

**Problem 1**

To evaluate a new test for detecting Hansen's disease, a group of people 5% of which are known to have Hansen's disease are tested. The test finds Hansen's disease among 98% of those with the disease and 3% of those who don't. What is the probability that someone testing positive for Hansen's disease under this new test actually has it?

**Solution**

A: known to have Hansen's disease

B: known to have no Hansen's disease

H: Testing positive for Hansen's disease

$$P(A) = 0.05$$

$$P(B) = 0.95$$

$$P(H | A) = 0.98$$

$$P(H | B) = 0.03$$

The probability that someone testing positive for Hansen's disease under this new test actually has the disease is:

$$\begin{aligned} P(A|H) &= \frac{P(H|A) \times P(A)}{P(H)} \\ &= \frac{P(H|A) \times P(A)}{P(H|A) \times P(A) + P(H|B) \times P(B)} \end{aligned}$$

Substituting the values, so we have:

$$\begin{aligned} P(A|H) &= \frac{0.98 \times 0.05}{0.98 \times 0.05 + 0.03 \times 0.95} \\ P(A|H) &= \frac{0.049}{0.0775} \approx 0.6323 \end{aligned}$$

**Problem 2**

Proof the following distributions are normalized then calculate the mean and standard deviation of these distribution:

1. Univariate normal distribution.
- 2.(Optional) Multivariate normal distribution.

**Solution****1.Univariate normal distribution**

The area under the normal distribution curve should be equal to 1. Next is the proof:

$$\text{Put } I = \int e^{\frac{-x^2}{2}} dx, \text{ then we have } I^2 = \left( \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx \right) \left( \int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{-x^2+y^2}{2}} dx dy$$

$$\text{Set } x = r \cos \theta, y = r \sin \theta \text{ We have } \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

$$\text{Which } J = \left[ \frac{\partial(x, y)}{\partial(r, \theta)} \right] \text{ We have : } dx dy = \begin{bmatrix} \frac{\partial(x)}{\partial(r)} & \frac{\partial(x)}{\partial(\theta)} \\ \frac{\partial(y)}{\partial(r)} & \frac{\partial(y)}{\partial(\theta)} \end{bmatrix} dr d\theta = r dr d\theta$$

$$\text{So : } I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{x^2+y^2}{2}} dx dy = \int_0^{2\pi} \int_0^{\infty} e^{\frac{-r^2}{2}} r dr d\theta.$$

$$\text{So : } I^2 = \int_0^{2\pi} \int_0^{\infty} e^{\frac{-r^2}{2}} r dr d\theta = \int_0^{2\pi} [-e^{\frac{-r^2}{2}}]_0^{\infty} d\theta = \int_0^{2\pi} 1 d\theta = 2\pi \Rightarrow I = \sqrt{2\pi}$$

**Calculating mean:**

Set:

$$Z = \frac{(X - \mu)}{\sigma}$$

We have:

$$\begin{aligned} E[Z] &= \int_{-\infty}^{+\infty} x f_Z(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x e^{\frac{-x^2}{2}} dx \\ &= \frac{-1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \Big|_{-\infty}^{+\infty} \\ &= 0 \end{aligned}$$

Because :  $X = \mu + \sigma Z$ 

$$\Rightarrow E(X) = E(\mu) + E(\sigma Z)$$

$$\Leftrightarrow E(X) = \mu + E(Z)E(\sigma)$$

But:  $E(Z) = 0$ 

$$\Rightarrow E(X) = \mu$$

**Calculating variance:**

$$Var(Z) = E(Z^2) - ((E(Z))^2 = E(Z^2))$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{\frac{-x^2}{2}} dx$$

Let:

$$u = x, \quad dv = x e^{\frac{-x^2}{2}}$$

$$Var(Z) = \frac{1}{\sqrt{2\pi}} \left( -x e^{\frac{-x^2}{2}} \Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2}} dx = 1$$

But:  $X = \mu + \sigma Z$ 

$$\Leftrightarrow Var(X) = Var(\mu) + \sigma^2 Var(Z)$$

$$\Rightarrow Var(X) = 0 + 1\sigma^2$$

$$\Rightarrow Var(X) = \sigma^2$$

Standard deviation is  $S = \sigma$