T = QR$.

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 0 \\ 0 & 2/\sqrt{6} & 0 \\ 1/\sqrt{2} & -1/\sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & -3/\sqrt{2} \\ 0 & 3/\sqrt{6} & 3/\sqrt{6} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

4. $y = c + dx$.

given $c - 4d = 4$.

$c + d = 6$

$c + 2d = 10$

$c + 3d = 8$.

$$Ax = b \Rightarrow \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

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$$A^T A \hat{x} = A^T b \quad \Rightarrow \quad \hat{x} = (A^T A)^{-1} A^T b$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix}$$

$$(A^T A)^{-1} A^T = \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix}$$

$$= \frac{1}{116} \begin{bmatrix} 38 & 28 & 26 & -24 \\ -18 & 2 & 6 & 10 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} c \\ d \end{bmatrix} = (A^T A)^{-1} A^T b$$

$$= \frac{1}{116} \begin{bmatrix} 38 & 28 & 26 & 24 \\ -18 & 2 & 6 & 10 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$= \frac{1}{116} \begin{bmatrix} 772 \\ 80 \end{bmatrix} = \begin{bmatrix} 193/29 \\ 20/29 \end{bmatrix}$$

$\therefore$ the eq is:

$$y = \frac{193}{29} + \frac{20}{29} x$$

5 $x_1 + x_2 + 3x_3 + 4x_4 = 0$

$$\begin{bmatrix} 1 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_1 = -x_2 - 3x_3 - 4x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$P_1 = A(A^T A)^{-1} A^T$ & $Q = I - P$.

$$A^T A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 4 \\ 3 & 10 & 12 \\ 4 & 12 & 17 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 26/27 & -1/9 & -4/27 \\ -1/9 & 2/3 & -4/9 \\ -4/27 & -4/9 & 11/29 \end{bmatrix}$$

$$P = A(A^T A)^{-1} A^T = \begin{bmatrix} 26/27 & -1/27 & -1/9 & -4/27 \\ -1/27 & 26/27 & -1/9 & -4/27 \\ -1/9 & -3/27 & 6/9 & -12/27 \\ -4/27 & -4/27 & -4/9 & 11/27 \end{bmatrix}$$

$$= \frac{1}{29} \begin{bmatrix} 26 & -1 & -3 & -4 \\ -1 & 26 & -3 & -4 \\ -3 & -3 & 18 & -12 \\ -4 & -4 & -12 & 11 \end{bmatrix}$$

$$Q = I - P = \frac{1}{27} \begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 1 & 3 & 4 \\ 3 & 3 & 9 & 12 \\ 4 & 4 & 12 & 16 \end{bmatrix}$$

6 $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$

positive if pivots are positive.

echelon form.

$R_2 \leftarrow R_2 - \frac{2}{a} R_1$

$R_3 \leftarrow R_3 - \frac{2}{a} R_1$

$$\begin{bmatrix} a & 2 & 2 \\ 0 & a - 4/a & 2 - 4/a \\ 0 & 2 - 4/a & a - 4/a \end{bmatrix}$$

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$R_3 \leftarrow R_3 - \left(\frac{2a-4}{a^2-4}\right) R_2$.

$$\begin{bmatrix} a & 2 & 2 \\ 0 & \frac{a^2-4}{a} & \frac{2a-4}{a} \\ 0 & 0 & \frac{a^2-4}{a} - \frac{(2a-4)(2a-4)}{a(a^2-4)} \end{bmatrix}$$

$a > 0$.

$\frac{a^2-4}{a} > 0$

$\Rightarrow a^2-4>0$.

$\Rightarrow (a-2)(a+2)>0$.

$= (-\infty,-2) \cup (2,\infty)$

as $a>0$

$\frac{a^2-4}{a} - \frac{(2a-4)(2a-4)}{a(a^2-4)}$

$\Rightarrow (a^2-4)^2 - (2a-4)^2 > 0$.

$\Rightarrow (a^2-4-2a+4)(a^2-4+2a-4) > 0$

$\Rightarrow (a^2-2a)(a^2+2a-8) > 0$

$\therefore (a^2-2a) > 0$ & $(a^2+2a-8) > 0$

$a(a-2) > 0$

$a^2+4a-2a-8 > 0$

$a(a+4) - 2(a+4) > 0$

$(a+4)(a-2) > 0$

$\therefore$ we get $a>0 \cup a>2 \cup a>-4 \cup a>2$

$\therefore a > 2$

$\therefore$ Range of a is $(2,\infty)$

$f = x^T A x$.

$f = 2x_1^2 + 2x_2^2 + 2x_3^2 \mp 2x_1x_2 - 2x_2x_3$

let $x = (x_1\ x_2\ x_3)$

~~Ac seq e8~~

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

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$$x^T A x = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{11}xy + 2a_{13}xz + 2a_{23}yz$$

$(x = x_1, \; y = x_2, \; z = x_3)$

$a_{11} = 2 \quad a_{22} = 2 \quad a_{33} = 2.$

$a_{12} = -1 \quad a_{13} = 0, \quad a_{23} = -1$

$$= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

7. $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$ Find the SVD of A, $U \Sigma V^T$ where.

$$A^T A = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 81-\lambda & -27 \\ -27 & 9-\lambda \end{vmatrix} = 0 \Rightarrow (81-\lambda)(9-\lambda) - (-27)(-27) = 0$$

$\Rightarrow (729 - 81\lambda - 9\lambda + \lambda^2 - 729) = 0.$

$\Rightarrow -90\lambda - \lambda^2 = 0$

$\Rightarrow \lambda(\lambda - 90) = 0 \quad \Rightarrow \lambda = 90, 0.$

$\lambda_1 = 90 \qquad \lambda_2 = 0$

$\lambda_1 = 90$

$$\begin{bmatrix} -9 & \cancel{-20}-27 \\ -27 & -81 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

$R_2 \leftarrow R_2 - 3R_1$

$$\begin{bmatrix} -9 & -27 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 .$$

$x = -3y .$

$$x_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$\sqrt{(-3)^2 + (1)^2} = \sqrt{10}$

from $\lambda_2 = 0 .$

$$\begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 ,$$

$R_2 \leftarrow R_2 + \frac{1}{3} R_1$

$$\begin{bmatrix} 81 & -27 \\ -27 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$81x - 27y = 0$

$x = \frac{1}{3} y$

$$x_2 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \qquad \sqrt{(1/3)^2 + (1)^2} = \sqrt{10/3}$$

Normalising x_1 & x_2.

$$v_1 = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

Singular val of A $\sigma_1 = \sqrt{90}$ & $\sigma_2 = 0$.
eigen val of AA^T (order 3×3) are $90, 0, 0$.

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}}{\sqrt{10}}$$

$$= \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix}$$

$$u_1^T x = 0 \Rightarrow \frac{x_1}{3} - \frac{2x_2}{3} - \frac{2x_3}{3} = 0$$

$$\Rightarrow x - 2x_2 - 2x_3 = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 = 2x_2 + 2x_3$$

$$\Rightarrow x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{a_2} + x_3 \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{a_3}$$

vectors $(2,1,0)$ & $(2,0,1)$

$$v_2 = \frac{a_2}{\|a_2\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{5}, 0\right)$$

$$e = a_2 - (u_2^T a_2) v_2 ..$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 2/5 & 1/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 8/5 \\ 4/5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/5 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 8/6 \\ 4/5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/5 \\ -4/5 \\ 1 \end{bmatrix}$$

$$\|e\| = \sqrt{\frac{4}{25} + \frac{16}{25} + 1} = \frac{\sqrt{45}}{5}$$

$$= \frac{3}{\sqrt{5}}$$

$$v_3 = \frac{e}{\|e\|} = \left(2/3\sqrt{5}, -4/3\sqrt{5}; \sqrt{5}/3\right) \text{ or}$$

$$\left(2/\sqrt{45}, -4/\sqrt{45}, 5/\sqrt{45}\right)$$

$$\Sigma = \begin{bmatrix} 90 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A = U \Sigma V^T$$

$$A = U \Sigma V^T$$

$$\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/\sqrt{5} & 2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & -4/\sqrt{45} \\ -2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$\therefore \underline{\underline{A = U \Sigma V^T}}$$