Fourier Continuation Framework for Higher-Order PDE Solvers

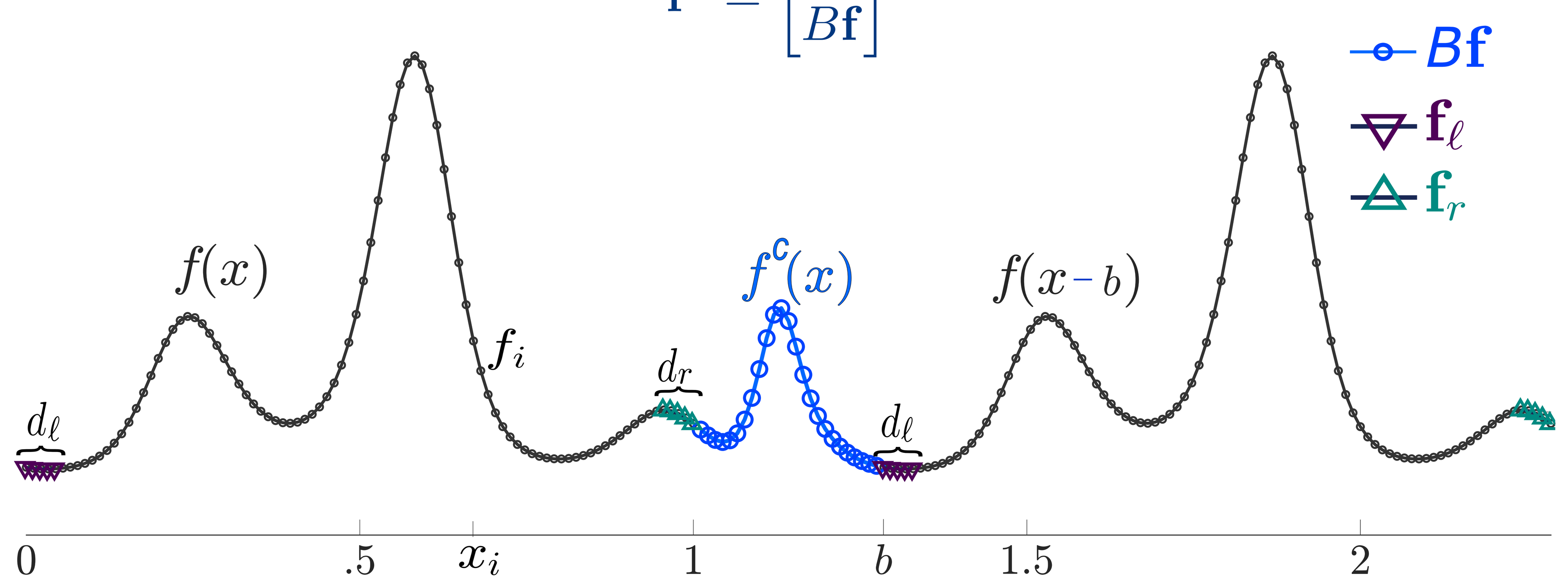
Ivan Méndez, Faisal Amlani
ivan.mendez_cruz@ens-paris-saclay.fr, faisal.amlani@ens-paris-saclay.fr
ENS Paris-Saclay, LMPS - Laboratoire de Mécanique Paris-Saclay, CNRS, Gif-sur-Yvette, France

Fourier Continuation (FC)

Problem. Given N values f_i of $f:[0,1]\to\mathbb{R}$ on a uniform mesh of $[0,1]$, find a trigonometric polynomial f^c with period $b>1$ that interpolates f.

Periodic Extension by Blending. Construction of an operator B that preserves both the first d_ℓ values, $\mathbf{f}_\ell$, and the last d_r values, $\mathbf{f_r}$. The values of the Fourier Continuation are given by

$$\mathbf{f^c} = \begin{bmatrix} \mathbf{f} \\ B\mathbf{f} \end{bmatrix}$$

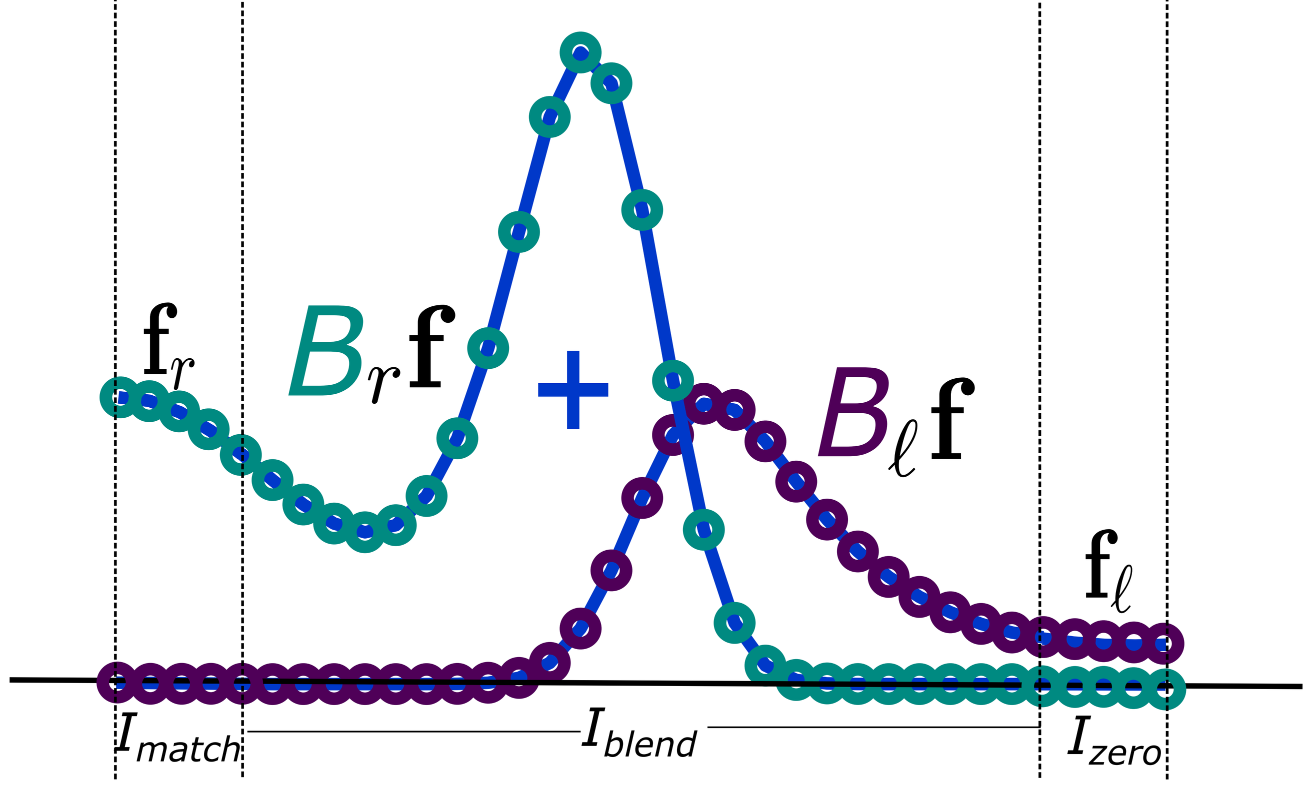


Accelerated Fourier Continuation

Blend-to-Zero Operator. Separate the blending as

$$B = B_\ell + B_r,$$

where B_r preserves $\mathbf{f_r}$ and makes $\mathbf{f}_\ell$ zero while B_ℓ does the opposite.



FC(Gram) Method. The rightward operator B_r projects the function f on a basis of Gram polynomials q_j on I_{match} which are blended by trigonometric interpolants p_j (via least squares) that vanish at I_{zero}:

$$B_r\mathbf{f} = PQ^T\mathbf{f}_r$$
$$P = \begin{bmatrix} p_1(I_{\text{blend}}) & \cdots & p_{d_r}(I_{\text{blend}}) \end{bmatrix}$$

QR Factorization of Vandermonde Matrix on I_{match}	$\longrightarrow$	values of q_j
Singular Value Decomposition of Sinusoidal Basis Matrix on $I_{\text{match}} \cup I_{\text{zero}}$	$\longrightarrow$	coefficients of p_j

High-precision arithmetic and oversampling to mitigate ill-conditioning.

- The leftward operator can be obtained as

$$B_\ell\mathbf{f} = \text{reverse}(PQ^T\text{reverse}(\mathbf{f}_\ell))$$

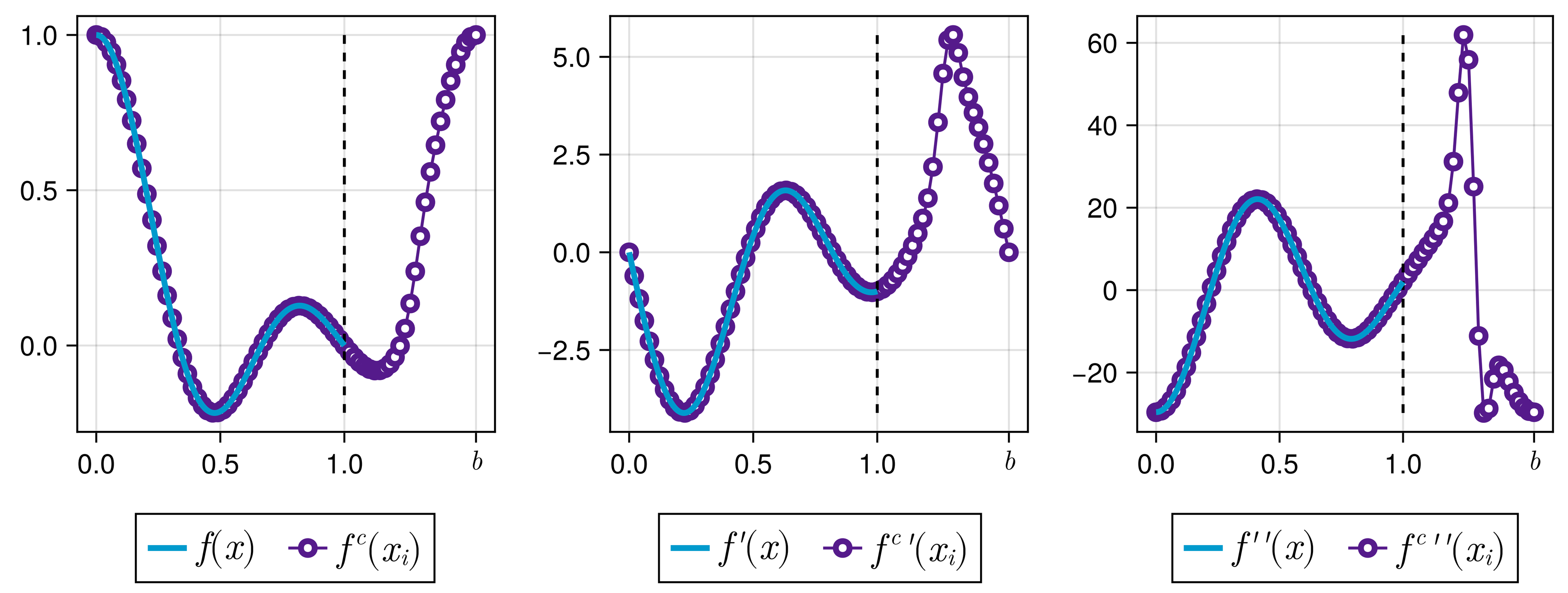
- The matrix PQ^T is independent of the step size and the function.

Higher-Order Spectral Differentiation

The values of higher-order derivatives on a uniform mesh are approximated by FFT-based differentiation of the Fourier Continuation:

$$\mathbf{f}^{(k)} \approx \mathbf{f}^{c(k)} = \text{IFFT}\left(\left(\tfrac{2\pi i}{b}\cdot\text{freq}(\mathbf{f^c})\right)^k \text{filter}(\text{FFT}(\mathbf{f^c}))\right)$$

Fourier Continuation of $f(x) = \text{sinc}(3\pi x)$ on $[0,1]$ and its derivatives:



Solving Wave Equations

A (modified) non-linear viscous Burger's Equation,

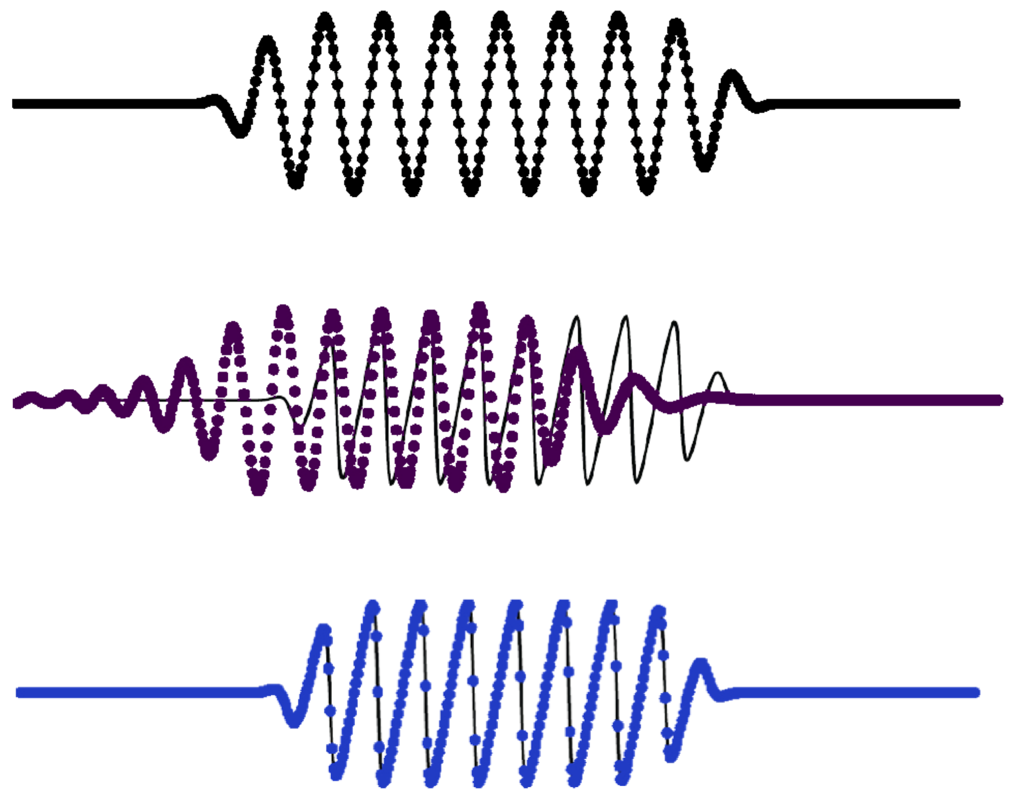
$$u_t + (u+1)u_x = \nu u_{xx}, \quad \nu > 0,$$

subject to initial and Dirichlet boundary conditions on $[0,1]$ is discretized in space by Fourier Continuation and in time by an explicit multistep method to get a numerical solution without numerical dispersion.

Exact Solution
$t = 0$

Finite Differences
$t \gg 0 \quad \mathcal{O}(N)$

Fourier Continuation
$t \gg 0 \quad \mathcal{O}(N\log N)$



Use a confluent Vandermonde matrix in FC(Gram) to treat Neumann boundary conditions.

Future Work

- Higher-order periodic extensions by extrapolation and blending.
- Implementation of 2D and 3D Fourier Continuation by higher-order structured meshing.

References

1. F. Amlani & O. P. Bruno (2016) *J. Comput. Phys. 307:333–354.* `10.1016/j.jcp.2015.11.060`
2. O. P. Bruno et al (2007) *J. Comput. Phys. 227:1094–1125.* `10.1016/j.jcp.2007.08.029`

FourierContinuation.jl