Team notebook

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Contents

1 Algorithms

1.1 Mo's Algorithm

```
/*
   https://www.spoj.com/problems/FREQ2/
*/
vector <int> MoQueries(int n, vector <query> Q){

    block_size = sqrt(n);
    sort(Q.begin(), Q.end(), [](const query &A,
        const query &B){
        return (A.l/block_size != B.l/block_size)?
            (A.l/block_size < B.l/block_size) :
            (A.r < B.r);
    });
    vector <int> res;
    res.resize((int)Q.size());

    int L = 1, R = 0;
    for(query q: Q){
        while (L > q.l) add(--L);
        while (R < q.r) add(++R);

        while (L < q.l) del(L++);
        while (R > q.r) del(R--);

        res[q.pos] = calc(1, R-L+1);
    }
    return res;
}
```

1.2 Mo's Algorithms on Trees

```
/*
Given a tree with N nodes and Q queries. Each
    node has an integer weight.
Each query provides two numbers u and v, ask for
    how many different integers weight of nodes
there are on path from u to v.

----------
Modify DFS:
----------
For each node u, maintain the start and the end
    DFS time. Let's call them ST(u) and EN(u).
=> For each query, a node is considered if its
    occurrence count is one.

--------------
Query solving:
--------------
Let's query be (u, v). Assume that ST(u) <=
    ST(v). Denotes P as LCA(u, v).

Case 1: P = u
Our query would be in range [ST(u), ST(v)].

Case 2: P != u
Our query would be in range [EN(u), ST(v)] +
    [ST(p), ST(p)]
*/

void update(int &L, int &R, int qL, int qR){
    while (L > qL) add(--L);
    while (R < qR) add(++R);
```

```
    while (L < qL) del(L++);
    while (R > qR) del(R--);
}

vector <int> MoQueries(int n, vector <query> Q){
    block_size = sqrt((int)nodes.size());
    sort(Q.begin(), Q.end(), [](const query &A,
         const query &B){
        return (ST[A.l]/block_size !=
            ST[B.l]/block_size)?
            (ST[A.l]/block_size <
            ST[B.l]/block_size) : (ST[A.r] <
            ST[B.r]);
    });
    vector <int> res;
    res.resize((int)Q.size());

    LCA lca;
    lca.initialize(n);

    int L = 1, R = 0;
    for(query q: Q){
        int u = q.l, v = q.r;
        if(ST[u] > ST[v]) swap(u, v); // assume
            that S[u] <= S[v]
        int parent = lca.get(u, v);

        if(parent == u){
            int qL = ST[u], qR = ST[v];
            update(L, R, qL, qR);
        }else{
            int qL = EN[u], qR = ST[v];
            update(L, R, qL, qR);
            if(cnt_val[a[parent]] == 0)
                res[q.pos] += 1;
        }

        res[q.pos] += cur_ans;
    }
    return res;
}
```

1.3 Parallel Binary Search

```
int lo[N], mid[N], hi[N];
vector<int> vec[N];

void clear() //Reset
{
        memset(bit, 0, sizeof(bit));
}

void apply(int idx) //Apply ith update/query
{
        if(ql[idx] <= qr[idx])
                update(ql[idx], qa[idx]),
                    update(qr[idx]+1, -qa[idx]);
        else
        {
                update(1, qa[idx]);
                update(qr[idx]+1, -qa[idx]);
                update(ql[idx], qa[idx]);
        }
}

bool check(int idx) //Check if the condition is
    satisfied
{
        int req=reqd[idx];
        for(auto &it:owns[idx])
        {
                req-=pref(it);
                if(req<0)
                        break;
        }
        if(req<=0)
                return 1;
        return 0;
}

void work()
{
        for(int i=1;i<=q;i++)
                vec[i].clear();
        for(int i=1;i<=n;i++)
                if(mid[i]>0)
                        vec[mid[i]].push_back(i);
        clear();
        for(int i=1;i<=q;i++)
        {
                apply(i);
                for(auto &it:vec[i]) //Add
                    appropriate check conditions
                {
                        if(check(it))
                                hi[it]=i;
                        else
                                lo[it]=i+1;
                }
        }
}

void parallel_binary()
{
        for(int i=1;i<=n;i++)
                lo[i]=1, hi[i]=q+1;
        bool changed = 1;
        while(changed)
        {
                changed=0;
                for(int i=1;i<=n;i++)
                {
                        if(lo[i]<hi[i])
                        {
                                changed=1;
                                mid[i]=(lo[i] +
                                    hi[i])/2;
                        }
                        else
                                mid[i]=-1;
                }
                work();
        }
}
```

2 Combinatorics

2.1 Factorial Approximate

Approximate Factorial:

$$n! = \sqrt{2.\pi.n}.(\frac{n}{e})^n \tag{1}$$

2.2 Factorial

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
n	11	12	13	14	15	16	17			
$n!$	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14			
n	20	25	30	40	50	100	150	171		
$n!$	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBL_MAX		

2.3 Fast Fourier Transform

```
/**
 * Fast Fourier Transform.
 * Useful to compute convolutions.
 * computes:
 *   C(f star g)[n] = sum_m(f[m] * g[n - m])
 * for all n.
 * test: icpc live archive, 6886 - Golf Bot
 * */


using namespace std;
#include <bits/stdc++.h>
#define D(x) cout << #x " = " << (x) << endl
#define endl '\n'

const int MN = 262144 << 1;
int d[MN + 10], d2[MN + 10];


const double PI = acos(-1.0);

struct cpx {
  double real, image;
  cpx(double _real, double _image) {
    real = _real;
    image = _image;
  }
  cpx(){}
};

cpx operator + (const cpx &c1, const cpx &c2) {
  return cpx(c1.real + c2.real, c1.image +
      c2.image);
}

cpx operator - (const cpx &c1, const cpx &c2) {
  return cpx(c1.real - c2.real, c1.image -
      c2.image);
}

cpx operator * (const cpx &c1, const cpx &c2) {
  return cpx(c1.real*c2.real - c1.image*c2.image,
      c1.real*c2.image + c1.image*c2.real);
}

int rev(int id, int len) {
  int ret = 0;
  for (int i = 0; (1 << i) < len; i++) {
    ret <<= 1;
    if (id & (1 << i)) ret |= 1;
  }
  return ret;
}

cpx A[1 << 20];

void FFT(cpx *a, int len, int DFT) {
  for (int i = 0; i < len; i++)
    A[rev(i, len)] = a[i];
  for (int s = 1; (1 << s) <= len; s++) {
    int m = (1 << s);
    cpx wm = cpx(cos( DFT * 2 * PI / m), sin(DFT
        * 2 * PI / m));
    for(int k = 0; k < len; k += m) {
      cpx w = cpx(1, 0);
      for(int j = 0; j < (m >> 1); j++) {
        cpx t = w * A[k + j + (m >> 1)];
        cpx u = A[k + j];
        A[k + j] = u + t;
        A[k + j + (m >> 1)] = u - t;
        w = w * wm;
      }
    }
  }
  if (DFT == -1) for (int i = 0; i < len; i++)
      A[i].real /= len, A[i].image /= len;
  for (int i = 0; i < len; i++) a[i] = A[i];
  return;
}

cpx in[1 << 20];

void solve(int n) {
  memset(d, 0, sizeof d);
  int t;
  for (int i = 0; i < n; ++i) {
    cin >> t;
    d[t] = true;
  }
  int m;
  cin  >> m;
  vector<int> q(m);
  for (int i = 0; i < m; ++i)
    cin >> q[i];

  for (int i = 0; i < MN; ++i) {
    if (d[i])
      in[i] = cpx(1, 0);
    else
      in[i] = cpx(0, 0);
  }

  FFT(in, MN, 1);
  for (int i = 0; i < MN; ++i) {
    in[i] = in[i] * in[i];
  }
  FFT(in, MN, -1);

  int ans = 0;
  for (int i = 0; i < q.size(); ++i) {
```

```
    if (in[q[i]].real > 0.5 || d[q[i]]) {
      ans++;
    }
  }
  cout << ans << endl;
}

int main() {
  ios_base::sync_with_stdio(false);cin.tie(NULL);
  int n;
  while (cin >> n)
    solve(n);
  return 0;
}
```

2.4 General purpose numbers

Bernoulli numbers
EGF of Bernoulli numbers is $B(t) = \frac{t}{e^t - 1}$ (FFT-able). $B[0, \ldots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots]$

Sums of powers:

$$\sum_{i=1}^{n} n^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\sum_{i=m}^{\infty} f(i) = \int_m^{\infty} f(x)dx - \sum_{k=1}^{\infty} \frac{B_k}{k!} f^{(k-1)}(m)$$

$$\approx \int_m^{\infty} f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m))$$

Stirling numbers of the first kind
Number of permutations on n items with k cycles.

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k), \; c(0, 0) = 1$$
$$\sum_{k=0}^{n} c(n, k)x^k = x(x+1) \ldots (x+n-1)$$

$c(8, k) = 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1$

Stirling numbers of the second kind
Partitions of n distinct elements into exactly k groups.

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

$$S(n, 1) = S(n, n) = 1$$

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

Eulerian numbers
Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.

$$E(n, k) = (n-k)E(n-1, k-1) + (k+1)E(n-1, k)$$

$$E(n, 0) = E(n, n-1) = 1$$

$$E(n, k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n$$

Bell numbers
Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \ldots$. For p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

Labeled unrooted trees
on n vertices: n^{n-2}
on k existing trees of size n_i: $n_1 n_2 \cdots n_k n^{k-2}$
with degrees d_i: $(n-2)!/((d_1-1)! \cdots (d_n-1)!)$

Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, \; C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \; C_{n+1} = \sum C_i C_{n-i}$$

$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$

[noitemsep]sub-diagonal monotone paths in an $n \times n$ grid. strings with n pairs of parenthesis, correctly nested. binary trees with with $n+1$ leaves (0 or 2 children). ordered trees with $n+1$ vertices. ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines. permutations of $[n]$ with no 3-term increasing subseq.

2.5 Lucas Theorem

For non-negative integers m and n and a prime p, the following congruence relation holds: :

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod{p},$$

where :

$$m = m_k p^k + m_{k-1} p^{k-1} + \cdots + m_1 p + m_0,$$

and :

$$n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_1 p + n_0$$

are the base p expansions of m and n respectively. This uses the convention that $\binom{m}{n} = 0$ if $m \leq n$.

2.6 Multinomial

```
/**
 * Description: Computes $\displaystyle
     \binom{k_1 + \dots + k_n}{k_1, k_2, \dots,
     k_n} = \frac{(\sum k_i)!}{k_1!k_2!...k_n!}$.
 * Status: Tested on kattis:lexicography
 */
#pragma once

long long multinomial(vector<int>& v) {
        long long c = 1, m = v.empty() ? 1 : v[0];
        for (long long i = 1; i < v.size(); i++) {
        for (long long j = 0; j < v[i]; j++) {
            c = c * ++m / (j + 1);
        }
    }
        return c;
}
```

2.7 Others

Cycles Let $g_S(n)$ be the number of n-permutations whose cycle lengths all belong to the set S. Then

$$\sum_{n=0}^{\infty} g_S(n)\frac{x^n}{n!} = \exp\left(\sum_{n\in S}\frac{x^n}{n}\right)$$

Derangements Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1)+D(n-2)) = nD(n-1)+(-1)^n = \left\lfloor\frac{n!}{e}\right\rceil$$

Burnside's lemma Given a group G of symmetries and a set X, the number of elements of X *up to symmetry* equals

$$\frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where X^g are the elements fixed by g ($g.x = x$).

If $f(n)$ counts "configurations" (of some sort) of length n, we can ignore rotational symmetry using $G = Z_n$ to get

$$g(n) = \frac{1}{n}\sum_{k=0}^{n-1} f(\gcd(n,k)) = \frac{1}{n}\sum_{k|n} f(k)\phi(n/k).$$

2.8 Permutation To Int

```
/**
 * Description: Permutation -> integer
     conversion. (Not order preserving.)
 * Integer -> permutation can use a lookup table.
 * Time: O(n)
 **/

int permToInt(vector<int>& v) {
        int use = 0, i = 0, r = 0;
        for(int x : v) r = r * ++i +
            __builtin_popcount(use & -(1<<x)),
                use |= 1 << x;                  //
                    (note: minus, not ~!)
        return r;
}
```

2.9 Sigma Function

The Sigma Function is defined as:

$$\sigma_x(n) = \sum_{d|n} d^x$$

when $x = 0$ is called the divisor function, that counts the number of positive divisors of n.

Now, we are interested in find

$$\sum_{d|n}\sigma_0(d)$$

If n is written as prime factorization:

$$n = \prod_{i=1}^{k} P_i^{e_k}$$

We can demonstrate that:

$$\sum_{d|n}\sigma_0(d) = \prod_{i=1}^{k} g(e_k+1)$$

where $g(x)$ is the sum of the first x positive numbers:

$$g(x) = (x*(x+1))/2$$

3 Data Structures

3.1 Disjoint Set Uninon (DSU)

```
class DSU{
public:
    vector <int> parent;
    void initialize(int n){
        parent.resize(n+1, -1);
    }

    int findSet(int u){
        while(parent[u] > 0)
            u = parent[u];
        return u;
    }

    void Union(int u, int v){
        int x = parent[u] + parent[v];
        if(parent[u] > parent[v]){
            parent[v] = x;
            parent[u] = v;
        }else{
            parent[u] = x;
            parent[v] = u;
        }
    }
};
```

3.2 Fenwick Tree

```
template <typename T>
class FenwickTree{
  vector <T> fenw;
  int n;
public:
  void initialize(int _n){
    this->n = _n;
    fenw.resize(n+1);
  }

  void update(int id, T val) {
    while (id <= n) {
      fenw[id] += val;
      id += id&(-id);
```

```
    }
  }

  T get(int id){
    T ans{};
    while(id >= 1){
      ans += fenw[id];
      id -= id&(-id);
    }
    return ans;
  }
};
```

3.3 Segment Tree

```
#include <bits/stdc++.h>
using namespace std;

const int N = 1e5 + 10;

int node[4*N];

void modify(int seg, int l, int r, int p, int
     val){
    if(l == r){
        node[seg] += val;
        return;
    }
    int mid = (l + r)/2;
    if(p <= mid){
        modify(2*seg + 1, l, mid, p, val);
    }else{
        modify(2*seg + 2, mid + 1, r, p, val);
    }
    node[seg] = node[2*seg + 1] + node[2*seg + 2];
}

int sum(int seg, int l, int r, int a, int b){
    if(l > b || r < a) return 0;
    if(l >= a && r <= b) return node[seg];
    int mid = (l + r)/2;
    return sum(2*seg + 1, l, mid, a, b) +
        sum(2*seg + 2, mid + 1, r, a, b);
}
```